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Skyrmions in the presence of isospin chemical potential

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Abstract

We analyze the existence of localized finite energy topological excitations on top of the perturbative pion vacuum within the Skyrme model at finite isospin chemical potential and finite pion mass. We show that there is a critical isospin chemical potential μ_I^c above which such solutions cease to exist. We find that μ_I^c is closely related to the value of the pion mass. In particular for vanishing pion mass we obtain $\mu_I^c = 0$ in contradiction with some results recently reported in the literature. We also find that below μ_I^c the skyrmion mass and baryon radius show, at least for the case of the hedgehog ansatz, only a mild dependence on the isospin chemical potential.

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Hadronic systems with vanishing baryon chemical potential μ_B and finite isospin chemical potential μ_I are unstable with respect to weak decays. However, if we are interested in the dynamics of the strong interaction alone, we can disregard the relative slow electroweak effects and consider them as stable. Moreover, although there are not yet precise lattice QCD calculations at finite baryon density due to the Fermion sign problem, it is in principle possible to perform lattice simulations at finite isospin density [1]. These remarks have led several groups to study the behavior of strongly interacting matter at $\mu_B = 0$ and finite μ_I. Effective Lagrangian analysis showed that there is a phase transition from normal phase to pion superfluidity at a critical isospin chemical potential which turns out to be equal to the pion mass in the vacuum [2]. This has been confirmed by lattice QCD calculations [3], random matrix method analysis [4], etc. Studies at finite temperature have been also performed [5]. Given these results it is of considerable interest to investigate on the behavior of baryon properties at finite isospin chemical potential. One model which is well suited to perform these studies is the Skyrme model. In the Skyrme model [6] and its generalizations, baryons arise as topological excitations of a non-linear chiral Lagrangian written in terms of meson fields. These type of models have been quite successful in describing the properties of octet and decouplet baryons (see e.g. Ref. [7]). In a series of recent articles [8] the skyrmion properties in the presence of the isospin chemical potential have been analyzed. It has been found that, in the case of vanishing pion mass, there is a critical chemical potential $\mu_I^c \approx 223$ MeV above which stable soliton solutions on top of the perturbative pion vacuum cease to exist. Moreover, according to Ref. [8] the skyrmion mass vanishes at $\mu = \mu_I^c$. In this Letter we re-examine the issue of the skyrmion stability for finite isospin chemical potential considering the possibility of having a finite pion mass. We show that although there is indeed a critical isospin chemical potential μ_I^c, its value is closely related to the value of the pion mass, and for vanishing pion mass one has $\mu_I^c = 0$. Moreover we find that, at least within

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the spherically symmetric hedgehog ansatz, the skyrmion mass and baryon radius remain rather stable up to the critical isospin chemical potential.

We start by considering the Lagrangian of the $SU(2)$ Skyrme model with quartic term stabilization and finite pion mass. It is given by

$$\mathcal{L} = -\frac{f_\pi^2}{4}\,\mathrm{Tr}\{L_\alpha L^\alpha\} + \frac{1}{32e^2}\,\mathrm{Tr}\{[L_\alpha, L_\beta]^2\} + \frac{m_\pi^2 f_\pi^2}{4}\,\mathrm{Tr}\{U + U^\dagger - 2\}. \tag{1}$$

Here, f_π is the pion decay constant whose empirical value is $f_\pi^{\mathrm{emp}} = 93$ MeV, e is the so-called Skyrme parameter and m_π is the pion mass which we will take at its empirical value $m_\pi^{\mathrm{emp}} = 139$ MeV. In our numerical calculations below we will use the standard set of values $f_\pi = 54$ MeV and $e = 4.84$ which leads to the empirical value of the nucleon and Δ masses within the rigid rotor approximation [9]. It should be stressed, however, that our main conclusions are expected to be independent of this particular choice of parameters. In Eq. (1), as usual, U represents the $SU(2)$ chiral field and the Maurier–Cartan operator L_α is defined by $L_\alpha = U^\dagger \partial_\alpha U$.

The isospin chemical potential μ_I is introduced by performing the replacement

$$\partial_\alpha U \longrightarrow \partial_\alpha U - i\frac{\mu_I}{2}[\tau_3, U] g_{\alpha 0}, \tag{2}$$

where $g_{\alpha\beta}$ is the metric tensor in Minkowski space and τ_3 is the third Pauli matrix. This leads to a modified Lagrangian which reads

$$\mathcal{L}(\mu_I) = \mathcal{L} + \frac{\mu_I^2 f_\pi^2}{16}\,\mathrm{Tr}\{\omega^2\} - \frac{\mu_I^2}{64e^2}\,\mathrm{Tr}\{[\omega, L_\alpha]^2\} + \frac{i\mu_I f_\pi^2}{4}\,\mathrm{Tr}\{\omega L_0\} - \frac{i\mu_I}{8e^2}\,\mathrm{Tr}\{\omega L_\alpha [L_0, L_\alpha]\} \tag{3}$$

where $\omega = U^\dagger \tau_3 U - \tau_3$.

In what follows we will be interested in static soliton configurations. The corresponding soliton mass reads

$$M(\mu_I) = -\int d^3x \left[\frac{f_\pi^2}{4}\,\mathrm{Tr}\{L_i L_i\} + \frac{1}{32e^2}\,\mathrm{Tr}\{[L_i, L_j]^2\} + \frac{m_\pi^2 f_\pi^2}{4}\,\mathrm{Tr}\{U + U^\dagger - 2\} + \frac{\mu_I^2 f_\pi^2}{16}\,\mathrm{Tr}\{\omega^2\} + \frac{\mu_I^2}{64e^2}\,\mathrm{Tr}\{[\omega, L_i]^2\}\right]. \tag{4}$$

It is not hard to see that the terms proportional to μ_I^2 are not invariant under isospin rotations. Namely, the isospin chemical potential introduces a preferred direction in isospin space which is expected to lead to an axially deformed soliton configuration. For the time being we will assume that such deformations are small for the range of values of μ_I considered here. Therefore we introduce the usual spherically symmetric hedgehog ansatz for the baryon number $B = 1$ configuration

$$U_H = \exp[i\vec{\tau}\cdot\hat{r}F(r)]. \tag{5}$$

In this case we obtain[1]

$$M_H(\mu_I) = \frac{f_\pi^2}{2}\int d^3x \left[F'^2 + 2\frac{s^2}{r^2}\left(1 + \frac{F'^2}{e^2 f_\pi^2}\right) + \frac{1}{e^2 f_\pi^2}\frac{s^4}{r^4} + 2m_\pi^2(1 - c) - \frac{2}{3}\mu_I^2 s^2\left(1 + \frac{F'^2}{e^2 f_\pi^2}\right) - \frac{2}{3}\mu_I^2 \frac{s^4}{e^2 f_\pi^2 r^2}\right], \tag{6}$$

where $s = \sin F$ and $c = \cos F$. The minimization of $M_H(\mu_I)$ leads to the following Euler–Lagrange equation for the soliton profile F,

$$F''\left[1 + \frac{2s^2}{e^2 f_\pi^2 r^2}\left(1 - \frac{\mu_I^2 r^2}{3}\right)\right] + 2\frac{F'}{r}\left(1 - \frac{2\mu_I^2 s^2}{3e^2 f_\pi^2}\right) - \frac{2sc}{r^2}\left(1 - \frac{F'^2}{e^2 f_\pi^2}\right)\left(1 - \frac{\mu_I^2 r^2}{3}\right) - \frac{2s^3 c}{e^2 f_\pi^2 r^4}\left(1 - \frac{2}{3}\mu_I^2 r^2\right) - m_\pi^2 s = 0. \tag{7}$$

As usual this differential equation is supplemented by the boundary conditions corresponding to $B = 1$ topological excitations on top of the perturbative pion vacuum, $F(0) = \pi$ and $F(\infty) = 0$. The associated baryon radius is given by

$$r_B(\mu_I) = \left(-\frac{2}{\pi}\int_0^\infty dr\, r^2 s^2 F'\right)^{1/2}. \tag{8}$$

Before presenting the numerical soliton solutions we will analyze the behavior of the profile $F(r)$ for large distances. Given the boundary conditions we can linearize Eq. (7) in that limit. We obtain

$$F'' + \frac{2F'}{r} - \left(m_\pi^2 - \frac{2}{3}\mu_I^2 + \frac{2}{r^2}\right)F = 0. \tag{9}$$

This equation implies that localized finite energy topological excitations on top of the perturbative pion vacuum exist only for $\mu_I \leqslant \mu_I^c$, where $\mu_I^c = \sqrt{3/2}\,m_\pi$. As it can be shown by solving Eq. (9), when $\mu_I > \mu_I^c$ the profile $F(r)$ behaves as a spherical Bessel function at large distances, implying that the usual criteria for having localized finite energy solutions [10] fail to be satisfied. In fact, we find that at large distances the corresponding energy density $\epsilon(r)$ defined by $M_H(\mu_I) = \int_0^\infty dr\, r^2 \epsilon(r)$ is given by

$$\epsilon(r) \sim \sin\left(\sqrt{2/3\mu_I^2 - m_\pi^2}\, r\right)/r^2 \tag{10}$$

which is clearly non-integrable. The situation is quite similar to the one found long time ago in the study of the stability of the skyrmion under spin–isospin rotations [11].

In Fig. 1 we plot the numerical solutions of Eq. (7) for some chosen values of μ_I. There we can clearly note the oscillatory behavior of $F(r)$ for $\mu_I > \mu_I^c$. For $\mu_I < \mu_I^c$, the skyrmion

[1] We have noticed the existence of some misprints in Eqs. (8)–(14) of second reference in [8]. The correct corresponding expressions can be obtained by taking the chiral limit of our Eqs. (6), (7).

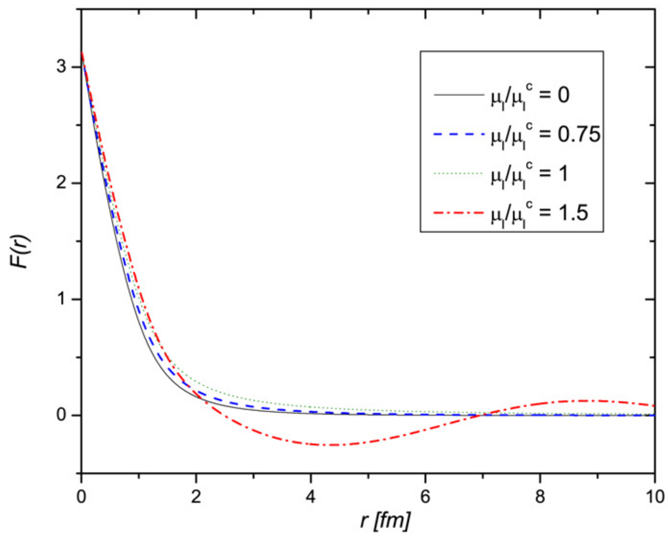

Fig. 1. (Color online.) Hedgehog profiles $F(r)$ for various values of the chemical potential μ_I. Note that localized solutions only exist for $\mu_I/\mu_I^c \leqslant 1$. Although always oscillatory at large distances, the detailed form of $F(r)$ for $\mu_I/\mu_I^c > 1$ depends on the maximum integration radius $R_{\max}$. Here we use $R_{\max} = 100$ fm. Of course, for $\mu_I/\mu_I^c \leqslant 1$ the profile function as well as the soliton properties are independent of $R_{\max}$, provided it is sufficiently large.

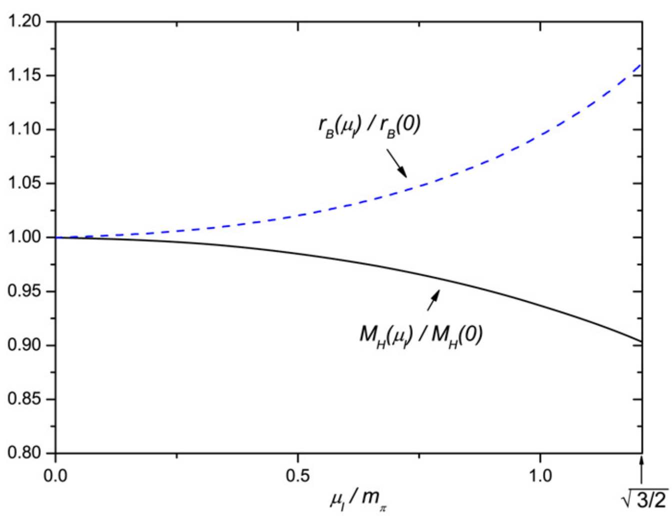

Fig. 2. (Color online.) Soliton mass (full line) and baryon radius (dashed line) as a function of the isospin chemical potential μ_I for the spherically symmetric hedgehog ansatz and taken with respect to their corresponding values at $\mu_I = 0$. For our parameter choice such values are $M_H(0) = 864$ MeV and $r_B(0) = 0.68$ fm [9].

is exponentially localized. Finally, when $\mu_I = \mu_I^c$ the solution displays a $1/r^2$ large distance behavior which is typical of the localized pion massless case.

In the region where localized solutions exist we can study the behavior of the soliton mass and baryon radius as a function of the isospin chemical potential. This is shown in Fig. 2. We observe that as μ_I increases, the soliton mass decreases while the radius increases. However, these effects are not too large (around 15% or less at $\mu_I = \mu_I^c$), and $M_H(\mu_I)$ never vanishes in such region.

As mentioned at the beginning of this Letter, general arguments indicate that in the meson sector there is a phase transition from the normal phase (perturbative pion vacuum) to a pion condensed phase at $\mu_I^c = m_\pi$. Comparing with the result obtained above we note that in the soliton sector it appears an extra factor $\sqrt{3/2}$ in the corresponding critical value. A similar factor has been found in the study of the stability of the hedgehog skyrmion under spin–isospin rotation [12]. As in that case, it reflects the fact that in the hedgehog approximation possible pion excitations are assumed to be spherical while, as already mentioned, the presence of the isospin chemical potential is expected to induce axially symmetric deformations. In order to account for this fact we introduce a general axial ansatz which, in cylindrical polar coordinates (ρ, ϕ, z), is given by [12,13]

$$U_{ax} = \psi_3 + i\tau_3\psi_2 + i\psi_1(\tau_1\cos\phi + \tau_2\sin\phi). \tag{11}$$

Here, ψ_a are the components of a unit vector $\vec{\psi}(\rho, z)$ that is independent of the angular variable ϕ. The boundary conditions for finite energy solutions are that $\vec{\psi} \to (0,0,1)$ as $\rho^2 + z^2 \to \infty$ and that on the symmetry axis $\rho = 0$ the equations $\psi_1 = 0$ and $\partial_\rho\psi_2 = \partial_\rho\psi_3 = 0$ must be satisfied. Using this ansatz, the mass for the static soliton configuration reads

$$\begin{aligned} M_{ax}(\mu_I) = \pi f_\pi^2 \int \Bigg[& \partial_i\vec{\psi}\cdot\partial_i\vec{\psi}\left(1 + \frac{1}{e^2 f_\pi^2}\frac{\psi_1^2}{\rho^2}\right) \\ & + \frac{1}{e^2 f_\pi^2}|\partial_\rho\vec{\psi}\times\partial_z\vec{\psi}|^2 + \frac{\psi_1^2}{\rho^2} + 2m_\pi^2(1-\psi_3) \\ & - \mu_I^2\psi_1^2\left(1 + \frac{1}{e^2 f_\pi^2}\partial_i\vec{\psi}\cdot\partial_i\vec{\psi}\right)\Bigg]\rho\, d\rho\, dz, \end{aligned} \tag{12}$$

where $i = \rho, z$. Minimizing this expression we obtain a set of coupled equations for the two independent functions that we take to be $\psi_1(\rho, z)$ and $\psi_2(\rho, z)$. Unfortunately the resolution of these equations implies a rather time-consuming numerical task. From the results obtained in the case of spin–isospin rotations [12], soliton properties are not expected to be too much affected by the axial deformations provided $\mu_I \leqslant \mu_I^c$. Thus we postpone this numerical analysis for a future work. It is important, however, to consider the linearized form of the equations for ψ_1 and ψ_2, which are valid at large distances. They read

$$\begin{aligned} &\partial_i\partial_i\psi_1 - \left(m_\pi^2 - \mu_I^2 + \frac{1}{\rho^2}\right)\psi_1 = 0, \\ &\partial_i\partial_i\psi_2 - m_\pi^2\psi_2 = 0. \end{aligned} \tag{13}$$

Thus, once axially symmetric configurations are considered, we have indeed $\mu_I^c = m_\pi$. Namely, for $\mu_I \leqslant \mu_I^c$ the behavior of the deformed skyrmion mass and radius as a function of the isospin chemical potential is expected to be very similar to that shown in Fig. 2, the only important difference being that the curve will end at $\mu_I/m_\pi = 1$ instead of $\mu_I/m_\pi = \sqrt{3/2}$.

In conclusion, we have re-examined the behavior of the skyrmion properties as a function of the isospin chemical potential μ_I. We have found that there is, indeed, a critical value of μ_I above which localized finite energy topological excitations on top of the perturbative pion vacuum cease to exist. Such critical value is closely related to the value of the pion mass. For the spherically symmetric hedgehog ansatz we find $\mu_I^c = \sqrt{3/2}m_\pi$, while when axially symmetric deformations are allowed we obtain $\mu_I^c = m_\pi$ as expected from general arguments in the meson sector. As in such sector [2], when μ_I exceeds this critical value it is energetically favorable to create pion excitations which, in turns, lead to a pion condensed phase. This implies that for $\mu_I > \mu_I^c$ one has to look for soliton excitations on top of a pion condensed vacuum. These results disagree with previous works [8] which, having overlooked the oscillatory behavior reported here, quote $\mu_I^c \approx 223$ MeV in the chiral limit ($m_\pi = 0$). Moreover, for $\mu_I \leqslant \mu_I^c$ we find only a mild dependence of the soliton mass and radius on μ_I. In particular, in contrast to the results in Ref. [8] our values of the soliton mass do not vanish for any value of $\mu_I \leqslant \mu_I^c$. It should be noted that these analyses of the soliton mass and radius dependence on μ_I have been performed using the spherically symmetric hedgehog ansatz. Although for that range of values of μ_I the axially symmetric deformations are expected to have only a minor effect on this dependence, the numerical resolution of the corresponding equations is required in order to confirm this expectation. This will also allow us to explore the region $\mu_I > \mu_I^c$ and, in particular, analyze the nature of the transition at $\mu_I = \mu_I^c$. We hope to report on these issues in forthcoming publications.

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