CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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*Cahiers de topologie et géométrie différentielle catégoriques*, tome 47, nº 2 (2006), p. 83-106

<http://www.numdam.org/item?id=CTGDC_2006__47_2_83_0>

*CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES* *Volume XLVII-2 (2006)*

# A CONSTRUCTION OF 2-FILTERED BICOLIMITS OF CATEGORIES

***by Eduardo J. DUBUC and Ross STREET***

A la mémoire de Charles Ehresmann

**RESUME.** Nous définissons la notion de 2-catégorie 2-filtrante et donnons une construction explicite de la bicolimite d'un 2-foncteur à valeurs dans les catégories. Une catégorie considérée comme étant une 2-catégorie triviale est 2-filtrante si et seulement si c'est une catégorie filtrante, et notre construction conduit à une catégorie équivalente à la catégorie qui s'obtient par la construction usuelle des colimites filtrantes de catégories. Pour cette construction des axiomes plus faibles suffisent, et nous appelons la notion correspondante 2-catégorie pré 2-filtrante. L'ensemble complet des axiomes est nécessaire pour montrer que les bicolimites 2-filtrantes ont les propriétés correspondantes aux propriétés essentielles des colimites.

## Introduction.

We define the notion of *2-filtered 2-category* and give an explicit construction of the bicolimit of a category valued 2-functor. A category considered as a trivial 2-category is 2-filtered if and only if it is a filtered category, and our construction yields a category equivalent to the category resulting from the usual construction of filtered colimits of categories. Weaker axioms suffice for this construction, and we call the corresponding notion *pre 2-filtered 2-category.* The full set of axioms is necessary to prove that 2-filtered bicolimits have the properties corresponding to the essential properties of filtered bicolimits.

In [3] Kennison already considers filterness conditions on a 2-category under the name of *bifiltered 2-category.* It is easy to check that a bifiltered 2-category is 2-filtered, so our results apply to bifiltered 2-categories. Actually Kennison's notion is equivalent to our's, but the other direction of this equivalence is not entirely trivial.

**Acknowledgement.** This paper gained its impetus from initial collaboration in Montréal amongst André Joyal and the authors. We are very grateful to André's input. As authors of this account of that work and some further developments, we remain responsible for any imperfections.

## 1. Pre 2-Filtered 2-Categories and the construction LL

**1.1 Definition.** A 2-category $\mathcal{A}$ is defined to be *pre-2-filtered* when it satisfies the following two axioms:

F1.

*Given* $\begin{tikzcd} & A \\ E \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd}$ *there exists invertible* $\begin{tikzcd} & A \arrow[dr, "u"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\gamma\Downarrow"] & & C \\ & B \arrow[ur, "v"'] & \end{tikzcd}$ .

F2.

*Given any* 2-*cells* $\begin{tikzcd} & A \arrow[dr, "u_1"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\gamma_1\Downarrow"] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd}$ , $\begin{tikzcd} & A \arrow[dr, "u_2"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\gamma_2\Downarrow"] & & C_2 \\ & B \arrow[ur, "v_2"'] & \end{tikzcd}$

*there exists* $\begin{tikzcd} C_1 \arrow[dr, "w_1"] & \\ & C \\ C_2 \arrow[ur, "w_2"'] & \end{tikzcd}$ *with invertible* 2-*cells* $\alpha$, $\beta$ *such that*

$$(1.2) \qquad \begin{tikzcd} & A \arrow[dr, "u_1"] & & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\gamma_1\Downarrow"] & & C_1 \arrow[dr, "w_1"] & \\ & B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\alpha\Downarrow"] & & C \\ & & \sim \arrow[ur, "w_2"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} & & C_1 \arrow[dr, "w_1"] & \\ & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\beta\Downarrow"] & & C \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\gamma_2\Downarrow"] & & C_2 \arrow[ur, "w_2"'] & \\ & B \arrow[ur, "v_2"'] & & \end{tikzcd} \ .$$

**1.3 Notation (the LL equation).** Given two pairs of 2-cells $(\gamma_1,\ \alpha)$, $(\gamma_2,\ \beta)$ as in F2, we shall call the equation 1.2 the *equation LL.*

Axioms F1 and F2 can be weakened if we add a third axiom:

WF1. Same axiom F1 but not requiring $\gamma$ to be invertible.

WF2. Same axiom F2 but only for invertible $\gamma_1$ and $\gamma_2$.

WF3.

*Given* $\begin{tikzcd} & A \arrow[dr, "u"] \arrow[dd, phantom, "\gamma\Downarrow"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] & & C \\ & B \arrow[ur, "v"'] & \end{tikzcd}$ *there exists invertible* 2*-cells* $\alpha$, $\beta$ *such that*

$$\begin{tikzcd} & A \arrow[dr, "u"] \arrow[dd, phantom, "\gamma\Downarrow"] & & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] & & C \arrow[dr, "h_2"] \arrow[dd, phantom, "\alpha\Downarrow"] & \\ & B \arrow[ur, "v"'] \arrow[dr, "s_2"'] & & D_2 \\ & & C_2 \arrow[ur, "w_2"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\beta\Downarrow"] & \\ & A \arrow[ur, "s_1"] \arrow[dr, "u"] \arrow[dd, phantom, "\gamma\Downarrow"] & & D_1 \\ E \arrow[ur, "f"] \arrow[dr, "g"'] & & C \arrow[ur, "h_1"'] & \\ & B \arrow[ur, "v"'] & & \end{tikzcd} \quad \textit{are invertible.}$$

We leave to the reader the proof of the following:

**1.4 Proposition.** *The set of axioms F1, F2 is equivalent to the set WF1 WF2 and WF3.* □

When $\mathcal{A}$ is a trivial 2-category (the only 2-cells are the identities), our axiom F1 corresponds to axiom PS1 in the definition of pseudofiltered category (cf [1] Exposé I), while axiom PS2 may not hold. Thus, a category which is pre 2-filtered as a trivial 2-category may not be pseudofiltered. Notice that our axioms F2 and WF3 are vacuous in this case.

**Construction LL**

Let $\mathcal{A}$ be a pre-2-filtered 2-category and $F : \mathcal{A} \to \mathcal{C}at$ a category valued 2-functor. We shall now construct a category which is to be the bicolimit (in the sense made precise in theorem 1.19) of $F$. This construction generalizes Grothendieck's construction of the category $\varinjlim_{\mathcal{A}} F$ for a filtered category $\mathcal{A}$ (cf [2], Exposé VI).

**1.5 Definition (Quasicategory $\mathcal{L}(F)$).**
i) An *object* is a pair $(x, A)$ with $x \in FA$.

ii) A *premorphism* $(x, A) \to (y, B)$ between two objects is a triple $(u, \xi, v)$, where and $A \xrightarrow{u} C$, $B \xrightarrow{v} C$ and $\xi : F(u)(x) \to F(v)(y)$ in $FC$.

iii) A *homotopy* between two premorphisms is a quadruple $(w_1, w_2, \alpha, \beta) : (u_1, \xi_1, v_1) \Rightarrow (u_2, \xi_2, v_2)$, where $C_1 \xrightarrow{w_1} C$, $C_2 \xrightarrow{w_2} C$ and $\alpha : w_1v_1 \xrightarrow{\cong} w_2v_2$, $\beta : w_1u_1 \xrightarrow{\cong} w_2u_2$ are invertible 2-cells such that

$$\begin{tikzcd} F(w_1)F(u_1)(x) = F(w_1u_1)(x) \arrow[r, "F(\beta)x"] \arrow[d, "F(w_1)(\xi_1)"] & F(w_2u_2)(x) = F(w_2)F(u_2)(x) \arrow[d, "F(w_2)(\xi_2)"] \\ F(w_1)F(v_1)(y) = F(w_1v_1)(y) \arrow[r, "F(\alpha)y"] & F(w_2v_2)(y) = F(w_2)F(v_2)(y) \end{tikzcd}$$

commutes in $FC$.

We shall formally introduce now an *abuse of notation*

**1.6 Notation.**
i) We omit the letter $F$ in denoting the action of $F$ on its arguments. Thus $\begin{tikzcd} A \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\alpha\Downarrow"] & B \end{tikzcd}$ indicates a 2-cell in $\mathcal{A}$ as well as the corresponding natural transformation $\begin{tikzcd} FA \arrow[r, bend left, "F(u)"] \arrow[r, bend right, "F(v)"'] \arrow[r, phantom, "F(\alpha)\Downarrow"] & FB \end{tikzcd}$ in $\mathcal{C}at$. In this way, the above commutative square becomes

$$\begin{tikzcd} w_1u_1x \arrow[r, "\beta x"] \arrow[d, "w_1\xi_1"] & w_2u_2x \arrow[d, "w_2\xi_2"] \\ w_1v_1y \arrow[r, "\alpha y"] & w_2v_2y \end{tikzcd}$$

ii) We write $F \xrightarrow{x} A$ in $(\mathcal{C}at^{\mathcal{A}})^{op}$ for the natural transformation $\mathcal{A}[A, -] \to F$ defined by $x_C(A \xrightarrow{u} C) = F(u)(x) \in FC$.

Notice that the notation in i) $F(u)(x) = ux$ is consistent with this since juxtaposition denotes composition. Also, in the same vein, given an object $x \in FA$ and a functor $FA \xrightarrow{h} \mathcal{X}$ into any other category, the composite $F \xrightarrow{x} A \xrightarrow{h} \mathcal{X}$ makes sense and we have $h(F(x)) = hx$.

In this notation then, a premorphism $(x, A) \to (y, B)$ is a triple $(u, \xi, v)$, where

$$\begin{tikzcd}[row sep=small] & A \arrow[dr, "u"] \arrow[dd, phantom, "\xi\Downarrow"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & & C \\ & B \arrow[ur, "v"'] & \end{tikzcd}$$

; that is $\xi : ux \longrightarrow vy$ in FC. A homotopy between two premorphisms

$$\begin{tikzcd}[row sep=small] & A \arrow[dr, "u_1"] \arrow[dd, phantom, "\xi_1\Downarrow"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd}, \quad \begin{tikzcd}[row sep=small] & A \arrow[dr, "u_2"] \arrow[dd, phantom, "\xi_2\Downarrow"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & & C_2 \\ & B \arrow[ur, "v_2"'] & \end{tikzcd}$$

is a pair of invertible 2-cells

$$\begin{tikzcd}[row sep=small] & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\alpha\Downarrow"] & \\ B \arrow[ur, "v_1"] \arrow[dr, "v_2"'] & & C \\ & C_2 \arrow[ur, "w_2"'] & \end{tikzcd}, \quad \begin{tikzcd}[row sep=small] & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\beta\Downarrow"] & \\ A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & & C \\ & C_2 \arrow[ur, "w_2"'] & \end{tikzcd}$$

satisfying the LL equation:

$$\begin{tikzcd}[row sep=small] & A \arrow[dr, "u_1"] \arrow[dd, phantom, "\xi_1\Downarrow"] & & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & & C_1 \arrow[dr, "w_1"] & \\ & B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\alpha\Downarrow"] & & C \\ & & C_2 \arrow[ur, "w_2"'] & \end{tikzcd} = \begin{tikzcd}[row sep=small] & & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\beta\Downarrow"] & \\ & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[dd, phantom, "\xi_2\Downarrow"] & & C \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & & C_2 \arrow[ur, "w_2"'] & \\ & B \arrow[ur, "v_2"'] & & \end{tikzcd}.$$

We shall simply write $(\alpha, \beta) : \xi_1 \Rightarrow \xi_2$ for all the data involved in an homotopy.

At this point it is convenient to introduce the following notation:

**1.7 Notation (LL-composition of 2-cells).** Given three 2-cells $\alpha$, $\beta$, and $\gamma$ that fit into a diagram as it follows, we write:

$$\beta \circ_\gamma \alpha \quad = \quad \begin{tikzcd}[row sep=small]
 & \cdot \arrow[dr] \arrow[dd, phantom, "\alpha\Downarrow"] & & \\
 & & \cdot \arrow[dr] \arrow[dd, phantom, "\gamma\Downarrow"] & \\
\cdot \arrow[uur] \arrow[r] \arrow[ddr] & \cdot \arrow[ur] \arrow[dr] & & \cdot \\
 & & \cdot \arrow[ur] & \\
 & \cdot \arrow[ur] \arrow[uu, phantom, "\beta\Downarrow"] & &
\end{tikzcd}$$

Thus, $\beta \circ_\gamma \alpha$ is our notation for the 2-cell between the top and the bottom composites of arrows. It should be thought of as the "composite of $\beta$ with $\alpha$ over $\gamma$".

Homotopies compose: Consider a third premorphism $\begin{tikzcd}[row sep=small, column sep=small]
 & A \arrow[dr, "u_3"] \arrow[dd, phantom, "\xi_3\Downarrow"] & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] & & C_3 \\
 & B \arrow[ur, "v_3"'] &
\end{tikzcd}$

and an homotopy $\begin{tikzcd}[row sep=small, column sep=small]
 & C_2 \arrow[dr, "w_2'"] \arrow[dd, phantom, "\alpha'\Downarrow"] & \\
B \arrow[ur, "v_2"] \arrow[dr, "v_3"'] & & C' \\
 & C_3 \arrow[ur, "w_3'"'] &
\end{tikzcd}$, $\begin{tikzcd}[row sep=small, column sep=small]
 & C_2 \arrow[dr, "w_2'"] \arrow[dd, phantom, "\beta'\Downarrow"] & \\
A \arrow[ur, "u_2"] \arrow[dr, "u_3"'] & & C' \\
 & C_3 \arrow[ur, "w_3'"'] &
\end{tikzcd}$, $(\alpha', \beta') : \xi_2 \Rightarrow \xi_3$. Use axiom F1 to obtain an invertible 2-cell $\begin{tikzcd}[row sep=small, column sep=small]
 & C \arrow[dr, "h"] \arrow[dd, phantom, "\gamma\Downarrow"] & \\
C_2 \arrow[ur, "w_2"] \arrow[dr, "w_2'"'] & & H \\
 & C' \arrow[ur, "h'"'] &
\end{tikzcd}$. This determines a pair of 2-cells:

$$\begin{tikzcd}[row sep=small]
 & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\alpha\Downarrow"] & & \\
 & & C \arrow[dr, "h"] \arrow[dd, phantom, "\gamma\Downarrow"] & \\
B \arrow[uur, "v_1"] \arrow[r, "v_2"] \arrow[ddr, "v_3"'] & C_2 \arrow[ur, "w_2"] \arrow[dr, "w_2'"'] & & H \\
 & & C' \arrow[ur, "h'"'] & \\
 & C_3 \arrow[ur, "w_3'"'] \arrow[uu, phantom, "\alpha'\Downarrow"] & &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[row sep=small]
 & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\beta\Downarrow"] & & \\
 & & C \arrow[dr, "h"] \arrow[dd, phantom, "\gamma\Downarrow"] & \\
A \arrow[uur, "u_1"] \arrow[r, "u_2"] \arrow[ddr, "u_3"'] & C_2 \arrow[ur, "w_2"] \arrow[dr, "w_2'"'] & & H \\
 & & C' \arrow[ur, "h'"'] & \\
 & C_3 \arrow[ur, "w_3'"'] \arrow[uu, phantom, "\beta'\Downarrow"] & &
\end{tikzcd}$$

which defines a homotopy $\xi_1 \Rightarrow \xi_3$. The corresponding LL equation follows easily from the LL equations for $(\alpha,\ \beta)$ and $(\alpha',\ \beta')$.

Using notation 1.7, we have:

**1.8 Proposition (vertical composition of homotopies).** *Given* $(\alpha,\ \beta) : \xi_1 \Rightarrow \xi_2$ *and* $(\alpha',\ \beta') : \xi_2 \Rightarrow \xi_3$, *there exists an appropriate* $\gamma$ *and a homotopy* $(\alpha' \circ_\gamma \alpha,\ \beta' \circ_\gamma \beta) : \xi_1 \Rightarrow \xi_3$. □

Homotopies are generated by composition out of two basic ones. The proof of the following is immediate:

**1.9 Proposition.** *Every pair of premorphisms* $\xi_1$, $\xi_2$ *and pair of 2-cells* $\alpha$, $\beta$ *that fit as follows determine two basic homotopies:*

$$(\alpha,\ id_1) : \begin{tikzcd} & A \arrow[dr, "u_1"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_1\Downarrow"] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & A \arrow[dr, "u_1"] & & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_1\Downarrow"] & & C_1 \arrow[dr, "w_1"] & \\ & B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\alpha\Downarrow"] & & C \\ & & C_2 \arrow[ur, "w_2"'] & \end{tikzcd}$$

$$(id_2,\ \beta) : \begin{tikzcd} & & C_1 \arrow[dr, "w_1"] & \\ & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[rr, phantom, "\beta\Downarrow"] & & C \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_2\Downarrow"] & & C_2 \arrow[ur, "w_2"'] & \\ & B \arrow[ur, "v_2"'] & & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & A \arrow[dr, "u_2"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_2\Downarrow"] & & C_2 \\ & B \arrow[ur, "v_2"'] & \end{tikzcd}$$

*where* $id_1$ *and* $id_2$ *are the identity 2-cells corresponding to the arrows* $w_1u_1$ *and* $w_2v_2$ *respectively. When the pair* $(\alpha,\ \beta)$ *satisfies the LL equation, then the composite (over the identity 2-cell of the identity arrow of* $C_2$*) of these basic homotopies is defined, and it is equal to the homotopy determined by* $(\alpha,\ \beta)$. □

Premorphisms compose: Given two premorphisms

$$(x,\, A) \xrightarrow{\ \xi\ } (y,\, B) \xrightarrow{\ \zeta\ } (z,\, C)\ ,\quad \begin{tikzcd}[column sep=small, row sep=small] & A \arrow[dr, "u"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & S \\ & B \arrow[ur, "v"'] & \end{tikzcd}\ ,\quad \begin{tikzcd}[column sep=small, row sep=small] & B \arrow[dr, "h"] & \\ F \arrow[ur, "y"] \arrow[dr, "z"'] \arrow[rr, phantom, "\zeta\Downarrow"] & & T \\ & C \arrow[ur, "k"'] & \end{tikzcd}\ , \text{ use}$$

axiom F1 to obtain invertible $\begin{tikzcd}[column sep=small, row sep=small] & S \arrow[dr, "s"] & \\ B \arrow[ur, "v"] \arrow[dr, "h"'] \arrow[rr, phantom, "\gamma\Downarrow"] & & H \\ & T \arrow[ur, "t"'] & \end{tikzcd}$. According to 1.7 this determines a premorphism $\zeta \circ_\gamma \xi$ between $(x,\, A)$ and $(z,\, C)$ that we take as a composite of $\xi$ with $\zeta$. Thus:

$$\zeta \circ_\gamma \xi \quad = \quad \begin{tikzcd} & A \arrow[dr, "u"] \arrow[d, phantom, "\xi\Downarrow"] & & \\ & {} & S \arrow[dr, "s"] & \\ F \arrow[uur, "x"] \arrow[r, "y"] \arrow[ddr, "z"'] & B \arrow[ur, "v"] \arrow[dr, "h"'] \arrow[rr, phantom, "\gamma\Downarrow"] \arrow[d, phantom, "\zeta\Downarrow"] & & H \\ & {} & T \arrow[ur, "t"'] & \\ & C \arrow[ur, "k"'] & & \end{tikzcd}$$

We have:

**1.10 Proposition (horizontal composition of premorphisms).** *Given $\xi : (x,\, A) \to (y,\, B)$ and $\zeta : (y,\, B) \to (z,\, C)$, there exists an appropriate $\gamma$ and $\zeta \circ_\gamma \xi : (x,\, A) \to (z,\, C)$.* □

Homotopies also compose horizontally:

**1.11 Proposition (horizontal composition of homotopies).** *Consider composable premorphisms and homotopies as follows:*

$$(x,\, A) \underset{\xi_2}{\overset{\xi_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\scriptstyle (\alpha,\beta)\Downarrow}}}} (y,\, B) \underset{\zeta_2}{\overset{\zeta_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\scriptstyle (\varepsilon,\delta)\Downarrow}}}} (z,\, C)\ .$$

*Then, given any two composites $\zeta_1 \circ_{\gamma_1} \xi_1$ and $\zeta_2 \circ_{\gamma_2} \xi_2$, there exists an homotopy $\zeta_1 \circ_{\gamma_1} \xi_1 \Rightarrow \zeta_2 \circ_{\gamma_2} \xi_2$.*

*Proof.* Consider the given homotopies and their LL equations:

$$
\begin{tikzcd}
& A \arrow[dr, "u_1"] & & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_1\Downarrow"] & & S_1 \arrow[dr, "s_1"] & \\
& B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\alpha\Downarrow"] & & S \\
& & S_2 \arrow[ur, "s_2"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & S_1 \arrow[dr, "s_1"] & \\
& A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\beta\Downarrow"] & & S \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_2\Downarrow"] & & S_2 \arrow[ur, "s_2"'] & \\
& B \arrow[ur, "v_2"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& B \arrow[dr, "h_1"] & & \\
F \arrow[ur, "y"] \arrow[dr, "z"'] \arrow[rr, phantom, "\zeta_1\Downarrow"] & & T_1 \arrow[dr, "t_1"] & \\
& C \arrow[ur, "k_1"'] \arrow[dr, "k_2"'] \arrow[rr, phantom, "\varepsilon\Downarrow"] & & T \\
& & T_2 \arrow[ur, "t_2"'] &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& & T_1 \arrow[dr, "t_1"] & \\
& B \arrow[ur, "h_1"] \arrow[dr, "h_2"'] \arrow[rr, phantom, "\delta\Downarrow"] & & T \\
F \arrow[ur, "y"] \arrow[dr, "z"'] \arrow[rr, phantom, "\zeta_2\Downarrow"] & & T_2 \arrow[ur, "t_2"'] & \\
& C \arrow[ur, "k_2"'] & &
\end{tikzcd}
$$

and consider the horizontal composites of the premorphisms:

$$
\begin{tikzcd}
& A \arrow[dr, "u_1"] \arrow[d, phantom, "\xi_1\Downarrow"] & & \\
& & S_1 \arrow[dr, "r_1"] \arrow[d, phantom, "\gamma_1\Downarrow"] & \\
F \arrow[uur, "x"] \arrow[r, "y"] \arrow[ddr, "z"'] & B \arrow[ur, "v_1"] \arrow[dr, "h_1"'] \arrow[d, phantom, "\zeta_1\Downarrow"] & & H_1 \\
& & T_1 \arrow[ur, "l_1"'] & \\
& C \arrow[ur, "k_1"'] & &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
& A \arrow[dr, "u_2"] \arrow[d, phantom, "\xi_2\Downarrow"] & & \\
& & S_2 \arrow[dr, "r_2"] \arrow[d, phantom, "\gamma_2\Downarrow"] & \\
F \arrow[uur, "x"] \arrow[r, "y"] \arrow[ddr, "z"'] & B \arrow[ur, "v_2"] \arrow[dr, "h_2"'] \arrow[d, phantom, "\zeta_2\Downarrow"] & & H_2 \\
& & T_2 \arrow[ur, "l_2"'] & \\
& C \arrow[ur, "k_2"'] & &
\end{tikzcd}
$$

We shall produce a homotopy between these composites.

First use axiom F1 to obtain $\phi_1$ , $\phi_2$ as follows:

$$\begin{tikzcd}
& S_1 \arrow[dr, "r_1"] \arrow[dd, phantom, "\gamma_1\Downarrow"] & & \\
& & H_1 \arrow[dr, "n_1"] \arrow[dd, phantom, "\phi_1\Downarrow"] & \\
B \arrow[uur, "v_1"] \arrow[r, "h_1"] \arrow[ddr, "h_2"'] & T_1 \arrow[ur, "l_1"] \arrow[dr, "t_1"'] \arrow[dd, phantom, "\delta\Downarrow"] & & U_1 \\
& & T \arrow[ur, "m_1"'] & \\
& T_2 \arrow[ur, "t_2"'] & &
\end{tikzcd} , \qquad
\begin{tikzcd}
& S_1 \arrow[dr, "s_1"] \arrow[dd, phantom, "\alpha\Downarrow"] & & \\
& & S \arrow[dr, "n_2"] \arrow[dd, phantom, "\phi_2\Downarrow"] & \\
B \arrow[uur, "v_1"] \arrow[r, "v_2"] \arrow[ddr, "h_2"'] & S_2 \arrow[ur, "s_2"] \arrow[dr, "r_2"'] \arrow[dd, phantom, "\gamma_2\Downarrow"] & & U_2 \\
& & H_2 \arrow[ur, "m_2"'] & \\
& T_2 \arrow[ur, "l_2"'] & &
\end{tikzcd}$$

Then use axiom F2 to obtain $\theta_1$ , $\theta_2$ satisfying the LL equation:

$$\begin{tikzcd}
& S_1 \arrow[dr, "r_1"] \arrow[dd, phantom, "\gamma_1\Downarrow"] & & & \\
& & H_1 \arrow[dr, "n_1"] \arrow[dd, phantom, "\phi_1\Downarrow"] & & \\
B \arrow[uur, "v_1"] \arrow[r, "h_1"] \arrow[ddr, "h_2"'] & T_1 \arrow[ur, "l_1"] \arrow[dr, "t_1"'] \arrow[dd, phantom, "\delta\Downarrow"] & & W_1 \arrow[ddr, "w_1"] \arrow[dddd, phantom, "\theta_1\Downarrow"] & \\
& & T \arrow[ur, "m_1"'] & & \\
& T_2 \arrow[ur, "t_2"'] \arrow[dr, "l_2"'] & & & W \\
& & H_2 \arrow[dr, "m_2"'] & & \\
& & & W_2 \arrow[uur, "w_2"'] &
\end{tikzcd} = 
\begin{tikzcd}
& & & W_1 \arrow[ddr, "w_1"] \arrow[dddd, phantom, "\theta_2\Downarrow"] & \\
& & H_1 \arrow[ur, "n_1"] & & \\
& S_1 \arrow[ur, "r_1"] \arrow[dr, "s_1"] \arrow[dd, phantom, "\alpha\Downarrow"] & & & W \\
& & S \arrow[dr, "n_2"] \arrow[dd, phantom, "\phi_2\Downarrow"] & & \\
B \arrow[uur, "v_1"] \arrow[r, "v_2"] \arrow[ddr, "h_2"'] & S_2 \arrow[ur, "s_2"] \arrow[dr, "r_2"'] \arrow[dd, phantom, "\gamma_2\Downarrow"] & & W_2 \arrow[uur, "w_2"'] & \\
& & H_2 \arrow[ur, "m_2"'] & & \\
& T_2 \arrow[ur, "l_2"'] & & &
\end{tikzcd}$$

The homotopy is given by the following pair of 2-cells:

$$\begin{tikzcd}
& & H_1 \arrow[dr, "n_1"] \arrow[dd, phantom, "\phi_1\Downarrow"] & & \\
& T_1 \arrow[ur, "l_1"] \arrow[dr, "t_1"'] \arrow[dd, phantom, "\varepsilon\Downarrow"] & & W_1 \arrow[ddr, "w_1"] \arrow[dddd, phantom, "\theta_1\Downarrow"] & \\
C \arrow[ur, "k_1"] \arrow[dr, "k_2"'] & & T \arrow[ur, "m_1"'] & & \\
& T_2 \arrow[ur, "t_2"'] \arrow[dr, "l_2"'] & & & W \\
& & H_2 \arrow[dr, "m_2"'] & & \\
& & & W_2 \arrow[uur, "w_2"'] &
\end{tikzcd} , \qquad
\begin{tikzcd}
& & & W_1 \arrow[ddr, "w_1"] \arrow[dddd, phantom, "\theta_2\Downarrow"] & \\
& & H_1 \arrow[ur, "n_1"] & & \\
& S_1 \arrow[ur, "r_1"] \arrow[dr, "s_1"] \arrow[dd, phantom, "\beta\Downarrow"] & & & W \\
A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] & & S \arrow[dr, "n_2"] \arrow[dd, phantom, "\phi_2\Downarrow"] & & \\
& S_2 \arrow[ur, "s_2"] \arrow[dr, "r_2"'] & & W_2 \arrow[uur, "w_2"'] & \\
& & H_2 \arrow[ur, "m_2"'] & &
\end{tikzcd}$$

The corresponding LL equation is :

$$
\begin{tikzcd}
 & A \arrow[dr, "u_1"] \arrow[dd, phantom, "\xi_1\Downarrow"] & & & & \\
 & & S_1 \arrow[dr, "r_1"] \arrow[dd, phantom, "\gamma_1\Downarrow"] & & & \\
F \arrow[uur, "x"] \arrow[r, "y"] \arrow[ddr, "z"'] & B \arrow[ur, "v_1"] \arrow[dr, "h_1"] \arrow[dd, phantom, "\zeta_1\Downarrow"] & & H_1 \arrow[dr, "n_1"] \arrow[dd, phantom, "\phi_1\Downarrow"] & & \\
 & & T_1 \arrow[ur, "l_1"'] \arrow[dr, "m_1"] \arrow[dd, phantom, "\varepsilon\Downarrow"] & & W_1 \arrow[ddr, "w_1"] & \\
 & C \arrow[ur, "k_1"'] \arrow[dr, "k_2"'] & & T \arrow[ur, "m_1"'] \arrow[rr, phantom, "\theta_1\Downarrow"] & & \\
 & & T_2 \arrow[ur, "t_2"'] \arrow[dr, "l_2"'] & & & W \\
 & & & H_2 \arrow[dr, "m_2"'] & & \\
 & & & & W_2 \arrow[uur, "w_2"'] &
\end{tikzcd}
\qquad =
$$

$$
= \qquad
\begin{tikzcd}
 & & & & W_1 \arrow[ddr, "w_1"] \arrow[dddd, phantom, "\theta_2\Downarrow"] & \\
 & & & H_1 \arrow[ur, "n_1"] & & \\
 & & S_1 \arrow[ur, "r_1"] \arrow[dr, "s_1"] \arrow[dd, phantom, "\beta\Downarrow"] & & & W \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[dd, phantom, "\xi_2\Downarrow"] & & S \arrow[dr, "n_2"] \arrow[dd, phantom, "\phi_2\Downarrow"] & & \\
 & & S_2 \arrow[ur, "s_2"] \arrow[dr, "r_2"] \arrow[dd, phantom, "\gamma_2\Downarrow"] & & W_2 \arrow[uur, "w_2"'] & \\
F \arrow[uur, "x"] \arrow[r, "y"] \arrow[ddr, "z"'] & B \arrow[ur, "v_2"] \arrow[dr, "h_2"'] \arrow[dd, phantom, "\zeta_2\Downarrow"] & & H_2 \arrow[ur, "m_2"'] & & \\
 & & T_2 \arrow[ur, "l_2"'] & & & \\
 & C \arrow[ur, "k_2"'] & & & &
\end{tikzcd}
$$

To pass from the left side to the right side of this equation use the LL equations of $(\varepsilon,\ \delta)$, $(\theta_1,\ \theta_2)$ and $(\alpha,\ \beta)$, in this order. $\square$

**1.12 Definition (equivalence of premorphisms).** Two premorphisms $\xi_1$, $\xi_2$ are said to be *equivalent* when there exists a homotopy $(\alpha,\ \beta) : \xi_1 \Rightarrow \xi_2$. We shall write $\xi_1 \sim \xi_2$.

Equivalence is indeed an equivalence relation. Proposition 1.8 shows transitivity, the inverse 2-cells define an homotopy $(\alpha^{-1},\ \beta^{-1}) : \xi_2 \Rightarrow \xi_1$ in the opposite direction, which shows symmetry, while reflexivity is obvious.

**1.13 Definition (Category $\mathcal{L}(F)$).** We define a category $\mathcal{L}(F)$ with objects pairs $(x,\ A)$, $x \in FA$. Morphisms are equivalence classes of premorphisms, and composition is defined by composing representative premorphisms.

It follows from 1.11 that composition is, up to equivalence, independent of the choice of representatives, and independent of the choice of the 2-cells given by axiom F1 when composing each pair of representatives. Since associativity holds and identities exist, this construction actually does define a category.

**Some lemmas on pre-2-filtered categories**

We establish now a few lemmas which are useful when proving the fundamental properties of the construction LL.

**1.14 Lemma.** *Given any pair of equivalent premorphisms*

$$\begin{tikzcd}[column sep=small, row sep=small] & A \arrow[dr, "u_1"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & \xi_1\Downarrow & C_1 \\ & A \arrow[ur, "v_1"'] & \end{tikzcd} \sim \begin{tikzcd}[column sep=small, row sep=small] & A \arrow[dr, "u_2"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & \xi_2\Downarrow & C_2 \\ & A \arrow[ur, "v_2"'] & \end{tikzcd}$$

*, if $u_1 = v_1$ and $u_2 = v_2$, then we can choose an homotopy $(\alpha,\ \beta) : \xi_1 \Rightarrow \xi_2$ with $\alpha = \beta$.*

*Proof.* It follows immediately from axiom F2. $\square$

**1.15 Lemma.** *Given a finite family of 2-cells*

$$\begin{tikzcd}[column sep=small, row sep=small] & A \arrow[dr, "u_i"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] & \gamma_i\Downarrow & C_i \\ & B \arrow[ur, "v_i"'] & \end{tikzcd},$$

*$i = 1 \ldots n$, there exists $A \xrightarrow{u} C$, $B \xrightarrow{v} C$, $C_i \xrightarrow{w_i} C$, $i = 1 \ldots n$, with invertible 2-cells $\alpha_i$, $\beta_i$, such that the 2-cells*

$$\begin{tikzcd} & A \arrow[dr, "u_i"'] \arrow[rr, bend left, "u"] & \arrow[d, phantom, "\beta_i\Downarrow"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] & \gamma_i\Downarrow & C_i \arrow[r, "w_i"] & C \\ & B \arrow[ur, "v_i"] \arrow[rr, bend right, "v"'] & \arrow[u, phantom, "\alpha_i\Downarrow"] & \end{tikzcd} \qquad i = 1 \ldots n$$

*are all equal.*

*Given a second family of 2-cells* $\begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dr, "u_i"] & \\ H \arrow[ur, "h"] \arrow[dr, "l"'] \arrow[rr, phantom, "\delta_i\Downarrow"] & & C_i \\ & B \arrow[ur, "v_i"'] & \end{tikzcd}$ *(with same* $u_i$*,* $v_i$*,* $C_i$ *), we can assume that the same* $u$*,* $v$*,* $w_i$*,* $\alpha_i$*,* $\beta_i$ *, also equalize the 2-cells of the second family.*

*Proof.* Axiom F2 provides the case $n = 2$ with $u = w_1u_1$ , $v = w_2v_2$ , $\alpha_1 = \alpha$ , $\beta_1 = id$ , $\alpha_2 = id$ , and $\beta_2 = \beta$ . Using this case, induction is straightforward. For the second part, if the 2-cells of the second family are not yet equalized, use the lemma again (and patch the new 2-cells also into the first family) ☐

From axiom F2 we deduce

**1.16 Lemma.** *Given any 2-cells* $\begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dr, "u_1"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\gamma_1\Downarrow"] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd}$ , $\begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dr, "u_2"] & \\ E \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[rr, phantom, "\gamma_2\Downarrow"] & & C_2 \\ & B \arrow[ur, "v_2"'] & \end{tikzcd}$ *and an object* $F \xrightarrow{x} E$ *, the premorphisms* $\begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dr, "u_1"] & \\ F \arrow[ur, "fx"] \arrow[dr, "gx"'] \arrow[rr, phantom, "\gamma_1 x\Downarrow"] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd}$ , $\begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dr, "u_2"] & \\ F \arrow[ur, "fx"] \arrow[dr, "gx"'] \arrow[rr, phantom, "\gamma_2 x\Downarrow"] & & C_2 \\ & B \arrow[ur, "v_2"'] & \end{tikzcd}$ *are equivalent.* ☐

From proposition 1.9 it follows that

**1.17 Lemma.** *Given a pair of premorphism* $\xi_1$ , $\xi_2$ *and a pair of invertible two cells* $\alpha$ , $\beta$ *that fit as follows, we have:*

$$\begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dr, "u_1"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_1\Downarrow"] & & C_1 \\ & B \arrow[ur, "v_1"'] & \end{tikzcd} \quad \sim \quad \begin{tikzcd}[row sep=small, column sep=small] & A \arrow[dr, "u_1"] & & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_1\Downarrow"] & & C_1 \arrow[dr, "w_1"] & \\ & B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] \arrow[rr, phantom, "\alpha\Downarrow"] & & C \\ & & C_2 \arrow[ur, "w_2"'] & \end{tikzcd}$$

*and*

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & C_1 \arrow[dr, "w_1"] & \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"'] \arrow[rr, phantom, "\beta\Downarrow"] & & C \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_2\Downarrow"] & & C_2 \arrow[ur, "w_2"'] & \\
 & B \arrow[ur, "v_2"'] & &
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}[column sep=small, row sep=small]
 & A \arrow[dr, "u_2"] & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi_2\Downarrow"] & & C_2 \\
 & B \arrow[ur, "v_2"'] &
\end{tikzcd}
$$

*When the pair $\alpha$, $\beta$ satisfy the LL equation, then transitivity applied to these two equivalences yields the equivalence $\xi_1 \sim \xi_2$.* □

From this lemma and transitivity of equivalence we deduce

**1.18 Lemma.** *Given a premorphism $\xi$, and a pair of invertible two cells $\alpha$, $\beta$ that fit as follows, we have:*

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & A \arrow[dr, "u"] & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \\
 & B \arrow[ur, "v"'] &
\end{tikzcd}
\quad \sim \quad
\begin{tikzcd}[column sep=small, row sep=small]
 & A \arrow[dr, "u"'] \arrow[drr, bend left, "s"] \arrow[dr, phantom, shift left=3ex, "\alpha\Downarrow"] & & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \arrow[r, "w"] & D \\
 & B \arrow[ur, "v"] \arrow[urr, bend right, "t"'] \arrow[ur, phantom, shift right=3ex, "\beta\Downarrow"] & &
\end{tikzcd}
$$

□

### The universal property of the construction LL

A *pseudocone* for a 2-functor $F$ with vertex the category $\mathcal{X}$ is a pseudonatural transformation $F \overset{h}{\Longrightarrow} \mathcal{X}$ from $F$ to the 2-functor which is constant at $\mathcal{X}$. Explicitly, it consists of a family of functors $(h_A : FA \to \mathcal{X})_{A \in \mathcal{A}}$, and a family of invertible natural transformations $(h_u : h_B \circ u \to h_A)_{(A \xrightarrow{u} B) \in \mathcal{A}}$. A morphism $h \overset{\varphi}{\Longrightarrow} l$ of pseudocones (with same vertex) is a modification; as such, it consists of a family of natural transformations $(h_A \overset{\varphi_A}{\Longrightarrow} l_A)_{A \in \mathcal{A}}$. In accordance with notation 1.6, we have

$$
\begin{tikzcd}
A \arrow[d, "u"'] \arrow[dr, "h_A"] \arrow[dr, phantom, shift right=3ex, "h_u\Uparrow"] & \\
B \arrow[r, "h_B"'] & \mathcal{X}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, bend left, "h_A"] \arrow[r, bend right, "l_A"'] \arrow[r, phantom, "\varphi_A\Downarrow"] & \mathcal{X}\,.
\end{tikzcd}
$$

This data is subject to the equations:

PC0. $\quad h_{id_A} = id_{h_A}$

PC1.

$$
\begin{tikzcd}
A \arrow[d, "u"'] \arrow[drr, "h_A"] \arrow[dr, phantom, "h_u\Uparrow"] & & \\
B \arrow[rr, "h_B"] \arrow[d, "v"'] \arrow[dr, phantom, "h_v\Uparrow"] & & \mathcal{X} \\
C \arrow[urr, "h_C"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[d, "u"'] \arrow[drr, "h_A"] & & \\
B \arrow[d, "v"'] \arrow[rr, phantom, "h_{vu}\Uparrow"] & & \mathcal{X} \\
C \arrow[urr, "h_C"'] & &
\end{tikzcd}
$$

PC2.

$$
\begin{tikzcd}
A \arrow[d, bend right, "u"'] \arrow[d, bend left, "v"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] \arrow[drr, "h_A"] \arrow[dr, phantom, "h_v\Uparrow"] & & \\
B \arrow[rr, "h_B"'] & & \mathcal{X}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[d, "u"'] \arrow[drr, "h_A"] \arrow[dr, phantom, "h_u\Uparrow"] & & \\
B \arrow[rr, "h_B"'] & & \mathcal{X}
\end{tikzcd}
$$

PCM.

$$
\begin{tikzcd}
A \arrow[d, "u"'] \arrow[drr, bend left, "l_A"] \arrow[drr, "h_A"'] \arrow[drr, phantom, bend left=15, "\varphi_A\Uparrow"] & & \\
B \arrow[rr, "h_B"'] \arrow[ur, phantom, "h_u\Uparrow"] & & \mathcal{X}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[d, "u"'] \arrow[drr, "l_A"] \arrow[dr, phantom, "l_u\Uparrow"] & & \\
B \arrow[rr, bend left, "l_B"] \arrow[rr, bend right, "h_B"'] \arrow[rr, phantom, "\varphi_B\Uparrow"] & & \mathcal{X}
\end{tikzcd}
$$

Given a pseudo cone $F \overset{h}{\Longrightarrow} \mathcal{Z}$ and a 2-functor $\mathcal{Z} \xrightarrow{s} \mathcal{X}$, it is clear and straightforward how to define a pseudocone $F \overset{sh}{\Longrightarrow} \mathcal{X}$ which is the composite $F \overset{h}{\Longrightarrow} \mathcal{Z} \xrightarrow{s} \mathcal{X}$; this determines a functor $\mathcal{C}at[\mathcal{Z}, \mathcal{X}] \times \mathcal{PC}[F, \mathcal{Z}] \longrightarrow \mathcal{PC}[F, \mathcal{X}]$.

**1.19 Theorem.** *Let $A \xrightarrow{u} B$ in $FA$ and $x \xrightarrow{\xi} y$ in $\mathcal{A}$. The following formulas define a pseudocone $F \overset{\lambda}{\Longrightarrow} \mathcal{L}(F)$:*

$$
\lambda_A(x) = F \xrightarrow{x} A\,, \qquad
\lambda_A(\xi) =
\begin{tikzcd}[column sep=small, row sep=small]
& A \arrow[dr, "id"] & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & A \\
& A \arrow[ur, "id"'] &
\end{tikzcd}\,, \qquad
\lambda_u(x) =
\begin{tikzcd}[column sep=small, row sep=small]
& B \arrow[dr, "id"] & \\
F \arrow[ur, "ux"] \arrow[dr, "x"'] \arrow[rr, phantom, "id\Downarrow"] & & B \\
& A \arrow[ur, "u"'] &
\end{tikzcd}
$$

*which induces by composition an equivalence of categories $\mathcal{C}at[\mathcal{L}(F), \mathcal{X}] \xrightarrow{\cong} \mathcal{PC}[F, \mathcal{X}]$, and this equivalence is actually an isomorphism. We have:*

*for all* $h$ *there exists a unique* $\widetilde{h}$ *such that* $\widetilde{h}\,\lambda = h$ *:*

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\lambda"] \arrow[dr, Rightarrow, "\forall h"'] & \mathcal{L}(F) \arrow[d, dashed, "\exists !\, \widetilde{h}"] \arrow[dl, phantom, "\equiv"] \\ & \mathcal{X} \end{tikzcd} .$$

*Proof.* Functoriality of $\lambda_A$, naturality of $\lambda_u$ and equation PC1 hold tautologically. The validity of PC2 means the following equivalence of premorphisms:

$$\begin{tikzcd}[row sep=small] & & B \arrow[dr, "id"] & & \\ & & \arrow[d, phantom, "\gamma x \Downarrow"] & B \arrow[dr, "id"] & \\ F \arrow[uurr, "ux"] \arrow[rr, "vx"] \arrow[ddrr, "x"'] & & B \arrow[ur, "id"] \arrow[dr, "id"'] & id\Downarrow & B \\ & & id\Downarrow & B \arrow[ur, "id"'] & \\ & & A \arrow[ur, "v"'] & & \end{tikzcd} \quad \sim \quad \begin{tikzcd} & B \arrow[dr, "id"] & \\ F \arrow[ur, "ux"] \arrow[dr, "x"'] & id\Downarrow & B \\ & A \arrow[ur, "u"'] & \end{tikzcd}$$

which is given by lemma 1.16.

We pass now to prove the universal property. Given $F \overset{h}{\Longrightarrow} \mathcal{X}$, define $\widetilde{h}$ by the formulas: for $\begin{tikzcd} & A \arrow[dr, "u"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & \xi\Downarrow & C \\ & B \arrow[ur, "v"'] & \end{tikzcd}$ in $\mathcal{L}(F)$,

$$\widetilde{h}\,(x) \;=\; (F \xrightarrow{x} A \xrightarrow{h_A} \mathcal{X}), \quad \widetilde{h}\,(\xi) \;=\; \begin{tikzcd} & A \arrow[dr, "u"] \arrow[drr, bend left, "h_A"] \arrow[dr, phantom, shift left=4ex, "h_u^{-1}\Downarrow"] & & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] & \xi\Downarrow & C \arrow[r, "h_C"] & \mathcal{X} \\ & B \arrow[ur, "v"'] \arrow[urr, bend right, "h_B"'] \arrow[ur, phantom, shift right=4ex, "h_v\Downarrow"] & & \end{tikzcd}$$

We have to show that the definition of $\widetilde{h}\,(\xi)$ is compatible with the equivalence of premorphisms. It is enough to consider the two cases in lemma 1.17. For the first case we have to show the equation

(1)
$$
\begin{tikzcd}
 & A \arrow[dr, "u"] \arrow[ddrr, bend left=50, "h_A"] \arrow[dr, phantom, shift left=6, "h_u^{-1}\Downarrow"] & & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \arrow[dr, "h_C"] & \\
 & B \arrow[ur, "v"'] \arrow[rr, bend right, "h_B"'] \arrow[rr, phantom, "h_v\Downarrow"] & & \mathcal{X}
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[dr, "u"] \arrow[dddrrr, bend left=50, "h_A"] & & & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \arrow[dr, "w"] \arrow[rr, phantom, "h_{wu}^{-1}\Downarrow"] & & \\
 & B \arrow[ur, "v"'] \arrow[dr, "s"'] \arrow[rr, phantom, "\alpha\Downarrow"] \arrow[drrr, bend right=60, "h_B"'] & & D \arrow[dr, "h_D"] & \\
 & & S \arrow[ur, "r"'] \arrow[rr, phantom, "h_{rs}\Downarrow"] & & \mathcal{X}
\end{tikzcd}
$$

Consider the following equation which follows from PC1:

(2)
$$
\begin{tikzcd}
 & A \arrow[dr, "u"] \arrow[ddrr, bend left=50, "h_A"] \arrow[dr, phantom, shift left=6, "h_u^{-1}\Downarrow"] & & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \arrow[dr, "w"] \arrow[ddr, bend left, "h_C"] \arrow[ddr, bend right=60, "h_C"'] \arrow[d, phantom, "h_w\Downarrow"] \arrow[dr, phantom, shift left=4, "h_w^{-1}\Downarrow"] & \\
 & B \arrow[ur, "v"'] \arrow[drr, bend right=60, "h_B"'] \arrow[drr, phantom, "h_v\Downarrow"] & & D \arrow[d, "h_D"] \\
 & & & \mathcal{X}
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[dr, "u"] \arrow[ddrr, bend left=50, "h_A"] & & & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \arrow[dr, "w"] \arrow[r, phantom, "h_{wu}^{-1}\Downarrow"] & \\
 & B \arrow[ur, "v"'] \arrow[drr, bend right=60, "h_B"'] \arrow[drr, phantom, "h_{wv}\Downarrow"] & & D \arrow[d, "h_D"] \\
 & & & \mathcal{X}
\end{tikzcd}
$$

The right hand sides of equations (1) and (2) are equal by PC2, while the left hand sides are clearly equal. The second case in lemma 1.17 is treated in a similar manner. Functoriality of $\tilde{h}\ (\xi)$ follows from PC1 and PC2 with the same type of techniques as above. Finally, the equation $\tilde{h}\,\lambda = h$ is immediate for the whole cone structure. $\square$

We finish this section with a lemma which follows from lemma 1.14

**1.20 Lemma.** *Given two arrows* $x \overset{\xi_1}{\underset{\xi_2}{\rightrightarrows}} y$ *in* $FA$, *if* $\lambda_A(x) = \lambda_A(y)$ *in* $\mathcal{L}(F)$, *then there exists* $A \xrightarrow{w} C$ *such that* $wx = wy$ *in* $FC$. $\square$

## 2. 2-Filtered 2-Categories

**2.1 Definition.** A 2-category $\mathcal{A}$ is defined to be *pseudo 2-filtered* when it is pre 2-filtered and satisfies the stronger form of axiom F1:

FF1.

$$\textit{Given}\ \begin{tikzcd}[row sep=small] & A \\ E_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & \\ & B \end{tikzcd}, \begin{tikzcd}[row sep=small] & A \\ E_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] & \\ & B \end{tikzcd}\ \textit{there exists}\ \begin{tikzcd}[row sep=small] & A \arrow[dr, "u"] & \\ E_1 \arrow[ur, "f_1"] \arrow[dr, "g_1"'] \arrow[rr, phantom, "\gamma_1\Downarrow"] & & C \\ & B \arrow[ur, "v"'] & \end{tikzcd}, \begin{tikzcd}[row sep=small] & A \arrow[dr, "u"] & \\ E_2 \arrow[ur, "f_2"] \arrow[dr, "g_2"'] \arrow[rr, phantom, "\gamma_2\Downarrow"] & & C \\ & B \arrow[ur, "v"'] & \end{tikzcd},$$

*with* $\gamma_1$ *and* $\gamma_2$ *invertible* 2- *cells*.

It is defined to be *2-filtered* when it is pseudo 2-filtered, non empty, and satisfies in addition the following axiom.

F0.

$$\textit{Given}\quad \begin{matrix} A \\ \\ B \end{matrix} \qquad \textit{there exists}\quad \begin{tikzcd}[row sep=small] A \arrow[dr, "u"] & \\ & C \\ B \arrow[ur, "v"'] & \end{tikzcd}.$$

As was the case for axiom F1, in the presence of axiom WF3, axiom FF1 can be replaced by the weaker version in which we do not require the 2-cells $\gamma_1$ and $\gamma_2$ to be invertible.

When $\mathcal{A}$ is a trivial 2-category (the only 2-cells are the identities), axiom F0 is the usual axiom in the definition of filtered category, while our axiom FF1 is equivalent to the conjunction of the two axioms PS1 and PS2 in the definition of pseudofiltered category (cf [1] Exposé I).

Two properties of the construction LL that follow for pseudo 2-filtered 2-categories and not for pre 2-filtered 2-categories are the following:

**2.2 Lemma.** *Given any morphism* $(x, A) \to (y, A)$ *in* $\mathcal{L}(F)$, *we can choose a representative premorphism* $\begin{tikzcd}[row sep=small] & A \arrow[dr, "u"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \\ & A \arrow[ur, "v"'] & \end{tikzcd}$ *with* $u = v$.

*Proof.* Consider $\begin{tikzcd}[row sep=small] & A \arrow[dr, "u"] & \\ F \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \\ & A \arrow[ur, "v"'] & \end{tikzcd}$ and apply axiom FF1 to obtain invertible

2-cells $\alpha$, $\beta$ as follows:

$$
\begin{tikzcd}[column sep=small]
 & A \arrow[dr, "s"] \arrow[dd, phantom, "\alpha\Downarrow"] & \\
A \arrow[ur, "id"] \arrow[dr, "v"'] & & D \\
 & C \arrow[ur, "t"'] &
\end{tikzcd}
,\quad
\begin{tikzcd}[column sep=small]
 & A \arrow[dr, "s"] \arrow[dd, phantom, "\beta\Downarrow"] & \\
A \arrow[ur, "id"] \arrow[dr, "u"'] & & D \\
 & C \arrow[ur, "t"'] &
\end{tikzcd}
:\quad
\begin{tikzcd}
 & A \arrow[dr, "u"] \arrow[dd, phantom, "\xi\Downarrow"] \arrow[rr, bend left, "s"] & \arrow[d, phantom, "\beta\Downarrow"] & \\
F \arrow[ur, "x"] \arrow[dr, "y"'] & & C \arrow[r, "t"] \arrow[d, phantom, "\alpha^{-1}\Downarrow"] & D \\
 & A \arrow[ur, "v"] \arrow[urr, bend right, "s"'] & &
\end{tikzcd}
$$

The proof follows by lemma 1.18. $\square$

**2.3 Lemma.** *Given a finite family of premorphisms* $\begin{tikzcd}[column sep=small, row sep=small] & A \arrow[dr, "u_i"] \arrow[dd, phantom, "\xi_i\Downarrow"] & \\ F \arrow[ur, "x_i"] \arrow[dr, "y_i"'] & & C_i \\ & B \arrow[ur, "v_i"'] & \end{tikzcd}$ ,

*$i = 1 \ldots n$, there exists $A \xrightarrow{u} C$, $B \xrightarrow{v} C$, $C_i \xrightarrow{w_i} C$, $i = 1 \ldots n$, with invertible 2-cells $\alpha_i$, $\beta_i$ as in the diagram:*

$$
\begin{tikzcd}
 & A \arrow[dr, "u_i"] \arrow[dd, phantom, "\xi_i\Downarrow"] \arrow[rr, bend left, "u"] & \arrow[d, phantom, "\beta_i\Downarrow"] & \\
F \arrow[ur, "x_i"] \arrow[dr, "y_i"'] & & C_i \arrow[r, "w_i"] \arrow[d, phantom, "\alpha_i\Downarrow"] & C \\
 & B \arrow[ur, "v_i"] \arrow[urr, bend right, "v"'] & &
\end{tikzcd}
\qquad i = 1 \ldots n.
$$

*When $A = B$ we can assume $u = v$.*

*Proof.* Given $\xi_1$, $\xi_2$, apply FF1 to obtain invertible $\alpha$, $\beta$ to fit as follows:

$$
\begin{tikzcd}[column sep=small]
 & A \arrow[dr, "u_1"] \arrow[dd, phantom, "\xi_1\Downarrow"] & & \\
F \arrow[ur, "x_1"] \arrow[dr, "y_1"'] & & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\alpha\Downarrow"] & \\
 & B \arrow[ur, "v_1"'] \arrow[dr, "v_2"'] & & C \\
 & & C_2 \arrow[ur, "w_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & C_1 \arrow[dr, "w_1"] \arrow[dd, phantom, "\beta\Downarrow"] & \\
 & A \arrow[ur, "u_1"] \arrow[dr, "u_2"] \arrow[dd, phantom, "\xi_2\Downarrow"] & & C \\
F \arrow[ur, "x_2"] \arrow[dr, "y_2"'] & & C_2 \arrow[ur, "w_2"'] & \\
 & B \arrow[ur, "v_2"'] & &
\end{tikzcd}
.
$$

This gives the case $n = 2$ with $u = w_1 u_1$, $v = w_2 v_2$, $\alpha_1 = \alpha$, $\beta_1 = id$, $\alpha_2 = id$, and $\beta_2 = \beta$. Using this case, induction is straightforward. For the second part, do as in the proof of lemma 2.2. $\square$

**2.4 Theorem.** *Let $\mathcal{A}$ be a pre 2-filtered 2-category, $F : \mathcal{A} \longrightarrow Cat$ a 2-functor, and $\mathcal{P}$ a finite category. Consider the 2-functor $F^{\mathcal{P}} : \mathcal{A} \longrightarrow Cat$ defined by $F^{\mathcal{P}}(A) = (FA)^{\mathcal{P}}$, and the canonical functor:*

$$\Diamond : \mathcal{L}(F^{\mathcal{P}}) \longrightarrow \mathcal{L}(F)^{\mathcal{P}} \qquad (\textit{given by theorem } 1.19).$$

*Then, $\Diamond$ is an equivalence of categories provided that $\mathcal{A}$ is 2-filtered or that $\mathcal{A}$ is pseudo 2-filtered and $\mathcal{P}$ is connected.*

*Proof.* Notice that an object $F^{\mathcal{P}} \longrightarrow A$ in $\mathcal{L}(F^{\mathcal{P}})$ is by definition a diagram $\mathcal{P} \longrightarrow FA$. We shall prove, in turn, that $\Diamond$ is (a) essentially surjective, (b) faithful, and (c) full.

(a) *essentially surjective*: We shall see that given a diagram $\mathcal{P} \to \mathcal{L}(F)$, there exists $A \in \mathcal{A}$ and a factorization (up to isomorphism):

$$\begin{tikzcd} & \mathcal{P} \arrow[dl] \arrow[d] \\ FA \arrow[r, "\lambda_A"'] \arrow[ur, phantom, "\cong"] & \mathcal{L}(F) \end{tikzcd}$$

Consider explicitly an object in $\mathcal{L}(F)^{\mathcal{P}}$:

$$F \xrightarrow{x_k} A_k,\ k \in \mathcal{P}, \qquad \begin{tikzcd} & A_p \arrow[dr, "u_f"] & \\ F \arrow[ur, "x_p"] \arrow[dr, "x_q"'] \arrow[rr, phantom, "\varphi_f \Downarrow"] & & A_f \\ & A_q \arrow[ur, "v_f"'] & \end{tikzcd}, \quad p \xrightarrow{f} q \in \mathcal{P}.$$

satisfying equations $\varphi_{f \circ g} \sim \varphi_f \circ_\gamma \varphi_g$ for all composable pairs $f$, $g$.

Let $\mathcal{Q} \subset \mathcal{P}$ be a part of $\mathcal{P}$ for which there exists $A$, $w_k$, $\psi_f$ such that:

$$F \xrightarrow{x_k} A_k,\ k \in \mathcal{Q}, \quad \begin{tikzcd} & A_p \arrow[dr, "u_f"] & \\ F \arrow[ur, "x_p"] \arrow[dr, "x_q"'] \arrow[rr, phantom, "\varphi_f \Downarrow"] & & A_f \\ & A_q \arrow[ur, "v_f"'] & \end{tikzcd} \sim \begin{tikzcd} & A_p \arrow[dr, "w_p"] & \\ F \arrow[ur, "x_p"] \arrow[dr, "x_q"'] \arrow[rr, phantom, "\psi_f \Downarrow"] & & A \\ & A_q \arrow[ur, "w_q"'] & \end{tikzcd}, \quad p \xrightarrow{f} q \in \mathcal{Q}.$$

$\mathcal{Q}$ is not necessarily a subcategory, but we agree that if $p \xrightarrow{f} q \in \mathcal{Q}$, then we consider $p \in \mathcal{Q}$ and $q \in \mathcal{Q}$. The equations $\psi_{f \circ g} \sim \psi_f \circ_\gamma \psi_g$

hold for all composable pairs $f$, $g$ in $\mathcal{Q}$ with $f \circ g$ also in $\mathcal{Q}$. By lemma 1.20 we can assume strict equality $\psi_{f \circ g} = \psi_f \circ \psi_g$ in $FA$.

We shall see that if $p \xrightarrow{g} q$ is not in $\mathcal{Q}$, we can add it to $\mathcal{Q}$ in such a way that the enlarged part retains the same property. In what it follows we use without mention lemma 1.18.

1) $p \in \mathcal{Q}$, $q \notin \mathcal{Q}$ or $q \in \mathcal{Q}$, $p \notin \mathcal{Q}$: In the first case apply axiom F1 to obtain an invertible 2-cell $\beta$ as follows:

$$\begin{tikzcd}
 & & A \arrow[dr, "h"] & \\
 & A_p \arrow[ur, "w_p"] \arrow[dr, "u_g"'] \arrow[rr, phantom, "\beta\Downarrow"] & & H \\
F \arrow[ur, "x_p"] \arrow[dr, "x_q"'] \arrow[rr, phantom, "\varphi_g\Downarrow"] & & A_g \arrow[ur, "l"'] & \\
 & A_q \arrow[ur, "v_g"'] & &
\end{tikzcd}$$

The new $A$ is $H$, the new $w_k$ are $hw_k$, all $k \in \mathcal{Q}$, the new $\psi_f$ are $h\psi_f$, all $f \in \mathcal{Q}$, and, finally, $w_q = lv_g$, and $\psi_g$ is the 2-cell above. The other case can be proved in the same way.

2) $p \in \mathcal{Q}$, $q \in \mathcal{Q}$: Apply axiom FF1 to obtain invertible 2-cells $\alpha$, $\beta$ as follows:

$$\begin{tikzcd}
 & A_g \arrow[dr, "s"] & \\
A_q \arrow[ur, "v_g"] \arrow[dr, "w_q"'] \arrow[rr, phantom, "\alpha\Downarrow"] & & H \\
 & A \arrow[ur, "h"'] &
\end{tikzcd}
,\quad
\begin{tikzcd}
 & A_g \arrow[dr, "s"] & \\
A_p \arrow[ur, "u_g"] \arrow[dr, "w_p"'] \arrow[rr, phantom, "\beta\Downarrow"] & & H \\
 & A \arrow[ur, "h"'] &
\end{tikzcd}
:\quad
\begin{tikzcd}
 & & A \arrow[ddr, "h"] & \\
 & A_p \arrow[ur, "w_p"] \arrow[dr, "u_g"'] \arrow[r, phantom, "\beta^{-1}\Downarrow"] & \, & \\
F \arrow[ur, "x_p"] \arrow[dr, "x_q"'] \arrow[rr, phantom, "\varphi_g\Downarrow"] & & A_g \arrow[r, "s"] & H \\
 & A_q \arrow[ur, "v_g"] \arrow[dr, "w_q"'] \arrow[r, phantom, "\alpha\Downarrow"] & \, & \\
 & & A \arrow[uur, "h"'] &
\end{tikzcd}$$

The new $A$ is $H$, the new $w_k$ are $hw_k$, all $k \in \mathcal{Q}$, the new $\psi_f$ are $h\psi_f$, all $f \in \mathcal{Q}$, and, finally, $\psi_g$ is the 2-cell above on the right. This proof holds whether $p \neq q$ or $p = q$.

3) $p \notin \mathcal{Q}$, $q \notin \mathcal{Q}$: Apply axiom F0 to obtain $A_g \xrightarrow{w} H$, $A \xrightarrow{h} H$. The new $A$ is $H$, the new $w_k$ are $hw_k$, all $k \in \mathcal{Q}$, the new $\psi_f$ are $h\psi_f$, all $f \in \mathcal{Q}$, and, finally, $w_p = lu_g$, $w_q = lv_g$, $\psi_g = l\varphi_g$. If $p = q$, use lemma 2.2 to assume $u_g = v_g$, and in this way $w_p$ is uniquely defined.

Notice that if $\mathcal{P}$ is connected, it is not necessary to consider this case since we can always choose $p \xrightarrow{g} q$ such that either $p$, or $q$, or both, are in $\mathcal{Q}$. Thus for connected $\mathcal{P}$ axiom F0 is not necessary.

It is clear that any singleton $\{(k,\ f = id_k)\}$ serves as an initial $\mathcal{Q}$, thus we can assume $\mathcal{Q} = \mathcal{P}$.

To finish the proof observe that given any arrow $ux \xrightarrow{\psi} vy$ in $FA$, the square

$$\begin{tikzcd}
(ux,\ A) \arrow[r, "(id,\, id,\, u)", "\cong"'] \arrow[d, "\lambda_A(\psi)"] & (x,\ A) \arrow[d, "(u,\, \psi,\, v)"] \\
(vy,\ A) \arrow[r, "(id,\, id,\, v)", "\cong"'] & (y,\ B)
\end{tikzcd}$$

commutes in $\mathcal{L}(F)$.

Notice that if $\mathcal{P}$ is a poset, case 2) cannot happen, so axiom FF1 is not necessary. For posets $\mathcal{P}$ the functor $\diamondsuit$ is essentially surjective also for pre 2-filtered 2-categories.

(b) *faithful:* Consider two premorphisms

$$\begin{tikzcd}
 & A \arrow[dr, "u"] & \\
F^{\mathcal{P}} \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\xi\Downarrow"] & & C \\
 & B \arrow[ur, "v"'] &
\end{tikzcd}, \quad
\begin{tikzcd}
 & A \arrow[dr, "s"] & \\
F^{\mathcal{P}} \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, phantom, "\eta\Downarrow"] & & D \\
 & B \arrow[ur, "t"'] &
\end{tikzcd}$$

in $\mathcal{L}(F^{\mathcal{P}})$,

$$\begin{tikzcd}
 & A \arrow[dr, "u"] & \\
F \arrow[ur, "x_k"] \arrow[dr, "y_k"'] \arrow[rr, phantom, "\xi_k\Downarrow"] & & C \\
 & B \arrow[ur, "v"'] &
\end{tikzcd}, \quad
\begin{tikzcd}
 & A \arrow[dr, "s"] & \\
F \arrow[ur, "x_k"] \arrow[dr, "y_k"'] \arrow[rr, phantom, "\eta_k\Downarrow"] & & D \\
 & B \arrow[ur, "t"'] &
\end{tikzcd}, \quad k \in \mathcal{P}.$$

To be equivalent in $\mathcal{L}(F)^{\mathcal{P}}$ means that there are homotopies $(\alpha_k,\ \beta_k) : \xi_k \Rightarrow \eta_k$ given by invertible 2-cells

$$\begin{tikzcd}
 & C \arrow[dr, "w_k"] & \\
B \arrow[ur, "v"] \arrow[dr, "t"'] \arrow[rr, phantom, "\alpha_k\Downarrow"] & & H_k \\
 & D \arrow[ur, "h_k"'] &
\end{tikzcd}, \quad
\begin{tikzcd}
 & C \arrow[dr, "w_k"] & \\
A \arrow[ur, "u"] \arrow[dr, "s"'] \arrow[rr, phantom, "\beta_k\Downarrow"] & & H_k \\
 & D \arrow[ur, "h_k"'] &
\end{tikzcd}, \quad k \in \mathcal{P}.$$

From lemma 1.15 it readily follows that we can assume there are single invertible 2-cells $\alpha$, $\beta$ which define all the homotopies $(\alpha,\ \beta) : \xi_k \Rightarrow \eta_k$. But this means that $\xi$ and $\eta$ are equivalent in $\mathcal{L}(F^{\mathcal{P}})$.

Notice that this proof of the faithfulness of the functor $\diamondsuit$ holds for pre 2-filtered 2-categories.

(c) *full*: Consider two objects $F^{\mathcal{P}} \xrightarrow{x} A$, $F^{\mathcal{P}} \xrightarrow{y} B$ in $\mathcal{L}(F^{\mathcal{P}})$.

A premorphism in $\mathcal{L}(F)^{\mathcal{P}}$ consists of a family
$$\begin{tikzcd} & A \arrow[dr, "u_k"] & \\ F \arrow[ur, "x_k"] \arrow[dr, "y_k"'] & \xi_k\Downarrow & C_k \\ & B \arrow[ur, "v_k"'] & \end{tikzcd}, \quad k \in \mathcal{P}.$$

From lemma 2.3 we can assume all the $u_k$, $v_k$, $C_k$ to be equal to a single $u$, $v$, $C$. But this is the data for a premorphism in $\mathcal{L}(F^{\mathcal{P}})$. For the naturality equations we proceed as in the proof of faithfulness in (b).

Here axiom FF1 is inevitable (lemma 2.3), and plays the role of axiom PS2 in the filtered category case. Notice that a function between sets viewed as a functor between trivial categories is injective precisely when it is a full functor. □

We state now an important corollary of theorem 2.4. Let $Cat_{f\ell}$ be the 2-category of finitely complete categories and finite limit preserving functors. We have:

**2.5 Theorem.** *Let $\mathcal{A}$ be a 2-filtered 2-category, and $\mathcal{A} \xrightarrow{F} Cat_{f\ell}$ a 2-functor. Then, the category $\mathcal{L}(F)$ has finite limits, the pseudocone functors $FA \xrightarrow{\lambda_A} \mathcal{L}(F)$ preserve finite limits and induce an equivalence of categories $Cat_{f\ell}[\mathcal{L}(F), \mathcal{X}] \xrightarrow{\cong} \mathcal{PC}_{f\ell}[F, \mathcal{X}]$; this equivalence is actually an isomorphism.* □

## Kennison notion of bifiltered 2-category

In [3], Kennison considers the following notion:

**2.6 Definition (Kennison).** A 2-category $\mathcal{A}$ is defined to be *bifiltered* when it satisfies the following three axioms:

BF0. Given two objects $A$, $B$, there exists $C$ and $A \to C$, $B \to C$.

BF1. Given two arrows $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$, there exists $B \xrightarrow{u} C$ and an invertible 2-cell $\gamma : uf \cong ug$.

BF2. Given two 2-cells $\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\gamma_1\Downarrow\,\gamma_2\Downarrow"] & B \end{tikzcd}$, there exists $B \xrightarrow{u} C$ such that $u\gamma_1 = u\gamma_2$.

This notion of bifiltered 2-category is equivalent to our notion of 2-filtered 2-category. The proof of this is elementary although not entirely trivial, and we leave it as an interesting exercise.

## References

[1] M. Artin, A. Grothendieck, J. L. Verdier, *SGA4, 1963/64*, Springer Lecture Notes Vol 269 (1972).

[2] M. Artin, A. Grothendieck, J. L. Verdier, *SGA4, 1963/64* Springer Lecture Notes Vol 270 (1972).

[3] J. Kennison, *The fundamental localic groupoid of a topos*, Journal of Pure and Applied Algebra 77 (1992).

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