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Boundary value problems on the half-line for a generalised Painlevé II equation

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ABSTRACT

Two types of boundary value problem on the infinite half-line $I = [0, +\infty]$ are investigated for a class of generalised Painlevé II equations. Existence results are obtained via the method of upper and lower solutions together with a diagonal argument.

1. Introduction

Hastings and McLeod [1] investigated a class of boundary value problems on a half-line for a Painlevé-type equation that arises in plasma physics (De Boer and Ludford [2]). Boundary value problems on the half-line $x \in [0, \infty)$ for the integrable Painlevé reduction with behaviour $\approx x^{\frac{1}{2}}$ as $x \to \infty$ have been investigated by Holmes and Spence [3], Heiffer and Weissler [4]. Bass [5], in 1963, derived the Painlevé II equation in the context of a steady electrolysis model arising out of a more general non-steady system [6]. This work was later subsumed in a multi-ion electrodiffusion model by Leuchtag [7]. The Painlevé structure underlying the latter nonlinear system has been recently investigated by Conte et al. [8]. Bass in [5,6] was also concerned with boundary value problems on the half-line $[0, \infty)$ and discussed critically the appropriate boundary conditions to be imposed. However, his analytic treatment only proceeded for a linearised version of the Painlevé II model. In fact, the integrability of the Painlevé II equations allows the sequential application of Bäcklund transformations to generate exact solutions of the Bass model (Rogers et al. [9]). The solvability of certain two-point boundary value problems for the latter has been investigated by Mariani et al. [10]. Recently, the method of upper and lower solutions [11] has been applied by Amster et al. in [12] to establish the solvability of Dirichlet or periodic boundary value problems for a generalised Painlevé equation originally derived in an electrodiffusion context by Leuchtag in [7]. Third-order problems arising in the three-ion case were recently investigated in [13]. Here, it is established that the method of upper and lower solutions may, in fact, be applied to establish the existence of solutions to basic classes of boundary value problems on the half-line for an extension of the Painlevé II equation. It should be noted, however, that the application of this method to such half-line problems is not straightforward. This is due to the fact that, unlike in the case of a bounded interval, the associated linear operator for the half-line problem does not have a compact inverse and a direct application of a Schauder fixed point argument seems inappropriate. This motivates our use here of a diagonal argument, which essentially consists in finding a solution u_N of the equation on the bounded interval $[0, N]$ under an appropriate boundary condition, and then proving the existence of a subsequence that converges to some limit function u, which solves the problem. The choice of this subsequence relies on the computation of accurate *a priori* bounds of the solution u_N restricted to the interval $[0, M]$, with $M \leq N$.

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In the above mentioned works [10,12], only the case of a bounded interval was considered. Here we study the extended Painlevé II$^{\pm}$ type of equations

$$p'' = a|p|^{\gamma} p + b\,|p|^{\delta}\, pp' \pm xp + C, \tag{1}$$

with constants $a > 0\gamma, \delta \geq 0$ and $b, C \in \mathbb{R}$, on the unbounded interval $[0, \infty)$, under the following boundary conditions:

I

$$p(0) = p_0, \qquad \lim_{x\to+\infty} p(x) = 0 \tag{2}$$

II

$$p(0) = p_0, \qquad \lim_{x\to+\infty} \frac{p(x)}{x} = 0. \tag{3}$$

Initially, we proceed with Case I for (1)$^+$. Case II is then dealt with at the end of the paper for problem (1)$^-$, for which the existence of solutions under the stronger condition (1)$^+$ cannot be ensured.

2. Upper and lower solutions and a diagonal argument

Our existence results are derived via the method of lower and upper solutions and a diagonal argument. Let us recall that a smooth function α is deemed to be a lower solution of problems (1)$^{\pm}$ when

$$\alpha'' \geq a\,|\alpha|^{\gamma}\,\alpha + b|\alpha|^{\delta}\alpha\alpha' \pm x\alpha + C.$$

Similarly, a smooth function β is said to be an upper solution of (1)$^{\pm}$ when

$$\beta'' \leq a|\beta|^{\gamma}\beta + b|\beta|^{\delta}\beta\beta' \pm x\beta + C.$$

If moreover $\alpha \leq \beta$, we shall say that (α, β) is an ordered couple consisting of a lower and an upper solution of (1)$^{\pm}$.

In order to investigate the solvability of the boundary value problems (1)$^{\pm}$ under conditions (2) or (3) on the half-line, we first recall a standard existence result for an associated Dirichlet boundary value problem on a bounded interval $[0, T]$ (cf. [11]).

Theorem 2.1. *Let (α, β) be an ordered couple consisting of a lower and an upper solution of the boundary value problem (1)$^+$ (respectively (1)$^+$), and let $p_0 \in [\alpha(0), \beta(0)]$, $p_T \in [\alpha(T), \beta(T)]$. Then, (1)$^+$ (resp. (1)$^-$) admits at least one solution $p : [0, T] \to \mathbb{R}$ with $\alpha \leq p \leq \beta$, satisfying the Dirichlet boundary conditions*

$$p(0) = p_0, \qquad p(T) = p_T.$$

It is noted that the nonlinear lower order term

$$g^{\pm}(x, p, p') := a\,|p|^{\gamma}\, p + b\,|p|^{\delta}\, pp' \pm xp + C$$

satisfies a Nagumo condition [14]: indeed, if $x \in [0, T]$ and $\alpha(x) \leq p \leq \beta(x)$, then

$$|g^{\pm}(x, p, q)| \leq A|q| + B$$

for some constants A and B and the method of upper and lower solutions applies.

Next, we introduce a diagonal argument which demonstrates the existence of a solution for (1)$^{\pm}$ under the boundary conditions (2) or (3).

Theorem 2.2. *Let (α, β) be an ordered couple consisting of a lower and an upper solution of (1)$^+$ (respectively (1)$^-$) such that $\alpha(0) \leq p_0 \leq \beta(0)$.*

Then (1)$^+$ (resp. (1)$^-$) admits at least one solution p with $\alpha \leq p \leq \beta$ such that $p(0) = p_0$. If moreover

$$\lim_{x\to+\infty} \alpha(x) = \lim_{x\to+\infty} \beta(x) = 0,$$

or

$$\lim_{x\to+\infty} \frac{\alpha(x)}{x} = \lim_{x\to+\infty} \frac{\beta(x)}{x} = 0,$$

then p satisfies condition (2) or (3), respectively.

Proof. For any $N \in \mathbb{N}$, consider the Dirichlet boundary value problem

$$\begin{cases} p'' = a\,|p|^{\gamma}\, p + b|p|^{\delta} pp' \pm xp + C & x \in (0, N) \\ p(0) = p_0, \qquad p(N) = (\alpha(N) + \beta(N))/2. \end{cases} \tag{4}$$

By Theorem 2.1, there exists at least one solution p_N of (4) such that $\alpha|_{[0,N]} \leq p_N \leq \beta|_{[0,N]}$. For fixed M and $N \geq M$, set

$$\varphi_N(x) = \frac{p_N(M) - p_0}{M} x + p_0$$

and

$$h_N(x) = -b \int_0^x |p_N(t)|^\delta p_N(t)\, \mathrm{d}t.$$

Then

$$\left(\mathrm{e}^{h_N}(p_N - \varphi_N)'\right)' := \theta_N(x),$$

with $\|\theta_N\|_{C([0,M])} \leq \bar{c}$ for some constant $\bar{c}$ depending only on M. Then

$$\int_0^M \mathrm{e}^{h_N(x)}[(p_N - \varphi_N)']^2(x)\mathrm{d}x = -\int_0^M \theta_N(x)(p_N - \varphi_N)(x)\mathrm{d}x \leq c$$

for some constant c independent of N. As p_N satisfies (4), it is seen that $\|p_N''\|_{L^2(0,M)}$ is bounded. It follows that $\|p_N - \varphi_N\|_{H^2(0,M)}$ is bounded by a constant c_M independent of N.

From the compact embedding $H^2(0, M) \hookrightarrow C^1([0, M])$, there exists a subsequence of $\{p_N\}_{N \geq M}$ that converges on $[0, M]$ for the C^1-norm. Thus, we proceed as follows. Take $M = 1$ and choose a subsequence, still denoted as $\{p_N\}$, which converges in $C^1([0, 1])$ to some function p^1. Repeating the procedure for $M = 2, 3, \ldots$, we may assume that $p_N|_{[0,M]}$ converges to some function p^M in the sense of C^1.

By construction, $p^{M+1}|_{[0,M]} = p^M$; thus, the function $p : [0, +\infty) \to \mathbb{R}$ given by $p(x) = p^M(x)$ if $0 \leq x \leq M$ is well-defined. Moreover, $p(0) = p_0$, and p_N'' converges uniformly in $[0, M]$ to

$$a|p|^\gamma p \pm xp + b|p|^\delta pp' + C.$$

Thus, for any test function $\xi \in C_0^\infty(0, M)$ it is seen that

$$\int_0^M (a|p|^\gamma p \pm xp + b|p|^\delta pp' + C)\xi = \lim_{N\to\infty} \int_0^M p_N''\xi = \lim_{N\to\infty} \int_0^M p_N \xi'' = \int_0^M p\xi'',$$

whence

$$p'' = a\,|p|^\gamma\, p \pm xp + b|p|^\delta pp' + C$$

in $[0, M]$, in the weak sense. By construction, p is continuously differentiable; thus, p is a classical solution of $(1)^\pm$ on $[0, +\infty)$, and the result follows. ■

3. Application to problems $(1)^\pm$

The above Theorem 2.2 is now applied to establish the following:

Theorem 3.1. *The boundary value problem* $(1)^+$–(2) *admits at least one solution.*

Proof. It suffices to construct an ordered couple (α, β) consisting of a lower and an upper solution of $(1)^+$ such that

$$\alpha(0) \leq p_0 \leq \beta(0), \qquad \lim_{x\to+\infty} \alpha(x) = \lim_{x\to+\infty} \beta(x) = 0.$$

Fix a non-increasing function $\psi \in C^2([0, 1])$ such that $0 \leq \psi \leq 1$, and

$$\psi|_{[0,\frac{1}{4}]} \equiv 1, \qquad \psi|_{[\frac{3}{4},1]} \equiv 0.$$

We shall define an upper solution β in the following way. Let $\{c_n\}$ be a non-increasing sequence such that $c_n \to 0$, and define

$$\beta(x) = (c_n - c_{n+1})\psi(x - n) + c_{n+1} \quad \text{for } n \leq x \leq n + 1.$$

It follows that $\beta : [0, +\infty) \longrightarrow [0, +\infty)$ is well-defined, non-increasing and twice continuously differentiable, with $\beta(x) \to 0$ as $x \to +\infty$. Moreover,

$$|\beta'(x)| = -(c_n - c_{n+1})\psi'(x - n) \leq (c_n - c_{n+1})R_1$$

and

$$\beta''(x) = (c_n - c_{n+1})\psi''(x - n) \leq (c_n - c_{n+1})R_2$$

for some positive constants R_1 and R_2. On the other hand,

$$a\beta(x)^{\gamma+1} + x\beta(x) + b\beta^{\delta+1}\beta' + C \geq ac_{n+1}^{\gamma+1} + nc_{n+1} - |b|c_0^{\delta+1}(c_n - c_{n+1})R_1 + C.$$

Thus, if for every n

$$ac_{n+1}^{\gamma+1} + (n+K)c_{n+1} + C \geq c_n K,$$

where $K = R_2 + |b|c_0^{\delta+1}R_1$, then β is an upper solution. For instance, we may consider $c_0 \geq \max\{p_0, 0\}$ such that $ac_0^{\gamma+1} + C \geq 0$, and fix n_0 such that

$$\frac{n+K}{\sqrt{n+1}}c_0 + C \geq c_0 K$$

for all $n \geq n_0$. It suffices to define

$$c_n = \begin{cases} c_0, & n \leq n_0 \\ \dfrac{c_0}{\sqrt{n}}, & n > n_0. \end{cases}$$

In the same way, we may define a lower solution $\alpha \leq 0$, and Theorem 2.2 applies. ■

Remark I. If $b = 0$, then the solution given by Theorem 3.1 is unique. Indeed, if p_1, p_2 are two solutions, define $W = p_1 - p_2$. Then, W satisfies

$$W'' = \phi(x)W + xW, \qquad W(0) = \lim_{x\to+\infty} W(x) = 0,$$

where $\phi(x) = a(\gamma+1)|\xi(x)|^\gamma$ for some mean value $\xi(x)$ between $p_1(x)$ and $p_2(x)$.

If $W(x_0) > 0$ for some $x_0 > 0$, we may assume that x_0 is a maximum, and so $W''(x_0) \leq 0$, a contradiction. It follows that $W \leq 0$, and in the same way we deduce that $W \geq 0$.

Remark II. If $\delta = 0$ and $\gamma = 2$ we recover the two-ion electrodiffusion model set down in Leuchtag [7]. In this case, an ordered couple (α, β) is readily obtained as follows. Set

$$\beta(x) = \frac{d}{x+K},$$

for some positive constants K and d, with d large enough that $\frac{d}{K} \geq p_0$, and that the inequality

$$\left[2 + bd - \frac{1}{2}ad^2\right]\frac{d}{(x+K)^3} - \frac{x}{x+K} + \frac{C}{d} \leq 0$$

is satisfied. In the same way, taking $d \ll 0$, a lower solution is constructed. The specific choice of the constants $a = 2$ and $b = 0$ yields the well-known Painlevé II^+ equation

$$p'' = 2p^3 + xp + C.$$

To conclude, we consider the boundary value problem (1). In this case, it is seen that the construction of the previous theorem fails; however, existence of solutions can be proved under the weaker condition of sub-linearity at infinity (3). The following result is established:

Theorem 3.2. *Assume that $\gamma > \max\{1, \delta\}$. Then the boundary value problem $(1)^-$, (3) admits at least one solution.*

Proof. It suffices to construct an ordered couple (α, β) consisting of a lower and an upper solution of $(1)^-$ satisfying

$$\alpha(0) \leq p_0 \leq \beta(0), \qquad \lim_{x\to+\infty}\frac{\alpha(x)}{x} = \lim_{x\to+\infty}\frac{\beta(x)}{x} = 0.$$

Let ψ be defined as in Theorem 3.1, and consider the function given by

$$\beta(x) = [c_n(x) - c_{n+1}(x)]\psi(x-n) + c_{n+1}(x) \quad \text{for } n \leq x \leq n+1,$$

where $c_n(x) = r_n x + s_n$ for some positive numbers

$$r_0 \geq r_1 \geq \cdots \to 0$$

and

$$s_0 \leq s_1 \leq \cdots \to +\infty$$

to be chosen. A simple computation shows that if $x \in [n, n+1]$ then

$$r_{n+1}x + s_n \le \beta(x) \le r_n x + s_{n+1},$$
$$|\beta'(x)| \le R_1[(r_n - r_{n+1})x + s_{n+1} - s_n] + r_n,$$

and

$$\beta''(x) \le R_2[(r_n - r_{n+1})x + s_{n+1} - s_n]$$

for some positive constants R_1, R_2.

For convenience, write

$$a\beta^{\gamma+1} - x\beta + b\beta^{\delta+1}\beta' + C = \beta\left(\frac{a}{2}\beta^{\gamma} - x\right) + \beta^{\delta+1}\left(\frac{a}{2}\beta^{\gamma-\delta} + b\beta'\right) + C,$$

and observe that if $x \in [n, n+1]$, then

$$\beta^{\gamma} \ge s_n^{\gamma}\left(\frac{r_{n+1}}{s_n}x + 1\right)^{\gamma} \ge s_n^{\gamma}\left(\gamma\frac{r_{n+1}}{s_n}x + 1\right) = s_n^{\gamma} + \gamma r_{n+1}s_n^{\gamma-1}x.$$

Thus, we may fix some appropriate value of $n_0 \in \mathbb{N}$ to be established and define

$$s_n = \begin{cases} \sqrt{n_0}, & n \le n_0 \\ \sqrt{n}, & n > n_0, \end{cases}$$
$$r_0 = r_1, \qquad r_{n+1} = \frac{2}{a\gamma s_n^{\gamma-1}}.$$

Then we have, for $x \in [n, n+1]$,

$$a\beta^{\gamma+1} - x\beta + b\beta^{\delta+1}\beta' + C \ge \frac{a}{2}(r_{n+1}x + s_n)s_n^{\gamma} + C,$$

provided that

$$\frac{a}{2}\beta^{\gamma-\delta}(x) + b\beta'(x) \ge 0. \tag{5}$$

If $n \ge n_0$, a simple computation shows that

$$s_{n+1} - s_n = \frac{1}{\sqrt{n} + \sqrt{n+1}},$$

and

$$r_n - r_{n+1} \le \frac{\gamma - 1}{a\gamma n^{(\gamma+1)/2}}.$$

Hence, if n_0 is large enough, then (5) is satisfied, and moreover

$$\beta'' \le \frac{a}{2}(r_{n+1}x + s_n)s_n^{\gamma} + C$$

on $[n_0, +\infty)$. Enlarging n_0 if necessary, in order to satisfy

$$n_0 \ge p_0^2,$$
$$\frac{a}{2}n_0^{(\gamma+1)/2} + C \ge 0,$$
$$\frac{a}{2}n_0^{(2\gamma-\delta-1)/2} + \frac{2b}{a\gamma} \ge 0,$$

it follows that β is also an upper solution over $[0, n_0]$, and $\beta(0) = \sqrt{n_0} \ge p_0$. Furthermore, for $[x] = n \ge n_0$

$$\frac{\beta(x)}{x} \le r_n + \frac{\sqrt{n+1}}{n} \to 0$$

as $n \to +\infty$. In a similar way, we define a lower solution $\alpha : [0, +\infty) \to \mathbb{R}_{\le 0}$ such that $\alpha(0) \le p_0$, and $\lim_{x\to\infty} \frac{\alpha(x)}{x} = 0$. The result now follows. ■

References

[1] S.P. Hastings, J.B. McLeod, A boundary value problem associated with the second Painlevé transcendent and the Korteweg–de Vries equation, Arch. Ration. Mech. Anal. 73 (1980) 31–51.

[2] P.C.T. De Boer, G.S.S. Ludford, Spherical electric probe in a continuum gas, Plasma Phys. 17 (1975) 29–43.

[3] P. Holmes, D. Spence, On a Painlevé-type boundary value problem, Q. J. Mech. Appl. Math. 37 (1984) 525–538.

[4] B. Heiffer, F.B. Weissler, On a family of solutions of the second Painlevé equation related to superconductivity, Eur. J. Appl. Math. 9 (1998) 223–243.
[5] L. Bass, Electrical structures of interfaces in steady electrolysis, Trans. Faraday Soc. 60 (1964) 1656–1663.
[6] L. Bass, Irreversible interactions between metals and electrolytes, Proc. Roy. Soc. London 277 (A) (1964) 125–136.
[7] H.R. Leuchtag, A family of differential equations arising from multi-ion electrodiffusion, J. Math. Phys. 22 (1981) 1317–1320.
[8] R. Conte, C. Rogers, W.K. Schief, Painlevé structure of a multi-ion electrodiffusion system, J. Phys. A 40 (2007) F1031–F1040.
[9] C. Rogers, A.P. Bassom, W.K. Schief, On a Painlevé II model in steady electrolysis: Application of a Bäcklund transformation, J. Math. Anal. Appl. 240 (1999) 367–381.
[10] M.C. Mariani, P. Amster, C. Rogers, Dirichlet and periodic-type boundary value problems for Painlevé II, J. Math. Anal. Appl. 265 (2002) 1–11.
[11] C. de Coster, P. Habets, Two-point boundary value problems: lower and upper solutions, in: Mathematics in Science and Engineering, Elsevier, 2006.
[12] P. Amster, M.C. Mariani, C. Rogers, C.C. Tisdell, On two-point boundary value problems in multi-ion electrodiffusion, J. Math. Anal. Appl. 289 (2004) 712–721.
[13] P. Amster, C. Rogers, On boundary value problems in three-ion electrodiffusion, J. Math. Anal. Appl. 333 (2007) 42–51.
[14] M. Nagumo, Uber die differentialgleichung $y'' = f(t, y, y')$, Proc. Phys.-Math. Soc. Japan 19 (1937) 861–866.