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AN ENTROPY BASED IN WAVELET LEADERS TO QUANTIFY THE LOCAL REGULARITY OF A SIGNAL AND ITS APPLICATION TO ANALIZE THE DOW JONES INDEX

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Local regularity analysis is useful in many fields, such as financial analysis, fluid mechanics, PDE theory, signal and image processing. Different quantifiers have been proposed to measure the local regularity of a function.

In this paper we present a new quantifier of local regularity of a signal: the *pointwise wavelet leaders entropy*. We define this new measure of regularity by combining the concept of entropy, coming from the information theory and statistical mechanics, with the wavelet leaders coefficients. Also we establish its inverse relation with one of the well-known regularity exponents, the pointwise Hölder exponent.

Finally, we apply this methodology to the financial data series of the Dow Jones Industrial Average Index, registered in the period 1928–2011, in order to compare the temporal evolution of the pointwise Hölder exponent and the pointwise wavelet leaders entropy. The analysis reveals that temporal variation of these quantifiers reflects the evolution of the Dow Jones Industrial Average Index and identifies historical crisis events.

We propose a new approach to analyze the local regularity variation of a signal and we apply this procedure to a financial data series, attempting to make a contribution to understand the dynamics of financial markets.

Keywords: Local regularity; entropy; wavelet leaders.

AMS Subject Classification: 65T60, 26A16, 28D20, 37M10

1. Introduction

Singularities and irregular structures often carry essential information in a signal or image. For example, in image processing, detecting the object contours is related to finding discontinuities and singularities in the image.[13] Another example is provided by electrocardiogram signals from which heart failure is associated with an increasing irregularity of the signal.[5]

To characterize these localized singular structures it is necessary to quantify the local regularity of a function $f(x)$. Different quantifiers have been proposed to measure the local regularity of a function.[17,18] The simplest one is the pointwise Hölder exponent which is defined in each $x_0 \in \mathrm{Dom}(f) \subseteq \mathbb{R}^d$, f a locally bounded function, as

$$H_f(x_0) = \sup_{0\leq\alpha<+\infty} \{\alpha : f \in \mathcal{C}^\alpha(x_0)\}, \tag{1.1}$$

recalling that a function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to the class $\mathcal{C}^\alpha(x_0)$ if there exists $C > 0$ and a polynomial $P_{x_0}(x)$ of degree less than α such that, near the point x_0, verifies

$$|f(x) - P_{x_0}(x)| \leq C|x - x_0|^\alpha. \tag{1.2}$$

This exponent captures the regularity variation of f quantifying how rugous or spiky is the graph of a function. A highly irregular point in a function is characterized by a low pointwise Hölder exponent, while the smooth portions of a function have higher exponents. On 2004, Jaffard[7] formulates a new characterization of the pointwise Hölder exponent through the study of the decay of the *wavelet leaders coefficients*, which are calculated from the local suprema of the wavelet coefficients of a signal $f \in L^2(\mathbb{R})$, reconcentrating its information and reorganizing its structure.

Wavelet analysis gives a time-scale decomposition of f, reflecting its scaling properties. Furthermore, the analysis of the decay of the amplitude and the local modulus maxima wavelet coefficients also provides an appropriate tool for studying topics from images and signals such as pattern recognition,[23] denoising,[13] edge detection[22] and other applications. Wavelet leaders exploits these properties to reveal the local regularity of f. Moreover the wavelet leaders gives an effective method for computing the *spectrum of singularities* or *multifractal spectrum* of numerical series and natural signals.[19,10] Analyzing multifractal features is closely related to look into the local regularity variation because multifractal framework provides a statistical description from the collection of local singularities. Indeed the *spectrum of singularities* computes the Hausdorff dimension of the set of singular points $\{x_0 \in \mathbb{R}^d : H_f(x_0) = H\}$, where H is a given Hölder exponent value taken by a multifractal function $f : \mathbb{R}^d \to \mathbb{R}$.

In this paper we propose a novel approach to analyze the dynamics of time series: studying the evolution of their local regularity through a new measure, the *pointwise wavelet leaders entropy*. For this purpose we define a discrete pointwise probability distribution $\mathcal{P}_{x_0} = \{\rho_j : j = 1, \ldots, m\}$, for each x_0, based on the computation of the wavelet leaders coefficients of a signal given by 2^m data and compute the entropy formulated by Shannon[21] to define the pointwise wavelet leaders entropy in x_0. An antecedent of combining the concept of entropy with the wavelet coefficients can be found in Refs. 24 and 1 for describing electroencephalogram series. Our new quantifier refines those formulated in these last references, where the entropies are defined in non-overlapping intervals of a given size, limited by the length and the frequency content of the signal. In addition, our new formulation is correlated with the local regularity analysis. We prove that pointwise wavelet leaders entropy takes values very close to the maximum when the pointwise Hölder exponent takes values close to zero, indicating that this quantifier also detects the singularities of a function.

Finally, we apply this methodology to the financial data series of the Dow Jones Industrial Average Index, registered in the period 1928–2011. There is evidence that signals from financial markets, such as stock indices, interest rates or commodities, follow scaling laws and have self-similar structures.[15] Furthermore, there exists models based on wavelet theory and self-similarity for approximating financial signals[4] and, in recent years, many efforts have been made to relate the inefficiency of markets with the multifractal nature of these corresponding signals. These features are closely connected to the local regularity analysis, thus in this view our procedure is an interesting alternative for analyzing financial data series.

2. Local Regularity and Wavelet Leaders

The decay of the wavelet transform amplitude across the scales is related to the local signal regularity. Measuring this asymptotic decay is equivalent to zooming into signal structures at fine scales.

For studying the local signal regularity additional properties are required for the real wavelet mother ψ. More precisely, we need an admissible orthogonal wavelet mother $\psi \in \mathcal{C}^r$, $r \in \mathbb{N}$, with derivatives that have a fast decay, and ψ has r vanishing moments, that is,

$$\int_{-\infty}^{+\infty} x^k \psi(x)dx = 0, \quad 0 \le k < r, \;\; k \in \mathbb{N}. \tag{2.1}$$

To measure the local regularity of a signal, vanishing moments are crucial. If the wavelet has r vanishing moments, the wavelet transform can be interpreted as a multiscale differential operator of order r. This yields a first relation between the differentiability of f and its wavelet transform decay at fine scales.[14] Also, if f

is a nowhere differentiable function, there is a direct connection between its local fractional differentiability and its local behavior.[9]

Using that $\mathcal{F} = \{2^{j/2}\psi(2^j x - k)\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis of $L^2(\mathbb{R})$ we can write a signal f in the space of signals having a finite energy $L^2(\mathbb{R})$ as,

$$f(x) = \sum_{j\in Z}\sum_{k\in Z} c_{j,k}\psi(2^j x - k), \tag{2.2}$$

where $c_{j,k} = \langle f, 2^j\psi(2^j x - k)\rangle$ are the *wavelet coefficients* of f, normalized in L^1 instead of the usual definition.

In Ref. 6, Jaffard finds a direct correlation between the wavelet coefficients $c_{j,k}$ and the pointwise Hölder regularity and proves that if f belongs to the class $\mathcal{C}^\alpha(x_0)$ then for all $j \geq 0$,

$$|c_{j,k}| \leq C2^{-j\alpha}(1 + |2^j x_0 - k|)^\alpha, \tag{2.3}$$

for some constant C. Moreover, when f has a non-oscillating singularity in x_0, like a cusp point, the significance coefficients are "localized" near the point x_0 and

$$|c_{j,k}| \approx C2^{-j\alpha}. \tag{2.4}$$

But this is not the case when the function has oscillating singularities, like *chirps*. Then, the maxima coefficients may be localized far from the singular point and the last property fails. In this view, Jaffard[7] gives a new formulation for this property, characterizing the local regularity in terms of the local suprema of the wavelet coefficients, the *wavelet leaders*. The notion of wavelet leaders were introduced in Ref. 8, finding a formula which yields the upper box dimension of a graph of a function.

We can suppose that ψ is essentially localized on the interval $[0, 1]$, thus $c_{j,k}$ has information about the signal related to the dyadic interval $I_{j,k} = [\frac{k}{2^j}, \frac{k+1}{2^j})$.

Then, *wavelet leaders* of a bounded function f are defined as follows:

$$d_{j,k} = \sup_{I_{l,h}\subset 3I_{j,k}} |c_{l,h}|, \tag{2.5}$$

where $3I_{j,k} = I_{j,k-1} \cup I_{j,k} \cup I_{j,k+1} = [\frac{k-1}{2^j}, \frac{k+2}{2^j})$ is the dilated interval.

We denote $I_j(x_0)$ the unique dyadic interval $I_{j,k}$ containing $x_0 \in R$ for the level j. Then the wavelet leader for x_0 in the level j is defined as

$$d_j(x_0) = \sup_{I_{l,h}\subset 3I_j(x_0)} |c_{l,h}|. \tag{2.6}$$

Figure 1 illustrates this definition.

Then, concentrating the wavelet coefficients information in the wavelet leaders Jaffard[7] proved the following general result about the "leaders" coefficients decay.

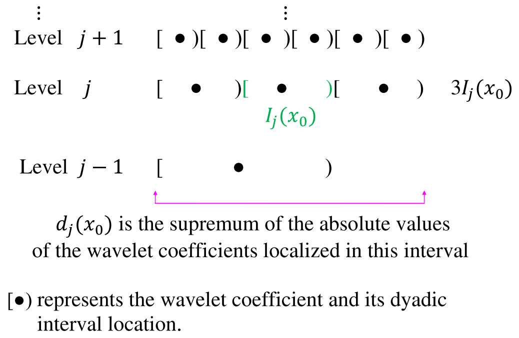

Fig. 1. A schematic illustration of the wavelet leaders coefficients definition.

Theorem 2.1. *Let f be a bounded function in $\mathcal{C}^\alpha(x_0)$, $\alpha > 0$. Then for all $j > 0$,*

$$d_j(x_0) \leq C2^{-j\alpha}, \tag{2.7}$$

for some constant C. Furthermore, if f is uniformly Hölder, the pointwise Hölder exponent of f can be computed using

$$H_f(x_0) = \liminf_{j\to+\infty} \frac{\log(d_j(x_0))}{\log(2^{-j})}. \tag{2.8}$$

Remark 2.1. A function f is uniformly Hölder if condition (1.2) takes place for all x_0 with the possibility of choosing C uniformly, i.e. for $0 < \alpha < 1$, f is uniformly Hölder if there exists C such that,

$$|f(x) - f(y)| \leq C|x - y|^\alpha \quad \forall x, y \in \mathbb{R}.$$

Remark 2.2. This property is independent of the wavelet mother election as long as ψ has the required conditions, $\psi \in \mathcal{C}^r$ with r vanishing moments and fast decay derivatives $\psi^{(n)}$, $0 \leq n \leq r$, with $r > \alpha$.

3. Wavelet Leaders Entropy

Entropy is a concept that arises from the thermodynamics and is reformulated by the information theory and the statistical mechanics.[2] It plays a central role in information theory as measure of information, choice and uncertainty. If X is a source of information which produces a sequence of symbols $\{x_1, \ldots, x_m\}$ whose probabilities of occurrence are $\{p_1, \ldots, p_m\}$ then the quantity

$$S = -\sum_{i=1}^{m} p_i \log_2(p_i) \tag{3.1}$$

is the Shannon[21] entropy of the discrete probability distribution $\{p_1, \ldots, p_m\}$. If $p_i = 0$, it is defined $p_i \log_2(p_i) = 0$.

The quantity S has properties which further validate it as a reasonable measure of choice or information.

- $S = 0$ if and only if $p_i = 1$ and $p_j = 0$ for all $j \neq i$. Thus, only when we are certain of the outcome, S vanishes. Otherwise S is positive.
- S is a maximum and equal to $\log_2(m)$ when all the p_i are equal. This is also intuitively the most uncertain situation.

To define the pointwise wavelet leaders entropy we determine a discrete probability distribution $\mathcal{P}_{x_0}$, for each $x_0 \in \mathrm{Dom}(f)$ and a resolution level m, as:

Definition 3.1. Let f be a bounded function and m a resolution level. It is defined $\mathcal{P}_{x_0} = \{\rho_1, \ldots, \rho_m\}$ such that:

$$\rho_i = \frac{d_i^2(x_0)}{\sum_{j=1}^{m} d_j^2(x_0)} \quad \text{if } d_i(x_0) \neq 0 \text{ and } \rho_i = 0 \text{ otherwise}, \tag{3.2}$$

for $i = 1, \ldots, m$ and recalling that $d_i(x_0)$ is the wavelet leader coefficient for x_0 in the level i defined in Eq. (2.6).

From these concepts:

Definition 3.2. Let f be a bounded function and let m a resolution level. The pointwise wavelet leaders entropy for $x_0 \in \mathrm{Dom}(f)$ is:

$$S_f(x_0) = S(\mathcal{P}_{x_0}) = -\sum_{i=1}^{m} \rho_i \log_2(\rho_i). \tag{3.3}$$

If $\rho_i = 0$, it is defined $\rho_i \log_2(\rho_i) = 0$.

If the highest resolution level j concentrates the higher wavelet coefficients in a neighborhood of x_0 then the wavelet leaders coefficients for x_0, across all levels, are equal and therefore $S_f(x_0)$ is maximum and equal to $\log_2(m)$. On the other hand, $S_f(x_0) = 0$ if the wavelet coefficients in a neighborhood of x_0 are zero except perhaps at the lowest resolution level j.

The following proposition proves an inverse relation between this new quantifier and the pointwise Hölder exponent.

Proposition 3.1. *Let f be a bounded function and let $H = H_f(x_0)$ be the pointwise Hölder exponent in $x_0 \in \mathrm{Dom}(f) \subseteq \mathbb{R}$ and $a > 1$. If $f \in \mathcal{C}^H(x_0)$ and f is uniformly Hölder then there exist a resolution level $m \in \mathbb{N}$ such that*:

$$4^{-(ma-1)H} \log_2(m4^{-(ma-1)H}) \leq S_f(x_0). \tag{3.4}$$

Proof. Without loss of generality we can assume the resolution level $m \in \mathbb{N}$ verifies that wavelet leader coefficient $d_m(x_0) > 0$. Since $(d_i(x_0))_{i\in\mathbb{N}}$ is a decreasing sequence, the probability distribution $\mathcal{P}_{x_0}$ defined in (3.2) verifies

$$1 > \rho_1 \geq \rho_2 \geq \cdots \geq \rho_m.$$

So that

$$m\rho_m \log_2\left(\frac{1}{\rho_1}\right) \leq \sum_{i=1}^{m} \rho_i \log_2\left(\frac{1}{\rho_i}\right) = S_f(x_0). \tag{3.5}$$

Using (3.2), we can write

$$m\rho_m \log_2\left(\frac{1}{\rho_1}\right) = m\frac{d_m^2(x_0)}{\sum_{j=1}^{m} d_j^2(x_0)} \log_2\left(\frac{\sum_{j=1}^{m} d_j^2(x_0)}{d_1^2(x_0)}\right).$$

In addition, $md_m^2(x_0) \leq \sum_{j=1}^{m} d_j^2(x_0) \leq md_1^2(x_0)$ then

$$\frac{d_m^2(x_0)}{d_1^2(x_0)} \log_2\left(m\frac{d_m^2(x_0)}{d_1^2(x_0)}\right) \leq m\rho_m \log_2\left(\frac{1}{\rho_1}\right)$$

or,

$$\left(\frac{d_m(x_0)}{d_1(x_0)}\right)^2 \log_2\left(m\left(\frac{d_m(x_0)}{d_1(x_0)}\right)^2\right) \leq S_f(x_0).$$

Since Eq. (2.7) and $f \in \mathcal{C}^H(x_0)$ there exists $C > 0$ such that $d_j(x_0) \leq C2^{-jH}$ for all $j > 0$. On the other hand, given that f is uniformly Hölder, Eq. (2.8) in Theorem (2.1) holds and it implies that for $a > 1$ there exists an infinite number of levels j such that

$$C2^{-jaH} \leq d_j(x_0). \tag{3.6}$$

By choosing a level m such that

$$C2^{-maH} \leq d_m(x_0) \leq C2^{-mH},$$

we have

$$\left(\frac{C2^{-maH}}{C2^{-H}}\right)^2 \log_2\left(m\left(\frac{C2^{-maH}}{C2^{-H}}\right)^2\right) \leq \left(\frac{d_m(x_0)}{d_1(x_0)}\right)^2 \log_2\left(m\left(\frac{d_m(x_0)}{d_1(x_0)}\right)^2\right)$$

or

$$4^{-(ma-1)H} \log_2(m4^{-(ma-1)H}) \leq S_f(x_0). \qquad \square$$

Corollary 3.1. *If Eq.* (3.4) *holds, it follows that*: $S_f(x_0)$ *takes values close to its maximum* $\log_2(m)$ *when* $H = H_f(x_0)$ *takes values close to zero.*

Remark 3.1. The use of the constant $a > 1$ in Eq. (3.4) is due to a technical issue to prove the theorem. It has no relevance to deduce the Corollary 3.1.

Remark 3.2. If the wavelet leaders coefficients verifies the equality

$$d_i(x_0) = C2^{-iH} \quad \forall\, i = 1, \ldots, m,$$

then it can be computed $S_f(x_0) = \sum_{i=1}^{m} \rho_i \log_2(\frac{1}{\rho_i})$, recalling the definition in Eq. (3.2) as:

$$S_f(x_0) = \log_2\left(\sum_{j=1}^{m} (4^{-H})^j\right) + \frac{2H\displaystyle\sum_{i=1}^{m} i(4^{-H})^i}{\displaystyle\sum_{j=1}^{m} (4^{-H})^j}.$$

Since $4^{-H} < 1$,

$$\begin{aligned}
S_f(x_0) &\leq \log_2\left(\frac{4^{-H}}{1-4^{-H}}\right) + 2H\frac{\dfrac{4^{-H}}{(4^{-H}-1)^2}}{\dfrac{(4^{-H})^{m+1}-4^{-H}}{4^{-H}-1}} \\
&= \log_2\left(\frac{4^{-H}}{1-4^{-H}}\right) + \frac{2H}{(1-4^{-H})(1-4^{-Hm})} \\
&\leq -\log_2(1-4^{-H}) - 2H + \frac{2H}{(1-4^{-H})^2} \\
&\leq -\log_2(1-4^{-H}) + 2H\left(\frac{1}{(1-4^{-H})^2} - 1\right).
\end{aligned}$$

Then $S_f(x_0)$ satisfies

$$S_f(x_0) \leq -\log_2(1-4^{-H}) + 2H4^{-H}\left(\frac{2-4^{-H}}{(1-4^{-H})^2}\right).$$

In this case $S_f(x_0)$ is near to zero when $H = H_f(x_0)$ is large enough.

4. Numerical Examples

To illustrate the inverse relation between the pointwise wavelet leaders entropy and the pointwise Hölder exponent we compute the pointwise wavelet leaders entropy to a synthetic signal whose Hölder exponent is predetermined. For this purpose, we consider the generalized Weierstrass function[3]:

$$F(t) = \sum_{k=0}^{+\infty} 3^{-kc(t)} \sin(3^k t), \tag{4.1}$$

which has exactly Hölder exponent $c(t)$ at every point, for $c(t)$ a continuous function such that $0 < c(t) < 1$.

We generate two numerical series data by computing the generalized Weierstrass function generated by different functions $c(t)$. In both cases the length of the series is 2^{15} data points.

In the present study, an orthogonal decimated discrete wavelet transform and a multiresolution analysis scheme is applied to compute the wavelet coefficients. Among several alternatives we select an orthogonal B-cubic spline function, with three vanishing moments, as mother wavelet. It combines in suitable proportion smoothness with numerical advantages.[25] We propose the following algorithm for estimating the pointwise wavelet leaders entropy:

Algorithm

(1) Via the Mallat algorithm,[12] compute the wavelet coefficients for the resolution levels $j = 1, \ldots, 14$, considering the data series $F(t)$ at the highest level.

(2) From the definition (2.6), estimate the wavelet leaders coefficients $(d_j(t))_{j=1,\ldots,14}$ using

$$d_j(t) = \sup\{|c_{l,h}| : I_{l,h} \subset 3I_j(t), 1 \le l \le 14\}.$$

The last definition indicates that to compute $d_j(t)$ we consider the data series $F(t)$ is localized on the dyadic interval $I_j(t) = [\frac{k}{2^j}, \frac{k+1}{2^j})$ and the indexes l, h such that:

$$2^{l-j}(k-1) \le h \le 2^{l-j}(k+2) \quad \text{for each } l \ge j-1 \text{ and } l \le 14.$$

(3) Calculate the pointwise wavelet leaders entropy $S_F(t)$ for the resolution level $m = 14$, using the formulas (3.2) and (3.3),

$$S_F(t) = -\sum_{i=1}^{14} \rho_i(t) \log_2(\rho_i(t)),$$

with $\rho_i(t) = \frac{d_i^2(t)}{\sum_{j=1}^{14} d_j^2(t)}$ if $d_i(t) \neq 0$ and $\rho_i(t) = 0$ otherwise.

We obtain the first time series data by estimating the generalized Weierstrass function generated by $c(t) = \frac{1}{4\pi}(t + 2\pi)$, t represents 2^{15} regularly spaced values in the interval $(-2\pi, 2\pi)$ (Figs. 2 and 3). The pointwise wavelet leaders entropy is displayed in Fig. 4.

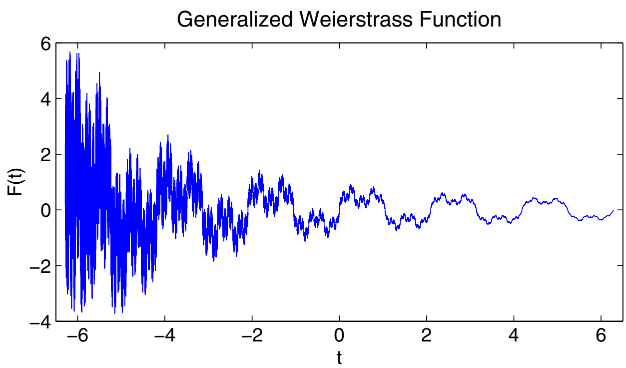

Fig. 2. Generalized Weierstrass function, $c(t) = \frac{1}{4\pi}(t + 2\pi)$.

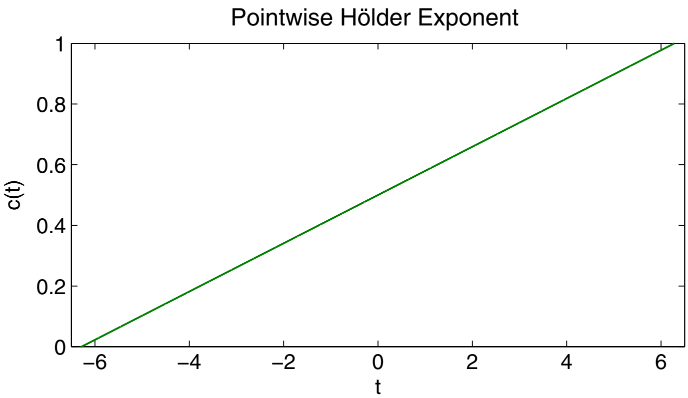

Fig. 3. The theoretical pointwise Hölder exponent $H_F(t) = \frac{1}{4\pi}(t + 2\pi)$.

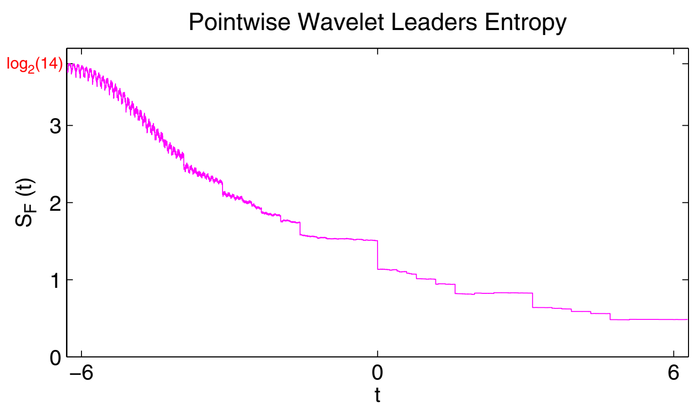

Fig. 4. The pointwise wavelet leaders entropy $S_F(t)$.

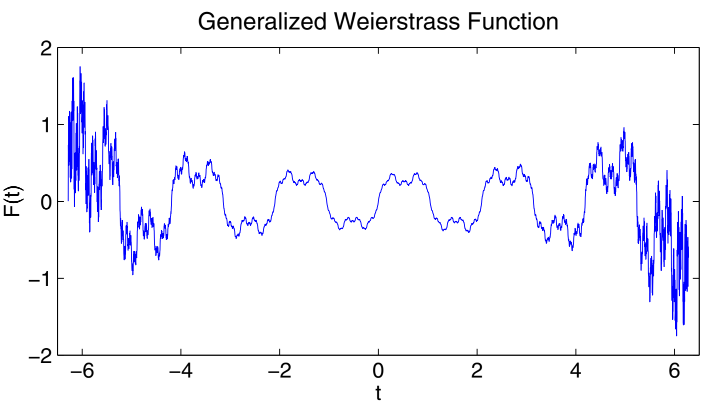

Fig. 5. Generalized Weierstrass function, $c(t) = \frac{2}{100}(49 - t^2)$.

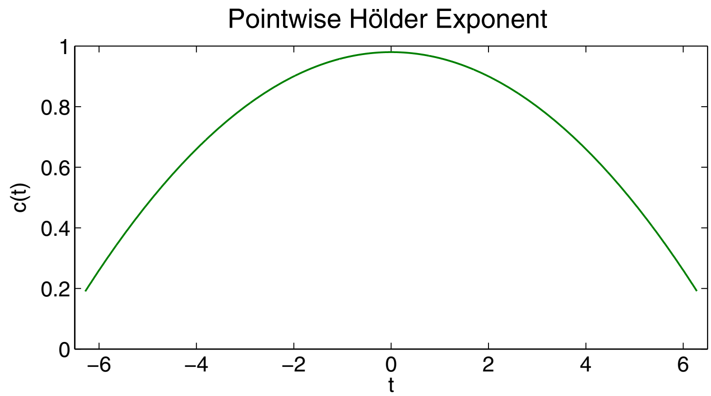

Fig. 6. The theoretical pointwise Hölder exponent $H_F(t) = \frac{2}{100}(49 - t^2)$.

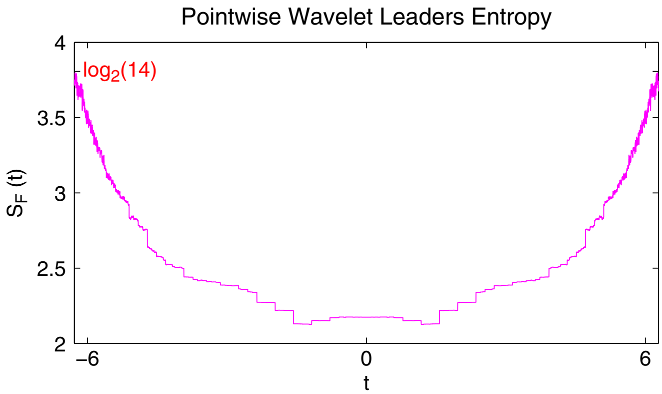

Fig. 7. The pointwise wavelet leaders entropy $S_F(t)$.

For computing the second synthetic data series we use $c(t) = \frac{2}{100}(49 - t^2)$, t represents 2^{15} regularly spaced values in the interval $(-2\pi, 2\pi)$ (Figs. 5 and 6). The pointwise wavelet leaders entropy is displayed in Fig. 7.

We can observe how the pointwise Hölder exponent evolution captures the regularity variation of the generalized Weierstrass function. The lower exponents characterize the spiky portions of the function, while the less irregular portions have higher Hölder exponents (Figs. 3 and 6). Also, the pointwise wavelet leaders entropy captures this regularity variation, in an inverse sense (see Figs. 4 and 7).

5. Application to a Financial Data Series

The Dow Jones Industrial Average (DJIA) is a price-weighted average of 30 blue-chip stocks that are generally the leaders in their industry in United States of

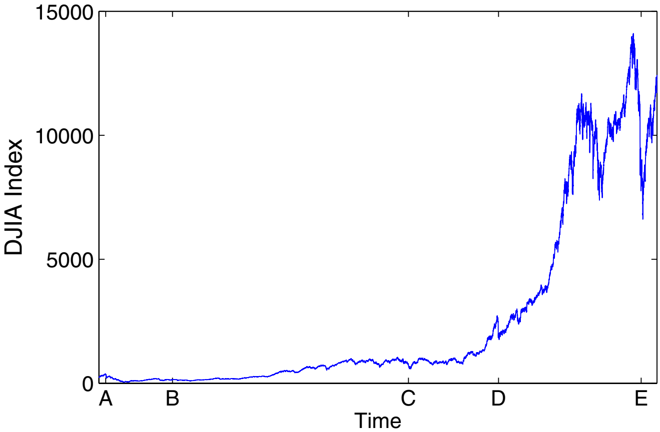

Fig. 8. Temporal evolution of the DJIA. (A) October 1, 1929; (B) September 1, 1939; (C) August 15, 1974; (D) November 3, 1987; (E) September 2, 2008.

America (USA). It has been a widely followed indicator of the stock market since October 1, 1928. The data were collected from the CME Group Index Services database (http://www.djindexes.com). We employ an average between the high and the low DJIA daily price beginning on October 1, 1928 and ending on May 12, 2011, obtaining 20,719 observations (Fig. 8).

We estimate the regularity quantifiers using appropriate numerical methods for calculating wavelet leaders coefficients, as in Sec. 4. These computations are performed under the hypothesis of a self-similar underlying structure of the analyzed series. This hypothesis is based on the evidence of the existence of scaling phenomena on stock market variations (see Ref. 15 for a deeper analysis in this topic).

Let $x(t)$ be the DJIA daily average price on a time t, the equity index returns rt are calculated as its logarithmic difference, $rt(t) = \log(x(t+1)/x(t))$. Using the same algorithm as in Sec. 4 we compute the wavelet leaders coefficients $(d_j(t))_{j=1,\ldots,14}$, associated with the data series rt, and estimate the temporal evolution of the pointwise wavelet leaders entropy $S_{rt}(t)$ (Fig. 10).

Also, we suppose

$$\log(d_j(t)) \approx \log(C) + H_{rt}(t)\log(2^{-j}), \quad j = 1, \ldots, 14 \tag{5.1}$$

and use a linear regression to estimate the pointwise Hölder exponent $H_{rt}(t)$ (Fig. 9). There are several techniques to estimate the pointwise Hölder exponent and this one is an efficient alternative, see Refs. 20 and 11 for developing this topic.

Values of the pointwise wavelet leaders entropy near the maximum $\log_2(14)$ indicate an irregularity in the signal, otherwise values of the pointwise Hölder exponent near zero indicate this fact.

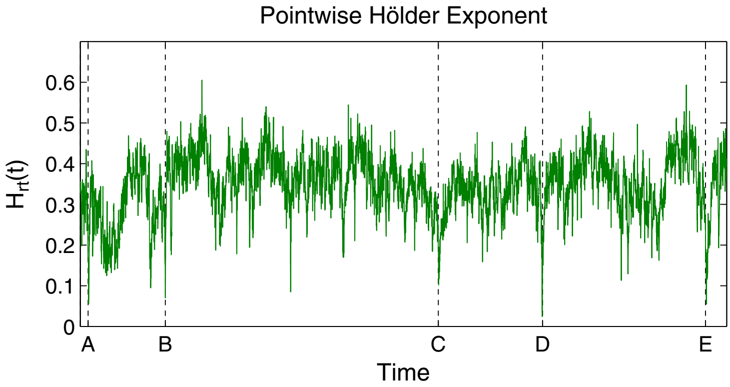

Fig. 9. (A) October 1, 1929; (B) September 1, 1939; (C) August 15, 1974; (D) November 3, 1987; (E) September 2, 2008.

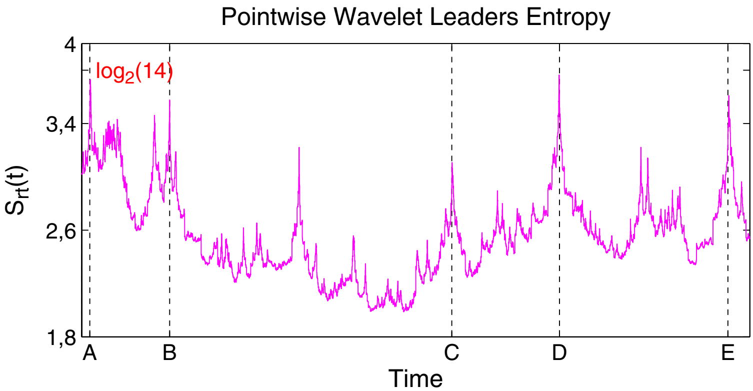

Fig. 10. (A) October 1, 1929; (B) September 1, 1939; (C) August 15, 1974; (D) November 3, 1987; (E) September 2, 2008.

From the temporal evolution of these regularity quantifiers we can identify historical crisis events as: (A) The stock market crash of 1929; (B) The beginning of the Second World War; (C) The 1973–1974 decline market compounded by the outbreak of the 1973 oil crisis; (D) The stock market crash of 1987, beginning at the end of October 1987 in Hong Kong and spreading west to Europe and USA; (E) The global financial crisis of 2008.

Observing Figs. 9 and 10, we can also conclude that DJIA regularity variation is not related with the daily price index magnitude (Fig. 8).

Although the computations performed in formula (5.1) and Step (3) of the algorithm in line 9 to estimate these quantifiers have different features, the inverse relation between the pointwise Hölder exponent $H_{rt}(t)$ and the pointwise wavelet leaders entropy $S_{rt}(t)$ is also displayed in Figs. 9 and 10 and zooming into those figures (see Figs. 11 and 12).

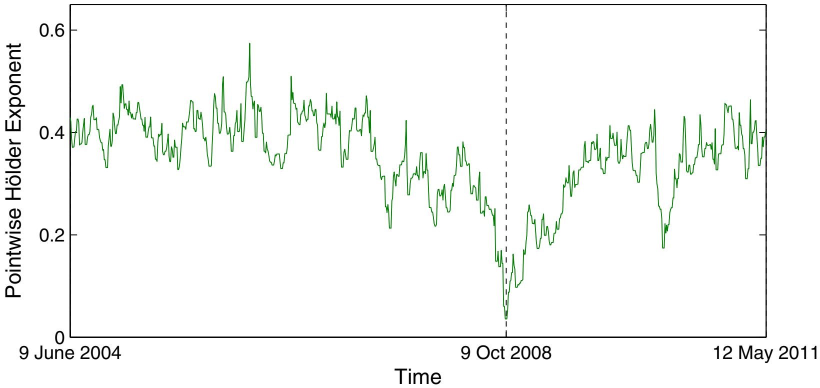

Fig. 11. Pointwise Hölder exponent evolution in global financial crisis of 2008.

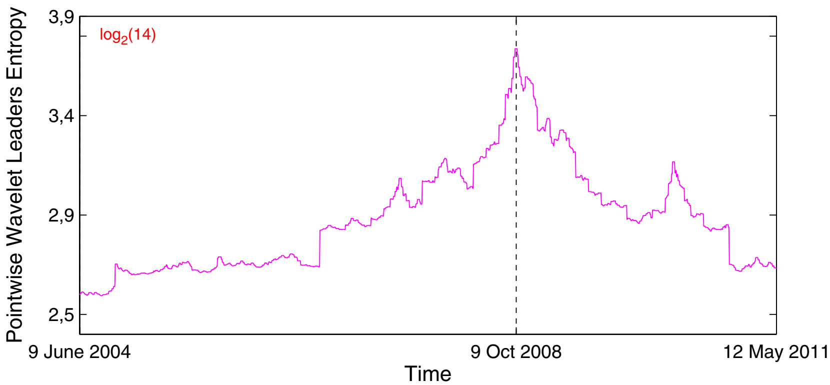

Fig. 12. Pointwise wavelet leaders entropy evolution in global financial crisis of 2008.

Both methods are appropriated for detecting the singularities of the DJIA data series but pointwise wavelet leaders entropy is more accurate than pointwise Hölder exponent for recognizing the crisis events from their graphical representations. The pointwise wavelet leaders entropy (Fig. 10) displays sharp peaks, distinguishing crisis events in the higher maxima points, while the pointwise Hölder exponent values (Fig. 9) are concentrated in the interval $[0.25, 0.42]$ taking on a few values close to 0.6 and a few values close to 0, indicating the crisis events in the lower minima points. This visual impression of the distribution of data is exhibited in both DJIA regularity variation histograms (Figs. 13 and 14). From these histograms we can notice that crisis events are related on the lower frequency rectangles in both cases, but the pointwise Hölder exponent histogram also has low frequencies for values larger than 0.5 which are not connected with the crisis events.

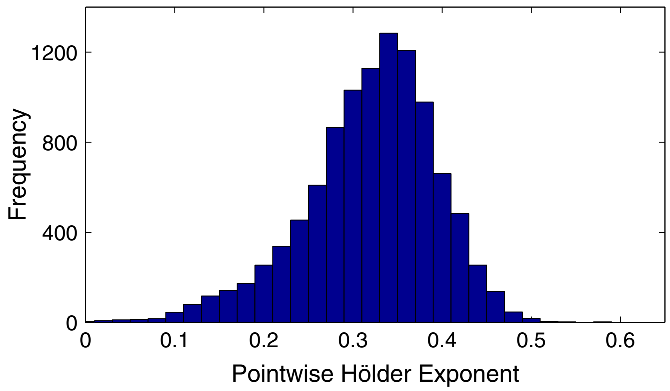

Fig. 13. Pointwise Hölder exponent histogram.

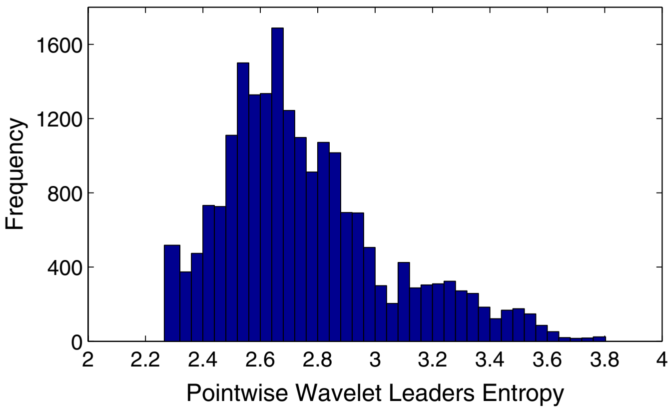

Fig. 14. Pointwise wavelet leaders entropy histogram.

6. Summary and Conclusions

- In this work we present a new estimator, based on the concept of entropy and wavelet leaders coefficients, to quantify the regularity signal variation and we prove its inverse relation with the well known regularity exponent, the pointwise Hölder exponent.
- We apply this methodology to the DJIA index data series, registered in the period 1928–2011. The analysis reveals that the temporal evolution of the pointwise wavelet leaders entropy accurately detects historical financial crisis events. This fact is evidenced from its higher maxima points which do not depend on the daily price index magnitude and volatility. According with information theory, entropy is maximum in the most uncertain situation which it may be interpreted, in this context, as an indicator of stock market instability. We also exhibit that pointwise wavelet leaders entropy is sharper than pointwise Hölder exponent for distinguishing historical financial crisis events from the DJIA data series.

- In summary, this is an interesting alternative for studying the transitions of a signal, through the regularity variation of the data. In future works we hope to find new applications for this methodology.

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