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Asymptotic profiles for inhomogeneous heat equations with memory

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Abstract

We study the large-time behavior in all L^p norms of solutions to an inhomogeneous nonlocal heat equation in $\mathbb{R}^N$ involving a Caputo α-time derivative and a power β of the Laplacian when the dimension is large, $N > 4\beta$. The asymptotic profiles depend strongly on the space-time scale and on the time behavior of the spatial L^1 norm of the forcing term.

Mathematics Subject Classification 35B40 · 35R11 · 45K05

1 Introduction and main results

1.1 Aim

The purpose of this paper is to give a precise description of the large-time behavior of solutions to the inhomogeneous *fully nonlocal* heat equation

$$\partial_t^\alpha u + (-\Delta)^\beta u = f \quad \text{in } Q := \mathbb{R}^N \times (0, \infty), \qquad u(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

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in the case of large dimensions, $N > 4\beta$, completing the analysis started by Kemppainen, Siljander and Zacher in [17]. Here, ∂_t^α, $\alpha \in (0, 1)$, denotes the so-called Caputo α-derivative, introduced independently by many authors using different points of view, see for instance [2, 13, 15, 16, 20, 22], defined for smooth functions by

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\,\partial_t \int_0^t \frac{u(x,\tau)-u(x,0)}{(t-\tau)^\alpha}\,\mathrm{d}\tau,$$

and $(-\Delta)^\beta$, with $\beta \in (0, 1]$, is the usual β power of the Laplacian, defined for smooth functions by $(-\Delta)^\beta = \mathcal{F}^{-1}(|\cdot|^{2\beta}\mathcal{F})$, where $\mathcal{F}$ stands for Fourier transform; see for instance [23]. Such equations, nonlocal both in space and time, are useful to model situations with long-range interactions and memory effects, and have been proposed for example to describe plasma transport [10, 11]; see also [3, 4, 21, 24] for further models that use such equations.

Problem (1.1) does not have in general a classical solution, unless the forcing term f is smooth enough. However, if $f \in L^\infty_{\mathrm{loc}}([0,\infty) : L^1(\mathbb{R}^N))$, it has a solution in a generalized sense, defined by Duhamel's type formula

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} Y(x-y, t-s) f(y,s)\,\mathrm{d}y\mathrm{d}s, \tag{1.2}$$

with $Y = \partial_t^{1-\alpha} Z$ outside the origin, where Z is the fundamental solution for the Cauchy problem,

$$\partial_t^\alpha u + (-\Delta)^\beta u = 0 \quad \text{in } Q, \qquad u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^N; \tag{1.3}$$

see [14, 17, 19]. Throughout the paper, by the solution to problem (1.1) we always mean the generalized solution given by (1.2).

The rate of decay/growth of the solution depends on the space-time scale under consideration, the L^p norm with which we measure the size of u, and the size of the right-hand side f; see [7], and also [8] for the case of small dimensions. Our goal here is to determine, under some assumptions on the forcing term f, the asymptotic profile of the solution, once it is normalized taking into account the decay/growth rate. Let us point out that even for the local case, $\alpha = 1$, $\beta = 1$, such study is not yet complete; see Sect. 1.4 below.

Notation. Let $g, h : \mathbb{R}_+ \to \mathbb{R}_+$. In what follows we write $g \simeq h$ if there are constants $\nu, \mu > 0$ such that $\nu \le g(t)/h(t) \le \mu$ for all $t \in \mathbb{R}_+$, and $g \succ h$ if $\lim_{t\to\infty} \frac{g(t)}{h(t)} = \infty$.

1.2 The kernel *Y*. Critical exponents

Our proofs depend on a good knowledge of the kernel Y, which, as mentioned above, is given by $Y = \partial_t^{1-\alpha} Z$. Let $\widehat{Z} = \widehat{Z}(\omega, t)$ denote the Fourier transform of the fundamental solution Z of problem (1.3) in the x variable. Then,

$$\partial_t^\alpha \widehat{Z}(\omega,t) = -|\omega|^{2\beta}\widehat{Z}(\omega,t), \qquad \widehat{Z}(\omega,0) = 1.$$

The solution to this ordinary fractional differential equation is

$$\widehat{Z}(\omega, t) = \mathbb{E}_\alpha(-|\omega|^{2\beta} t^\alpha),$$

where $\mathbb{E}_\alpha$ is the Mittag–Leffler function of order α,

$$\mathbb{E}_\alpha(s) = \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(1+k\alpha)}.$$

Inverting the Fourier transform, we obtain that Z has a self-similar form,

$$Z(x,t) = t^{-N\theta} F(\xi), \quad \xi = xt^{-\theta}, \quad \theta := \frac{\alpha}{2\beta}, \tag{1.4}$$

with a radially symmetric positive profile F that has an explicit expression in terms of certain Fox's H-functions. Hence, Y has also a self-similar form,

$$Y(x,t) = t^{-\sigma_*} G(\xi), \quad \xi = xt^{-\theta}, \qquad \sigma_* := 1 - \alpha + N\theta. \tag{1.5}$$

Its profile G is positive, radially symmetric, and smooth outside the origin, has integral 1, and, in the case of large dimensions that we are considering here, $N > 4\beta$, satisfies, for all $\beta \in (0, 1]$, the estimates

$$G(\xi) \simeq |\xi|^{4\beta - N}, \qquad |\xi| \le 1, \tag{1.6}$$

$$G(\xi) = O\big(|\xi|^{-(N+2\beta)}\big), \quad |DG(\xi)| = O\big(|\xi|^{-(N+2\beta+1)}\big), \qquad |\xi| \ge 1. \tag{1.7}$$

We have also the limit

$$|\xi|^{N-4\beta} G(\xi) \to \kappa \quad \text{as } |\xi| \to 0 \text{ for some constant } \kappa > 0, \tag{1.8}$$

which shows that the inner estimate (1.6) is sharp. The exterior estimates (1.7) are also sharp if $\beta \in (0, 1)$. In the special case $\beta = 1$, both G and $|DG|$ decay exponentially, but we do not need this fact in our calculations. All these estimates, and many others, are proved in [17, 18].

As a consequence of (1.6)–(1.7) we have the global bound

$$0 \le Y(x,t) \le Ct^{-(1+\alpha)} |x|^{4\beta - N} \quad \text{in } Q, \tag{1.9}$$

and also the exterior bounds, valid if $|x| \ge \nu t^\theta$, $t > 0$, for some $\nu > 0$,

$$0 \le Y(x,t) \le C_\nu t^{2\alpha - 1} |x|^{-(N+2\beta)}, \tag{1.10}$$

$$|DY(x,t)| \le C_\nu t^{2\alpha - 1} |x|^{-(N+2\beta+1)}, \tag{1.11}$$

$$|\partial_t Y(x,t)| \le C_\nu t^{2\alpha - 2} |x|^{-(N+2\beta)}. \tag{1.12}$$

Notice that $Y(\cdot,t)\in L^p(\mathbb{R}^N)$ if and only if $p\in[1,p_*)$, where $p_*:=N/(N-4\beta)$. Moreover,

$$\|Y(\cdot,t)\|_{L^p(\mathbb{R}^N)}=Ct^{-\sigma(p)}\quad\text{for all } t>0\quad\text{if } p\in[1,p_*),$$
$$\text{where } \sigma(p):=\sigma_*-\frac{N\theta}{p}. \tag{1.13}$$

Observe also that $\sigma(p)<1$, and hence $Y\in L^1_{\text{loc}}([0,\infty):L^p(\mathbb{R}^N))$, if and only if $p\in[1,p_{\rm c})$, with $p_{\rm c}:=N/(N-2\beta)$. Since the solution is given by a convolution of f with Y *both in space and time*, the threshold value that will mark the border between subcritical and supercritical behaviors will be $p_{\rm c}$, and not p_*.

The self-similar form of Y, see (1.5), stemming from the scaling invariance of the integro-differential operator, gives a hint of the special role played by *diffusive* scales, $|x|\simeq t^\theta$. As we will see, there is a marked difference between the behavior in compact sets and that in *outer* scales, $|x|\ge \nu t^\theta$ for some $\nu>0$, with intermediate behaviors in *intermediate* scales, $|x|\simeq\varphi(t)$, with $\varphi(t)\succ 1$, $\varphi(t)=o(t^\theta)$.

1.3 Assumptions on *f*

We always assume, no matter the space-time scale under consideration, the size hypothesis

$$\|f(\cdot,t)\|_{L^1(\mathbb{R}^N)}\le C(1+t)^{-\gamma}\quad\text{for some } \gamma\in\mathbb{R} \text{ and } C>0. \tag{1.14}$$

This condition guarantees that the function u given by Duhamel's type formula (1.2) is well defined, and moreover, that $u(\cdot,t)\in L^p(\mathbb{R}^N)$ for all $t>0$ in the *subcritical* range $p\in[1,p_{\rm c})$, though not for $p\ge p_{\rm c}$. In case we wish to analyze the large-time behavior of u when p is not subcritical, we will need some extra assumption on the spatial behavior of f to force u and the function giving its asymptotic behavior to be in the right space. The assumptions will depend on the scales, p, and γ.
Notation. Given $p\ge p_{\rm c}$, we define

$$q_{\rm c}(p):=\begin{cases}\dfrac{Np}{2\beta p+N}, & p\in[1,\infty),\\[2mm] \dfrac{N}{2\beta}, & p=\infty.\end{cases}$$

Compact sets. When dealing with the behavior in compact sets, if $p\ge p_{\rm c}$ we will assume that

$$\text{there is } q\in(q_{\rm c}(p),p] \text{ such that } \|f(\cdot,t)\|_{L^q(K)}\le C_K(1+t)^{-\gamma} \text{ for each } K\subset\subset\mathbb{R}^N. \tag{1.15}$$

On the other hand, if the time decay of the L^1 norm is not fast enough, namely, if $\gamma\le 1+\alpha$, in order to obtain a limit profile we will need to assume that f is asymptotically a function in separate variables in the following precise sense,

$$\text{there exists } g \in L^1(\mathbb{R}^N) \text{ such that } \|f(\cdot,t)(1+t)^\gamma - g\|_{L^1(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty, \tag{1.16}$$

again with an extra assumption if $p \geq p_c$,

$$\begin{aligned} &g \in L^q_{\mathrm{loc}}(\mathbb{R}^N) \text{ for some } q \in (q_c(p), p], \\ &\quad \|f(\cdot,t)(1+t)^\gamma - g\|_{L^q(K)} \to 0 \text{ as } t \to \infty \text{ if } K \subset\subset \mathbb{R}^N. \end{aligned} \tag{1.17}$$

INTERMEDIATE SCALES. For intermediate space-time scales, unless they are *fast* (see Sect. 1.5 for a precise definition) and $\gamma > 1$, we have to assume that f is asymptotically a function in separate variables, hypothesis (1.16). If $\gamma = 1$ and the scale is not *slow* (see Sect. 1.5 for a definition) we will require the tail control condition

$$\sup_{t>0}\big((1+t)^\gamma \|f(\cdot,t)\|_{L^1(\{|x|>R\})}\big) = o(1) \quad \text{as } R \to \infty. \tag{1.18}$$

Remark Condition (1.18) is satisfied, for instance, if $|f(x,t)| \leq h(x)(1+t)^{-\gamma}$, for some $h \in L^1(\mathbb{R}^N)$.

Finally, if $p \geq p_c$ we assume moreover the uniform tail control condition

$$\begin{aligned} &\sup_{t>0}\big((1+t)^\gamma \|f(\cdot,t)\|_{L^q(\{|x|>R\})}\big) \\ &\quad = O\big(R^{-N(1-\frac{1}{q})}\big) \quad \text{as } R \to \infty \text{ for some } q \in (q_c(p), p]. \end{aligned} \tag{1.19}$$

Remark Condition (1.19) is satisfied, for instance, if $|f(x,t)| \leq C|x|^{-N}(1+t)^{-\gamma}$ for some $C > 0$.

OUTER SCALES. For outer space-time scales and $\gamma \leq 1$, we assume the uniform tail control condition (1.19) if p is subcritical, and

$$\begin{aligned} &\sup_{t>0}\big((1+t)^\gamma \|f(\cdot,t)\|_{L^q(\{|x|>R)}\big) \\ &\quad = o\big(R^{-N(1-\frac{1}{q})}\big) \quad \text{for some } q \in (q_c(p), p] \text{ as } R \to \infty \end{aligned} \tag{1.20}$$

otherwise.

Remark Condition (1.20) is satisfied, for instance, if $|f(x,t)| \leq h(x)(1+t)^{-\gamma}$ with $h(x) = o(|x|^{-N})$.

We do not claim that the above conditions are optimal; but they are not too restrictive, and are easy enough to keep the proofs simple.

1.4 Precedents

A full description of the large-time behavior of the homogeneous problem (1.27) for a nontrivial initial data $u_0 \in L^1(\mathbb{R}^N)$ was recently given in [5, 6]; see also [17]. The first precedent for the inhomogeneous problem (1.1) is [17], where the authors study the problem in the *integrable in time* case $\gamma > 1$ and prove, for all $p \in [1, \infty]$ if $1 \le N < 2\beta$, and for $p \in [1, p_c)$ otherwise, that

$$\lim_{t\to\infty} t^{\sigma(p)}\|u(\cdot,t) - M_\infty Y(\cdot,t)\|_{L^p(\mathbb{R}^N)} = 0,$$
$$\text{where } M_\infty := \int_0^\infty \int_{\mathbb{R}^N} f(x,t)\,\mathrm{d}x\mathrm{d}t < \infty. \tag{1.21}$$

This result is also known to be valid for the local case, $\alpha = 1$, $\beta = 1$, if $p = 1$; see [1, 12]. In this special local situation $Y = Z$ is the well-known fundamental solution of the heat equation, whose profile does not have a spatial singularity and belongs to all L^p spaces. But a complete analysis for $\alpha = 1$, $\beta \in (0, 1]$ is still missing, and will be given elsewhere [9].

The above result (1.21) *cannot* hold when $N > 4\beta$ if $p \ge p_c$, even if we impose additional conditions on f to guarantee that $u(\cdot,t) \in L^p(\mathbb{R}^N)$ and $\gamma > 1$, since $Y(\cdot,t) \notin L^p(\mathbb{R}^N)$ in that range. On the other hand, in the subcritical range $p \in [1, p_c)$, the result does not give information on the shape of the solution in *inner regions*, that is, sets of the form $\{|x| \le g(t)\}$ with $g(t) = o(t^\theta)$, since $\|Y(\cdot,t)\|_{L^p(\{|x|\le g(t)\})} = o(t^{-\sigma(p)})$ in that case. We will tackle these two difficulties along the paper.

A first step towards the understanding of the large-time behavior of solutions to (1.1) in different space-time scales and for all possible values of p was the determination of the decay/growth rates under the above assumptions on f. This was done in [7] for the case of large dimensions that we are considering here, and in [8] for low dimensions.

1.5 Main results

As we have already mentioned, the decay/growth rates of solutions and their asymptotic profiles depend on the space-time scale under consideration.

Compact sets. Given $\mu \in (0, N)$, and h satisfying suitable integrability assumptions, let

$$E_\mu(x) = |x|^{\mu - N}, \quad I_\mu[h](x) = \int_{\mathbb{R}^N} h(x-y)E_\mu(y)\,\mathrm{d}y. \tag{1.22}$$

The large-time behavior in compact sets will be described in terms of $I_{2\beta}[g]$ and $I_{4\beta}[F]$, where g is the asymptotic spatial factor of the forcing term f, and

$$F(x) = \int_0^\infty f(x,s)\,\mathrm{d}s. \tag{1.23}$$

Remark Let $c_\mu := \Gamma\big((N-\mu)/2\big)/(\pi^{N/2}2^\mu\Gamma(\mu/2))$. Then $c_\mu I_\mu[h]$ is the μ-Riesz potential of h, so that $(-\Delta)^{\mu/2}(c_\mu I_\mu[h]) = h$.

Theorem 1.1 *Let f satisfy (1.14), and also (1.15) if $p \geq p_c$. If $\gamma \leq 1+\alpha$ we assume moreover (1.16), and also (1.17) if $p \geq p_c$. Let u be solution to (1.1). Given $K \subset\subset \mathbb{R}^N$,*

$$\|t^{\min\{\gamma,1+\alpha\}}u(\cdot,t) - \mathcal{L}\|_{L^p(K)} \to 0 \quad \text{as } t \to \infty, \quad \text{where} \tag{1.24}$$

$$\mathcal{L} = \begin{cases} c_{2\beta}I_{2\beta}[g] & \text{if } \gamma < 1+\alpha, \\ c_{2\beta}I_{2\beta}[g] + \kappa I_{4\beta}[F] & \text{if } \gamma = 1+\alpha, \quad \text{with } \kappa \text{ as in (1.8)}, \\ \kappa I_{4\beta}[F] & \text{if } \gamma > 1+\alpha. \end{cases} \tag{1.25}$$

Remarks (a) We already knew from [7] that $\|u(\cdot,t)\|_{L^p(K)} = O(t^{\min\{\gamma,1+\alpha\}})$ for any $K \subset\subset \mathbb{R}^N$. Theorem 1.1 shows that this rate is sharp, $\|u(\cdot,t)\|_{L^p(K)} \simeq t^{\min\{\gamma,1+\alpha\}}$.

(b) Under the hypotheses of Theorem 1.1, if in addition $f(x,t) = g(x)(1+t)^{-\gamma}$, the asymptotic profile $\mathcal{L}$ simplifies to

$$\mathcal{L} = \begin{cases} c_{2\beta}I_{2\beta}[g] + \frac{\kappa}{\alpha}I_{4\beta}[g] & \text{if } \gamma = 1+\alpha, \\ \frac{\kappa}{\gamma-1}I_{4\beta}[g] & \text{if } \gamma > 1+\alpha. \end{cases}$$

(c) When the forcing term is independent of time, $f(\cdot,t) = g \in L^1(\mathbb{R}^N)$ for all $t > 0$, Theorem 1.1 yields

$$\|u(\cdot,t) - c_{2\beta}I_{2\beta}[g]\|_{L^p(K)} \to 0 \quad \text{as } t \to \infty \tag{1.26}$$

for all $p \in [1,\infty]$ (assuming also $g \in L^q_{\text{loc}}(\mathbb{R}^N)$ for some $q \in (q_c(p), p]$ if $p \geq p_c$). Hence, the limit profile in compact sets is a *stationary* solution of the equation. This convergence result cannot be extended to the whole space if $p \in [1, p_c]$, since $I_{2\beta}[g] \notin L^p(\mathbb{R}^N)$ in this range.
The convergence result (1.26) for forcing terms independent of time also holds for solutions to

$$\partial_t^\alpha u + (-\Delta)^\beta u = f \quad \text{in } Q, \qquad u(\cdot,0) = u_0 \quad \text{in } \mathbb{R}^N \tag{1.27}$$

for any initial datum $u_0 \in L^1(\mathbb{R}^N)$ (with the additional assumption $u_0 \in L^q_{\text{loc}}(\mathbb{R}^N)$ for some $q \in (q_c(p), p]$ if $p \geq p_c$). This follows from the linearity of the problem, since the generalized solution v to (1.3) with $v(\cdot,0) = u_0$, given by $v(\cdot,t) = Z(\cdot,t) * u_0$, satisfies $\|v(\cdot,t)\|_{L^p(K)} \simeq t^{-\alpha}$ for every compact $K \subset\subset \mathbb{R}^N$; see [5, 6],

(d) Under some integrability assumptions on h,

$$I_\mu[h] \approx \Big(\int_{\mathbb{R}^N} h\Big)E_\mu \quad \text{as } |x| \to \infty;$$

see Theorem A.1 in the Appendix for the details. Hence, the "outer limit", $|x| \to \infty$, of the function describing the large-time behavior in compact sets is given by

$$t^{-\min\{\gamma,1+\alpha\}}\mathcal{L}(x) \approx \begin{cases} t^{-\gamma}M_0c_{2\beta}E_{2\beta}(x), & \gamma < 1+\alpha, \\ t^{-(1+\alpha)}M_\infty\kappa E_{4\beta}(x), & \gamma \geq 1+\alpha, \end{cases} \quad \text{as } |x| \to \infty, \tag{1.28}$$
$$\text{where } M_0 = \int_{\mathbb{R}^N} g, \quad M_\infty = \int_0^\infty \int_{\mathbb{R}^N} f.$$

INTERMEDIATE SCALES. These are scales for which $|x| \simeq \varphi(t)$, with $\varphi(t) \succ 1$, $\varphi = o(t^\theta)$. We will make a distinction among the different intermediate scales according to their speed, measured against the value of the decay/growth exponent γ. Thus, we have

$$\gamma < 1;\ \text{or } \gamma = 1,\ \varphi(t) = o\big(t^\theta/(\log t)^{\frac{1}{2\beta}}\big);\ \text{or } \gamma \in (1, 1+\alpha),\ \varphi(t) = o\big(t^{\frac{1+\alpha-\gamma}{2\beta}}\big); \tag{S}$$

$$\gamma = 1, \varphi(t) \simeq t^\theta/(\log t)^{\frac{1}{2\beta}}; \tag{C$_1$}$$

$$\gamma \in (1, 1+\alpha),\ \varphi(t) \simeq t^{\frac{1+\alpha-\gamma}{2\beta}}; \tag{C}$$

$$\gamma = 1,\ \varphi(t) \succ t^\theta/(\log t)^{\frac{1}{2\beta}}; \tag{F$_1$}$$

$$\gamma \in (1, 1+\alpha),\ \varphi(t) \succ t^{\frac{1+\alpha-\gamma}{2\beta}};\ \text{or } \gamma \geq 1+\alpha. \tag{F}$$

In *slow* scales, satisfying (S), the large-time behavior coincides with the outer limit of the behavior in compact sets, being given in terms of $E_{2\beta}$. Notice that when $\gamma < 1$ all scales are slow. In *fast* scales, satisfying (F) or (F$_1$), the behavior coincides with the inner limit of the behavior in outer scales, and is given in terms of $E_{4\beta}$. Notice that when $\gamma \geq 1+\alpha$ all intermediate scales are fast. In the critical cases, (C$_1$) and (C), the large time behavior involves both $E_{2\beta}$ and $E_{4\beta}$. In the cases (C$_1$) and (F$_1$) (in both of them $\gamma = 1$; that is the reason for the subscript) a factor $\log t$ is involved.

Let us recall from [7] that $\|u(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = O(\phi(t))$, where

$$\phi(t) = \begin{cases} t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}} & \text{if (S), (C}_1\text{), or (C)}, \\ t^{-(1+\alpha)}\log t\,\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}} & \text{if (F}_1\text{)}, \\ t^{-(1+\alpha)}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}} & \text{if (F)}. \end{cases} \tag{1.29}$$

It is worth noticing that $\sigma(p) < 1$ if and only if $p \in [1, p_c)$, and $\sigma(p) < 1+\alpha$ if and only if $p \in [1, p_*)$.

Theorem 1.2 *Let $\varphi(t) \succ 1$, $\varphi = o(t^\theta)$. Let f satisfy (1.14). We assume moreover (1.16) if (S) or (C) hold, and both (1.16) and (1.18) if (C_1) or (F_1) hold. When $p \geq p_c$ we assume further (1.19). Let u be the solution to (1.1). Given $0 < \nu < \mu < \infty$,*

$$\lim_{t\to\infty} \frac{1}{\phi(t)} \|u(\cdot,t) - \mathcal{L}(t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = 0, \quad \textit{with } \phi \textit{ as in (1.29), and}$$

$$\mathcal{L}(t) = \begin{cases} t^{-\gamma} M_0 c_{2\beta} E_{2\beta} & \textit{if } (S), \\ t^{-1} M_0 c_{2\beta} E_{2\beta} + t^{-(1+\alpha)} \log t \, M_0 \kappa E_{4\beta} & \textit{if } (C_1), \\ t^{-\gamma} M_0 c_{2\beta} E_{2\beta} + t^{-(1+\alpha)} M_\infty \kappa E_{4\beta} & \textit{if } (C), \\ t^{-(1+\alpha)} \log t \, M_0 \kappa E_{4\beta} & \textit{if } (F_1), \\ t^{-(1+\alpha)} M_\infty \kappa E_{4\beta} & \textit{if } (F). \end{cases}$$

Remarks (a) If $M_0, M_\infty \neq 0$, then $\|\mathcal{L}(t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} \simeq \phi(t)$ in all cases, since

$$\|E_{2\beta}\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} \simeq \varphi(t)^{\frac{1-\sigma(p)}{\theta}}, \qquad \|E_{4\beta}\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} \simeq \varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}. \tag{1.30}$$

As a corollary, $\|u(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} \simeq \phi(t)$.

(b) The behavior in "inner" intermediate scales is given by

$$t^{-\gamma} M_0 c_{2\beta} E_{2\beta} \quad \text{if } \gamma < 1+\alpha, \qquad t^{-(1+\alpha)} M_\infty \kappa E_{4\beta} \quad \text{if } \gamma \geq 1+\alpha.$$

This coincides with the "outer limit" of the behavior in compact sets; see (1.28).

Exterior scales. These are scales for which $|x| \geq \nu t^\theta$, $\nu > 0$. We already know from [7] that

$$\|u(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} = O(\phi(t)), \quad \text{where } \phi(t) = \begin{cases} t^{1-\gamma-\sigma(p)}, & \gamma < 1, \\ t^{-\sigma(p)} \log t, & \gamma = 1, \\ t^{-\sigma(p)}, & \gamma > 1. \end{cases} \tag{1.31}$$

The asymptotic behavior of u in such scales is given by a time convolution of $Y(\cdot,t)$ with the "mass" of f at time t,

$$M_f(t) := \int_{\mathbb{R}^N} f(x,t)\,\mathrm{d}x.$$

Theorem 1.3 *Let f satisfy (1.14). If $\gamma \leq 1$, assume moreover (1.18) if $p \in [1, p_c)$, and (1.20) if $p \geq p_c$. Then, given $\nu > 0$,*

$$\lim_{t\to\infty} \frac{1}{\phi(t)} \left\| u(\cdot,t) - \int_0^t M_f(s) Y(\cdot, t-s)\,\mathrm{d}s \right\|_{L^p(\{|x|>\nu t^\theta\})} = 0, \quad \textit{with } \phi \textit{ as in (1.31).}$$

Remarks (a) Notice that $|x|(t-s)^{-\theta} \geq \nu$ if $s \in (0,t)$ and $|x| > \nu t^{\theta}$. Therefore, using (1.10),

$$\|Y(\cdot, t-s)\|_{L^p(\{|x|>\nu t^\theta\})} \leq C t^{-\alpha - N\theta(1-\frac{1}{p})}(t-s)^{2\alpha-1} \quad \text{for all } s \in (0,t).$$

On the other hand, (1.14) yields $|M_f(s)| \leq C(1+s)^{-\gamma}$, and we conclude easily that

$$\begin{aligned}&\Big\|\int_0^t M_f(s) Y(\cdot, t-s)\,\mathrm{d}s\Big\|_{L^p(\{|x|>\nu t^\theta\})}\\ &\quad\leq C t^{-\alpha - N\theta(1-\frac{1}{p})}\int_0^t (1+s)^{-\gamma}(t-s)^{2\alpha-1}\,\mathrm{d}s = O(\phi(t)).\end{aligned}$$

Hence, $\|u(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} = O(\phi(t))$.

(b) Assume that $M_f(s)(1+s)^{\gamma} \geq c$ or $M_f(s)(1+s)^{\gamma} \leq -c$ for some $c > 0$, a condition that is satisfied, for instance, if f is of separate variables, $f(x,t) = g(x)(1+t)^{-\gamma}$, and $\int_{\mathbb{R}^N} g \neq 0$. If $\beta \in (0,1)$, then $G(\xi) \simeq E_{-2\beta}(\xi)$ for $|\xi| > \nu > 0$; see [17]. Therefore, for some constants $c_\nu > 0$, which may change from line to line,

$$\begin{aligned}&\Big\|\int_0^t M_f(s) Y(\cdot, t-s)\,\mathrm{d}s\Big\|_{L^p(\{|x|>\nu t^\theta\})}\\ &\quad\geq c_\nu \|E_{-2\beta}\|_{L^p(\{|x|>\nu t^\theta\})}\int_0^t (1+s)^{-\gamma}(t-s)^{2\alpha-1}\,\mathrm{d}s\\ &\quad\geq c_\nu t^{-\alpha - N\theta(1-\frac{1}{p})}\int_0^t (1+s)^{-\gamma}(t-s)^{2\alpha-1}\,\mathrm{d}s \simeq \phi(t).\end{aligned}$$

We conclude that

$$\Big\|\int_0^t M_f(s) Y(\cdot, t-s)\,\mathrm{d}s\Big\|_{L^p(\{|x|>\nu t^\theta\})} \simeq \phi(t),$$

and hence we have the sharp rate $\|u(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} \simeq \phi(t)$.

When $\gamma \geq 1$, we can avoid the time convolution in the description of the large time behavior, which is now given by $M(t)t^{1-\alpha}Y(\cdot,t)$, where $M(t)$ is the mass of the solution u at time t,

$$M(t) := \int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x.$$

Let us remark that Fubini's theorem plus the fact that $\int_{\mathbb{R}^N} Y(x,t)\,\mathrm{d}x = t^{\alpha-1}$ yield

$$M(t) = \int_0^t M_f(s)(t-s)^{\alpha-1}\,\mathrm{d}s.$$

Hence, $M(t)$ can be computed directly in terms of the forcing term f without determining u.

Theorem 1.4 *Under the assumptions of Theorem 1.3, if $\gamma \geq 1$, then*

$$\frac{1}{\phi(t)}\left\| \int_0^t M_f(s)Y(\cdot, t-s)\,\mathrm{d}s - M(t)t^{1-\alpha}Y(\cdot, t)\right\|_{L^p(\{|x|>\nu t^\theta\})} \to 0 \quad \text{as } t \to \infty.$$

As a corollary,

$$\lim_{t\to\infty} \frac{1}{\phi(t)}\|u(\cdot, t) - M(t)t^{1-\alpha}Y(\cdot, t)\|_{L^p(\{|x|>\nu t^\theta\})} = 0.$$

When $\gamma > 1$ things simplify even further, since $M(t)t^{1-\alpha}$ has a computable limit.

Theorem 1.5 *Let $\gamma > 1$. Under the assumptions of Theorem 1.3, $\lim_{t\to\infty} M(t)t^{1-\alpha} = M_\infty$. As a corollary,*

$$\lim_{t\to\infty} t^{\sigma(p)}\|u(\cdot, t) - M_\infty Y(\cdot, t)\|_{L^p(\{|x|>\nu t^\theta\})} = 0. \tag{1.32}$$

Remarks (a) The result (1.32) was already known when $p \in [1, p_c)$; see [17].

(b) Since $\|Y(\cdot, t)\|_{L^p(\{|x|>\nu t^\theta\})} = Ct^{-\sigma(p)}$ (see Sect. 1.2), if $\gamma > 1$ and $M_\infty \neq 0$ we obtain as a corollary that $\|u(\cdot, t)\|_{L^p(\{|x|>\nu t^\theta\})} \simeq t^{-\sigma(p)}$, without assuming that f is of separate variables.

(c) When $\gamma > 1$, the inner limit, $|\xi| \to 0$, of the outer profile is given by

$$M_\infty Y(x, t) \approx M_\infty t^{-\sigma_*}\kappa E_{4\beta}(\xi) = t^{-(1+\alpha)}M_\infty\kappa E_{4\beta}(x),$$

which coincides with the behavior for "outer" intermediate scales; see Theorem 1.2, case (F).

The asymptotic limit can also be simplified when $\gamma = 1$, at the expense of asking f to be asymptotically of separate variables.

Theorem 1.6 *Let $\gamma = 1$. Under the assumption (1.16), $M(t) = M_0 t^{\alpha-1}\log t(1 + o(1))$. As a corollary,*

$$\lim_{t\to\infty} \frac{t^{\sigma(p)}}{\log t}\|u(\cdot, t) - M_0 \log t\, Y(\cdot, t)\|_{L^p(\{|x|>\nu t^\theta\})} = 0.$$

Remarks (a) If $M_0 \neq 0$ and $\gamma = 1$, we have the sharp decay rate $\|u(\cdot, t)\|_{L^p(\{|x|>\nu t^\theta\})} \simeq t^{-\sigma(p)}\log t$ assuming only that f is asymptotically of separate variables.

(b) If f satisfies (1.16) with $\gamma = 1$, the inner limit, $|\xi| \to 0$, of the outer profile is given by

$$M_0 \log t\, Y(x, t) \approx M_0 t^{-\sigma_*}\log t\, \kappa E_{4\beta}(\xi) = t^{-(1+\alpha)}\log t\, M_0\kappa E_{4\beta}(x),$$

which coincides with the behavior for "outer" intermediate scales; see Theorem 1.2, case (F_1).

The purpose of our last result is to check that for $\gamma < 1$ the inner limit of the outer profile also coincides with the behavior for "outer" intermediate scales, given by Theorem 1.2, case (S), when f is asymptotically of separate variables.

Proposition 1.1 *Let $\gamma < 1$. If f satisfies (1.16) and the hypotheses of Theorem 1.3, then*

$$\int_0^t M_f(s)Y(x,t-s)\,\mathrm{d}s = \big(1+o(1)\big)t^{-\gamma}M_0 c_{2\beta}E_{2\beta}(x) \quad \textit{if } |\xi| = o(1) \textit{ as } t\to\infty.$$

Remarks (a) The limit behavior (1.8) for the profile G yields

$$Y(x,t) = (\kappa + o(1))t^{-(1+\alpha)}E_{4\beta}(x) \quad \text{if } |\xi| \to 0. \tag{1.33}$$

Hence, $M(t)t^{1-\alpha}Y(x,t) \approx M(t)t^{-2\alpha}\kappa E_{4\beta}(x)$ as $|\xi| \to 0$. Therefore, if $\gamma < 1$ the limit profile in outer regions, $\int_0^t M_f(s)Y(\cdot,t-s)\,\mathrm{d}s$, does not coincide with $M(t)t^{1-\alpha}Y(x,t)$, in contrast with the case $\gamma \geq 1$.

(b) Under the assumptions of Proposition 1.1, if $M_0 \neq 0$ and $\delta > 0$ is small enough, then

$$\Big|\int_0^t M_f(s)Y(x,t-s)\,\mathrm{d}s\Big| \simeq t^{-\gamma}E_{2\beta}(x) \quad \text{if } |x|t^{-\theta} < \delta.$$

Therefore, if $\nu < \delta$, with $\delta > 0$ small enough, we have

$$\begin{aligned}
&\Big\|\int_0^t M_f(s)Y(\cdot,t-s)\,\mathrm{d}s\Big\|_{L^p(\{|x|>\nu t^\theta\})}\\
&\geq \Big\|\int_0^t M_f(s)Y(\cdot,t-s)\,\mathrm{d}s\Big\|_{L^p(\{\delta t^\theta>|x|>\nu t^\theta\})}\\
&\simeq t^{-\gamma}\|E_{2\beta}\|_{L^p(\{\delta t^\theta>|x|>\nu t^\theta\})} \simeq t^{1-\gamma-\sigma(p)}.
\end{aligned}$$

We conclude that if $\gamma < 1$, under suitable assumptions on f,

$$\Big\|\int_0^t M_f(s)Y(\cdot,t-s)\,\mathrm{d}s\Big\|_{L^p(\{|x|>\nu t^\theta\})} \simeq t^{1-\gamma-\sigma(p)} \quad \text{if } \nu \text{ is small,}$$

and hence we have the sharp rate $\|u(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} \simeq t^{1-\gamma-\sigma(p)}$.

2 Compact sets

The goal of this section is to obtain the large-time behavior in compact sets, Theorem 1.1. We start by checking that, under the assumptions of that theorem, the functions giving the large-time behavior are in the desired spaces.

Lemma 2.1 (i) *Let $g \in L^1(\mathbb{R}^N)$. If $p \geq p_c$, assume in addition that $g \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for some $q \in (q_c(p), p]$. Then $I_{2\beta}[g] \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$.*

(ii) *Let* $\gamma > 1$. *Let* f *satisfy* (1.14). *If* $p \geq p_c$, *assume in addition* (1.15). *Then* F *given by* (1.23) *satisfies* $I_{4\beta}[F] \in L^p_{\rm loc}(\mathbb{R}^N)$.

Proof (i) We make the estimate $|I_{2\beta}[g]| \leq \mathrm{I} + \mathrm{II}$, where

$$\mathrm{I}(x) = \int_{|y|<1} |g(x-y)| E_{2\beta}(y)\,\mathrm{d}y, \qquad \mathrm{II}(x) = \int_{|y|>1} |g(x-y)|\,\mathrm{d}y.$$

In order to estimate I we take

$$\begin{aligned} &q = 1 \text{ if } p \in [1, p_c), \quad q \in (q_c(p), p] \text{ as in the hypothesis if } p \geq p_c, \\ &1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}. \end{aligned} \tag{2.1}$$

Notice that $r \in [1, p_c)$. Then, using the hypotheses on g, we have

$$\|\mathrm{I}\|_{L^p(K)} \leq \|g\|_{L^q(K+B_1)} \|E_{2\beta}\|_{L^r(B_1)} < \infty.$$

On the other hand, $\mathrm{II}(x) \leq \|g\|_{L^1(\mathbb{R}^N)}$, hence the result.

(ii) Splitting the spatial integral as before, we have $|I_{4\beta}[F]| \leq \mathrm{I} + \mathrm{II}$, where

$$\mathrm{I}(x) = \int_0^\infty \int_{|y|<1} |f(x-y)| E_{4\beta}(y)\,\mathrm{d}y\mathrm{d}s,$$
$$\mathrm{II}(x) = \int_0^\infty \int_{|y|>1} |f(x-y)|\,\mathrm{d}y\mathrm{d}s.$$

Taking q and r as in (2.1), and using (1.14), and also (1.15) if $p \geq p_c$,

$$\|\mathrm{I}\|_{L^p(K)} \leq \|E_{4\beta}\|_{L^r(B_1)} \int_0^\infty \|f(\cdot, s)\|_{L^q(K+B_1)}\,\mathrm{d}s \leq C \int_0^\infty (1+s)^{-\gamma}\,\mathrm{d}s < \infty,$$

since $r \in [1, p_*)$, and $\gamma > 1$. On the other hand, using the size hypothesis (1.14), $\mathrm{II}(x) \leq M_\infty$, whence the result.

□

We now proceed to the proof of Theorem 1.1. As a first step we prove the result substituting the constant $c_{2\beta}$ in the definition (1.25) of $\mathcal{L}$ by the constant

$$A := \frac{1}{\theta\omega_N} \int_{\mathbb{R}^N} \frac{G(\xi)}{|\xi|^{2\beta}}\,\mathrm{d}\xi = \frac{1}{\theta} \int_0^\infty \rho^{N-1-2\beta} \tilde{G}(\rho)\,\mathrm{d}\rho, \tag{2.2}$$

where ω_N denotes de measure of $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$, the unit sphere in $\mathbb{R}^N$, and $\tilde{G}(\rho) = G(\rho\zeta)$ for any $\zeta \in \mathbb{S}^{N-1}$ (remember that G is radially symmetric). We observe that the estimates (1.6)–(1.7) on the profile G guarantee that A is a finite number.

Proposition 2.1 *Under the assumptions of Theorem 1.1, the convergence result (1.24) is true with the constant $c_{2\beta}$ in the definition (1.25) of $\mathcal{L}$ substituted by the constant A given in (2.2).*

Proof As we will see, the value of f at points that are far away from x will not contribute to the behavior of the solution at that point. Hence, we estimate the error as

$$
\begin{aligned}
|t^{\min\{\gamma,1+\alpha\}}u(x,t)-\mathcal{L}(x)| &\le t^{\min\{\gamma,1+\alpha\}}(\mathrm{I}(x,t)+\mathrm{II}(x,t)), \quad \text{where}\\
\mathrm{I}(x,t) &= \int_0^t\int_{|y|>L}|f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{II}(x,t) &= \Big|\int_0^t\int_{|y|<L} f(x-y,t-s)Y(y,s)\,\mathrm{d}y\mathrm{d}s - t^{-\min\{\gamma,1+\alpha\}}\mathcal{L}(x)\Big|,
\end{aligned}
$$

with $L>0$ large to be chosen later.

In order to estimate I, for every $t>1$ we split it as $\mathrm{I}=\mathrm{I}_1+\mathrm{I}_2$, where

$$
\begin{aligned}
\mathrm{I}_1(x,t) &= \int_0^1\int_{|y|>L}|f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_2(x,t) &= \int_1^t\int_{|y|>L}|f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

We start with I_1. Using the exterior bound (1.10) and the decay assumption (1.14),

$$
\mathrm{I}_1(x,t)\le C\int_0^1\int_{|y|>L}|f(x-y,t-s)|\frac{s^{2\alpha-1}}{|y|^{N+2\beta}}\,\mathrm{d}y\mathrm{d}s \le \frac{Ct^{-\gamma}}{L^{N+2\beta}}\int_0^1 s^{2\alpha-1}\,\mathrm{d}s < \varepsilon t^{-\gamma}
$$

for all $t>1$ if $L>0$ is large enough.

In order to estimate I_2 we use now the global estimate (1.9), and again the decay assumption (1.14) to obtain, for all $t>1$,

$$
\begin{aligned}
\mathrm{I}_2(x,t) &\le C\int_1^t\int_{|y|>L}|f(x-y,t-s)|s^{-(1+\alpha)}E_{4\beta}(y)\,\mathrm{d}y\mathrm{d}s\\
&\le \frac{Ct^{-\gamma}}{L^{N-4\beta}}\int_1^{t/2}s^{-(1+\alpha)}\,\mathrm{d}s + \frac{Ct^{-(1+\alpha)}}{L^{N-4\beta}}\int_{t/2}^t(1+t-s)^{-\gamma}\,\mathrm{d}s\\
&\le \frac{Ct^{-\gamma}}{L^{N-4\beta}} + \frac{Ct^{-(1+\alpha)}}{L^{N-4\beta}}\int_{t/2}^t(1+t-s)^{-\gamma}\,\mathrm{d}s.
\end{aligned}
$$

Since

$$
\int_{\lambda t}^t(1+t-s)^{-\gamma}\,\mathrm{d}s = \int_0^{(1-\lambda)t}(1+s)^{-\gamma}\,\mathrm{d}s \le C_\lambda\begin{cases} t^{1-\gamma}, & \gamma<1,\\ \log t, & \gamma=1,\\ 1, & \gamma>1,\end{cases} \tag{2.3}
$$

for any $\lambda \in (0,1)$, then $\mathrm{I}(x,t) \le C\varepsilon t^{-\min\{\gamma,1+\alpha\}}$ for every $t > 1$ if $L > 0$ is large enough.

It will turn out that the values of f at times close to t only contribute to the asymptotic behavior of u if $\gamma \le 1+\alpha$, while its values at times in the interval $(0, t/2)$ only matter if $\gamma \ge 1+\alpha$ (notice, however, that this interval expands to the whole $\mathbb{R}_+$ as $t \to \infty$). Therefore, we make the estimate $\mathrm{II} \le \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3$, where

$$\mathrm{II}_1(x,t) = \begin{cases} \Big| \displaystyle\int_0^{\delta t} \int_{|y|<L} f(x-y,t-s) Y(y,s)\,\mathrm{d}y\mathrm{d}s - t^{-\gamma} A I_{2\beta}[g](x) \Big|, & \gamma \le 1+\alpha, \\ \displaystyle\int_0^{\delta t} \int_{|y|<L} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s, & \gamma > 1+\alpha, \end{cases}$$

$$\mathrm{II}_2(x,t) = \int_{\delta t}^{t/2} \int_{|y|<L} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s,$$

$$\mathrm{II}_3(x,t) = \begin{cases} \displaystyle\int_{t/2}^{t} \int_{|y|<L} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s, & \gamma < 1+\alpha, \\ \Big| \displaystyle\int_{t/2}^{t} \int_{|y|<L} f(x-y,t-s) Y(y,s)\,\mathrm{d}y\mathrm{d}s - t^{-(1+\alpha)} \kappa I_{4\beta}[F](x) \Big|, & \gamma \ge 1+\alpha, \end{cases}$$

for some small value $\delta \in (0, 1/2)$ to be chosen later.

We start by estimating II_1 when $\gamma \le 1+\alpha$. We have $\mathrm{II}_1 \le \sum_{i=1}^4 \mathrm{II}_{1i}$, where

$$\begin{aligned} \mathrm{II}_{11}(x,t) &= \int_0^{\delta t} \int_{|y|<L} |f(x-y,t-s)(1+t-s)^{\gamma} - g(x-y)| \\ &\quad \times (1+t-s)^{-\gamma} Y(y,s)\,\mathrm{d}y\mathrm{d}s, \\ \mathrm{II}_{12}(x,t) &= \int_0^{\delta t} \big|(1+t-s)^{-\gamma} - t^{-\gamma}\big| \int_{|y|<L} |g(x-y)| Y(y,s)\,\mathrm{d}y\mathrm{d}s, \\ \mathrm{II}_{13}(x,t) &= t^{-\gamma} \int_{|y|<L} |g(x-y)| E_{2\beta}(y) \Big| \int_0^{\delta t} \frac{Y(y,s)}{E_{2\beta}(y)}\,\mathrm{d}s - A \Big|\,\mathrm{d}y, \\ \mathrm{II}_{14}(x,t) &= t^{-\gamma} A \int_{|y|>L} |g(x-y)| E_{2\beta}(y)\,\mathrm{d}y. \end{aligned}$$

We take q and r as in (2.1). Using the L^r norm of the kernel (1.13) if $s < 1$, and the global estimate (1.9) when $s > 1$, together with the size assumption (1.16) on f if p is subcritical or (1.17) otherwise, for all $t > 1/\delta$ we have

$$\begin{aligned} \|\mathrm{II}_{11}(\cdot,t)\|_{L^p(K)} \le{}& C t^{-\gamma} \int_0^1 \|f(\cdot,t-s)(1+t-s)^{\gamma} - g\|_{L^q(K+B_L)} \\ & \times \|Y(\cdot,s)\|_{L^r(B_L)}\,\mathrm{d}s \\ & + C t^{-\gamma} \int_1^{\delta t} \|f(\cdot,t-s)(1+t-s)^{\gamma} - g\|_{L^q(K+B_L)} \\ & \times s^{-(1+\alpha)} \|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s \end{aligned}$$

$$\leq C\varepsilon t^{-\gamma}\Big(\int_0^1 s^{-\sigma(r)}\,\mathrm{d}s+\int_1^{\delta t} s^{-(1+\alpha)}\,\mathrm{d}s\Big)\leq C\varepsilon t^{-\gamma}.$$

Given $\varepsilon>0$, there exists a small constant $\delta=\delta(\varepsilon)>0$ and a time T_ε such that

$$|(1+t-s)^{-\gamma}-t^{-\gamma}|<\varepsilon t^{-\gamma}\quad \text{if } 0<s<\delta t \text{ and } t\geq T_\varepsilon.$$

Therefore, taking q and r as in (2.1),

$$\begin{aligned}\|\mathrm{II}_{12}(\cdot,t)\|_{L^p(K)}&\leq \varepsilon t^{-\gamma}\int_0^1 \|g\|_{L^q(K+B_L)}\|Y(\cdot,s)\|_{L^r(B_L)}\,\mathrm{d}s\\ &\quad+\varepsilon t^{-\gamma}\int_1^{\delta t}\|g\|_{L^q(K+B_L)}s^{-(1+\alpha)}\|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s\leq C\varepsilon t^{-\gamma}.\end{aligned}$$

As for II_{13}, making the change of variables $\rho=|y|s^{-\theta}$ we get

$$\int_0^{\delta t}\frac{Y(y,s)}{E_{2\beta}(y)}\,\mathrm{d}s=\frac{1}{\theta}\int_{|y|/(\delta t)^\theta}^{\infty}\rho^{N-1-2\beta}G\Big(\frac{y}{|y|}\rho\Big)\,\mathrm{d}\rho. \tag{2.4}$$

Therefore, due to the definition (2.2) of A, for any fixed $L>0$,

$$\Big|\int_0^{\delta t}\frac{Y(y,s)}{E_{2\beta}(y)}\,\mathrm{d}s-A\Big|=o(1)\quad\text{uniformly in } |y|<L.$$

Hence, taking q and r as before,

$$\|\mathrm{II}_{13}(\cdot,t)\|_{L^p(K)}\leq o(t^{-\gamma})\|g\|_{L^q(K+B_L)}\|E_{2\beta}\|_{L^r(B_L)}=o(t^{-\gamma}).$$

We finally observe that

$$\mathrm{II}_{14}(x,t)\leq\frac{A\|g\|_{L^1(\mathbb{R}^N)}}{L^{N-2\beta}}t^{-\gamma}\leq\varepsilon t^{-\gamma}$$

if $L>0$ is large enough.

If $\gamma>1+\alpha$ the estimate for II_1 is easier. Let q and r be as above. Using the L^p norm of the kernel (1.13) if $s<1$ and the global estimate (1.9) when $s>1$, for all

$t > 1/\delta$ we have, thanks to the assumptions on the size of f,

$$\begin{aligned}\|\mathrm{II}_1(\cdot,t)\|_{L^p(K)} &\le \int_0^1 \|f(\cdot,t-s)\|_{L^q(K+B_L)}\|Y(\cdot,s)\|_{L^r(B_L)}\,\mathrm{d}s\\ &\quad+\int_1^{\delta t}\|f(\cdot,t-s)\|_{L^q(K+B_L)}s^{-(1+\alpha)}\|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s\\ &\le C\int_0^1(1+t-s)^{-\gamma}s^{-\sigma(r)}\,\mathrm{d}s+\int_1^{\delta t}(1+t-s)^{-\gamma}s^{-(1+\alpha)}\,\mathrm{d}s\\ &\le Ct^{-\gamma}\Big(\int_0^1 s^{-\sigma(r)}\,\mathrm{d}s+\int_1^{\delta t}s^{-(1+\alpha)}\,\mathrm{d}s\Big)\le Ct^{-\gamma}.\end{aligned}$$

Now we turn our attention to II_2. Taking q and r as before,

$$\begin{aligned}\|\mathrm{II}_2(\cdot,t)\|_{L^p(K)} &\le \int_{\delta t}^{t/2}\|f(\cdot,t-s)\|_{L^q(K+B_L)}s^{-(1+\alpha)}\|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s\\ &\le C\int_{\delta t}^{t/2}(1+t-s)^{-\gamma}s^{-(1+\alpha)}\,\mathrm{d}s\le Ct^{-\gamma-\alpha}=o(t^{-\min\{\gamma,1+\alpha\}}).\end{aligned}$$

Finally, we turn our attention to II_3. We start with the case $\gamma < 1+\alpha$. Using the global estimate (1.9) and then the size assumptions on f we get, with q and r as above,

$$\begin{aligned}\|\mathrm{II}_3(\cdot,t)\|_{L^p(K)} &\le \int_{t/2}^{t}\|f(\cdot,t-s)\|_{L^q(K+B_L)}s^{-(1+\alpha)}\|E_{4\beta}\|_{L^r(B_L)}\\ &\le Ct^{-(1+\alpha)}\int_{t/2}^{t}(1+t-s)^{-\gamma}\,\mathrm{d}s,\end{aligned}$$

from where we get, using (2.3), that $\|\mathrm{II}_3(\cdot,t)\|_{L^p(K)} = t^{-\min\{\gamma,1+\alpha\}}o(1)$ in this range, as desired.

When $\gamma \ge 1+\alpha$ the estimate of II_3 is more involved. We have $\mathrm{II}_3 \le \sum_{i=1}^5 \mathrm{II}_{3i}$, where

$$\begin{aligned}\mathrm{II}_{31}(x,t) &= \int_{t/2}^{t}\int_{|y|<L}|f(x-y,t-s)|\big|Y(y,s)-\kappa E_{4\beta}(y)s^{-(1+\alpha)}\big|\,\mathrm{d}y\mathrm{d}s,\\ \mathrm{II}_{32}(x,t) &= \kappa\int_{t/2}^{t-\sqrt{t}}\int_{|y|<L}|f(x-y,t-s)|E_{4\beta}(y)\big(s^{-(1+\alpha)}-t^{-(1+\alpha)}\big)\,\mathrm{d}y\mathrm{d}s,\\ \mathrm{II}_{33}(x,t) &= \kappa\int_{t-\sqrt{t}}^{t}\int_{|y|<L}|f(x-y,t-s)|E_{4\beta}(y)\big(s^{-(1+\alpha)}-t^{-(1+\alpha)}\big)\,\mathrm{d}y\mathrm{d}s,\\ \mathrm{II}_{34}(x,t) &= \kappa t^{-(1+\alpha)}\int_{|y|<L}E_{4\beta}(y)\Big|\int_{t/2}^{t}f(x-y,t-s)\,\mathrm{d}s-\int_0^\infty f(x-y,s)\,\mathrm{d}s\Big|\,\mathrm{d}y,\\ \mathrm{II}_{35}(x,t) &= \kappa t^{-(1+\alpha)}\int_0^\infty\int_{|y|>L}|f(x-y,s)|E_{4\beta}(y)\,\mathrm{d}y\mathrm{d}s.\end{aligned}$$

We first observe that

$$|Y(y,s)-\kappa E_{4\beta}(y)s^{-(1+\alpha)}|=s^{-(1+\alpha)}E_{4\beta}(y)\Big|\frac{G(ys^{-\theta})}{E_{4\beta}(ys^{-\theta})}-\kappa\Big|.$$

Therefore, thanks to (1.8), given $\varepsilon>0$ and $L>0$ there is a time $T=T(\varepsilon,L)$ such that

$$|Y(y,s)-\kappa E_{4\beta}(y)s^{-(1+\alpha)}|<\varepsilon s^{-(1+\alpha)}E_{4\beta}(y)\quad \text{if } |y|<L,\ t\ge T.$$

Hence, if $t\ge T$ we have, for q and r chosen as above, and using (2.3) and the size assumptions on f,

$$\|\mathrm{II}_{31}(\cdot,t)\|_{L^p(K)}\le C\varepsilon t^{-(1+\alpha)}\int_{t/2}^{t}\|f(\cdot,t-s)\|_{L^q(K+B_L)}\|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s\le C\varepsilon t^{-(1+\alpha)},$$

so that $\|\mathrm{II}_{31}(\cdot,t)\|_{L^p(K)}\le C\varepsilon t^{-\min\{\gamma,(1+\alpha)\}}$, as desired.

On the other hand, with q and r as always, using the size assumptions on f,

$$\begin{aligned}\|\mathrm{II}_{32}(\cdot,t)\|_{L^p(K)}&\le Ct^{-(1+\alpha)}\int_{t/2}^{t-\sqrt{t}}\|f(\cdot,t-s)\|_{L^q(K+B_L)}\|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s\\&\le Ct^{-(1+\alpha)}\int_{t/2}^{t-\sqrt{t}}(1+t-s)^{-\gamma}\,\mathrm{d}s\le Ct^{-(1+\alpha)}t^{\frac{1-\gamma}{2}}=o(t^{-(1+\alpha)}).\end{aligned}$$

By the Mean Value Theorem, $0\le s^{-(1+\alpha)}-t^{-(1+\alpha)}\le(1+\alpha)s^{-(2+\alpha)}(t-s)$ if $s<t$. Therefore,

$$\begin{aligned}\|\mathrm{II}_{33}(\cdot,t)\|_{L^p(K)}&\le Ct^{-(2+\alpha)}\sqrt{t}\int_{t-\sqrt{t}}^{t}\|f(\cdot,t-s)\|_{L^q(K+B_L)}\|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s\\&\le Ct^{-(2+\alpha)}\sqrt{t}\int_0^{\sqrt{t}}(1+s)^{-\gamma}\,\mathrm{d}s\le Ct^{-(1+\alpha)}t^{-1/2}=o(t^{-(1+\alpha)}).\end{aligned}$$

Now we notice that

$$\Big|\int_{t/2}^{t}f(x-y,t-s)\,\mathrm{d}s-\int_0^\infty f(x-y,s)\,\mathrm{d}s\Big|=\Big|\int_{t/2}^\infty f(x-y,s)\,\mathrm{d}s\Big|.$$

Therefore, for t large enough,

$$\begin{aligned}\|\mathrm{II}_{34}(\cdot,t)\|_{L^p(K)}&\le Ct^{-(1+\alpha)}\int_{t/2}^\infty\|f(\cdot,s)\|_{L^q(K+B_L)}\|E_{4\beta}\|_{L^r(B_L)}\,\mathrm{d}s\\&\le Ct^{-(1+\alpha)}\int_{t/2}^\infty(1+s)^{-\gamma}\,\mathrm{d}s\le Ct^{-(\gamma+\alpha)}.\end{aligned}$$

Finally,

$$\mathrm{II}_{35}(x,t) \le \frac{Ct^{-(1+\alpha)}}{L^{N-4\beta}} \int_0^\infty \int_{\mathbb{R}^N} |f(y,s)|\,\mathrm{d}y\mathrm{d}s < \varepsilon t^{-(1+\alpha)}$$

if L is large enough, and hence $\|\mathrm{II}_{35}(\cdot,t)\|_{L^p(K)} \le C\varepsilon t^{-(1+\alpha)}$. □

The proof of Theorem 1.1 will then be complete if we are able to show that $A = c_{2\beta}$, a somewhat surprising result that is interesting on its own. The idea to prove this fact is to consider a particular forcing term for which the computations are "explicit", namely a stationary one, $g \in C_c^\infty(\mathbb{R}^N)$, with g nonnegative. We expect the solution u of (1.1) with such a right-hand side to resemble for large times the stationary solution $S := c_{2\beta} I_{2\beta}[g]$. Hence, we will study the difference, $U = S - u$. This function is a solution (in a generalized sense) to (1.3), given by the formula $U(\cdot,t) = Z(\cdot,t) * S$. To check this last assert it is enough to observe that S is a bounded classical solution to (1.27) with $f(\cdot,t) = g$ for all $t > 0$ and $u_0 = S$. But bounded classical solutions to (1.27) are unique, and represented by

$$u(x,t) = \int_{\mathbb{R}^N} Z(x-y,t)u_0(y)\,\mathrm{d}y + \int_0^t \int_{\mathbb{R}^N} Y(x-y,t-s)f(y,s)\,\mathrm{d}y\mathrm{d}s,$$

hence generalized solutions; see [14, 17, 19].

Our first aim is to prove that U vanishes asymptotically, so that u indeed resembles S.

Proposition 2.2 *Let U be the generalized solution to (1.3) with initial datum $U(\cdot,0) = c_{2\beta} I_{2\beta}[g]$ for some nonnegative $g \in C_c^\infty(\mathbb{R}^N)$. Then, $\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = o(1)$ as $t \to \infty$.*

Proof We recall that the kernel Z has a self-similar form; see (1.4). Its profile F belongs to $L^p(\mathbb{R}^N)$ if and only if $p \in [1, p_c)$; see, for instance, [17]. Hence, $\|Z(\cdot,t)\|_{L^p(\mathbb{R}^N)} = Ct^{-N\theta(1-\frac{1}{p})}$ for p in that range. On the other hand, $0 \le U(x,0) \le C(1+|x|)^{-(N-2\beta)}$ (see, for instance, Theorem A.1 in the Appendix), so that $U(\cdot,0) \in L^p(\mathbb{R}^N)$ for all $p > p_c$. Let $p > N/(2\beta)$. This guarantees, on the one hand, that $p > p_c$, since $N > 4\beta$, and on the other hand that $p/(p-1) < p_c$. Therefore, for any such p, Young's inequality implies

$$\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le \|Z(\cdot,t)\|_{L^{p/(p-1)}(\mathbb{R}^N)} \|U(\cdot,0)\|_{L^p(\mathbb{R}^N)} = Ct^{-\frac{N\theta}{p}},$$

whence the result. □

We are now ready to prove the equality of the constants, and hence the validity of Theorem 1.1.

Corollary 2.1 *The constant A defined in (2.2) coincides with $c_{2\beta}$. Therefore, Proposition 2.1 implies Theorem 1.1.*

Proof Let u be the solution to problem (1.1) with a non-negative and non-trivial right-hand side $g \in C_c^\infty(\mathbb{R})$ independent of time. Then, as mentioned above, $U = S - u$ is a generalized solution to (1.3). Thus, given $K \subset\subset \mathbb{R}^N$,

$$\begin{aligned}
&|c_{2\beta} - A| \|I_{2\beta}[g]\|_{L^\infty(K)} \\
&\quad = \|c_{2\beta} I_{2\beta}[g] - u(\cdot, t) + u(\cdot, t) - A I_{2\beta}[g]\|_{L^\infty(K)} \\
&\quad \le \|U(\cdot, t)\|_{L^\infty(K)} + \|u(\cdot, t) - A I_{2\beta}[g]\|_{L^\infty(K)} \to 0 \quad \text{as } t \to \infty,
\end{aligned}$$

due to propositions 2.1 and 2.2, hence the result, since $\|I_{2\beta}[g]\|_{L^\infty(K)} \neq 0$. □

3 Intermediate scales

In this section, we study the limit profile in regions where $|x| \simeq \varphi(t)$ with $\varphi(t) \succ 1$ and $\varphi(t) = o(t^\theta)$ as $t \to \infty$.

Proof of Theorem 1.2 We want to show that $u(\cdot, t) - \mathcal{L}(t) = o(\phi(t))$. It will turn out that in these scales the large-time behavior at a point x comes in first approximation from the behavior of $f(\cdot, t)$ at points that are relatively close to x, as compared with $|x|$. Hence, we estimate the error as $|u - \mathcal{L}| \le \mathrm{I} + \mathrm{II} + \mathrm{III}$, where

$$\begin{aligned}
\mathrm{I}(x, t) &= \left| \int_0^t \int_{|x-y| \le \ell(t)|x|} f(x - y, t - s) Y(y, s)\, \mathrm{d}y \mathrm{d}s - \mathcal{L}(t)(x) \right|, \\
\mathrm{II}(x, t) &= \int_0^{t/2} \int_{|x-y| > \ell(t)|x|} |f(x - y, t - s)| Y(y, s)\, \mathrm{d}y \mathrm{d}s, \\
\mathrm{III}(x, t) &= \int_{t/2}^t \int_{|x-y| > \ell(t)|x|} |f(x - y, t - s) Y(y, s)|\, \mathrm{d}y \mathrm{d}s,
\end{aligned}$$

with $\ell(t) = o(1)$ such that $\ell(t) \in (0, 1/2)$ for all $t > 0$ to be further especified later.

The times that are closer to t will contribute with a term involving $E_{2\beta}$ in slow and critical scales, while times which are closer to 0 will contribute with a term involving $E_{4\beta}$ in fast and critical scales. Therefore, we make an estimate of the form $\mathrm{I} \le \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3$, where

$$
\mathrm{I}_1(x,t)=\begin{cases}\displaystyle\int_0^{t/2}\int_{|x-y|\le\ell(t)|x|}|f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s \text{ if (F}_1)/(\mathrm{F}),\\ \displaystyle\Big|\int_0^{t/2}\int_{|x-y|\le\ell(t)|x|}f(x-y,t-s)Y(y,s)\,\mathrm{d}y\mathrm{d}s-M_0At^{-\gamma}E_{2\beta}(x)\Big| \text{ if (S)/(C}_1)/(\mathrm{C}),\end{cases}
$$

$$
\mathrm{I}_2(x,t)=\int_{t/2}^{t-t^{1-\delta(t)}}\int_{|x-y|\le\ell(t)|x|}|f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s,
$$

$$
\mathrm{I}_3(x,t)=\begin{cases}\displaystyle\int_{t-t^{1-\delta(t)}}^{t}\int_{|x-y|\le\ell(t)|x|}|f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s \quad \text{if (S)},\\ \displaystyle\Big|\int_{t-t^{1-\delta(t)}}^{t}\int_{|x-y|\le\ell(t)|x|}f(x-y,t-s)Y(y,s)\,\mathrm{d}y\mathrm{d}s\\ \qquad\qquad -M_0\kappa t^{-(1+\alpha)}\log t\,E_{4\beta}(x)\Big| \quad \text{if (C}_1)/(\mathrm{F}_1),\\ \displaystyle\Big|\int_{t-t^{1-\delta(t)}}^{t}\int_{|x-y|\le\ell(t)|x|}f(x-y,t-s)Y(y,s)\,\mathrm{d}y\mathrm{d}s-M_\infty\kappa t^{-(1+\alpha)}E_{4\beta}(x)\Big| \quad \text{if (C)/(F)},\end{cases}
$$

with $\delta(t)\in(\log 2/\log t,1/2)$ to be further specified later. The lower bound for $\delta(t)$ guarantees that $t/2<t-t^{1-\delta(t)}$.

Since $\ell(t)\in(0,1/2)$, $\frac{|x|}{2}<|y|<\frac{3|x|}{2}$ if $|x-y|<\ell(t)|x|$. Hence, the global estimate (1.9) yields

$$
0\le Y(y,s)\le Cs^{-(1+\alpha)}E_{4\beta}(x),\quad (y,s)\in Q, \tag{3.1}
$$

a bound that will be used several times when estimating I_i, $i=1,2,3$.

Let (F$_1$)/(F) hold. If $\nu\varphi(t)<|x|<\mu\varphi(t)$, for large times we have on the one hand $|x|\ne 0$, since $\varphi(t)\succ 1$, and on the other hand $(|x|/2)^{1/\theta}<t/2$ for large times, since $\varphi(t)=o(t^\theta)$. If $s<(|x|/2)^{1/\theta}$ and $|y|>\frac{|x|}{2}$, we have $|y|>s^\theta$, and we may use the exterior estimate (1.10) for the kernel. If $s>(|x|/2)^{1/\theta}$ we are away from the singularity in time, and we may use the global estimate (3.1). Therefore, using also the size assumption (1.14) on f, we obtain

$$
\begin{aligned}
\mathrm{I}_1(x,t)&\le C\int_0^{(|x|/2)^{1/\theta}}\int_{|y|>|x|/2}|f(x-y,t-s)|s^{2\alpha-1}E_{-2\beta}(y)\,\mathrm{d}y\mathrm{d}s\\
&\quad+C\int_{(|x|/2)^{1/\theta}}^{t/2}\int_{|y|>|x|/2}|f(x-y,t-s)|s^{-(1+\alpha)}E_{4\beta}(y)\,\mathrm{d}y\mathrm{d}s\\
&\le Ct^{-\gamma}\Big(E_{-2\beta}(x)\int_0^{(|x|/2)^{1/\theta}}s^{2\alpha-1}\,\mathrm{d}s+E_{4\beta}(x)\int_{(|x|/2)^{1/\theta}}^{t/2}s^{-(1+\alpha)}\,\mathrm{d}s\Big)\\
&=O(t^{-\gamma})E_{2\beta}(x),
\end{aligned}
$$

which combined with (1.30) yields $\|\mathrm{I}_1(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}=O\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big)=o(\phi(t))$ if (F$_1$) or (F) hold.

When (S)/(C_1)/(C) hold, we make the estimate $\mathrm{I}_1 \le \sum_{i=1}^5 \mathrm{I}_{1i}$, where

$$\begin{aligned}
\mathrm{I}_{11}(x,t) &= \int_0^{(|x|/2)^{1/\theta}} \int_{|x-y|<\ell(t)|x|} |f(x-y,t-s)(1+t-s)^\gamma \\
&\qquad - g(x-y)|(1+t-s)^{-\gamma} Y(y,s)\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_{12}(x,t) &= \int_{(|x|/2)^{1/\theta}}^{t/2} \int_{|x-y|<\ell(t)|x|} |f(x-y,t-s)(1+t-s)^\gamma \\
&\qquad - g(x-y)|(1+t-s)^{-\gamma} Y(y,s)\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_{13}(x,t) &= \int_0^{\delta t} \left|(1+t-s)^{-\gamma} - t^{-\gamma}\right| \int_{|x-y|<\ell(t)|x|} |g(x-y)|Y(y,s)\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_{14}(x,t) &= \int_{\delta t}^{t/2} \left|(1+t-s)^{-\gamma} - t^{-\gamma}\right| \int_{|x-y|<\ell(t)|x|} |g(x-y)|Y(y,s)\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_{15}(x,t) &= t^{-\gamma} \int_{|x-y|<\ell(t)|x|} |g(x-y)|E_{2\beta}(y) \left| \int_0^{t/2} \frac{Y(y,s)}{E_{2\beta}(y)}\,\mathrm{d}s - A \right| \mathrm{d}y,\\
\mathrm{I}_{16}(x,t) &= t^{-\gamma} A \int_{|x-y|<\ell(t)|x|} |g(x-y)||E_{2\beta}(y) - E_{2\beta}(x)|\,\mathrm{d}y,\\
\mathrm{I}_{17}(x,t) &= t^{-\gamma} A E_{2\beta}(x) \int_{|x-y|\ge\ell(t)|x|} |g(x-y)|\,\mathrm{d}y.
\end{aligned}$$

Since $s^\theta < |x|/2 < |y|$ in the region of integration of I_{11}, we may use on the one hand the exterior estimate (1.10) for the kernel, and on the other hand that $E_{-2\beta}(y) \le C E_{-2\beta}(x)$. Hence,

$$\mathrm{I}_{11}(x,t) \le C t^{-\gamma} v(t) E_{-2\beta}(x) \int_0^{(|x|/2)^{1/\theta}} s^{2\alpha-1}\,\mathrm{d}s,$$

with $v(t) = \sup_{\tau>t/2} \left\| f(\cdot,\tau)(1+\tau)^\gamma - g \right\|_{L^1(\mathbb{R}^N)}$. But we know from (1.16) that $v(t) = o(1)$. Therefore, $\mathrm{I}_{11}(x,t) = o(t^{-\gamma})E_{2\beta}(x)$, whence, using (1.30),

$$\|\mathrm{I}_{11}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o\Big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\Big) = o(\phi(t)).$$

The region of integration in I_{12} avoids the singularity in time. Hence, we may use the global estimate (3.1) to obtain

$$\mathrm{I}_{12}(x,t) \le C t^{-\gamma} v(t) E_{4\beta}(x) \int_{(|x|/2)^{1/\theta}}^{t/2} s^{-(1+\alpha)}\,\mathrm{d}s = o(t^{-\gamma})E_{2\beta}(x),$$

whence $\|\mathrm{I}_{12}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o(\phi(t))$.

To estimate I_{13} we note that $\left|(1+t-s)^{-\gamma} - t^{-\gamma}\right| \le \varepsilon t^{-\gamma}$ if $s \in (0,\delta t)$ with δ small and t large. Therefore, changing the order of integration,

$$\mathrm{I}_{13}(x,t) \le C\varepsilon t^{-\gamma} \int_{|x-y|<\ell(t)|x|} |g(x-y)| \int_0^{\delta t} Y(y,s)\,\mathrm{d}s\mathrm{d}y.$$

But (2.2) and (2.4) yield $\int_0^{\delta t} Y(y,s)\,\mathrm{d}s \le AE_{2\beta}(y)$. Hence, since $E_{2\beta}(y) \le CE_{2\beta}(x)$ for $|y| > |x|/2$, and using also the integrability of g,

$$\mathrm{I}_{13}(x,t) \le C\varepsilon t^{-\gamma} E_{2\beta}(x) \int_{\frac{1}{2}|x|<|y|<\frac{3}{2}|x|} |g(x-y)|\,\mathrm{d}y \le C\varepsilon t^{-\gamma} E_{2\beta}(x),$$

which combined with (1.30) yields $\|\mathrm{I}_{12}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = C\varepsilon t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}} = C\varepsilon\phi(t)$.

In the estimate of I_{14} we may use once more the global estimate (3.1), since we are away from the singularity in time, to obtain

$$\mathrm{I}_{14}(x,t) \le Ct^{-\gamma} E_{4\beta}(x) \int_{\delta t}^{t/2} s^{-(1+\alpha)}\,\mathrm{d}s = O\big(t^{-(\gamma+\alpha)}\big) E_{4\beta}(x).$$

Thus, using (1.30) and also that $\varphi(t) = o(t^\theta)$, we arrive at

$$\|\mathrm{I}_{12}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = O\big(t^{-(\gamma+\alpha)}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big) = o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).$$

As for II_{15}, since $|y| \le 3|x|/2$ in the region of integration, it follows from (2.4) that

$$\Big|\int_0^{t/2} \frac{Y(y,s)}{E_{2\beta}(y)}\,\mathrm{d}s - A\Big| < \int_0^{\frac{3|x|}{2(t/2)^\theta}} \rho^{N-1-2\beta} G\Big(\frac{y}{|y|}\rho\Big)\,\mathrm{d}\rho = o(1).$$

Hence, using also that $|y| \ge |x|/2$ and the integrability of g, we get $\mathrm{I}_{15}(x,t) = o(t^{-\gamma})E_{2\beta}(x)$, whence

$$\|\mathrm{I}_{15}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).$$

If $|x-y| < \ell(t)|x|$, then $\big||y|-|x|\big| \le |x-y| \le \ell(t)|x|$. Thus, using the Mean Value Theorem,

$$|E_{2\beta}(y) - E_{2\beta}(x)| \le \frac{(N-2\beta)\ell(t)}{(1-\ell(t))^{N-2\beta+1}} E_{2\beta}(x) \le C\ell(t)E_{2\beta}(x).$$

Thus, $\mathrm{I}_{16}(x,t) = o(t^{-\gamma})E_{2\beta}(x)$, whence

$$\|\mathrm{I}_{16}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).$$

As for I_{17}, if $|x| > \nu\varphi(t)$ with $\varphi(t) \succ 1$, taking $\ell(t) \succ 1/\varphi(t)$,

$$\int_{|x-y|\ge\ell(t)|x|} |g(x-y)|\,\mathrm{d}y \le \int_{|z|\ge\ell(t)\nu\varphi(t)} |g(z)|\,\mathrm{d}z = o(1) \quad \text{as } t\to\infty,$$

since $g \in L^1(\mathbb{R}^N)$. Therefore, $\|\mathrm{I}_{17}(\cdot,t)\|_{L^p(\{\nu\le|x|/\varphi(t)\le\mu\})} = o(\phi(t))$.

Summarizing, if $\ell(t) = o(1)$ is such that $\ell(t) \succ 1/\varphi(t)$, then

$$\|\mathrm{I}_1(\cdot,t)\|_{L^p(\{\nu\le|x|/\varphi(t)\le\ \mu\})} = o(\phi(t)).$$

We now turn our attention to I_2. Using again (3.1) and the assumption (1.14) on the time decay of f, we have, since $s > t/2$ in the region of integration,

$$\mathrm{I}_2(x,t) \le Ct^{-(1+\alpha)} E_{4\beta}(x) \int_{t/2}^{t-t^{1-\delta(t)}} (1+t-s)^{-\gamma}\,\mathrm{d}s.$$

On the other hand (remember that $\delta(t) < 1/2$, so that $t^{1-\delta(t)} \to \infty$),

$$\begin{aligned}\int_{t/2}^{t-t^{1-\delta(t)}} (1+t-s)^{-\gamma}\,\mathrm{d}s &= \int_{t^{1-\delta(t)}}^{t/2} (1+s)^{-\gamma}\,\mathrm{d}s \\ &= \begin{cases} O(t^{1-\gamma}), & \gamma < 1, \\ \log \frac{1+(t/2)}{1+t^{1-\delta(t)}} \le C\log(t^{\delta(t)}/2), & \gamma = 1, \\ o(1), & \gamma > 1. \end{cases}\end{aligned}$$

If $\delta(t) \succ 1/\log t$, then $t^{\delta(t)} \to \infty$, and hence $\log(t^{\delta(t)}/2) < C\log t^{\delta(t)} = C\delta(t)\log t = o(\log t)$ if $\delta(t) = o(1)$. With these additional assumptions on $\delta(t)$, we have then

$$\|\mathrm{I}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = \begin{cases} O\big(t^{-(\gamma+\alpha)}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big) = o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big), & \gamma < 1, \\ o\big(t^{-(1+\alpha)}\log t\,\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big), & \gamma = 1, \\ o\big(t^{-(1+\alpha)}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big), & \gamma > 1, \end{cases}$$

where we have used that $\varphi(t) = o(t^\theta)$ in the last equality of the case $\gamma < 1$. This estimate yields $\|\mathrm{I}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o(\phi(t))$ in all cases.

As for I_3, if (S), we use once more (3.1) and (1.14) to obtain, since $s > t/2$ in this case,

$$\begin{aligned}\mathrm{I}_3(x,t) &\le E_{4\beta}(x) \int_{t-t^{1-\delta(t)}}^{t} s^{-(1+\alpha)}(1+t-s)^{-\gamma}\,\mathrm{d}s \\ &\le Ct^{-(1+\alpha)}E_{4\beta}(x) \int_{t-t^{1-\delta(t)}}^{t} (1+t-s)^{-\gamma}\,\mathrm{d}s \\ &\le Ct^{-(1+\alpha)}E_{4\beta}(x) \begin{cases} O\big(t^{(1-\delta(t))(1-\gamma)}\big) = o(t^{1-\gamma}), & \gamma < 1, \\ (1-\delta(t))\log t = O(\log t), & \gamma = 1, \\ O(1), & \gamma > 1, \end{cases}\end{aligned}$$

where we have used that $\delta(t) \succ 1/\log t$ to show that $t^{-\delta(t)} = o(1)$ in the case $\gamma < 1$. From here it is easily checked, using also that $\varphi(t) = o(t^\theta)$ when $\gamma < 1$, that for all the cases included in (S) we have

$$\|\mathrm{I}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).$$

If (C_1)/(F_1), we make the estimate $\mathrm{I}_3 \le \sum_{i=1}^5 \mathrm{I}_{3i}$, where

$$\begin{aligned}
\mathrm{I}_{31}(x,t) &= \int_{t-t^{1-\delta(t)}}^t \int_{|x-y|\le\ell(t)|x|} |f(x-y,t-s)||Y(y,s)-\kappa s^{-(1+\alpha)}E_{4\beta}(y)|\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_{32}(x,t) &= \kappa\int_{t-t^{1-\delta(t)}}^t \int_{|x-y|\le\ell(t)|x|} |f(x-y,t-s)|s^{-(1+\alpha)}|E_{4\beta}(y)-E_{4\beta}(x)|\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_{33}(x,t) &= \kappa E_{4\beta}(x)\int_{t-t^{1-\delta(t)}}^t \int_{|x-y|\le\ell(t)|x|} |f(x-y,t-s)||s^{-(1+\alpha)}-t^{-(1+\alpha)}|\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{I}_{34}(x,t) &= \kappa t^{-(1+\alpha)}E_{4\beta}(x)\Big|\int_{t-t^{1-\delta(t)}}^t \int_{|x-y|<\ell(t)|x|} f(x-y,t-s)\,\mathrm{d}y\mathrm{d}s - M_0\log(1+t)\Big|,\\
\mathrm{I}_{35}(x,t) &= \kappa M_0 t^{-(1+\alpha)}E_{4\beta}(x)|\log(1+t)-\log t|.
\end{aligned}$$

Since $|y| > |x|/2$ and $s > t/2$ in the integration region for I_{31},

$$\begin{aligned}
\mathrm{I}_{31}(x,t) &= \int_{t-t^{1-\delta(t)}}^t \int_{|x-y|\le\ell(t)|x|} |f(x-y,t-s)|s^{-(1+\alpha)}E_{4\beta}(y)\Big|\frac{G(ys^{-\theta})}{E_{4\beta}(ys^{-\theta})}-\kappa\Big|\,\mathrm{d}y\mathrm{d}s\\
&\le Ct^{-(1+\alpha)}E_{4\beta}(x)\int_{t-t^{1-\delta(t)}}^t \int_{|x-y|\le\ell(t)|x|} |f(x-y,t-s)|\Big|\frac{G(ys^{-\theta})}{E_{4\beta}(ys^{-\theta})}-\kappa\Big|\,\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

But $|y| \le 3|x|/2 \le 3\mu\varphi(t)/2$ and $s > t/2$ imply that $|y|s^{-\theta} \le C\varphi(t)t^{-\theta} = o(1)$. Hence, (1.8) yields

$$\Big|\frac{G(ys^{-\theta})}{E_{4\beta}(ys^{-\theta})}-\kappa\Big| = o(1) \quad \text{as } t\to\infty.$$

Therefore, since $\gamma = 1$ in this case, using the assumption (1.14) on f,

$$\mathrm{I}_{31}(x,t) = o(t^{-(1+\alpha)})E_{4\beta}(x)\int_{t-t^{1-\delta(t)}}^t (1+t-s)^{-1}\,\mathrm{d}s = o\big(t^{-(1+\alpha)}\log t\big)E_{4\beta}(x).$$

By the Mean Value Theorem, since $\big||y|-|x|\big| \le |x-y| \le \ell(t)|x|$ in the region of integration of I_{32},

$$|E_{4\beta}(y)-E_{4\beta}(x)| \le \frac{(N-4\beta)\ell(t)}{(1-\ell(t))^{N-4\beta+1}}E_{4\beta}(x) \le C\ell(t)E_{4\beta}(x)$$

there, since $\ell(t) < 1/2$. Thus, using also the assumption (1.14),

$$\mathrm{I}_{32}(x,t) \le C\ell(t)t^{-(1+\alpha)}E_{4\beta}(x)\int_{t-t^{1-\delta(t)}}^t (1+t-s)^{-1}\,\mathrm{d}s = o\big(t^{-(1+\alpha)}\log t\big)E_{4\beta}(x).$$

In order to estimate I_{33}, we apply the Mean Value Theorem to obtain that

$$|s^{-(1+\alpha)} - t^{-(1+\alpha)}| \le Ct^{-(1+\alpha)}t^{-\delta(t)} = o(t^{-(1+\alpha)})$$

for all $s \in (t - t^{1-\delta(t)}, t)$. Thus, using the assumption (1.14),

$$\mathrm{I}_{33}(x,t) = o(t^{-(1+\alpha)})E_{4\beta}(x)\int_{t-t^{1-\delta(t)}}^{t}(1+t-s)^{-1}\,\mathrm{d}s = o\big(t^{-(1+\alpha)}\log t\big)E_{4\beta}(x).$$

As for I_{34}, we observe that

$$\begin{aligned}
\mathrm{I}_{34}(x,t) &= \kappa t^{-(1+\alpha)}E_{4\beta}(x)\Big|\int_0^{t^{1-\delta(t)}}\int_{|y|<\ell(t)|x|} f(y,s)\,\mathrm{d}y\mathrm{d}s \\
&\quad - \int_0^t\int_{\mathbb{R}^N} g(y)(1+s)^{-1}\,\mathrm{d}y\mathrm{d}s\Big| \\
&\le \kappa t^{-(1+\alpha)}E_{4\beta}(x)\Big(\int_0^{t^{\delta(t)}}\|f(\cdot,s)\|_{L^1(\mathbb{R}^N)}\,\mathrm{d}s + \|g\|_{L^1(\mathbb{R}^N)}\int_0^{t^{\delta(t)}}(1+s)^{-1}\,\mathrm{d}s \\
&\quad + \int_{t^{\delta(t)}}^{t^{1-\delta(t)}}\sup_{s>t^{\delta(t)}}\|f(\cdot,s)(1+s)-g\|_{L^1(\mathbb{R}^N)}(1+s)^{-1}\,\mathrm{d}s \\
&\quad + \int_{|y|>\ell(t)|x|}|g(y)|\int_{t^{\delta(t)}}^{t^{1-\delta(t)}}(1+s)^{-1}\,\mathrm{d}s + \|g\|_{L^1(\mathbb{R}^N)}\int_{t^{1-\delta(t)}}^{t}(1+s)^{-1}\,\mathrm{d}s\Big).
\end{aligned}$$

Notice that $t^{\delta(t)} < t^{1-\delta(t)}$, since $\delta(t) < 1/2$. From (1.16) we get that

$$\sup_{s>t^{\delta(t)}}\|f(\cdot,s)(1+s)-g\|_{L^1(\mathbb{R}^N)} = o(1),$$

since, due to the condition $\delta(t) \succ 1/\log t$, $t^{\delta(t)} \to \infty$ as $t \to \infty$. Moreover, if $|x| \ge \nu\varphi(t)$,

$$\int_{|y|>\ell(t)|x|}|g(y)|\,\mathrm{d}y \le \int_{|y|>\ell(t)\nu\varphi(t)}|g(y)|\,\mathrm{d}y = o(1),$$

since g is integrable and $\ell(t)\varphi(t) \to \infty$ (remember that $\ell(t) \succ 1/\varphi(t)$). Therefore, using also the assumption (1.14),

$$\mathrm{I}_{34}(x,t) \le Ct^{-(1+\alpha)}E_{4\beta}(x)(\delta(t)\log t + o(1)\log t)) = o(t^{-(1+\alpha)}\log t)E_{4\beta}(x),$$

since $\delta(t) = o(t)$.

It is immediate to check that

$$\mathrm{I}_{35}(x,t) = \kappa M_0 t^{-(1+\alpha)}\log t\,E_{4\beta}(x)\left|\frac{\log(1+t)}{\log t} - 1\right| = o(t^{-(1+\alpha)}\log t)E_{4\beta}(x).$$

Summarizing, $\mathrm{I}_3(x,t)=o(t^{-(1+\alpha)}\log t)E_{4\beta}(x)$, whence

$$\|\mathrm{I}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}=o\big(t^{-(1+\alpha)}\log t\,\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big),$$

from where it is easily checked that $\|\mathrm{I}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}=o(\phi(t))$ if (C$_1$)/(F$_1$).

If (C)/(F), we make the estimate $\mathrm{I}_3\le\sum_{i=1}^6\mathrm{I}_{3i}$, with I_{3i}, $i=1,2,3$ as for (C$_1$)/(F$_1$), and

$$\mathrm{I}_{34}(x,t)=\kappa t^{-(1+\alpha)}E_{4\beta}(x)\int_0^{t-t^{1-\delta(t)}}\int_{|x-y|\le\ell(t)|x|}|f(x-y,t-s)|\,\mathrm{d}y\mathrm{d}s,$$
$$\mathrm{I}_{35}(x,t)=\kappa t^{-(1+\alpha)}E_{4\beta}(x)\int_0^{t}\int_{|x-y|>\ell(t)|x|}|f(x-y,t-s)|\,\mathrm{d}y\mathrm{d}s,$$
$$\mathrm{I}_{36}(x,t)=\kappa t^{-(1+\alpha)}E_{4\beta}(x)\int_t^{\infty}\int_{\mathbb{R}^N}|f(y,s)|\,\mathrm{d}s.$$

Reasoning as for the cases (C$_1$)/(F$_1$), we get (notice that now $\gamma>1$),

$$\begin{aligned}\mathrm{I}_{3i}(x,t)&=o(t^{-(1+\alpha)})E_{4\beta}(x)\int_{t-t^{1-\delta(t)}}^{t}(1+t-s)^{-1}\,\mathrm{d}s\\&=o(t^{-(1+\alpha)})E_{4\beta}(x),\quad i=1,2,3.\end{aligned}$$

On the other hand, using the hypothesis (1.14) on f,

$$\mathrm{I}_{34}(x,t)\le Ct^{-(1+\alpha)}E_{4\beta}(x)\int_0^{t-t^{1-\delta(t)}}(1+t-s)^{-\gamma}\,\mathrm{d}s\le Ct^{-(\gamma+\alpha)}E_{4\beta}(x).$$

Finally, as $f\in L^1(Q)$ and $\ell(t)\succ 1/\phi(t)$,

$$\begin{aligned}&\int_0^t\int_{|x-y|>\ell(t)|x|}|f(x-y,t-s)|\,\mathrm{d}y\mathrm{d}s\\&\quad\le\int_0^\infty\int_{|y|>\ell(t)\nu\varphi(t)}|f(y,s)|\,\mathrm{d}y\mathrm{d}s=o(1)\quad\text{for }|x|>\nu\varphi(t),\\&\int_t^\infty\int_{\mathbb{R}^N}|f(y,s)|\,\mathrm{d}y\mathrm{d}s=o(1).\end{aligned}$$

Therefore, $\mathrm{I}_{3i}(x,t)=o(t^{-(1+\alpha)})E_{4\beta}(x)$, $i=1,6$.

Summarizing, $\mathrm{I}_3(x,t)=o(t^{-(1+\alpha)})E_{4\beta}(x)$, whence

$$\|\mathrm{I}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}=o\big(t^{-(1+\alpha)}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big),$$

from where it is easily checked that $\|\mathrm{I}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}=o(\phi(t))$ if (C)/(F).

Now we analyze II. We make the decomposition $\mathrm{II} = \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3$, where

$$\mathrm{II}_1(x,t) = \int_0^{t/2} \int_{\substack{|y|<k(t)|x|\\ |x-y|>\ell(t)|x|}} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s,$$
$$\mathrm{II}_2(x,t) = \int_0^{(k(t)|x|)^{1/\theta}} \int_{\substack{|y|>k(t)|x|\\ |x-y|>\ell(t)|x|}} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s,$$
$$\mathrm{II}_3(x,t) = \int_{(k(t)|x|)^{/\theta}}^{t/2} \int_{\substack{|y|>k(t)|x|\\ |x-y|>\ell(t)|x|}} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s,$$

with $k(t) = o(1)$ such that $k(t) \in (0, 1/2)$ for all $t > 0$ to be further especified later.

We estimate II_1 as $\mathrm{II}_1 \le \mathrm{II}_{11} + \mathrm{II}_{12}$, where

$$\mathrm{II}_{11}(x,t) = \int_0^{t/2} \int_{|y|<\min\{k(t)|x|,s^\theta\}} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s,$$
$$\mathrm{II}_{12}(x,t) = \int_0^{(k(t)|x|)^{1/\theta}} \int_{s^\theta<|y|<k(t)|x|} |f(x-y,t-s)| Y(y,s)\,\mathrm{d}y\mathrm{d}s.$$

Since $k(t) < 1/2$, $|x-y| \ge |x|/2 > \nu\varphi(t)/2$. Hence, taking q and r as in (2.1), and using the global bound (1.9),

$$\begin{aligned}
&\|\mathrm{II}_{11}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}\\
&\le \int_0^{t/2} \|f(\cdot,t-s)\|_{L^q(\{|x|>\frac{\nu}{2}\varphi(t)\}} \|Y(\cdot,s)\|_{L^r(\{|x|<\min\{\mu k(t)\varphi(t),s^\theta\}\})}\,\mathrm{d}s\\
&\le Cm(t)t^{-\gamma} \int_0^{t/2} s^{-(1+\alpha)} (\min\{\mu k(t)\varphi(t), s^\theta\})^{\frac{1+\alpha-\sigma(r)}{\theta}}\,\mathrm{d}s,
\end{aligned}$$

where $m(t) := \sup_{\tau>0} \|f(\cdot,\tau)(1+\tau)^\gamma\|_{L^q(\{|x|>\frac{\nu}{2}\varphi(t)\})}$, since $\|E_{4\beta}\|_{L^r(\{|x|<a\})} \le Ca^{\frac{1+\alpha-\sigma(r)}{\theta}}$. Thanks to assumptions (1.14) and (1.19),

$$m(t) = O\Big(\varphi(t)^{-N\left(1-\frac{1}{q}\right)}\Big) = O\Big(\varphi(t)^{\frac{\sigma(r)-\sigma(p)}{\theta}}\Big),$$

and hence, since $k(t) = o(1)$ and $\sigma(r) < 1$,

$$\begin{aligned}
&\|\mathrm{II}_{11}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}\\
&\le Cm(t)t^{-\gamma}\Big(\int_0^{(k(t)\mu\varphi(t))^{1/\theta}} s^{-\sigma(r)}\,\mathrm{d}s + (k(t)\mu\varphi(t))^{\frac{1+\alpha-\sigma(r)}{\theta}} \int_{(k(t)\mu\varphi(t)))^{1/\theta}}^{t/2} s^{-(1+\alpha)}\,\mathrm{d}s\Big)\\
&\le Cm(t)t^{-\gamma}(k(t)\varphi(t))^{\frac{1-\sigma(r)}{\theta}} \le k(t)^{\frac{1-\sigma(r)}{\theta}} O\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big)\\
&= o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).
\end{aligned}$$

As for II_{12}, since $|y|/s^{\theta} > 1$ in the integration range, we may use the exterior estimate (1.10) for the kernel to obtain

$$\mathrm{II}_{12}(x,t) \le C \int_0^{(k(t)|x|)^{1/\theta}} \int_{s^{\theta}<|y|<k(t)|x|} |f(x-y,t-s)| s^{2\alpha-1} E_{-2\beta}(y)\,\mathrm{d}y\mathrm{d}s.$$

On the other hand, since $k(t) < 1/2$, $|x-y| \ge |x|/2 > \nu\varphi(t)/2$. Hence, taking q and r as in (2.1),

$$\begin{aligned}
&\|\mathrm{II}_{12}(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} \\
&\le C \int_0^{(\mu k(t)\varphi(t))^{1/\theta}} \|f(\cdot,t-s)\|_{L^q(\{|x|>\frac{\nu}{2}\varphi(t)\})} s^{2\alpha-1} \|E_{-2\beta}\|_{L^r(\{|x|>s^{\theta}\})}\,\mathrm{d}s \\
&\le Cm(t)t^{-\gamma} \int_0^{(\mu k(t)\varphi(t)))^{1/\theta}} s^{-\sigma(r)}\,\mathrm{d}s \le k(t)^{\frac{1-\sigma(r)}{\theta}} O\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) \\
&= o(\phi(t)).
\end{aligned}$$

In order to estimate II_2, we observe that in the region of integration $s^{\theta} < k(t)|x| < |y|$. Therefore, we may use the outer estimate (1.10), and hence

$$\mathrm{II}_2(x,t) \le C \int_0^{(k(t)|x|)^{1/\theta}} \int_{\substack{|y|>s^{\theta} \\ |x-y|>\ell(t)|x|}} |f(x-y,t-s)| s^{2\alpha-1} E_{-2\beta}(y)\,\mathrm{d}y\mathrm{d}s.$$

Therefore, taking q and r as in (2.1),

$$\begin{aligned}
&\|\mathrm{II}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} \\
&\le C \int_0^{(\mu k(t)\varphi(t))^{1/\theta}} \|f(\cdot,t-s)\|_{L^q(\{|x|>\nu\ell(t)\varphi(t)\})} s^{2\alpha-1} \|E_{-2\beta}\|_{L^r(\{|x|>s^{\theta}\})}\,\mathrm{d}s \\
&\le Cn(t)t^{-\gamma} \int_0^{(\mu k(t)\varphi(t)))^{1/\theta}} s^{-\sigma(r)}\,\mathrm{d}s = Cn(t)t^{-\gamma}(k(t)\varphi(t))^{\frac{1-\sigma(r)}{\theta}},
\end{aligned}$$

where $n(t) := \sup_{\tau>0} \|f(\cdot,\tau)(1+\tau)^{\gamma}\|_{L^q(\{|x|>\nu\ell(t)\varphi(t)\})}$. Thanks to the assumptions (1.14) and (1.19),

$$n(t) = O\big((\ell(t)\varphi(t))^{-N(1-\frac{1}{q})}\big) = \ell(t)^{-N(1-\frac{1}{q})} O\big(\varphi(t)^{\frac{\sigma(r)-\sigma(p)}{\theta}}\big),$$

whence, if $\ell(t)$ satisfies $\ell(t) \succ k(t)^{(1-\sigma(r))/(N\theta(1-\frac{1}{q}))}$, in addition to $\ell(t) \succ 1/\varphi(t)$,

$$\begin{aligned}
\|\mathrm{II}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} &= \ell(t)^{-N(1-\frac{1}{q})} k(t)^{\frac{1-\sigma(r)}{\theta}} O\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) \\
&= o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).
\end{aligned}$$

To estimate II_3, we use the global bound (1.9). Then, if $|x| > \nu\varphi(t)$,

$$\begin{aligned}\mathrm{II}_3(x,t) &\le C\int_{(k(t)|x|)^{1/\theta}}^{t/2} s^{-(1+\alpha)}\int_{\substack{|y|>k(t)|x|\\|x-y|>\ell(t)|x|}} |f(x-y,t-s)|E_{4\beta}(y)\,\mathrm{d}y\mathrm{d}s\\
&\le Ck(t)^{4\beta-N}t^{-\gamma}E_{4\beta}(x)\\
&\quad\times\int_{(k(t)|x|)^{1/\theta}}^{t/2} s^{-(1+\alpha)}(1+t-s)^{\gamma}\|f(\cdot,t-s)\|_{L^1(\{|x|>\nu\ell(t)\varphi(t)\})}\,\mathrm{d}y\mathrm{d}s\\
&\le Cv(t)k(t)^{4\beta-N}E_{4\beta}(x)t^{-\gamma}\int_{(k(t)|x|)^{1/\theta}}^{t/2} s^{-(1+\alpha)}\,\mathrm{d}s\\
&= Cv(t)k(t)^{2\beta-N}t^{-\gamma}E_{2\beta}(x),\end{aligned}$$

where $v(t) := \sup_{\tau>t/2}\|(1+\tau)^{\gamma}f(\cdot,\tau)\|_{L^1(\{|x|>\nu\ell(t)\varphi(t)\})}$ is a bounded function, thanks to the size assumption (1.14). Using (1.30),

$$\|\mathrm{II}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = Cv(t)k(t)^{2\beta-N}t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}.$$

Since v is bounded, in fast scales (F) we have, see (1.29),

$$\|\mathrm{II}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = C\phi(t)k(t)^{2\beta-N}t^{1+\alpha-\gamma}\varphi(t)^{-2\beta}.$$

On the other hand it is readily checked that in these scales $t^{1+\alpha-\gamma}\varphi(t)^{-2\beta} = o(1)$. Therefore, if we take $k(t) \succ \big(t^{1+\alpha-\gamma}\varphi(t)^{-2\beta}\big)^{1/(N-2\beta)}$, we finally arrive at $\|\mathrm{II}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o(\phi(t))$, as desired.

For the scales (S), (C_1), (C), and (F_1) we use assumption (1.16) to show that, since $\ell(t) \succ 1/\varphi(t)$,

$$v(t) \le \sup_{\tau>t/2}\|(1+\tau)^{\gamma}f(\cdot,\tau)-g\|_{L^1(\{|x|>\nu\ell(t)\varphi(t)\})} + \|g\|_{L^1(\{|x|>\nu\ell(t)\varphi(t)\})} = o(1).$$

Therefore, if we take $k(t) \succ v(t)^{1/(N-2\beta)}$, we get

$$\|\mathrm{II}_3(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).$$

We now consider the last term, III. We have $\mathrm{III} = \mathrm{III}_1 + \mathrm{III}_2$, where

$$\begin{aligned}\mathrm{III}_1(x,t) &= \int_{t/2}^{t}\int_{\substack{|y|<h(t)|x|\\|x-y|>\ell(t)|x|}} |f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s,\\
\mathrm{III}_2(x,t) &= \int_{t/2}^{t}\int_{\substack{|y|>h(t)|x|\\|x-y|>\ell(t)|x|}} |f(x-y,t-s)|Y(y,s)\,\mathrm{d}y\mathrm{d}s,\end{aligned}$$

with $h(t) = o(1)$ such that $h(t) \in (0, 1/2)$ for all $t > 0$ to be further especified later.

Since $h(t) < 1/2$, $|x-y| \ge |x|/2 > \nu\varphi(t)/2$. Hence, taking q and r as in (2.1), and using the global estimate (1.9),

$$\begin{aligned}
&\|\mathrm{III}_1(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})}\\
&\quad\le \int_{t/2}^t \|f(\cdot,t-s)\|_{L^q(\{|x|>\frac{\nu}{2}\varphi(t)\})}\|Y(\cdot,s)\|_{L^r(\{|x|<h(t)\mu\varphi(t)\})}\,\mathrm{d}s\\
&\quad\le Cm(t)t^{-(1+\alpha)}\|E_{4\beta}\|_{L^r(\{|x|<h(t)\mu\varphi(t)\})}\int_{t/2}^t (1+t-s)^{-\gamma}\,\mathrm{d}s,
\end{aligned}$$

with $m(t)$ as above. Then, since $\sigma(r) < 1$ and $h(t) = o(1)$, and using also (2.3), we conclude that

$$\begin{aligned}
\|\mathrm{III}_1(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} &\le Ct^{-(1+\alpha)}h(t)^{\frac{1+\alpha-\sigma(r)}{\theta}}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\int_{t/2}^t (1+t-s)^{-\gamma}\,\mathrm{d}s\\
&= o\big(\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big)\int_{t/2}^t (1+t-s)^{-\gamma}\,\mathrm{d}s = o(\phi(t)).
\end{aligned}$$

To estimate III_2, we use the global bound (1.9). Then, if $|x| > \nu\varphi(t)$,

$$\begin{aligned}
\mathrm{III}_2(x,t) &\le C\int_{t/2}^t \int_{\substack{|y|>h(t)|x|\\ |x-y|>\ell(t)|x|}} |f(x-y,t-s)|s^{-(1+\alpha)}E_{4\beta}(y)\,\mathrm{d}y\mathrm{d}s\\
&\le Ch(t)^{4\beta-N}t^{-(1+\alpha)}w(t)E_{4\beta}(x),
\end{aligned}$$

where $w(t) = \int_0^{t/2} \|f(\cdot,\tau)\|_{L^1(\{|x|\ge\ell(t)\nu\varphi(t)\})}\,\mathrm{d}\tau$, so that, using (1.30),

$$\|\mathrm{III}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} \le Ch(t)^{4\beta-N}w(t)t^{-(1+\alpha)}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}.$$

If $\gamma > 1$, then $w(t) = o(1)$. Hence, taking $h(t) \succ w(t)^{1/(N-4\beta)}$,

$$\|\mathrm{III}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o\big(t^{-(1+\alpha)}\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big),$$

whence it is easy to check that $\|\mathrm{III}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} = o(\phi(t))$ in the scales (S), (C), and (F) when $\gamma > 1$.

If $\gamma < 1$, something which only happens in the case (S), hypothesis (1.14) yields $w(t) = O(t^{1-\gamma})$. Remember that $\varphi(t) = o(t^\theta)$. Hence, taking $h(t) \succ (\varphi(t)/t^\theta)^{\alpha/(\theta(N-4\beta))} = o(1)$,

$$\begin{aligned}
\|\mathrm{III}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} &\le Ch(t)^{4\beta-N}(\varphi(t)/t^\theta)^{\frac{\alpha}{\theta}}t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\\
&= o\big(t^{-\gamma}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)).
\end{aligned}$$

If $\gamma = 1$, the size hypothesis (1.14) yields $w(t) = O(\log t)$. If $\varphi(t) = o\big(t^\theta/(\log t)^{\frac{1}{2\beta}}\big)$, then, taking $h(t) \succ (\varphi(t)/(t^\theta/(\log t)^{\frac{1}{2\beta}}))^{\frac{\alpha}{\theta(N-4\beta)}} = o(1)$,

$$\begin{aligned}\|\mathrm{III}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} &= Ch(t)^{4\beta-N}(\varphi(t)/(t^\theta/(\log t)^{\frac{1}{2\beta}}))^{\frac{\alpha}{\theta}}t^{-1}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\\ &= o\big(t^{-1}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big) = o(\phi(t)),\end{aligned}$$

which completes the analysis of the case (S).

For the remaining cases with $\gamma = 1$, namely (C_1) and (F_1), we require the tail control hypothesis (1.18), that yields $w(t) = o(\log t)$. Taking $h(t) \succ (w(t)/\log t)^{1/(N-4\beta)}$, for ($F_1$) we have

$$\begin{aligned}\|\mathrm{III}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} &\le Ch(t)^{4\beta-N}\frac{w(t)}{\log t}t^{-(1+\alpha)}\log t\,\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\\ &= o\big(t^{-(1+\alpha)}\log t\,\varphi(t)^{\frac{1+\alpha-\sigma(p)}{\theta}}\big) = o(\phi(t)),\end{aligned}$$

and in the case (C_1), for which $\varphi(t) \simeq t^\theta/(\log t)^{\frac{1}{2\beta}}$,

$$\begin{aligned}\|\mathrm{III}_2(\cdot,t)\|_{L^p(\{\nu<|x|/\varphi(t)<\mu\})} &\le Ch(t)^{4\beta-N}\frac{w(t)}{\log t}t^{-1}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}(\varphi(t)(\log t)^{\frac{1}{2\beta}}/t^\theta)^{\frac{\alpha}{\theta}}\\ &\simeq h(t)^{4\beta-N}\frac{w(t)}{\log t}t^{-1}\varphi(t)^{\frac{1-\sigma(p)}{\theta}} = o\big(t^{-1}\varphi(t)^{\frac{1-\sigma(p)}{\theta}}\big)\\ &= o(\phi(t)).\end{aligned}$$

□

4 Exterior regions

This section is devoted to prove the results concerning the large-time behavior in exterior regions, $\{|x| > \nu t^\theta\}$ for some $\nu > 0$, Theorems 1.3–1.6 and Proposition 1.1.

Proof of Theorem 1.3 We make the decomposition

$$\begin{aligned}&\Big|u(x,t) - \int_0^t M_f(s)Y(x,t-s)\,\mathrm{d}s\Big| \le \mathrm{I}(x,t) + \mathrm{II}(x,t), \quad \text{where}\\ &\mathrm{I}(x,t) = \int_0^t\int_{|y|<\delta|x|}|f(y,s)||Y(x-y,t-s) - Y(x,t-s)|\,\mathrm{d}y\mathrm{d}s,\\ &\mathrm{II}(x,t) = \int_0^t\int_{|y|>\delta|x|}|f(y,s)||Y(x-y,t-s) - Y(x,t-s)|\,\mathrm{d}y\mathrm{d}s,\end{aligned}$$

with $\delta \in (0, 1/2)$ to be fixed later.

By the Mean Value Theorem, for each x, y, t and s there is a value $\lambda \in (0,1)$ such that

$$|Y(x-y,t-s)-Y(x,t-s)| = |DY(x-\lambda y,t-s)||y|.$$

But, if $|y| < \delta|x|$ with $\delta \in (0,1/2)$ and $\lambda \in (0,1)$, with $|x| \geq \nu t^{\theta}$ and $s \in (0,t)$, then

$$|x-\lambda y| > |x|/2, \qquad |x-\lambda y|(t-s)^{-\theta} > (1-\delta)|x|t^{-\theta} > \nu/2.$$

Therefore, using the estimate (1.11) for the gradient of Y and the size assumption (1.14),

$$\begin{aligned}
\mathrm{I}(x,t) &\leq C\int_0^t\int_{|y|<\delta|x|}|f(y,s)|(t-s)^{2\alpha-1}|x-ry|^{-(N+2\beta+1)}|y|\,\mathrm{d}y\mathrm{d}s\\
&\leq C\delta E_{-2\beta}(x)\int_0^t\int_{|y|<\delta|x|}|f(y,s)|(t-s)^{2\alpha-1}\,\mathrm{d}y\mathrm{d}s\\
&\leq C\delta E_{-2\beta}(x)\int_0^t(1+s)^{-\gamma}(t-s)^{2\alpha-1}\,\mathrm{d}s.
\end{aligned}$$

Thus, since

$$\|E_{-2\beta}\|_{L^p(\{|x|>\nu t^{\theta}\})} = Ct^{-\sigma(p)-2\alpha+1}, \tag{4.1}$$

we get

$$\begin{aligned}
&\|\mathrm{I}(\cdot,t)\|_{L^p(\{|x|>\nu t^{\theta}\})}\\
&\quad\leq C\delta t^{-\sigma(p)-2\alpha+1}\Big(t^{2\alpha-1}\int_0^{t/2}(1+s)^{-\gamma}\,\mathrm{d}s + t^{-\gamma}\int_{t/2}^t(t-s)^{2\alpha-1}\,\mathrm{d}s\Big),
\end{aligned} \tag{4.2}$$

which combined with (2.3) yields $\|\mathrm{I}(\cdot,t)\|_{L^p(\{|x|>\nu t^{\theta}\})} \leq C\delta\phi(t)$. From now on we fix $\delta \in (0,1/2)$ so that $C\delta < \varepsilon$.

We now turn our attention to II. If $p \in [1,p_c)$, using (1.13),

$$\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^{\theta}\})} \leq C\int_0^t\|f(\cdot,s)\|_{L^1(\{|x|>\delta\nu t^{\theta}\})}(t-s)^{-\sigma(p)}\,\mathrm{d}s.$$

If moreover $\gamma > 1$, then $f \in L^1(\mathbb{R}^N\times(0,\infty))$, and hence

$$\int_0^{t/2}\|f(\cdot,s)\|_{L^1(\{|x|>\delta\nu t^{\theta}\})}\mathrm{d}s = o(1),$$

so that, using also assumption (1.14) to estimate the integral over $(t/2, t)$,

$$\begin{aligned}\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} &\le Ct^{-\sigma(p)}\int_0^{t/2}\|f(\cdot,s)\|_{L^1(\{|x|>\delta\nu t^\theta\})}\,\mathrm{d}s\\ &\quad + Ct^{-\gamma}\int_{t/2}^t (t-s)^{-\sigma(p)}\,\mathrm{d}s\\ &= o(t^{-\sigma(p)}) + O(t^{1-\gamma-\sigma(p)}) = o(t^{-\sigma(p)}) = o(\phi(t)).\end{aligned}$$

Still in the subcritical case, if $\gamma \le 1$, using the tail control assumption (1.18) we have

$$\begin{aligned}\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} &\le C\sup_{t>0}\big((1+t)^\gamma\|f(\cdot,t)\|_{L^1(\{|x|>\delta\nu t^\theta\})}\big)\\ &\quad\times\int_0^t (1+s)^{-\gamma}(t-s)^{-\sigma(p)}\,\mathrm{d}s\\ &= o\Big(t^{-\sigma(p)}\int_0^{t/2}(1+s)^{-\gamma}\,\mathrm{d}s + Ct^{-\gamma}\int_{t/2}^t (t-s)^{-\sigma(p)}\,\mathrm{d}s\Big)\\ &= \begin{cases} o(t^{1-\gamma-\sigma(p)}), & \gamma<1,\\ o(t^{-\sigma(p)}\log t) & \gamma=1,\end{cases}\end{aligned}$$

and therefore $\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} = o(\phi(t))$.

If p is not subcritical, we take q and r as in (2.1). Then, since r is subcritical, using (1.13),

$$\begin{aligned}\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} &\le C\int_0^t \|f(\cdot,s)\|_{L^q(\{|x|>\delta\nu t^\theta\})}(t-s)^{-\sigma(r)}\,\mathrm{d}s.\\ &\le Cv(t)\Big(t^{-\sigma(r)}\int_0^{t/2}(1+s)^{-\gamma}\,\mathrm{d}s + t^{-\gamma}\int_{t/2}^t (t-s)^{-\sigma(r)}\,\mathrm{d}s\Big),\end{aligned} \tag{4.3}$$

where

$$v(t) = \sup_{s>0}\Big((1+s)^\gamma\|f(\cdot,s)\|_{L^q(\{|x|>\delta\nu t^\theta\})}\Big) = o(t^{-N\theta\left(1-\frac{1}{q}\right)}), \tag{4.4}$$

thanks to the uniform tail control assumption (1.20). Therefore

$$\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} = \begin{cases} o(t^{1-\gamma-\sigma(p)}), & \gamma<1,\\ o(t^{-\sigma(p)}\log t), & \gamma=1,\\ o(t^{-\sigma(p)}, & \gamma>1,\end{cases}$$

whence $\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} = o(\phi(t))$. □

Proof of Theorem 1.4 Take $\delta \in (0, 1/2)$. Using hypothesis (1.14) on the size of f, we have

$$\Big|\int_0^t M_f(s)Y(x,t-s)\,\mathrm{d}s - M(t)t^{1-\alpha}Y(x,t)\Big| \le \mathrm{I}(x,t)+\mathrm{II}(x,t)+\mathrm{III}(x,t),$$

where

$$\mathrm{I}(x,t) = \int_0^{\delta t}(1+s)^{-\gamma}(t-s)^{\alpha-1}|(t-s)^{1-\alpha}Y(x,t-s)-t^{1-\alpha}Y(x,t)|\,\mathrm{d}s,$$
$$\mathrm{II}(x,t) = \int_{\delta t}^t (1+s)^{-\gamma}Y(x,t-s)\,\mathrm{d}s,$$
$$\mathrm{III}(x,t) = t^{1-\alpha}Y(x,t)\int_{\delta t}^t (1+s)^{-\gamma}(t-s)^{\alpha-1}\,\mathrm{d}s,$$

for some $\delta \in (0, 1/2)$ to be fixed later.

By the Mean Value Theorem, for each x, t and $s \in (0,t)$ there exists $\lambda \in (0,1)$ such that

$$|(t-s)^{1-\alpha}Y(x,t-s)-t^{1-\alpha}Y(x,t)| = s|\partial_t H(x,t-\lambda s)|,$$
$$\text{where } H(x,t)=t^{1-\alpha}Y(x,t).$$

From estimates (1.10) and (1.12), if $|x|t^{-\theta}\ge \nu$, $t>0$, for some $\nu>0$, then

$$|\partial_t H(x,t)| \le C_\nu t^{\alpha-1}E_{-2\beta}(x).$$

But, if $|x|>\nu t^{\theta}$, with $\nu>0$, $s\in(0,\delta t)$, with $\delta\in(0,1/2)$, and $\lambda\in(0,1)$, then

$$t-\lambda s > t/2, \quad |x|(t-\lambda s)^{-\theta} \ge |x|t^{-\theta} \ge \nu.$$

Therefore, we have

$$|(t-s)^{1-\alpha}Y(x,t-s)-t^{1-\alpha}Y(x,t)| \le Cs(t-\lambda s)^{\alpha-1}E_{-2\beta}(x) \le C\delta t^{\alpha}E_{-2\beta}(x),$$

so that

$$\mathrm{I}(x,t) \le C\delta t^{\alpha}E_{-2\beta}(x)\int_0^{\delta t}(t-s)^{\alpha-1}(1+s)^{-\gamma}\,\mathrm{d}s$$
$$\le C\delta t^{2\alpha-1}E_{-2\beta}(x)\int_0^{\delta t}(1+s)^{-\gamma}\,\mathrm{d}s.$$

Using (4.1), we finally get $\|\mathrm{I}(\cdot,t)\|_{L^p(\{|x|>\nu t^{\theta}\})} \le \varepsilon\phi(t)$ if we choose $\delta\in(0,1/2)$ small enough.

Once the value of δ is fixed, we have, using the exterior bound (1.10) for the kernel,

$$\mathrm{II}(x,t) \le C E_{-2\beta}(x) \int_{\delta t}^{t} (1+s)^{-\gamma}(t-s)^{2\alpha-1}\,\mathrm{d}s \le C t^{2\alpha-\gamma} E_{-2\beta}(x),$$
$$\mathrm{III}(x,t) \le C t^{2\alpha-1} E_{-2\beta}(x) \int_{\delta t}^{t} (1+s)^{-\gamma}\,\mathrm{d}s \le C t^{2\alpha-\gamma} E_{-2\beta}(x),$$

so that $\|\mathrm{II}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})}$, $\|\mathrm{III}(\cdot,t)\|_{L^p(\{|x|>\nu t^\theta\})} \le C t^{1-\gamma-\sigma(p)} = o(\phi(t))$ if $\gamma \ge 1$. □

Proof of Theorem 1.5 Let $\delta \in (0,1/2)$ to be chosen later. We have

$$\begin{aligned}|M(t)t^{1-\alpha} - M_\infty| \le{}& t^{1-\alpha} \int_0^{\delta t}\int_{\mathbb{R}^N} |f(y,s)|\big((t-s)^{\alpha-1} - t^{\alpha-1}\big)\,\mathrm{d}y\mathrm{d}s \\ &+ t^{1-\alpha}\int_{\delta t}^{t}\int_{\mathbb{R}^N} |f(y,s)|(t-s)^{\alpha-1}\,\mathrm{d}y\mathrm{d}s \\ &+ \int_{\delta t}^{\infty}\int_{\mathbb{R}^N} |f(y,s)|\,\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Since, by the Mean Value Theorem, $0 \le (t-s)^{\alpha-1} - t^{\alpha-1} \le C\delta t^{\alpha-1}$ if $s \in (0,\delta t)$, using also the size condition (1.14) on f with $\gamma > 1$ we conclude that

$$\begin{aligned}|M(t)t^{1-\alpha} - M_\infty| \le{}& C\delta \int_0^{\delta t} (1+s)^{-\gamma}\,\mathrm{d}s \\ &+ C t^{1-\alpha-\gamma}\int_{\delta t}^{t} (t-s)^{\alpha-1}\,\mathrm{d}s + \int_{\delta t}^{\infty}\int_{\mathbb{R}^N} |f(y,s)|\,\mathrm{d}y\mathrm{d}s \\ \le{}& C\delta + C t^{1-\gamma} + \int_{\delta t}^{\infty}\int_{\mathbb{R}^N} |f(y,s)|\,\mathrm{d}y\mathrm{d}s \le \varepsilon,\end{aligned}$$

if we fix δ small enough and then take t large. □

Proof of Theorem 1.6 We make the estimate $|M(t) - M_0 t^{\alpha-1}\log(1+t)| \le \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t)$, where

$$\mathrm{I}(t) = \int_0^t (t-s)^{\alpha-1}(1+s)^{-1}|(1+s)M_f(s) - M_0|\,\mathrm{d}s,$$
$$\mathrm{II}(t) = |M_0| \int_0^t \big|(t-s)^{\alpha-1} - t^{\alpha-1}\big|(1+s)^{-1}\,\mathrm{d}s,$$
$$\mathrm{III}(t) = |M_0| t^{\alpha-1}\log\frac{1+t}{t}.$$

From assumption (1.16) we know that there is a time τ_ε such that

$$|\big(1+s)M_f(s) - M_0| \le \|(1+s)f(\cdot,s) - g\|_{L^1(\mathbb{R}^N)} < \varepsilon \quad \text{for all } s \ge \tau_\varepsilon. \tag{4.5}$$

With this in mind, we make the estimate $\mathrm{I}(t) \le \mathrm{I}_1(t) + \mathrm{I}_2(t)$, where

$$\mathrm{I}_1(t) = \int_0^{\tau_\varepsilon} (t-s)^{\alpha-1}(1+s)^{-1}|(1+s)M_f(s) - M_0|\,\mathrm{d}s,$$
$$\mathrm{I}_2(t) = \varepsilon \int_{\tau_\varepsilon}^t (t-s)^{\alpha-1}(1+s)^{-1}\,\mathrm{d}s,$$

valid for $t > \tau_\varepsilon$. On the one hand, the size assumption (1.14) yields $(1+s)|M_f(s)| \le C$, so that

$$\mathrm{I}_1(t) \le C \int_0^{\tau_\varepsilon} (t-s)^{\alpha-1}(1+s)^{-1}\,\mathrm{d}s \le C_\varepsilon t^{\alpha-1} \int_0^{\tau_\varepsilon} (1+s)^{-1}\,\mathrm{d}s$$
$$= C_\varepsilon t^{\alpha-1}\log(1+\tau_\varepsilon) \le \varepsilon t^{\alpha-1}\log(1+t)$$

if t is large enough. On the other hand, from (4.5), for all large t,

$$\mathrm{I}_2(t) \le C\varepsilon\Big(t^{\alpha-1}\int_{\tau_\varepsilon}^{t/2}(1+s)^{-1}\,\mathrm{d}s + t^{-1}\int_{t/2}^t (t-s)^{\alpha-1}\,\mathrm{d}s\Big)$$
$$\le C\varepsilon(t^{\alpha-1}\log(1+t) + t^{\alpha-1}) \le C\varepsilon t^{\alpha-1}\log t.$$

As for II, we estimate it as $\mathrm{II}(t) \le \mathrm{II}_1(t) + \mathrm{II}_2(t)$, where

$$\mathrm{II}_1(t) = |M_0| \int_0^{\delta t} \big|(t-s)^{\alpha-1} - t^{\alpha-1}\big|(1+s)^{-1}\,\mathrm{d}s,$$
$$\mathrm{II}_2(t) = |M_0| \int_{\delta t}^{t} \big|(t-s)^{\alpha-1} - t^{\alpha-1}\big|(1+s)^{-1}\,\mathrm{d}s,$$

for some $\delta \in (0, 1/2)$ to be chosen. Given $\varepsilon > 0$, there exists a small constant $\delta = \delta(\varepsilon) > 0$ such that

$$|(t-s)^{\alpha-1} - t^{\alpha-1}| < \varepsilon t^{\alpha-1} \quad \text{if } s \in (0, \delta t).$$

We fix such δ. Then, if t is large enough,

$$\mathrm{II}_1(t) \le |M_0|\varepsilon t^{\alpha-1}\int_0^{\delta t}(1+s)^{-1}\,\mathrm{d}s = |M_0|\varepsilon t^{\alpha-1}\log(1+\delta t) \le C\varepsilon t^{\alpha-1}\log t.$$

On the other hand, for t large enough,

$$\mathrm{II}_2(t) \le Ct^{-1}\int_{\delta t}^t \big((t-s)^{\alpha-1} + t^{\alpha-1}\big)\,\mathrm{d}s \le Ct^{\alpha-1} \le \varepsilon t^{\alpha-1}\log t.$$

Finally, since $\log\frac{1+t}{t} = o(1) = o(\log t)$ as $t \to \infty$, we get immediately that $\mathrm{III}(t) = o\big(t^{\alpha-1}\log t\big)$. □

Proof of Proposition 1.1 Let $\delta \in (0, 1/2)$ to be fixed later. We have

$$\Big|\int_0^t M_f(s)Y(x,t-s)\,\mathrm{d}s - t^{-\gamma}M_0c_{2\beta}E_{2\beta}(x)\Big| \le \mathrm{I}(x,t)+\mathrm{II}(x,t), \quad \text{where}$$

$$\mathrm{I}(x,t) = \Big|\int_0^{\delta t}\int_{\mathbb{R}^N} f(y,t-s)Y(x,s)\,\mathrm{d}y\mathrm{d}s - t^{-\gamma}M_0c_{2\beta}E_{2\beta}(x)\Big|,$$

$$\mathrm{II}(x,t) = \int_{\delta t}^t\int_{\mathbb{R}^N} |f(y,t-s)|Y(x,s)\,\mathrm{d}y\mathrm{d}s.$$

We estimate I as $\mathrm{I} \le \mathrm{I}_1 + \mathrm{I}_2$, where

$$\mathrm{I}_1(x,t) = \int_0^{\delta t}(1+t-s)^{-\gamma}Y(x,s)\int_{\mathbb{R}^N}|f(y,t-s)(1+t-s)^{\gamma}-g(y)|\,\mathrm{d}y\mathrm{d}s,$$

$$\mathrm{I}_2(x,t) = \Big|M_0\int_0^{\delta t}(1+t-s)^{-\gamma}Y(x,s)\,\mathrm{d}s - t^{-\gamma}M_0c_{2\beta}E_{2\beta}(x)\Big|.$$

Let $\varepsilon > 0$. Since $t-s \ge t/2$ for $s \in (0,\delta t)$, using hypothesis (1.16) we get

$$\int_{\mathbb{R}^N}|f(y,t-s)(1+t-s)^{\gamma}-g(y)|\,\mathrm{d}y \le \varepsilon|M_0| \quad \text{for } s \in (0,\delta t)$$

for t large enough, how big not depending on δ, so that

$$\mathrm{I}_1(x,t) \le \varepsilon t^{-\gamma}|M_0|E_{2\beta}(x)\int_0^{\delta t}\frac{Y(x,s)}{E_{2\beta}(x)}\,\mathrm{d}s.$$

We recall now that

$$c_{2\beta} = \int_0^{\infty}\frac{Y(y,s)}{E_{2\beta}(y)}\,\mathrm{d}s. \tag{4.6}$$

Therefore, $\mathrm{I}_1(x,t) \le \varepsilon t^{-\gamma}|M_0|c_{2\beta}E_{2\beta}(x)$.

Using again (4.6), we have $\mathrm{I}_2 \le \mathrm{I}_{21} + \mathrm{I}_{22}$, where

$$\mathrm{I}_{21}(x,t) = |M_0|E_{2\beta}(x)\int_0^{\delta t}|(1+t-s)^{-\gamma}-t^{-\gamma}|\,\frac{Y(x,s)}{E_{2\beta}(x)}\,\mathrm{d}s,$$

$$\mathrm{I}_{22}(x,t) = |M_0|t^{-\gamma}\int_{\delta t}^{\infty}Y(x,s)\,\mathrm{d}s.$$

If $s \in (0,\delta t)$, then

$$1-\delta < \frac{1+(1-\delta)t}{t} < \frac{1+t-s}{t} < \frac{1+t}{t} = 1+\frac{1}{t},$$

so that $\left|(1+t-s)^{-\gamma}-t^{-\gamma}\right| \le \varepsilon t^{-\gamma}$ if t is large and δ small. Thus, using once more (4.6),

$$\mathrm{I}_{21}(x,t) \le \varepsilon t^{-\gamma}|M_0|E_{2\beta}(x)\int_0^{\delta t}\frac{Y(y,s)}{E_{2\beta}(y)}\,\mathrm{d}s \le \varepsilon t^{-\gamma}|M_0|c_{2\beta}E_{2\beta}(x).$$

Once we fix δ as above, using the global estimate (1.9) for the kernel, if $|\xi| = |x|t^{-\theta}$ is small enough,

$$\begin{aligned}\mathrm{I}_{22}(x,t) &\le Ct^{-\gamma}E_{4\beta}(x)\int_{\delta t}^{\infty}s^{-(1+\alpha)}\,\mathrm{d}s \le C_\delta(|x|t^{-\theta})^{2\beta}|M_0|c_{2\beta}t^{-\gamma}E_{2\beta}(x)\\ &\le \varepsilon t^{-\gamma}|M_0|c_{2\beta}E_{2\beta}(x).\end{aligned}$$

Similarly, using also the size hypothesis (1.14), if $|\xi| = |x|t^{-\theta}$ is small enough,

$$\begin{aligned}\mathrm{II}(x,t) &\le CE_{4\beta}(x)\int_{\delta t}^{t}(1+t-s)^{-\gamma}s^{-(1+\alpha)}\,\mathrm{d}s \le C_\delta t^{-(1+\alpha)}E_{4\beta}(x)\int_{\delta t}^{t}(1+t-s)^{-\gamma}\,\mathrm{d}s\\ &\le C_\delta(|x|t^{-\theta})^{2\beta}|M_0|c_{2\beta}t^{-\gamma}E_{2\beta}(x) \le \varepsilon t^{-\gamma}|M_0|c_{2\beta}E_{2\beta}(x).\end{aligned}$$

□

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Declarations

Conflict of interest On behalf of all authors, Fernando Quirós states that there is no conflict of interest.

Appendix

We study here the behavior at infinity of Riesz potentials, using only integral assumptions, a result of independent interest.

Theorem A.1 *Let* $\mu \in (0, N)$*. Let* $g \in L^1(\mathbb{R}^N)$ *and* $M = \int_{\mathbb{R}^N} g$*. If* $p \geq p_\mu := N/(N-\mu)$ *we assume in addition the tail control condition*

$$\|g\|_{L^q(\{|x|>R\})} = O\big(R^{-N(1-\frac{1}{q})}\big) \quad \text{as } R \to \infty \text{ for some } q \in (q_\mu(p), p],$$

$$q_\mu(p) := \begin{cases} \dfrac{Np}{\mu p + N}, & p \in [1, \infty), \\ \dfrac{N}{\mu}, & p = \infty. \end{cases}$$

Let E_μ *and* I_μ *as in* (1.22)*. Then, if* $0 < \nu \leq \mu < \infty$*, for any* $p \in [1, \infty]$ *we have*

$$R^{N\left(1-\frac{1}{p}\right)-\mu} \|I_\mu[g] - ME_\mu\|_{L^p(\{\nu<|x|/R<\mu\})} \to 0 \quad \text{as } R \to \infty.$$

Proof We may assume without loss of generality that $g \neq 0$. We have $|I_\mu[g] - ME_\mu| \leq \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}$, where

$$\begin{aligned} \mathrm{I}(x) &= E_\mu(x) \int_{|y|<\gamma|x|} \left| \frac{|x|^{N-\mu}}{|x-y|^{N-\mu}} - 1 \right| |g(y)|\,\mathrm{d}y, \\ \mathrm{II}(x) &= E_\mu(x) \int_{|y|>\gamma|x|} |g(y)|\,\mathrm{d}y, \\ \mathrm{III}(x) &= \int_{\substack{|y|>\gamma|x| \\ |x-y|<\delta|x|}} \frac{|g(y)|}{|x-y|^{N-\mu}}\,\mathrm{d}y, \\ \mathrm{IV}(x) &= \int_{\substack{|y|>\gamma|x| \\ |x-y|>\delta|x|}} \frac{|g(y)|}{|x-y|^{N-\mu}}\,\mathrm{d}y, \end{aligned}$$

with $\gamma, \delta > 0$ to be chosen later.

On the one hand, if $|y| < \gamma|x|$, with $\gamma \in (0, 1)$,

$$\frac{1}{(1+\gamma)^{N-\mu}} \leq \frac{|x|^{N-\mu}}{(|x|+|y|)^{N-\mu}} \leq \frac{|x|^{N-\mu}}{|x-y|^{N-\mu}} \leq \frac{|x|^{N-\mu}}{(|x|-|y|)^{N-\mu}} \leq \frac{1}{(1-\gamma)^{N-\mu}}.$$

Hence, $\left| \frac{|x|^{N-\mu}}{|x-y|^{N-\mu}} - 1 \right| < \varepsilon/\|g\|_{L^1(\mathbb{R}^N)}$ if γ is small enough, and therefore $\mathrm{I}(x) \leq \varepsilon E_\mu(x)$, whence

$$\|\mathrm{I}\|_{L^p(\{\nu<|x|/R<\mu\})} \leq \varepsilon \|E_\mu\|_{L^p(\{\nu<|x|/R<\mu\})} \leq \varepsilon R^{-N\left(1-\frac{1}{p}\right)+\mu}$$

for all values of R. From now on γ is assumed to be fixed.

Since $g \in L^1(\mathbb{R}^N)$, $\|g\|_{L^1(\{|x|\geq \nu R\})} \leq \varepsilon$ if R is large enough. Hence,

$$\|\mathrm{II}\|_{L^p(\{\nu<|x|/R<\mu\})} \leq \varepsilon \|E_\mu\|_{L^p(\{\nu<|x|/R<\mu\})} \leq \varepsilon R^{-N\left(1-\frac{1}{p}\right)+\mu}$$

if R is large enough.

To estimate III, we choose

$$q = 1 \text{ if } p \in [1, p_\mu), \quad q \in (q_\mu(p), p] \text{ as in the hypothesis if } p \geq p_\mu,$$
$$1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}.$$

Notice that $r \in [1, p_\mu)$ in all cases. Then, using the integrability of g if $p \in [1, p_\mu)$, or the tail control condition otherwise,

$$\begin{aligned}\|\mathrm{III}\|_{L^p(\{\nu<|x|/R<\mu\})} &\leq \|g\|_{L^q(\{|x|>\gamma\nu R\})}\|E_\mu\|_{L^r(\{|x|<\delta\mu R\})}\\ &\leq C_{\nu,\mu}\gamma^{-N(1-\frac{1}{q})}\delta^{-N(1-\frac{1}{r})+\mu}R^{-N\left(1-\frac{1}{p}\right)+\mu}.\end{aligned}$$

Since $r \in [1, p_\mu)$, then $-N(1-\frac{1}{r}) + \mu > 0$. Therefore, taking $\delta > 0$ small enough,

$$\|\mathrm{II}\|_{L^p(\{\nu<|x|/R<\mu\})} \leq \varepsilon R^{-N\left(1-\frac{1}{p}\right)+\mu}.$$

Finally, once γ and δ are fixed, since

$$\mathrm{IV}(x) \leq \delta^{\mu-N} E_\mu(x) \int_{|y|>\gamma|x|} |g(y)|\,\mathrm{d}y,$$

we have, using the integrability of g,

$$\|\mathrm{IV}\|_{L^p(\{\nu<|x|/R<\mu\})} \leq C_\delta \|g\|_{L^1(\{|x|>\gamma\nu R\})}\|E_\mu\|_{L^p(\{\nu<|x|/R<\delta\})} \leq \varepsilon R^{-N\left(1-\frac{1}{p}\right)+\mu},$$

if R is large enough. □

Remark The tail control assumption in Theorem A.1 is satisfied, for instance, if $|g(x)| \leq |x|^{-N}$.

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