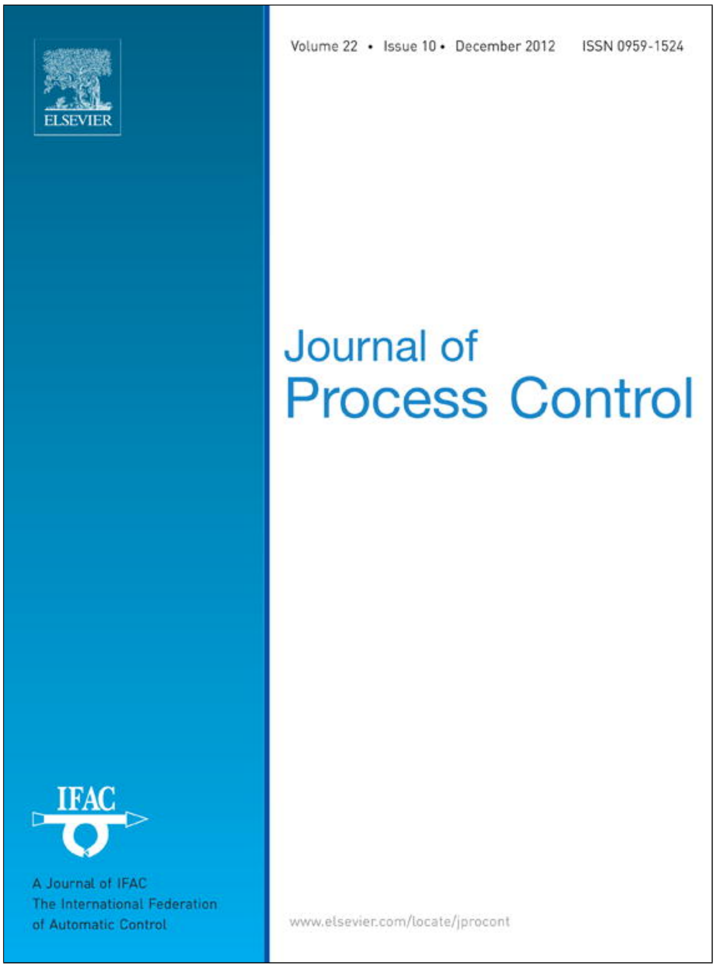
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Robust MPC for tracking zone regions based on nominal predictions ☆

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ABSTRACT

This paper deals with the problem of robust tracking of target sets using a model predictive control (MPC) law. Real industries applications often require a control strategy in which some system outputs are controlled within specified ranges or zones (zone control), while some others variables – possibly including input variables – are steered to fixed target or setpoint. From a theoretical point of view, the control objective of this kind of problem can be seen as a target set (in the output space) instead of a target point, since inside the zones there are no preferences between one point or another. This problem is particularly interesting in case of additive disturbances which might push the outputs out of the zones. In this work, a stable robust MPC formulation for constrained linear systems, based on nominal predictions is presented. The main features of this controller are the use of nominal predictions, restricted constraints and the concept of distance from a point to a set as offset cost function. The controller ensures both recursive feasibility and local optimality. The properties of the controller are shown in a simulation test, in which we consider a subsystem of an industrial FCC system.

1. Introduction

1.1. Set-interval control

In modern processing plants, MPC controllers are usually implemented as part of a multilevel hierarchy of control functions [1,2]. At the intermediary levels of this control structure, the process unit optimizer computes an optimal economic steady state and provides this information to the MPC in a lower level for implementation. The role of the MPC is then to drive the plant to the most profitable operating condition, fulfilling the constraints and minimizing the dynamic error along the path.

In many cases, the optimal economic steady state operating condition is not given by a point in the output space (fixed setpoint), but by a region or zone into which the output should lie most of the time. Conceptually, these output zones can be seen as a generalization of the output targets from a point to a set (i.e., a generalized setpoint) rather than an output constraint, since they are desired steady state sets that can be transitorily disregarded, while the constraints must be fulfilled at each time step. In this way, the concept of degrees of freedom is substantially altered. In fact, it is generalized in such a way that even systems with more outputs than inputs allow (economic) targets for some inputs. A kind of hierarchy of objectives arises in the MPC control problem, in which the first one is to find a feasible solution (i.e. one that fulfills the input and output constraints), the second one is to reach and maintain the outputs inside their corresponding zones and the third one is to steer the inputs as close as possible to the desired economic targets. Only once a higher priority objective is reached, the remainder "*degrees of freedom*" can be used to reach the lower one.

From the point of view of real systems, the zone control may appear in different kind of dynamic systems: (i) process systems with highly correlated outputs to be controlled, in which there are not enough inputs to control all the outputs; (ii) process systems with problems to use the surge capacity of tanks to smooth out the operation of a process unit (in this case, it is desired to let the level of the tank to float between limits, as necessary, to buffer disturbances between sections of a plant); (iii) biological systems, such as diabetes patient, in which tracking a given output setpoint (glycemic level) could demand an excessive and unnecessary control effort (insulin administration) while maintaining the glycemic level in a given safety interval is sufficient to guaranty the control objectives [3].

Several approaches have been proposed to account for the MPC set-interval or zone control. [4] shows how some commercial MPC controllers are adapted to account for the zone control problem,

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including the so-called *funnel* strategy. In [5] and [3] it can be found simple approaches to tackle the zone control problem: they penalize an output into the MPC cost function only if it is inside the zone. Although this kind of switching control has shown to be plausible to be applied in real control systems (as diverse as process system control and biological system control), stability and recursive feasibility cannot be proved under their formulation framework. A closed-loop stable and recursively feasible MPC controller is presented in [6]. In this approach, the authors develop a controller that incorporates steady state economic targets for input and output in the control cost function. Assuming open-loop stable systems, classical stability proofs are extended to the zone control strategy by considering artificial output setpoints as additional decision variables. This controller, however, is formulated only for open-loop stable systems, and since it considers a null controller as local controller, it does not achieve local optimality.

1.2. Robust set-interval control

The explicit consideration of model uncertainties or disturbances is quite different in the context of set-interval control. Since the uncertainty affects the system gains, it also affects the compatibility between the available input set (given by the input constraints) and the desired output set (given by the output zones). Thus, an efficient robust design should take these problems into account in order to avoid unfeasibilities, even unfeasibilities at steady state. An extension to the robust case of the strategy presented in [6] (considering multi-model uncertainty) was proposed in [7]. Although these approaches account for the tracking of non-zero targets and the second one considers time delayed systems, they also fail to guarantee local optimality and they are only formulated for open-loop stable systems.

1.3. MPC and the tracking problem

Most of the rigorous MPC stability, feasibility and optimality results consider the regulation problem, that is steering the system to a fixed steady state (typically the origin) [8,9]. If, for a given non-zero set point, a suitable choice of the steady state is taken, the problem can be posed as a regulation problem translating the state and input of the system [10]. The steady state target is usually determined by solving an optimization problem that can be formulated as different mathematical programs for the cases of perfect target tracking or non-square systems [11], or by solving a unique problem for both situations [12]. However, since the stabilizing choice of the terminal cost and constraints depends on the desired steady state, when the target operating point changes, the feasibility of the controller may be lost and the controller fails to track the reference [13–16], thus requiring to re-design the MPC at each change of the reference.

In [17], a rigorous MPC formulation for tracking is proposed, which is able to steer the system to any admissible setpoint in an admissible way, by considering the steady conditions as optimization variable of the MPC problem. This controller ensures both, recursive feasibility and convergence to the target (if admissible) for any change of the steady state target. Furthermore, if the target is not admissible, the system is steered to the closest admissible steady state. In [18], the MPC for tracking is extended considering a general offset cost function. Under some mild sufficient assumptions, the new offset cost function ensures the local optimality property, letting the controller achieve optimal closed-loop performance. In [19] this controller is extended to the case of tracking target sets (a generalized set-interval control) by using the concept of distance of a point to a set. In contrast to the approach presented in [6], this strategy allows local optimality an it is suitable for non-stable systems.

In this paper, the controller presented in [19] is extended to cope with the problem of robust tracking of target sets in presence of additive disturbance. Although here we consider a different uncertainty representation than the one used in [7], the proposed controller constitutes an improved robust MPC formulation for the zone control problem (i.e. a robust control suitable for non-stable systems, which preserves local optimality). Based on some of the results presented in [20], we propose here an MPC based on nominal predictions and restricted constraints, which ensures stability, robust satisfaction of the constraints, recursive feasibility and local optimality.

The paper is organized as follows. In Section 2 the control problem is stated. Sections 3 and 4 present the proposed controller and its main properties, respectively. In Section 5 the properties of the controller are shown in a simulation test, in which we consider a subsystem of an industrial FCC system. Finally, in Section 6, some conclusions are drawn.

Notation: A positive definite symmetric matrix T is denoted as $T>0$ and $T>P$ denotes that $T-P>0$. For a given symmetric matrix $P>0$, $\|x\|_P$ denotes the weighted Euclidean norm of x, i.e. $\|x\|_P = \sqrt{x'Px}$. Consider $a \in \mathbf{R}^{n_a}$ and $b \in \mathbf{R}^{n_b}$, the vector made from stacking both vectors is defined as $(a, b) \triangleq [a', b']' \mathbf{R}^{n_a+n_b}$; for a set $\Gamma \subset \mathbf{R}^{n_a+n_b}$, the projection of Γ onto a is defined as $Proj_a(\Gamma) = \{a \in \mathbf{R}^{n_a} : \exists b \in \mathbf{R}^{n_b}, (a, b) \in \Gamma\}$. A vector $\mathbf{t}$ in bold denotes a finite sequence of vectors, that is, a vector defined as $\{t(0), t(1), \ldots, t(N)\}$, where N is deduced from the context. The norm of a signal $\mathbf{t}$ is defined as $\|t\|_\infty = \sup_{k\geq 0}(t(k))$. A matrix $\mathbf{0}_{n,m} \in \mathbf{R}^{n\times m}$ denotes a matrix of zeros and $I_n \in \mathbf{R}^{n\times n}$ denotes the identity matrix. Given two sets $\mathcal{U}$ and $\mathcal{V}$, such that $\mathcal{U} \subset \mathbf{R}^n$ and $\mathcal{V} \subset \mathbf{R}^n$, the Minkowski sum is defined by $\mathcal{U} \oplus \mathcal{V} \triangleq \{u+v : u \in \mathcal{U}, v \in \mathcal{V}\}$, the Pontryagin set difference is: $\mathcal{U} \ominus \mathcal{V} \triangleq \{u : u \oplus \mathcal{V} \subseteq \mathcal{U}\}$; given a matrix $M \in \mathbf{R}^{p\times n}$, the set $M\mathcal{U} \subset \mathbf{R}^p$ is defined as $M\mathcal{U} \triangleq \{Mu : u \in \mathcal{U}\}$; for a given λ, $\lambda\mathcal{U} \triangleq (\lambda I_n)\mathcal{U}$.

2. Problem statement

Consider a plant described by the following uncertain discrete-time LTI system

$$
\begin{aligned}
x^+ &= Ax + Bu + w \\
y &= Cx + Du
\end{aligned} \tag{1}
$$

where $x \in \mathbb{R}^n$ is the state of the system at the current time instant, x^+ denotes the successor state, that is, the state of the system at next sampling time, $u \in \mathbb{R}^m$ is the manipulated control input, $y \in \mathbb{R}^p$ is the controlled variables and $w \in \mathbb{R}^n$ is an unknown but bounded state disturbance. In what follows, $x(k)$, $u(k)$, $y(k)$ and $w(k)$ denote the state, the manipulable variable, controlled variable and the disturbance respectively, at sampling time k.

The plant is subject to hard constraints on state and control:

$$
(x(k), u(k)) \in \mathcal{Z} \tag{2}
$$

where $\mathcal{Z} = \mathcal{X} \times \mathcal{U}$ is a compact convex polyhedron containing the origin in its interior.

Define also the plant nominal model, given by (1) neglecting the disturbance input w:

$$
\begin{aligned}
\bar{x}^+ &= A\bar{x} + B\bar{u} \\
\bar{y} &= C\bar{x} + D\bar{u}
\end{aligned} \tag{3}
$$

The plant model is assumed to fulfil the following assumption:

Assumption 1.

- The pair (A, B) is controllable.

- The uncertainty vector w is bounded and lies in a compact convex polyhedron containing the origin in its interior

$$\mathcal{W} = \{w \in \mathbb{R}^n : A_w w \le b_w\} \quad (4)$$

- The state of the system is measured, and hence $x(k)$ is known at each sample time.

It is remarkable that no assumption is considered on the number of inputs m and outputs p, allowing thin plants ($p > m$), square plants ($p = m$) and flat plants ($p < m$). Moreover, it is not assumed that (A, B, C, D) is a minimal realization of the state-space model. This allows us to use state-space models derived from input–output models, that is, using as state a collection of past inputs and outputs of the plant [8]. The necessity of an observer is also avoided while the global uncertainty and the noise can be posed as additive uncertainties in the state-space model (1).

The aim of this paper is to find a control law $u(k) = \kappa_N(x(k), \Gamma_t)$ such that the system is steered into a (possibly time varying) region Γ_t, which defines the range into which the controlled outputs should remain fulfilling the plant constraints $(x(k), u(k)) \in \mathcal{Z}$, despite the uncertainties.

3. Robust MPC for tracking zone regions based on nominal predictions

In this section the proposed controller is presented. The proposed controller is an extension to the robust case of the MPC for tracking zone regions [19], using the concepts presented in [20].

3.1. Preliminaries: the robust MPC based on nominal predictions and restricted constraints

The keystone of the robust MPC presented in [20] is to use predictions based on the nominal system for the MPC cost (i.e., predictions that neglect the disturbance input w), and to restrict the constraints set $\mathcal{X}$ and $\mathcal{U}$ at any step of the prediction horizon.

The controller is based on a pre-stabilization of the plant using a state feedback control gain K, such that $A_K = A + BK$ has all its eigenvalues in the interior of the unit circle. The controlled system is then given by

$$x(k+1) = A_K x(k) + Bc(k) + w(k)$$
$$u(k) = Kx(k) + c(k)$$

Neglecting the disturbances w, the nominal prediction model is then given by:

$$\bar{x}(k+1) = A_K \bar{x}(k) + Bc(k)$$
$$\bar{u}(k) = K\bar{x}(k) + c(k)$$

The notion of robust positively invariant (RPI) set [21,22] plays an important role in the design of robust controllers for constrained systems. This is defined as follows:

Definition 1. A set Ω is called a robust positively invariant (RPI) set for the uncertain system $x(k+1) = A_K x(k) + w(k)$ with $w(k) \in \mathcal{W}$ if $A_K \Omega \oplus \mathcal{W} \subseteq \Omega$.

It is also necessary to define the so-called reachable sets, that represents the forced response of the system due to the uncertainty.

Definition 2. The reachable set in j steps, $\mathcal{R}_j$, is given by

$$\mathcal{R}_j \triangleq \bigoplus_{i=0}^{j-1} A_K^i \mathcal{W}$$

This is the set of states of the nominal closed-loop systems which are reachable in j steps from the origin, under the disturbance input w [20]. This set satisfies the following properties:

(i) It is given by the recursion $\mathcal{R}_j \oplus A_K^j \mathcal{W} = \mathcal{R}_{j+1}$ with $\mathcal{R}_1 = \mathcal{W}$.
(ii) $A_K \mathcal{R}_j \oplus \mathcal{W} = \mathcal{R}_{j+1} = \mathcal{R}_j \oplus A_K^j \mathcal{W}$
(iii) $\mathcal{R}_j \subseteq \mathcal{R}_{j+1}$
(iv) The sequence of sets $\mathcal{R}_j$ has a limit $\mathcal{R}_\infty$ as $j \to \infty$, and $\mathcal{R}_\infty$ is a robust positive invariant set.
(v) $\mathcal{R}_\infty$ is the minimal RPI set.

Based on this, the sets of restricted constraints on the nominal predictions considered in the optimization problem are given by:

$$\bar{\mathcal{X}}_j \triangleq \mathcal{X} \ominus \mathcal{R}_j$$
$$\bar{\mathcal{U}}_j \triangleq \mathcal{U} \ominus K\mathcal{R}_j \quad (5)$$

These sets are non-empty if the following assumption holds

Assumption 2. $\mathcal{R}_\infty \subset \mathcal{X}$ and $K\mathcal{R}_\infty \subset \mathcal{U}$.

Remark 1. The calculation of the restricted constraints $\mathcal{X} \ominus \mathcal{R}_j$ is not an easy task, due to the complexity of the calculation of the set $\mathcal{R}_j$, based on a series of Minkowsky's sums. The computational burden can be reduced if the set $\mathcal{W}$ is given by an interval or an affine map of an hypercube [23]. It is also important to note that the calculation of such sets is made off-line, so it has no practical effects on the MPC problem.

Remark 2. An important parameter on the design of this controller, is the control gain K. This parameter determines the dynamic of the closed-loop system in presence of disturbances and hence, it has to ensure that Assumption 2 holds. In [24] it is proposed an LMI-based method for the calculation of the control gain K which ensures that Assumption 2 holds and that the set $\mathcal{R}_\infty$ is minimized.

3.2. Preliminaries: characterization of the steady state

Every nominal steady state and input $z_s = (x_s, u_s)$ is a solution of the equation

$$[A - I_n \quad B] \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \mathbf{0}_{n,1} \quad (6)$$

Therefore, there exists a matrix $M_\theta \in \mathbb{R}^{(n+m)\times m}$ such that every nominal steady state and input can be posed as

$$z_s = M_\theta \theta \quad (7)$$

for certain $\theta \in \mathbb{R}^m$ [17]. The nominal steady outputs are then given by

$$y_s = N_\theta \theta \quad (8)$$

where $N_\theta \triangleq [C\,D] M_\theta$.

Defining $\bar{\mathcal{Z}} \triangleq \bar{\mathcal{X}}_N \times \bar{\mathcal{U}}_N$, the set of admissible nominal steady states and inputs and the set of admissible nominal controlled variables are given by

$$\bar{\mathcal{Z}}_s \triangleq \{(x, u) \in \bar{\mathcal{Z}} : (A - I_n)x + Bu = \mathbf{0}_{n,1}\}$$
$$\bar{\mathcal{Y}}_s \triangleq \{Cx + Du : (x, u) \in \lambda \bar{\mathcal{Z}}_s\}$$

where $\lambda \in (0, 1)$ is a given parameter added to avoid those steady states and inputs that provide active constraints.

3.3. The proposed controller

As in [19], the proposed controller maintains the main ingredients of the MPC for tracking target sets: the steady state conditions of the system are decision variables in the optimization problem (artificial reference), the stage cost is a measure of the distance to the artificial reference, the so-called *offset cost* function is added in order to penalize the deviation between the artificial reference and the target, the terminal constraint is an invariant set for tracking.

The proposed controllers is derived following the results presented in Section 3.1. Therefore, the plant is pre-stabilized by the following control law

$$u(k) = Kx(k) + L\theta + c(k) \tag{9}$$

where $L = [-K\, I_m]M_\theta$. Then the nominal system can be rewritten as follows:

$$\begin{aligned} \bar{x}^+ &= A_K\bar{x} + BL\theta + Bc \\ \bar{u} &= Kx + L\theta + c \end{aligned} \tag{10}$$

The cost function to minimize is given by:

$$V_N(x, \Gamma_t; \mathbf{c}, \theta) \triangleq \sum_{j=0}^{N-1} \|c(j)\|_\Psi^2 + V_O(y_s, \Gamma_t) \tag{11}$$

where $\mathbf{c} = \{c(0), c(1), \ldots, c(N-1)\}$, $\Psi = \Psi' > 0$, the pair $(x_s, u_s) = M_\theta\theta$ is the artificial steady state and input and $y_s = N_\theta\theta$ the artificial output, all of them parameterized by θ; Γ_t is the zone in which the controlled variables have to be steered. $V_O(y_s, \Gamma_t)$ is the so-called offset cost function and it is such that the following assumption is ensured

Assumption 3.

1. Γ_t is a compact convex set.
2. $V_O(y_s, \Gamma_t)$ is convex w.r.t. y_s.
3. If $y_s \in \Gamma_t$, then $V_O(y_s, \Gamma_t) \geq 0$. Otherwise, $V_O(y_s, \Gamma_t) > 0$.

Remark 3. Following same arguments as in [20,25], it is possible to prove that, in the case that K is the gain of the LQR, minimizing $V_N(x, \Gamma_t; \mathbf{c}, \theta)$ is equivalent to minimizing the following cost function

$$\tilde{V}_N(x, \Gamma_t; \mathbf{c}, \theta) = \sum_{j=0}^{N-1} \|\bar{x}(j) - x_s\|_Q^2 + \|\bar{u}(j) - u_s\|_R^2 + \|\bar{x}(N) - x_s\|_P^2 + V_O(y_s, \Gamma_t) \tag{12}$$

where $\bar{x}(j)$ is the nominal prediction of the model for $\bar{u}(j) = K\bar{x}(j) + L\theta + c(j)$; Q is a real symmetric positive semidefinite matrix such that the couple $(Q^{1/2}, A)$ is detectable; R is a real symmetric positive definite matrix; P is the unique solution of the Riccati equation

$$(A + BK)'P(A + BK) - P = -(Q + K'RK)$$

In fact, if Ψ is chosen as $\Psi = R + B'PB$, the equivalence between cost (11) and (12) holds since

$$\tilde{V}_N(x, \Gamma_t; \mathbf{c}, \theta) = V_N(x, \Gamma_t; \mathbf{c}, \theta) + \|\bar{x}(0) - x_s\|_P^2$$

Then, taking $K = K_{LQR}$, minimizing the cost (11) is equivalent to minimize the cost of the predicted nominal trajectory.

The optimization problem $P_N(x, \Gamma_t)$ is now given by:

$$\begin{aligned} \min_{\mathbf{c},\theta} \quad & V_N(x, \Gamma_t; \mathbf{c}, \theta) \\ s.t. \quad & \bar{x}(0) = x, && (13) \\ & \bar{x}(j+1) = A\bar{x}(j) + B\bar{u}(j),\ j \in \mathbb{I}_{[0,N-1]} && (14) \\ & \bar{u}(j) = K\bar{x}(j) + L\theta + c(j),\ j \in \mathbb{I}_{[0,N-1]} && (15) \\ & \bar{x}(j) \in \bar{\mathcal{X}}_j, \quad j \in \mathbb{I}_{[0,N-1]} && (16) \\ & \bar{u}(j) \in \bar{\mathcal{U}}_j, \quad j \in \mathbb{I}_{[0,N-1]} && (17) \\ & y_s = N_\theta\theta && (18) \\ & (\bar{x}(N), \theta) \in \Omega_t^a && (19) \end{aligned}$$

where Ω_t^a is a suitable polyhedral set. Notice that the decision variables are: (i) the sequence of the future actions of the nominal system $\mathbf{c}$ and (ii) the parameter vector θ that determines the artificial target steady state, input and output (x_s, u_s, y_s).

Considering the receding horizon policy, the control law is given by

$$\kappa_N(x, \Gamma_t) \triangleq Kx + L\theta^0(x, \Gamma_t) + c^0(0; x, \Gamma_t)$$

where $c^0(0; x, \Gamma_t)$ is the first element of the control sequence $\mathbf{c}^0(x, \Gamma_t)$ which is the optimal solution of problem $P_N(x, \Gamma_t)$. Notice also that, in the following, the optimal value of the cost function will be denoted as $V_N^0(x, \Gamma_t)$, the optimal value of the other decision variable as $\theta^0(x, \Gamma_t)$, the nominal optimal state trajectory as $\bar{\mathbf{x}}^0(x, \Gamma_t)$ and the optimal artificial reference $(x_s^0(x, \Gamma_t), u_s^0(x, \Gamma_t), y_s^0(x, \Gamma_t))$.

Since the set of constraints of $P_N(x, \Gamma_t)$ does not depend on Γ_t, its feasibility region does not depend on the target region Γ_t. The feasible set of the proposed controller is a polyhedral region $\mathcal{X}_N \subseteq \mathbb{R}^n$ given by the set of initial states that can be steered into $\Omega_t = Proj_x(\Omega_t^a)$ in N steps fulfilling the constraint (16), for all admissible disturbances.

3.4. Stability of the proposed controller

Consider the following assumption on the controller parameters:

Assumption 4.

1. Define the extended state $x_a = (x, \theta)$, and

$$A_a = \begin{bmatrix} A + BK & BL \\ 0 & I_m \end{bmatrix}$$

where $L = [-K\, I_m]M_\theta$. Define also

$$\mathcal{X}_a^i = \{(x, \theta) : x \in \bar{\mathcal{X}}_i,\ Kx + L\theta \in \bar{\mathcal{U}}_i,\ M_\theta\theta \in \lambda\mathcal{Z}_s\}$$

and

$$\Sigma_t = \{x_a : A_a^i x_a \in \mathcal{X}_a^i, \quad \text{for } i \geq 0\}$$

Then

$$\Omega_t^a = \Sigma_t \ominus (\mathcal{R}_N \times \{0\})$$

In the following theorem, stability and constraints satisfaction of the controlled system are stated.

Theorem 1 (Stability). *Consider that Assumptions 1–4 hold and consider a given target operation zone Γ_t. The system controlled by the proposed MPC controller $\kappa_N(x, \Gamma_t)$ is such that:*

(i) *For all initial condition $x(0) \in \mathcal{X}_N$ and for every Γ_t, the evolution of the system is robustly feasible and admissible, that is, $x(j) \in \mathcal{X}_N$ and $(x(j), \kappa_N(x(j), \Gamma_t)) \in \mathcal{Z}$, $\forall w(k) \in \mathcal{W}$, $k = 0, 1, \cdots, j-1$.*

(ii) $\lim_{k\to\infty} c(k) = 0$

(iii) *If $\Gamma_t \cap \bar{\mathcal{Y}}_s \neq \emptyset$ then the closed-loop system asymptotically converges to a set $\bar{y}(\infty) \oplus (C + DK)\mathcal{R}_\infty$, such that $\bar{y}(\infty) \in \Gamma_t$.*

(iv) *If $\Gamma_t \cap \bar{\mathcal{Y}}_s = \emptyset$, the closed-loop system asymptotically converges to a set $y_s^* \oplus (C + DK)\mathcal{R}_\infty$, where y_s^* is the reachable nominal steady output such that*

$$y_s^* \triangleq \arg\min_{y_s \in \bar{\mathcal{Y}}_s} V_O(y_s, \Gamma_t)$$

4. Properties of the proposed controller

The proposed controller is a robust formulation of the MPC for tracking target sets presented in [19]. As a consequence, it inherits all the good properties of that controllers:

- **Steady state optimization**. The offset cost function can be considered as a steady state target optimizer (SSTO) built in the same MPC, since the proposed controller drives the system to a neighborhood of the optimal operating point minimizing the offset cost function $V_O(y_s, \Gamma_t)$.
- **Feasibility for any reachable target zone**. Since the set of constraint of the proposed controller does not depend on the target set Γ_t, feasibility is ensured for any Γ_t and for any prediction horizon N. Therefore, if the initial condition is an admissible equilibrium point, the proposed controller is able to drive the system to any admissible target zone (i.e. $\Gamma_t \cap \overline{\mathcal{Y}}_s \neq \emptyset$) even for $N = 1$.
  Moreover, if Γ_t varies with the time, the results of Theorem 1 still hold.
- **Input target**. The proposed controller can be formulated considering input targets of the form $u_{min} \leq u_t \leq u_{max}$, by defining an offset cost function $V_O(u_s, \Gamma_{u,t})$ convex w.r.t. u_s, where $\Gamma_{u,t}$ is a convex polyhedron.
- **Enlargement of the domain of attraction**. The terminal constraint of the proposed controller is an invariant set for *any* equilibrium point. In standard MPC, the invariant set is calculated for a fixed equilibrium point. Therefore, the terminal constraint, and as a consequence the domain of attraction of the proposed controller are (potentially) larger than in standard MPC. This property allows to consider small values of the control horizon.
- **Optimization problem posed as a QP**. Since all the ingredients (functions and sets) of the optimization problem $P_N(x, \Gamma_t)$ are convex, then it derives that $P_N(x, \Gamma_t)$ is a convex mathematical programming problem that can be efficiently solved in polynomial time by specialized Algorithms [26,27]. As in [19], this problem can be re-casted as a standard QP problem, choosing one of the following formulations of the offset cost function:
  (i) distance from a set as ∞-norm

$$V_O(y_s, \Gamma_t) \triangleq \min_{y \in \Gamma_t} \|y_s - y\|_\infty \tag{20}$$

  (ii) distance from a set as 1-norm

$$V_O(y_s, \Gamma_t) \triangleq \min_{y \in \Gamma_t} \|y_s - y\|_1 \tag{21}$$

  (iii) distance from a set as a scaling factor: in this implementation, the target region is defined as
  $\Gamma_t \triangleq y_t \oplus \Xi_t$
  where y_t is a desired target point and Ξ_t is a polyhedron that defines the zone. Then

$$\begin{aligned} V_O(y_s, \Gamma_t) \;=\; & \min_{\lambda, y} \;\; \lambda \\ & s.t. \;\; \lambda \geq 0 \\ & \quad y - y_t \in \lambda \Xi_t \end{aligned} \tag{22}$$

  Notice that, this measure is such that, if $y \notin \Gamma_t$ then $\lambda > 1$, and if $y \in \Gamma_t$ then $\lambda \in [0, 1]$. In particular, if $y = y_t$, hence $\lambda = 0$. Therefore, λ has the double role of measuring the distance to a set and to a point.

4.1. Robust convergence to the target zone

The objective of the robust MPC for tracking zone regions proposed in this paper is to ensure that the output of the system y will robustly converge to the target zone Γ_t. Since Theorem 1 ensures that the output y converges to the set $y_s^* \oplus (C + DK)\mathcal{R}_\infty$, then

$$y_s^* \oplus (C + DK)\mathcal{R}_\infty \subseteq \Gamma_t$$

Hence y converges to a point $y_s^* \in \tilde{\Gamma}_t$, where

$$\tilde{\Gamma}_t = \Gamma_t \ominus (C + DK)\mathcal{R}_\infty$$

Due to this fact, the robust convergence of the closed-loop system to the target zone Γ_t is ensured if the proposed controller control law is given by $\kappa_N(x, \tilde{\Gamma}_t)$. In particular

- If $\tilde{\Gamma}_t \cap \overline{\mathcal{Y}}_s \neq \emptyset$ then the closed-loop system asymptotically converges to Γ_t.
- If $\tilde{\Gamma}_t \cap \overline{\mathcal{Y}}_s = \emptyset$, the closed-loop system asymptotically converges to $y_s^* \oplus (C + DK)\mathcal{R}_\infty$.

Remark 4. The calculation of $\mathcal{R}_\infty$ is not trivial. In [22,23] approximation methods are proposed based on outer estimations.

In the case of the formulation based on the scaling factor, robust convergence to Γ_t is ensured without calculating the set $\mathcal{R}_\infty$ if $(C + DK)\mathcal{R}_\infty \subseteq \Xi_t$ and $y_t \in \overline{\mathcal{Y}}_s$. In this case the closed-loop system converges to $y_t \oplus (C + DK)\mathcal{R}_\infty$.

5. Simulation results

To test the proposed control strategy, a subsystem of a fluid catalytic cracking (FCC) unit, presented in [28], will be used. The main objective of this simplified choice is to clearly show the ability of the proposed robust controller to handle both, persistent disturbance rejection and output zone control in systems with more output than inputs. The original system has two manipulated inputs (u_1 represents the air flow rate to the catalyst regenerator and u_2 represents the opening of the regenerated catalyst valve) and three controlled outputs (y_1 represents the riser temperature, y_2 the regenerator dense phase temperature and y_3 the regenerator dilute phase temperature), while the selected subsystem only consider the second input and the first two controlled outputs.

5.1. Nominal system description

The nominal linear model of the selected subsystem is given by:

$$G(S) = \begin{bmatrix} \dfrac{0.2033}{1.7187s + 1} \\ \dfrac{0.1886s + 3.8087}{17.7347s^2 + 10.8348s + 1} \end{bmatrix} \tag{23}$$

For a sample time of $T = 1$, the following discrete state space model is obtained:

$$A = \begin{bmatrix} 0.5589 & 0 & 0 \\ 0 & 0.5240 & -0.1672 \\ 0 & 0.1853 & 0.9769 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1895 \\ 0.7413 \\ 0.1025 \end{bmatrix} \tag{24}$$

and

$$C = \begin{bmatrix} 0.4731 & 0 & 0 \\ 0 & 0.0106 & -0.8590 \end{bmatrix} \tag{25}$$

To complete the system description, the manipulated input will be constrained to be in $\mathcal{U} = \{u \in \mathbf{R} : \|u\|_\infty \leq 5\}$. Notice that for this (2×1) system, the set of admissible nominal steady state output, $\overline{\mathcal{Y}_s}$, is in a subspace of dimension 1. Therefore, from the operation point of view, only the output desired zones with a non-empty intersection with $\overline{\mathcal{Y}_s}$ will be reachable at steady state. The sequence of desired zones proposed for the simulations is given by 4 sets of the form $\Gamma_t = \{y_{min} \leq y \leq y_{max}\}$, which are shown in Table 1. Fig. 1

Table 1
Target zones used in the simulation example.

Γ_t	y_{min}	y_{max}
$\Gamma_{t,1}$	(− 1.6, 12.5)	(0, 17.5)
$\Gamma_{t,2}$	(0, − 17.5)	(1.6, − 13.5)
$\Gamma_{t,3}$	(− 1.8, − 13)	(− 0.2, − 8.5)
$\Gamma_{t,4}$	(− 0.8, − 2.5)	(0.8, 2.5)

shows these desired set together with the set of admissible nominal steady state output, $\overline{\mathcal{Y}_s}$. The intersection of these sets constitutes the nominal reachable desired outputs zones. Notice that the sets $\Gamma_{t,i}$, for $i=1, 2, 3, 4$, constitute disjoint sets of the output space. Furthermore, as can be seen, the third target set is unreachable for the nominal system.

5.2. Disturbance description

The set $\mathcal{W}$ of possible disturbance realizations is given by $\mathcal{W}=\{w \in \mathbf{R}^3 : \|w\|_\infty \leq 0.5\}$. This choice allows a possible disturbance of 2 percent of the maximal state excursion selected for the simulation, which means that it can be, in many cases, the same order of the current system state. The sets $\mathcal{W}^y = C\mathcal{W}$ and $\mathcal{R}^y_\infty = C\mathcal{R}_\infty$ (placed in the output space), which derive from the set $\mathcal{W}$, are shown in Fig. 2. Notice that the set $\mathcal{W}$ is such that the nominal MPC controller, even if it is designed to handle target zones, as the one presented in [19], cannot reject the disturbance realization used in these simulations.

5.3. Dynamic simulations

The simulation starts at $x_0=(0, 0)$. The parameters of the proposed MPC are as follows: $N=3$, $Q=100I_3$ and $R=I_2$. This particular choice of Q and R is motivated by the fact that it provides a reasonably small $\mathcal{R}_\infty$, thus reducing the conservatism of the controller. The gain matrix K of the local controller is given by the LQR and matrix P is the solution of the Riccati equation.

As it was already said, the simulation consists in the four output target (zone) changes shown in Table 1. Furthermore, a persistent disturbance w that remain switching between extreme points of $\mathcal{W}$ is injected to the system along the complete simulation. To clearly show that the disturbance w is in fact difficult to reject (given that it has not a stationary behavior), we simulate the closed-loop under a nominal controller; i.e., a controller that accounts for the zone control but does not include the disturbance model. As can be seen in Fig. 3 the closed-loop performance is clearly unacceptable for the second output, while the input saturates at different time intervals. In a second simulation stage, we simulate the closed-loop under the proposed robust controller, using an offset cost $V_O(y_s, \Gamma_t)=\min_{y\in\Gamma_t}\|y_s-y\|_1$. The system evolution in the output space is shown in Fig. 4. As can be seen, the nature of the offset cost V_O is crucial when the target zone is not reachable, as it does occur in the third change. Fig. 4 clearly shows that the controller steers the system to the corresponding output zone, if possible, and to a region around a steady state point which minimizes the 1-norm, if not. The corresponding time evolutions of the input and outputs are shown in Fig. 5.

Finally, the same simulation sequence is repeated for the proposed controller, but now using an offset cost given by $V_O(y_s, \Gamma_t)=\min_{y\in\Gamma_t}\|y_s-y\|_\infty$. Figs. 6 and 7 show the system evolution in the output space and the input and outputs time evolution. The main difference between this simulation and the previous one, is that for the third change the distance to the unreachable target is determined by the ∞-norm.

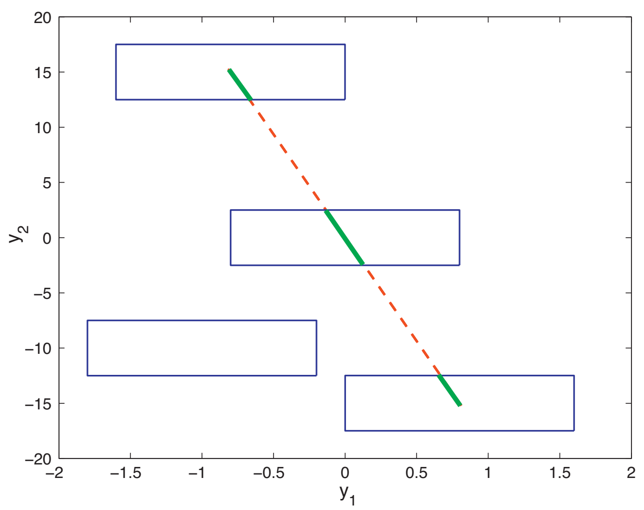

Fig. 1. Set $\overline{\mathcal{Y}_s}$ (red-dashed line), the desired output sets $\Gamma_{t,1}$, $\Gamma_{t,2}$, $\Gamma_{t,3}$ and $\Gamma_{t,4}$ (blue-solid line) and the intersection of these sets (green-solid line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

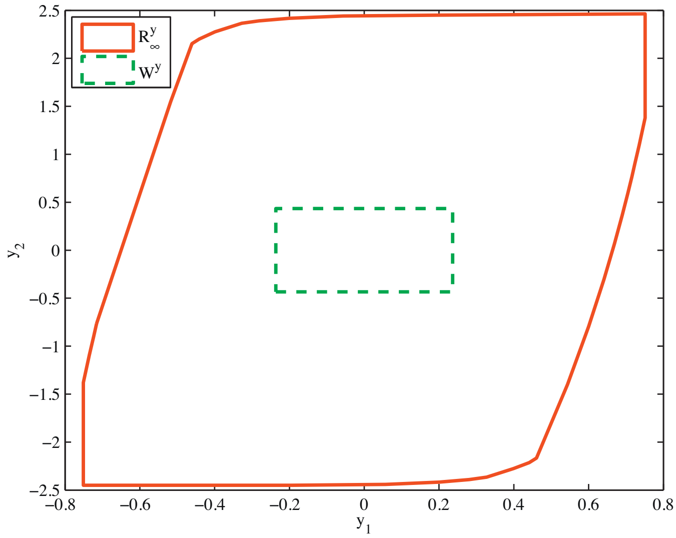

Fig. 2. The sets $\mathcal{W}^y$ and R^y_∞ for the selected disturbance set $\mathcal{W}$.

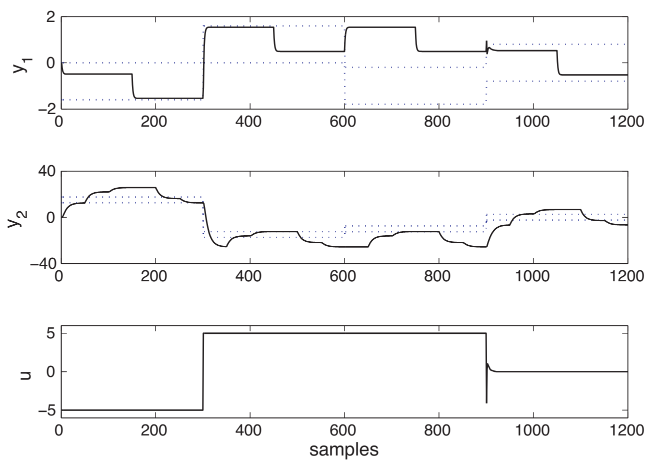

Fig. 3. Input and the outputs time evolutions for the nominal controller.

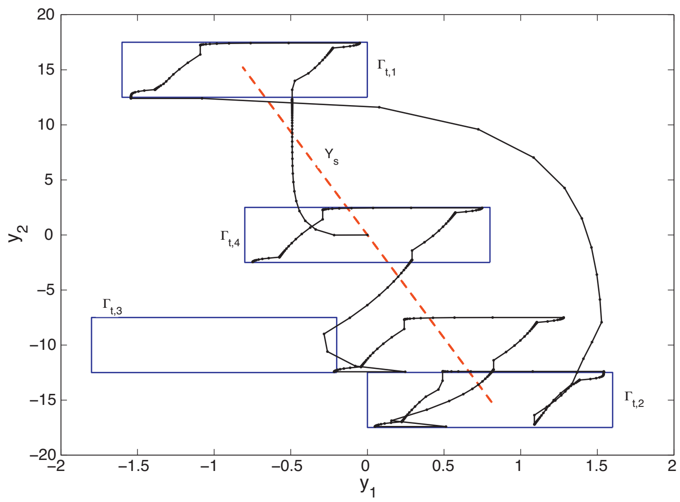

Fig. 4. System evolution in the output space, for $V_O(y_s, \Gamma_t) = \min_{y \in \Gamma_t} \|y_s - y\|_1$.

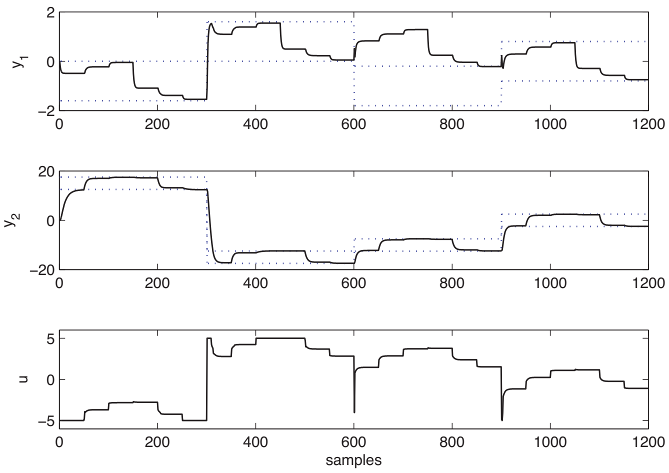

Fig. 5. Input and the outputs time evolutions, for $V_O(y_s, \Gamma_t) = \min_{y \in \Gamma_t} \|y_s - y\|_1$.

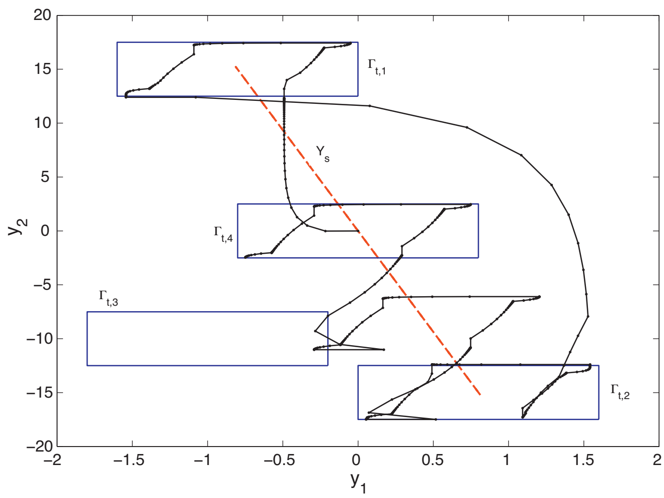

Fig. 6. System evolution in the output space, for $V_O(y_s, \Gamma_t) = \min_{y \in \Gamma_t} \|y_s - y\|_\infty$.

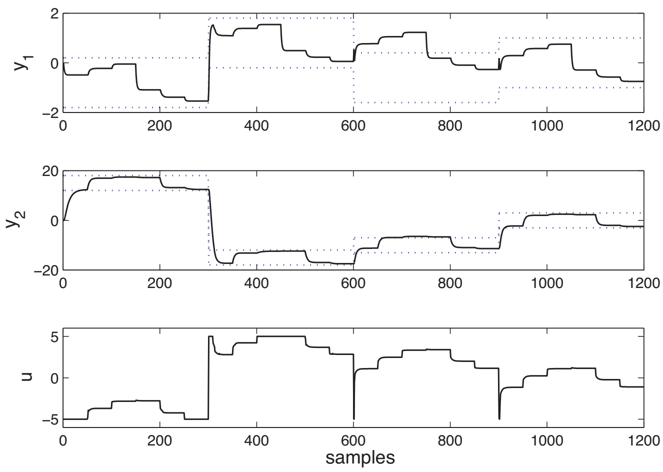

Fig. 7. Input and the outputs time evolutions, for $V_O(y_s, \Gamma_t) = \min_{y \in \Gamma_t} \|y_s - y\|_\infty$.

5.4. Comment about the nature of the disturbance and the robust controller conservatism

This subsection is devoted to clearly elucidate the meaning of $\mathcal{R}_\infty^y$ and the relation with the conservatism of the proposed strategy. The set $\mathcal{R}_\infty^y$ corresponds to the complete dynamic of the system output under the effect of the disturbance, i.e., it includes both, the stationary and the transitory regime of the evolution. Here we assume for the simulation a disturbance realization with a permanent variation along the time (i.e., with no steady state). Most of the disturbance models, however, assume a permanent but constant disturbance, since this assumption has sense in real application as they account for model mismatches usually more significant than the disturbance itself. In fact, a more realistic (and less conservative) situation is to consider a constant disturbance (maybe, plus a small variable signal) once the system reach a given target zone. In this case, the system output will be stabilized at a fixed point inside the desired zone (if reachable), and mainly, this desired zone could be too much tighter (i.e., the zone will conserve the reachability condition for tighter limits). The reason for that is that the set to be subtracted from the output zone to obtain $\tilde{\Gamma}_t$ is no longer $\mathcal{R}_\infty^y$, but an approximation of $\mathcal{W}^y$, which is clearly smaller (see Fig. 2). This could be an important point since it shows that the conservatism of the proposed strategy could be significantly reduced for some frequent application cases.

6. Conclusion

The zone control strategy is implemented in applications where the exact values of the controlled outputs are not important, as long as they remain inside a range with specified limits. In this paper, a robust extension of the MPC for tracking zone regions control has been presented, based on nominal predictions and restricted constraints. From a tracking point of view, the controller considers a set, instead of a point, as target. The concept of deviation between two points used in the offset cost function has been generalized to the concept of distance from a point to a set. A characterization of the offset cost function has been given as the minimal distance between the output and some point inside the target set. The controller ensures recursive feasibility and robust satisfaction of the constraints by using nominal predictions and restricted constraints.

Appendix A. Stability proof

In this section, the stability proof of Theorem 1 is presented. Firstly, it is necessary to introduce some lemmas. To this aim, define as $(\mathbf{c}^0(x(k), \Gamma_t), \theta^0(x(k), \Gamma_t))$ the optimal solution of problem $P_N(x, \Gamma_t)$ at the time instant k, where

$$\mathbf{c}^0(x(k), \Gamma_t) = \{c^0(0; x(k), \Gamma_t), c^0(1; x(k), \Gamma_t), \ldots, c^0(N-1; x(k), \Gamma_t)\}$$

Define the control sequence

$$\tilde{\mathbf{c}}(x(k+1), \Gamma_t) = \{c^0(1; x(k), \Gamma_t), \ldots, c^0(N-1; x(k), \Gamma_t), 0\}$$

and define $\tilde{\theta}(x(k+1), \Gamma_t) = \theta^0(x(k), \Gamma_t)$. Moreover, define as $\tilde{x}(j; x(k+1), \Gamma_t)$ the jth step prediction, given $x(k+1)$. Hence

$$\tilde{x}(j; x(k+1), \Gamma_t) = A_K^j x(k+1) + \sum_{i=0}^{j-1} A_K^i B[\tilde{c}(j-i-1; x(k+1), \Gamma_t) + L\tilde{\theta}(x(k+1), \Gamma_t)]$$

In what follows, the dependence from (x, Γ_t) will be omitted for the sake of clarity, namely, $x(j; k)$ will denote $x(j; x(k), \Gamma_t)$.

Lemma 1. *For all $j = 0, \ldots, N-1$*

$$\tilde{x}(j; k+1) - \bar{x}(j+1; k) = A_K^j w(k)$$

Proof. Since

$$\bar{x}^0(j+1; k) = A_K^j \bar{x}^0(1; k) + \sum_{i=0}^{j-1} A_K^i B[c^0(j-i; k) + L\theta^0(k)]$$

and

$$\tilde{x}(j; k+1) = A_K^j x(k+1) + \sum_{i=0}^{j-1} A_K^i B[\tilde{c}(j-i-1; k+1) + L\tilde{\theta}(k+1)]$$
$$= A_K^j x(k+1) + \sum_{i=0}^{j-1} A_K^i B[c^0(j-i; k) + L\theta^0(k)]$$

hence

$$\tilde{x}(j; k+1) - \bar{x}(j+1; k) = A_K^j [x(k+1) - \bar{x}^0(1; k)] = A_K^j w(k)$$

□

Lemma 2. *If $\bar{x}^0(j; k) \in \bar{\mathcal{X}}_j$, then $\tilde{x}(j-1; k+1) \in \bar{\mathcal{X}}_{j-1}$, for all $j = 0, \ldots, N$.*

Proof. Since $\tilde{x}(j-1; k+1) = \bar{x}^0(j; k) + A_K^{j-1} w(k)$, then

$$\tilde{x}(j-1; k+1) \in \bar{\mathcal{X}}_j \oplus A_K^{j-1}\mathcal{W} = \mathcal{X} \ominus [\bigoplus_{i=0}^{j-1} A_K^i \mathcal{W}] \oplus A_K^{j-1}\mathcal{W}$$
$$\subseteq \mathcal{X} \ominus [\bigoplus_{i=0}^{j-2} A_K^i \mathcal{W}]$$
$$\subseteq \bar{\mathcal{X}}_{j-1}$$

□

Lemma 3. *If $K\bar{x}^0(j; k) + c^0(j; k) + L\theta^0(k) \in \bar{\mathcal{U}}_j$, then $K\tilde{x}(j-1; k+1) + \tilde{c}(j-1; k+1) + L\tilde{\theta}(k+1) \in \bar{\mathcal{U}}_{j-1}$, for all $j = 1, \ldots, N-1$.*

Proof. Taking into account that

$$K\bar{x}^0(j; k) + c^0(j; k) + L\theta^0(k) = K\tilde{x}(j-1; k+1) - KA_K^{j-1} w(k) + \tilde{c}(j-1; k+1) + L\tilde{\theta}(k+1)$$

hence

$$K\tilde{x}(j-1; k+1) + \tilde{c}(j-1; k+1) + L\tilde{\theta}(k+1) \in \bar{\mathcal{U}}_j \oplus KA_K^{j-1}\mathcal{W}$$

and

$$\bar{\mathcal{U}}_j \oplus KA_K^{j-1}\mathcal{W} = \mathcal{U} \ominus K\mathcal{R}_j \oplus KA_K^{j-1}\mathcal{W} = \mathcal{U} \ominus K\mathcal{R}_{j-1} = \bar{\mathcal{U}}_{j-1}$$

□

Lemma 4. *[Recursive feasibility of the terminal constraint]For all $k \geq 0$,*

$$(\bar{x}^0(N; k), \theta^0(k)) \in \Omega_t^a$$

Proof. Consider that at time k $(\bar{x}^0(N; k), \theta^0(k)) \in \Omega_t^a$. Since $\Omega_t^a = \Sigma_t \ominus (\mathcal{R}_N \times 0)$, hence

$$(\bar{x}^0(N-1; k+1), \theta^0(k+1)) \in \Sigma_t \ominus (\mathcal{R}_N \times 0) \oplus (A_K^{N-1}\mathcal{W} \times 0)$$

Then, since $(\bar{x}^0(N; k+1), \theta^0(k+1)) = A_a(\bar{x}^0(N-1; k+1), \theta^0(k+1))$, hence

$$(\bar{x}^0(N; k+1), \theta^0(k+1)) \in A_a(\Sigma_t \ominus (\mathcal{R}_N \times 0) \oplus (A_K^{N-1}\mathcal{W} \times 0))$$

Taking into account that

$$A_a(\Sigma_t \ominus (\mathcal{R}_N \times 0) \oplus (A_K^{N-1}\mathcal{W} \times 0))$$
$$= A_a\Sigma_t \ominus (A_K\mathcal{R}_N \times 0) \oplus (A_K^N\mathcal{W} \times 0)$$
$$= A_a\Sigma_t \ominus \left(\bigoplus_{j=1}^{N} A_K^j\mathcal{W} \times 0\right) \oplus (A_K^N\mathcal{W} \times 0)$$
$$= A_a\Sigma_t \ominus \left(\bigoplus_{j=1}^{N-1} A_K^j\mathcal{W} \times 0\right) \ominus (A_K^N\mathcal{W} \times 0) \oplus (A_K^N\mathcal{W} \times 0)$$
$$\subseteq A_a\Sigma_t \ominus \left(\bigoplus_{j=1}^{N-1} A_K^j\mathcal{W} \times 0\right)$$
$$\subseteq (\Sigma_t \ominus (\mathcal{W} \times 0)) \ominus \left(\bigoplus_{j=1}^{N-1} A_K^j\mathcal{W} \times 0\right)$$
$$= \Sigma_t \ominus \left(\bigoplus_{j=0}^{N-1} A_K^j\mathcal{W} \times 0\right)$$
$$= \Sigma_t \ominus (\mathcal{R}_N \times 0)$$

where the second from last equality comes from $A_a\Sigma_t \oplus (\mathcal{W} \times 0) \subseteq \Sigma_t \Leftrightarrow A_a\Sigma_t \subseteq \Sigma_t \ominus (\mathcal{W} \times 0)$.

Hence,

$$(\bar{x}^0(N; k+1), \theta^0(k+1)) \in \Sigma_t \ominus (\mathcal{R}_N \times 0) = \Omega_t^a$$

□

A.1. Proof of Theorem 1

Before starting with the proof, we introduce the notion of Input-to-State Stability (ISS) [29]. To this aim, consider a closed-loop disturbed system

$$x(k+1) = f_K(x(k), w(k)) \tag{19}$$

where $f_K(x, w) \triangleq f(x, K(x), w)$. The solution of this equation at sampling time k, for the initial state $x(0)$ and the sequence of disturbances $\mathbf{w}$, is denoted as $\phi_K(k; x(0), \mathbf{w})$.

Definition 3. System (19) is ISS for all initial conditions $x(0)$ and sequence of disturbances $\mathbf{w}$ if there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that

$$|\phi_K(k; x(0), \mathbf{w})| \leq \beta(|x(0)|, k) + \gamma(\|\mathbf{w}\|)$$

In what follows, it will be proved that the closed-loop system is ISS for all $x(0) \in \mathcal{X}_N$.

Proof. From Lemmas 1–4, it is derived that the couple $(\tilde{\mathbf{c}}(k+1), \tilde{\theta}(k+1))$ is a feasible solution of problem $P_N(x, \Gamma_t)$.

Consider now the optimal value of the cost function $V_N^0(x(k), \Gamma_t)$, due to the optimal solution of problem $P_N(x(k), \Gamma_t)$, given by $(\mathbf{c}^0(k), \theta^0(k))$. Define

$$\tilde{V}_N(x(k+1), \Gamma_t; \tilde{\mathbf{c}}, \tilde{\theta}) = \sum_{j=0}^{N-1} \|\tilde{c}(j; k+1)\|_\Psi^2 + V_O(y_s, \Gamma_t)$$

Comparing $\tilde{V}_N(x(k+1), \Gamma_t; \tilde{\mathbf{c}}, \tilde{\theta})$ with $V_N^0(x(k), \Gamma_t)$, we have that

$$\tilde{V}_N(x(k+1), \Gamma_t; \tilde{\mathbf{c}}, \tilde{\theta}) - V_N^0(x(k), \Gamma_t) = -\|c^0(0; k)\|_\Psi^2$$

and hence, by optimality:

$$V_N^0(x(k+1), \Gamma_t) - V_N^0(x(k), \Gamma_t) \leq -\|c^0(0; k)\|_\Psi^2$$

Since $\Psi > 0$, we can state that:

$$\lim_{k\to\infty} c^0(0; k) = 0$$

and (*ii*) is proved.

The fact that $c^0(0\,; k) \to 0$ implies that $u(k) \to K(x(k) - x_s^0(k)) + u_s^0(k)$, and hence:

$$x(k) \to x_s^0(k) \oplus \mathcal{R}_\infty, \quad u(k) \to u_s^0(k) \oplus K\mathcal{R}_\infty$$

Using the same arguments as in [19], it can be proved that $(x_s^0(k), u_s^0(k))$ converges to the optimal equilibrium point (x_s^*, u_s^*) which is the minimizer of the offset cost function $V_O(y_s, \Gamma_t)$.

Now, the stability of the equilibrium point will be proved. If the uncertainty is null, then (following [19]) the system is asymptotically stable in (x_s^*, u_s^*). If $w \neq 0$, the continuity of the control law provides that the closed-loop system is such that the closed-loop prediction $\phi_{cl}(j; x, w) = \phi(j; x, k_N(x, \Gamma_t), w)$ is continuous in x and w. Then, resorting to ISS arguments [29], it can be proved that there exist a $\mathcal{KL}$ function β and a $\mathcal{K}$ function γ such that

$$|x(k) - x_s^*| \leq \beta(|x(0) - x_s^*|, k) + \gamma(\|w\|)$$

for all initial state $x(0) \in \mathcal{X}_N$ and all disturbances $w(k)$. □

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