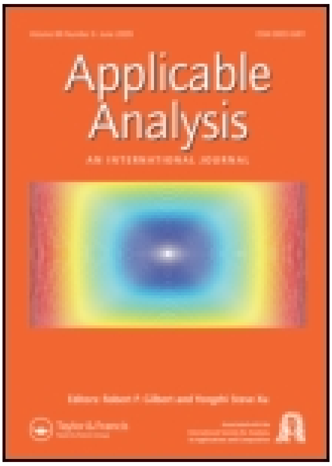

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A general RLC system with complex values

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A general RLC system with complex values

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We study a semilinear second-order ordinary differential equation for a complex valued function Q which describes the evolution of a generalized RLC system over an interval $[0, T]$. We solve the Dirichlet and periodic problems under appropriate conditions. Moreover, we give conditions in order to ensure that any solution satisfying an initial condition $Q(0) = Q_0$, $Q'(0) = I_0$ is defined over $[0, T]$.

Keywords: RLC-systems; Fixed point methods

1991 Mathematics Subject Classifications: 34B15; 34C25

1. Introduction

In this article we study the semilinear second-order ordinary differential equation

$$L(t)Q''(t) + R(t)Q'(t) + F(C(t), Q(t)) = E(t) \tag{1.1}$$

for a complex valued function Q describing the evolution of a generalized RLC system over an interval $[0, T]$. The coefficient $L \in C([0, T], \mathbb{R}^+)$ is the inductance, the friction term $R \in C([0, T], \mathbb{R}^+)$ is the resistence, and $C \in C([0, T], \mathbb{R})$ the capacity. The function $F : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ generalizes the linear case for an RLC system, where $F(C, Q) = Q(t)/C$.

The forcing term $E(t)$, often a T-periodic function, corresponds to an external electric field. We recall that usually the line integral of E over the circuit gives the electromotive force of the system [1–3].

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We study equation (1.1) under Dirichlet conditions, namely:

$$Q(0) = Q_0, \qquad Q(T) = Q_T \tag{1.2}$$

or periodic

$$Q(0) = Q(T), \qquad Q'(0) = Q'(T) \tag{1.3}$$

(1.1)–(1.3) represents the case in which the charge and the current are coincident at the initial and final times. The corresponding boundary value problems for the real case are studied by several authors (see [4–6]).

Moreover, we give conditions in order to ensure that any solution satisfying a Cauchy condition for the initial charge Q_0 and the initial current I_0 is defined over $[0, T]$. This ensures the nonexistence of resonant type effects.

In the second section we make a brief review of the RLC model. In the third section we establish the basic assumptions and results concerning the Dirichlet problem associated to (1.1).

In the fourth section we define a fixed point operator in order to solve the periodic problem. From the physical point of view, the existence of a periodic solution of (1.1) implies that the action of the external force compensates the effect of the dissipative term. This fact is reflected in the condition $(E/L) \perp p$, where the real function p is constructed uniquely from R.

Finally, in the last section we prove, for fixed Q_0, that the set of complex values I_0 such that a solution of (1.1) under the initial conditions $Q(0) = Q_0$, $Q'(0) = I_0$ is defined over the interval $[0, T]$ is a simply connected subset of $\mathbb{C}$. More precisely, there exists at least one solution defined over $[0, T]$ if and only if the equation $\psi(s) = I_0$ is solvable, where $\psi : \mathbb{C} \to \mathbb{C}$ is a continuous function depending on Q_0. Furthermore, if F is locally Lipschitz on $[0, T] \times \mathbb{C}$ then the disjoint union over Q_0 of the sets $\{Q_0\} \times \text{Range}(\psi_{Q_0})$ is an open domain of $\mathbb{C}^2$.

2. Brief review of the model

If two coils of wire are placed near each other, a changing current in one will induce an electromotive force (emf) in the other, and according to Faraday's law, the emf induced is proportional to the rate of change of flux passing through it. When considering an isolated single coil of N turns (or solenoid), we will find that a changing current passing through it will produce a changing magnetic flux inside the coil, and will induce an emf. This induced emf opposes the change in flux (Lenz's law). Defining the self-inductance $L = (N\Phi/I)$, where Φ is the magnetic flow, and I the current, we can conclude from Faraday's law that the emf induced can be computed as:

$$\mathcal{E} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

Any inductor will have some resistance, so it will be represented by its inductance L and its resistance R. When a DC source of voltage V is connected in series to the LR circuit, the emf's in the circuit are the battery V and the emf $\mathcal{E} = -L(\mathrm{d}I/\mathrm{d}t)$ in the inductor, and applying Kirchhoff's loop rule to the circuit, we obtain the following differential equation for the current I:

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + RI = V$$

More generally, in any electric circuit there can be three basic components: resistance, capacitance (C), and inductance. If we consider an LC circuit, at any instant, the potential difference across the capacitor will be $V = (Q/C)$, (where Q is the charge on the capacitor at that instant), and it will be equal to that across the inductor, so:

$$\frac{Q}{C} = -L\frac{\mathrm{d}I}{\mathrm{d}t}$$

The current I is due solely to the flow of charge from the capacitor and so $I = (\mathrm{d}Q/\mathrm{d}t)$. We can conclude that the charge Q will be determined from the differential equation:

$$\frac{\mathrm{d}^2Q}{\mathrm{d}^2t} + \frac{1}{LC}Q = 0$$

This is the differential equation for harmonic motion, with frequency given by:

$$\omega = \sqrt{\frac{1}{LC}}$$

This LC circuit is an idealization, taking account of the resistance R, we obtain the following differential equation:

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}^2t} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = 0$$

Thus the RLC circuit will correspond to the damped harmonic oscillator. In more realistic models, we also have that L, R, and C will not be constant anymore.

3. Basic assumptions and unique solvability of the Dirichlet problem

Let $\Omega = (0, T)$ and $\mathcal{S} : H^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ be the semilinear operator given by

$$\mathcal{S}Q = Q'' + \frac{R}{L}Q' + \frac{1}{L}F(C, Q)$$

We shall assume throughout the article that F is continuous and satisfies the growth condition

(F1) There exists a positive function μ such that

$$\frac{\mu}{L}\mathrm{Re}\left(\frac{F(C,Q) - F(C,P)}{Q - P}\right) + \frac{\tilde{\mu}'}{2} \leq c < \lambda_1$$

for any $C \in \mathbb{R}$ and $Q \neq P \in \mathbb{C}$, where $\tilde{\mu} = \mu' - (R/L)\mu$ and λ_1 is the first eigenvalue of the problem

$$(-\mu Q')' = \lambda Q, \qquad Q|_{\partial\Omega} = 0$$

Writing $\mathcal{S}Q = Q'' + (R/L)Q' + (1/L)F(C, Q)$, a simple computation shows that (F1) implies, for any $Q, P \in H^2(\Omega, \mathbb{C})$ such that $Q = P$ on $\partial\Omega$:

$$\left(\int_0^T \mu|Q' - P'|^2\right)^{1/2} \leq \frac{\sqrt{\lambda_1}}{\lambda_1 - c}\|\mu(\mathcal{S}Q - \mathcal{S}P)\|_{L^2} \tag{3.1}$$

THEOREM 3.1 *Assume that (F1) holds. Then the Dirichlet problem*

$$\begin{cases} L(t)Q''(t) + R(t)Q'(t) + F(C(t), Q(t)) = E(t) & \text{in } \Omega \\ Q(0) = Q_0, \quad Q(T) = Q_T \end{cases}$$

is uniquely solvable in $H^2(\Omega, \mathbb{C})$ *for any* $E \in L^2(\Omega, \mathbb{C})$, $Q_0, Q_T \in \mathbb{C}$.

Proof Let us consider $A = [0, 1] \times B_M$, where $B_M \subset H^1(\Omega, \mathbb{C})$ is the open ball of radius M centered at 0, and $T : \overline{A} \to H^1(\Omega, \mathbb{C})$ given by $T(\sigma, \hat{Q}) = Q$, with Q the unique solution of the linear problem

$$\begin{cases} L(t)Q''(t) + \sigma\Big(R(t)Q'(t) + F(C(t), \hat{Q}(t))\Big) = E(t) & \text{in } \Omega \\ Q(0) = Q_0, \quad Q(T) = Q_T \end{cases}$$

As (3.1) holds for the operators $\mathcal{S}_\sigma Q = Q'' + \sigma((R/L)Q' + (1/L)F(C, Q))$, it is immediate that T is compact. Moreover, $T_0 = T(0, \cdot)$ is constant and for $T(\sigma, \hat{Q}) = \hat{Q}$ we have that

$$\|\hat{Q} - T_0\|_{H^1} \leq \tilde{c}\sigma\left\|\frac{\mu}{L}\left(\frac{R}{L}T'_0 + \frac{1}{L}F(C, T_0)\right)\right\|_{L^2} \leq K.$$

Hence, choosing M large enough we conclude that $T(\sigma, \hat{Q}) \neq \hat{Q}$ for any $\hat{Q} \in \partial B_M$. By definition of the Leray–Schauder degree (see [7]),

$$\deg_{LS}(id - T_0, B_M, 0) = \deg_B(id - T_0|_X, B_M \cap X, 0)$$

where $X = \operatorname{span}\{T_0\}$, and hence $\deg_{LS}(id - T_0, B_M, 0) = 1$ for $M > \|T_0\|_{H^1}$. By homotopy invariance, we conclude that $\deg_{LS}(id - T_1, B_M, 0) = 1$, proving that $T(1, Q) = Q$ for some $Q \in B_M$. ■

THEOREM 3.2 *Assume that (F1) holds. Let* $E \in L^2(\Omega, \mathbb{C})$ *and denote by* $Tr : \mathcal{S}^{-1}(E) \to \mathbb{C}^2$ *the restriction of the usual trace function, i.e.* $Tr(Q) = (Q(0), Q(T))$. *Then Tr is an homeomorphism.*

Proof From the previous theorem, Tr is bijective, and its continuity is clear. Conversely, if $(Q_0)_n \to Q_0$ and $(Q_T)_n \to Q_T$ set $Q_n = Tr^{-1}((Q_0)_n, (Q_T)_n)$, $Q = Tr^{-1}(Q_0, Q_T)$ and then

$$0 = \int_0^T (\mathcal{S}Q_n - \mathcal{S}Q)(\overline{Q}_n - \overline{Q}) = (Q'_n - Q')(\overline{Q}_n - \overline{Q})\big|_0^T - \int_0^T |Q'_n - Q'|^2 + \int_0^T \frac{R}{L}(Q'_n - Q')(\overline{Q}_n - \overline{Q}) + \int_0^T \frac{1}{L}(F(C,Q_n) - F(C,Q))(\overline{Q}_n - \overline{Q})$$

Then

$$\int_0^T |Q'_n - Q'|^2 \leq \left|(Q'_n - Q')(\overline{Q}_n - \overline{Q})\big|_0^T\right| + \int_0^T \frac{R}{L}\mathrm{Re}(Q'_n - Q')(\overline{Q}_n - \overline{Q}) + \int_0^T \frac{1}{L}\mathrm{Re}\left(\frac{F(C,Q_n) - F(C,Q)}{Q_n - Q}\right)|Q_n - Q|^2$$

As

$$\int_0^T \frac{R}{L}\mathrm{Re}(Q'_n - Q')(\overline{Q}_n - \overline{Q}) = \frac{1}{2}\left(\frac{R}{L}|Q_n - Q|^2\big|_0^T - \int_0^T \left(\frac{R}{L}\right)'|Q_n - Q|^2\right),$$

by (F1) and Poincaré's inequality it suffices to prove that $|Q'_n - Q'|$ is bounded on $\partial\Omega$. As

$$\|Q''_n - Q''\|_{L^2} \leq \left\|\frac{R}{L}(Q'_n - Q') + \frac{1}{L}(F(C,Q_n) - F(C,Q))\right\|_{L^2}$$

it is easy to conclude that $Q_n - Q$ is bounded in $H^2(\Omega, \mathbb{C})$ and the result follows. ■

4. Applications to the periodic problem

In this section we apply the previous results to the periodic problem (1.1)–(1.3):

$$\begin{cases} L(t)Q''(t) + R(t)Q'(t) + F(C(t), Q(t)) = E(t) & \text{in } \Omega \\ Q(T) - Q(0) = Q'(T) - Q'(0) = 0 \end{cases}$$

Let F satisfy (F1). By Theorem 3.1, for any $z \in \mathbb{C}$ we may define $\mathbf{Q}_z$ as the unique solution of the problem

$$\begin{cases} L(t)Q''(t) + R(t)Q'(t) + F(C(t), Q(t)) = E(t) & \text{in } \Omega \\ Q(0) = Q(T) = z \end{cases}$$

By Theorem 3.2, $\mathcal{C} = \{\mathbf{Q}_z : z \in \mathbb{C}\}$ is an embedded curve for the H^2-norm. Clearly $\mathcal{C}$ is unbounded for $\|\cdot\|_{L^\infty}$.

Remark 4.1 Let us note that there exists a unique (up to a constant factor) positive p such that

$$p' = \frac{R}{L}p - k, \qquad p(0) = p(T)$$

for some constant k.

Indeed, from the equation $p' = (R/L)p - k$ we obtain that

$$p(t) = \left(c_0 - k\int_0^t e^{-\int_0^s (R/L)}\mathrm{d}s\right)e^{\int_0^t (R/L)}$$

Without loss of generality we may assume that $p(0) = c_0 = 1$, and as $p(0) = p(T)$ we deduce that

$$k = \frac{e^{\int_0^T (R/L)} - 1}{\int_0^T e^{\int_s^T (R/L)}\mathrm{d}s}$$

As $k > 0$, if p vanishes in Ω there exists $t_0 \in \Omega$ such that $p(t_0) = 0$ and $p'(t_0) \geq 0$. Then $k = -p'(t_0) \leq 0$, a contradiction.

In particular, if $(R/L) \perp 1$ then $p(t) = e^{\int_0^t (R/L)}$, and if (R/L) is constant then $p \equiv 1$.

Let $\mathrm{Int} : H^2(\Omega, \mathbb{C}) \to \mathbb{C}$ given by $\mathrm{Int}(Q) = \int_0^T (p/L)F(C, Q)$. Then we have:

THEOREM 4.1 *Let F satisfy* ($F1$) *and* $E \in L^2(\Omega, \mathbb{C})$. *Then the following statements are equivalent*:

(i) (1.1)–(1.3) *admits at least one solution*
(ii) *There exists* $Q \in \mathcal{C}$ *such that* $\mathrm{Int}(Q) = \int_0^T (p/L)E$

Proof By construction of p it holds that $Q \in \mathcal{C}$ if and only if

$$(pQ')' + kQ' + \frac{p}{L}F(C, Q) = \frac{p}{L}E$$

and $Q(0) = Q(T)$. Integrating over $\overline{\Omega}$, we obtain that

$$pQ'\big|_0^T + \mathrm{Int}(Q) = \int_0^T \frac{p}{L}E$$

and the result follows since $p(0) = p(T) = 1$. ■

Remark 4.2 By the previous theorem, solutions of (1.1)–(1.3) may be regarded as the zeroes of the continuous mapping $\psi : \mathbb{C} \to \mathbb{C}$ given by $\psi(z) = \mathrm{Int}(\mathbf{Q}_z) - \int_0^T (p/L)E$.

Thus, if we define $\mathbf{Q}_z^\sigma$ as the unique solution in $z + H^2 \cap H_0^1(\Omega)$ of the equation

$$Q'' + \sigma\left(\frac{R}{L}Q' + \frac{1}{L}F(C, Q)\right) = \frac{E}{L},$$

we may use a degree argument in order to obtain solutions of (1.1)–(1.3).

THEOREM 4.2 *Assume that* ($F1$) *holds, and let* $a = \int_0^T (p/L)E$ *and*

$$T_0(z) = \int_0^T \frac{p}{L} F(C, z + \varphi)$$

where φ *is the function given by*

$$\varphi(t) = \int_0^t \int_0^s \frac{E}{L} \mathrm{d}s - \frac{t}{T} \int_0^T \int_0^s \frac{E}{L} \mathrm{d}s$$

Further, assume that there exists a bounded set $A \subset [0, 1] \times \mathbb{C}$ *such that*

$$A_\sigma := \{z \in \mathbb{C} \colon (\sigma, z) \in A\}$$

is nonempty for every $\sigma \in [0, 1]$, *and that*

$$\mathrm{Int}(\mathbf{Q}_z^\sigma) \neq a$$

for any $z \in \partial A_\sigma$. *Then, if*

$$\deg_B(T_0, A_0, a) \neq 0,$$

the periodic problem (1.1)–(1.3) *admits at least one solution with* $Q(0) \in A_1$.

Proof Let $T \colon [0, 1] \times \mathbb{C} \to \mathbb{C}$ be the continuous mapping given by

$$T(\sigma, z) = \mathrm{Int}(\mathbf{Q}_z^\sigma)$$

A simple computation shows that $\mathbf{Q}_z^0(t) = z + \varphi(t)$, which implies that $T(0, \cdot) = T_0$. As $T(\sigma, z) \neq a$ for any $z \in \partial A_\sigma$, the result follows from the homotopy invariance of the Brouwer degree. ■

COROLLARY 4.3 *Let* ($F1$) *hold and assume, using the notation of the previous theorem, that* $\deg_B(T_0, B_M(0), a) \neq 0$ *for* M *large. Then* (1.1)–(1.3) *is solvable in the following cases*:

(i) $a = 0$ *and*

- $\liminf_{|z| \to \infty} |\mathrm{Re}(F(C, z))| + \liminf_{|z| \to \infty} |\mathrm{Im}(F(C, z))| > 0$
- $\limsup_{|z| \to \infty} (|F(C, z)|)/L|z| < (\inf_t \sqrt{\mu(t)}(\lambda_1 - c))/\sqrt{\lambda_1 T}\, \|\mu\|_{L^\infty}$

(ii) $a \neq 0$ *and*

$$\limsup_{|z| \to \infty} |F(C, z)| < \frac{|a|}{\int (p/L)}$$

Proof From (3.1) we have, for any $\sigma \in [0, 1]$:

$$\|\mathbf{Q}_z^\sigma - z\|_{L^\infty} \leq \frac{\sqrt{T}}{\inf_t \sqrt{\mu(t)}} \left(\int_0^T \mu |(\mathbf{Q}_z^\sigma - z)'|^2 \right)^{1/2} \leq \frac{\sqrt{\lambda_1 T}}{\inf_t \sqrt{\mu(t)}(\lambda_1 - c)} \left\| \frac{\mu}{L} (E - \sigma F(C, z)) \right\|_{L^2}$$

Hence in both cases we obtain, for $|z| = M$ large, that

$$|\mathbf{Q}_z^\sigma| > \gamma M - \delta$$

for some positive constants γ, δ.

Using (i), we obtain

$$|\mathrm{Re}(F(C, \mathbf{Q}_z^\sigma))| \geq r > 0 \quad \text{or} \quad |\mathrm{Im}(F(C, \mathbf{Q}_z^\sigma))| \geq r > 0$$

This proves that $\mathrm{Int}(\mathbf{Q}_z^\sigma) \neq 0 = a$.

On the other hand, if (ii) holds, for large M it follows that

$$|\mathrm{Int}(\mathbf{Q}_z^\sigma)| \neq |a|$$

and the proof is complete. ∎

THEOREM 4.4 *Let* $(F1)$ *hold, and assume that there exists a constant* $M > 0$ *such that*

$$\begin{aligned} \mathrm{Re}\big(F(C,z) - E(t)\big)\mathrm{sgn}(\mathrm{Re}\, z) < 0 \qquad &\text{if} \quad |\mathrm{Re}\, z| \geq M, \\ \mathrm{Im}\big(F(C,z) - E(t)\big)\mathrm{sgn}(\mathrm{Im}\, z) < 0 \qquad &\text{if} \quad |\mathrm{Im}\, z| \geq M \end{aligned}$$

Then, if $E \perp (p/L)$, (1.1)–(1.3) *has at least one solution* Q *with* $|\mathrm{Re}\, Q|, |\mathrm{Im}\, Q| \leq M$.

In particular, if

$$\mathrm{Re}\, F(C,z)\mathrm{sgn}(\mathrm{Re}\, z), \mathrm{Im}\, F(C,z)\mathrm{sgn}(\mathrm{Im}\, z) \to -\infty \quad \text{as} \quad |z| \to \infty$$

then (1.1)–(1.3) *has at least one solution for any* $E \perp (p/L)$.

Proof With the previous notations, assume that $\mathrm{Re}\big(\mathbf{Q}_z(t)\big) > \mathrm{Re}\, z$ for some z with $\mathrm{Re}\, z \geq M$. We may suppose that t is a maximum, and then

$$\mathrm{Re}\big(p\mathbf{Q}_z''(t)\big) = \mathrm{Re}\Big(\frac{p}{L}\big[E(t) - F(C, \mathbf{Q}_z)\big]\Big) > 0,$$

a contradiction. It follows that $\mathrm{Re}\big(\mathbf{Q}_z(t)\big) \leq \mathrm{Re}\, z$, and hence

$$\mathrm{Re}\big(\mathbf{Q}_z'(T)\big) - \mathrm{Re}\big(\mathbf{Q}_z'(0)\big) \geq 0$$

In the same way, if $\mathrm{Re}\, z \leq -M$ we have that

$$\mathrm{Re}\big(\mathbf{Q}_z'(T)\big) - \mathrm{Re}\big(\mathbf{Q}_z'(0)\big) \leq 0,$$

and a similar result holds for $\mathrm{Im}\, z$. By the generalized intermediate value theorem there exists z with $|\mathrm{Re}\, z|, |\mathrm{Im}\, z| \leq M$ such that $\mathbf{Q}_z'(T) - \mathbf{Q}_z'(0) = 0$, and so completes the proof. ∎

5. Some results concerning the initial value problem

In this section we study the behavior of the solutions of the initial value problem

$$\begin{cases} L(t)Q''(t) + R(t)Q'(t) + F(C(t), Q(t)) = E(t) & \text{in } \Omega \\ Q(0) = Q_0, \quad Q'(0) = I_0 \end{cases} \tag{5.1}$$

As in the previous section, for every $z \in \mathbb{C}$ we define $\mathbf{Q}_z \in H^2(\Omega, \mathbb{C})$ as a solution of a two-point boundary value problem. In this case, we may consider $\varphi_z(t) = zt + Q_0$ and $\mathbf{Q}_z$ the unique solution of

$$\begin{cases} L(t)Q''(t) + R(t)Q'(t) + F(C(t), Q(t)) = E(t) & \text{in } \Omega \\ Q(0) = Q_0, \quad Q(T) = \varphi_z(T) \end{cases}$$

Then we have:

THEOREM 5.1 *Let* $(F1)$ *hold and consider* $\psi_{Q_0} : \mathbb{C} \to \mathbb{C}$ *given by*

$$\psi_{Q_0}(z) = z + \int_0^T \frac{1}{L}(E - R\mathbf{Q}_z' - F(\theta, \mathbf{Q}_z))\frac{\theta - T}{T}\mathrm{d}\theta$$

Then (5.1) *admits a solution defined over* $\overline{\Omega}$ *if and only if* $I_0 \in \text{Range}(\psi_{Q_0})$.

Proof Let $Q \in H^2(\Omega, \mathbb{C})$ be a solution of (5.1) and let $z = (Q(T) - Q_0)/T$. Then

$$\mathbf{Q}_z(t) - \varphi_z(t) = \int_0^T \frac{1}{L}\big(E - R\mathbf{Q}_z' - F(\theta, \mathbf{Q}_z))G(t, \theta\big)\mathrm{d}\theta$$

where G is the Green's function given by

$$G(t, s) = \begin{cases} \dfrac{t(s - T)}{T} & \text{if } s \geq t \\ \dfrac{(t - T)s}{T} & \text{if } s \leq t \end{cases}$$

By simple computation we obtain that $\mathbf{Q}_z'(0) = \psi_{Q_0}(z)$, which proves that $I_0 \in \text{Range}(\psi_{Q_0})$. Conversely, if $I_0 = \psi_{Q_0}(z)$, then the corresponding $\mathbf{Q}_z$ is a solution of (5.1). ■

COROLLARY 5.2 *Let* $(F1)$ *hold and define*

$$\mathcal{I}(Q_0) = \{I_0 \in \mathbb{C}\colon (5.1) \textit{ admits a solution in } H^2(\Omega, \mathbb{C})\}$$

Further, assume that F *is locally Lipschitz on* $\overline{\Omega} \times \mathbb{C}$ *with respect to* Q. *Then* $\mathcal{I}(Q_0)$ *is a simply connected open subset of* $\mathbb{C}$.

Proof It follows immediately from the previous theorem that $\mathcal{I}(Q_0)$ is simply connected. Moreover, if $\psi_{Q_0}(z_1) = \psi_{Q_0}(z_2)$ then $\mathbf{Q}'_{z_1}(0) = \mathbf{Q}'_{z_2}(0)$, and by uniqueness we conclude that $z_1 = z_2$. Then ψ_{Q_0} is injective and hence $\psi_{Q_0}(\mathbb{C})$ is open. ∎

THEOREM 5.3 *Let* $(F1)$ *hold and assume that* F *is locally Lipschitz on* $\overline{\Omega} \times A$, *where* A *is an open domain of* $\mathbb{C}$. *Then*

$$\bigcup_{Q_0 \in A} \{Q_0\} \times \mathcal{I}(Q_0)$$

is an open domain of $\mathbb{C}^2$.

Proof Let $\mathcal{S}_A = \{u \in H^2(\Omega, \mathbb{C}) \colon \mathcal{S}Q = (E/L), Q(0) \in A\}$, and consider the continuous mapping $\rho : \mathcal{S}_A \to \mathbb{C}^2$ given by $\rho(Q) = (Q(0), Q'(0))$. As F is locally Lipschitz, ρ is injective, and hence $\rho \circ Tr^{-1}(A \times \mathbb{C})$ is open and connected. ∎

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