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A parabolic problem arising in Financial Mathematics

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ABSTRACT

We study a parabolic problem arising in Financial Mathematics. Under suitable conditions, we prove the existence and uniqueness of solutions in a general domain using the method of upper and lower solutions and a diagonal argument.

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1. Introduction

In recent years there has been an increasing interest in problems arising in Financial Mathematics and in particular on option pricing. The standard approach to this problem leads to the study of equations of a parabolic type.

An option is a contract that gives the holder the right to trade in the future at a previously agreed price. A European *call option* is a right to buy a particular asset for an agreed amount at a specific time in future. A *put option* is the right to sell a particular asset for an agreed amount at a specific time in future.

In Financial Mathematics, usually the Black–Scholes model [1] is used to price these contracts, by means of a reversed-time parabolic partial differential equation. In this model, an important quantity is the so-called volatility. Volatility is a measure of the amount of fluctuation in the asset prices: a measure of randomness. It has a major impact on the value of the option; in mathematical terms, it corresponds to the diffusion coefficient in the Black–Scholes equation.

In the standard Black–Scholes model, a basic assumption is that the volatility is constant. Several models that have been proposed in recent years, however, allowed the volatility to be non-constant or a stochastic variable. For instance, in [2] a model with stochastic volatility is proposed. In this model the underlying security S follows, as in the standard Black–Scholes model, an stochastic process

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}Z_t,$$

where Z is a standard Brownian motion. Unlike the classical model, the variance $v(t) = \sigma^2(t)$ also follows stochastic process given by

$$\mathrm{d}v_t = \kappa(\theta - v(t))\mathrm{d}t + \gamma\sqrt{v_t}\mathrm{d}W_t,$$

where W is another standard Brownian motion. The correlation coefficient between W and Z is denoted by ρ:

$$E(\mathrm{d}Z_t, \mathrm{d}W_t) = \rho \mathrm{d}t.$$

This leads to a generalized Black–Scholes equation:

$$\frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \rho\gamma vS\frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2}v\gamma^2\frac{\partial^2 U}{\partial v^2} + rS\frac{\partial U}{\partial S} + [\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} - rU + \frac{\partial U}{\partial t} = 0.$$

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If we introduce the change of variables given by $y = \log S, x = \frac{v}{\gamma}, \tau = T - t$ the following problem for $u(x, y) = U(S, v)$ is obtained:

$$u_\tau = \frac{1}{2}\gamma x\left[\Delta u + 2\rho\frac{\partial^2 u}{\partial x\partial y}\right] + \frac{1}{\gamma}\left[\kappa(\theta - \gamma x) - \lambda\gamma x\right]\frac{\partial u}{\partial x} + \left(r - \frac{\gamma x}{2}\right)\frac{\partial u}{\partial y} - ru$$

in a cylindrical domain $\Omega \times (0, T)$, with $\Omega \subset \mathbb{R}^2$. A similar model has been considered in [3], for which the stationary equation has been studied in [4].

More general models with stochastic volatility have been considered for example in [5], where the following problem is derived from the Feynman–Kac relation:

$$\begin{cases} u_t = \frac{1}{2}Tr\left(M(x, \tau)D^2u\right) + q(x, \tau)\cdot Du, \\ u(x, 0) = u_0(x) \end{cases}$$

for some diffusion matrix M and a payoff function u_0.

This discussion motivates us to consider the general parabolic problem

$$\begin{cases} Lu - u_t = g(u, x, t) & \text{in } \Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{on } \Omega \times \{0\} \\ u(x, t) = h(x, t) & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{1.1}$$

We shall assume that $\Omega \subset \mathbb{R}^d$ is an unbounded smooth domain, $g : [0, +\infty) \times \overline{\Omega} \times [0, T] \to [0, +\infty)$ is continuous and continuously differentiable with respect to u, L is a second order elliptic operator in non-divergence form, namely

$$Lu := \sum_{i,j=1}^{d} a^{ij}(x, t)u_{x_i x_j} + \sum_{i=1}^{d} b^i(x, t)u_{x_i} + c(x, t)u,$$

where the coefficients of L belong to the Hölder Space $C^{\delta,\delta/2}\left(\overline{\Omega} \times [0, T]\right)$ and satisfy the following conditions

$$\Lambda|v|^2 \geq \sum_{i,j=1}^{d} a^{ij}(x, t)v_i v_j \geq \lambda|v|^2 \quad (\Lambda \geq \lambda > 0)$$

$$|b^i(x, t)| \leq C, \qquad c(x, t) \leq 0.$$

Furthermore, we shall assume that $u_0 \in C^{2+\delta}(\overline{\Omega})$, $h \in C^{2+\delta,1+\delta/2}(\overline{\Omega} \times [0, T])$ and satisfy the following compatibility condition

$$h(x, 0) = u_0(x) \quad \forall\, x \in \partial\Omega. \tag{1.2}$$

Our main result reads as follows:

Theorem 1.1. *Let L be the elliptic operator defined as above, and assume that $g(0, x, t) = 0$. Then for any $T > 0$ there exists $\theta_0 = \theta_0(\Lambda, d, \|b\|_\infty, T)$ such that if $\theta < \theta_0$, then for any initial and boundary conditions u_0 and h satisfying*

$$0 \leq u_0(x) \leq kT^{-\frac{d}{2}}e^{\frac{\theta}{T}|x|^2}$$

and

$$0 \leq h(x, t) \leq k(T - t)^{-\frac{d}{2}}e^{\frac{\theta}{T-t}|x|^2} \quad \text{for } x \in \partial\Omega, 0 \leq t < T$$

for some constant k, there exists at least one solution u of the problem (1.1) satisfying

$$0 \leq u(x, t) \leq k(T - t)^{-\frac{d}{2}}e^{\frac{\theta}{T-t}|x|^2}.$$

We give a proof of Theorem 1.1 in Section 2, using the method of upper and lower solutions. We recall that u is called an upper (lower) solution of problem (1.1) if

$$\begin{cases} Lu - u_t \leq (\geq)g(u, x, t) & \text{in } \Omega \times (0, T) \\ u(x, 0) \geq (\leq)u_0(x) & \text{on } \Omega \times \{0\} \\ u(x, t) \geq (\leq)h(x, t) & \text{on } \partial\Omega \times (0, T). \end{cases}$$

On the other hand, we obtain a uniqueness result, which can be deduced immediately from the following version of the maximum principle.

Theorem 1.2. *In the situation of Theorem 1.1, let $\tilde{T} < T$ and let u be a lower solution of (1.1) in the domain $V = \Omega \times (0, \tilde{T})$ such that $0 \le u \le K e^{A|x|^2}$ for some constants A and K. Furthermore, assume that g is nondecreasing in u. Then*

$$\sup_V u = \sup_{\partial' V} u,$$

where $\partial' V = (\Omega \times \{0\}) \cup (\partial\Omega \times [0, \tilde{T}])$ denotes the parabolic boundary of the domain V.

A proof of Theorem 1.2 is given in Section 3.

2. The method of upper and lower solutions

In order to prove Theorem 1.1, we shall apply the method of upper and lower solutions. More precisely, we shall consider $\alpha \equiv 0$ and $\beta = k(T-t)^{-\frac{d}{2}} e^{\frac{\theta}{T-t}|x|^2}$. Indeed, from the hypothesis it is clear that α is a lower solution, and a straightforward computation shows that β satisfies:

$$L\beta - \beta_t = \beta \left\{ \left(\frac{2\theta}{T-t} \right)^2 \sum_{i,j=1}^{d} a^{ij} x_i x_j + \frac{2\theta}{T-t} \sum_{i=1}^{d} a^{ii} + \frac{2\theta}{T-t} \sum_{i=1}^{d} b^i x_i + c - \left[\frac{d}{2(T-t)} + \frac{\theta}{(T-t)^2} |x|^2 \right] \right\}.$$

From our assumptions, and using the fact that $\sum_{i=1}^{d} a^{ii} \le \Lambda$, and that $2\sum_{i=1}^{d} b^i x_i \le \varepsilon|x|^2 + \frac{1}{\varepsilon}\|b\|_\infty^2$, we deduce that

$$\frac{1}{\beta}(L\beta - \beta_t) \le (4\theta\Lambda - 1 + \varepsilon(T-t)) \frac{\theta|x|^2}{(T-t)^2} + \frac{1}{T-t}\left[2\theta\Lambda - \frac{d}{2} + \frac{1}{\varepsilon}\theta\|b\|_\infty^2 + c \right].$$

Taking $\varepsilon < \frac{1}{T}$, and

$$\theta \le \min\left\{ \frac{1 - T\varepsilon}{4\Lambda}, \frac{d\varepsilon}{2\|b\|_\infty^2 + 4\Lambda} \right\},$$

it follows that

$$L\beta - \beta_t \le 0 \le g(\beta).$$

As $u_0(x) \le \beta(x, 0)$ and $h(x, t) \le \beta(x, t)$ for $x \in \partial\Omega$, we conclude that β is an upper solution of the problem.

Remark 2.1. If U is a smooth and bounded subset of Ω, by [6, Thm. 10.4.1], and the compatibility condition (1.2), there exists a unique function $\varphi_U \in C^{2+\delta,1+\delta/2}(\overline{U} \times [0, T])$ such that

$$\begin{cases} L\varphi_U - (\varphi_U)_t = 0, & \\ \varphi_U(x, 0) = u_0(x) & x \in U \\ \varphi_U(x, t) = h(x, t) & (x, t) \in \partial U \times [0, T]. \end{cases}$$

By the standard maximum principle,

$$0 \le \varphi_U(x, t) \le \beta(x, t)$$

for $(x, t) \in \overline{U} \times [0, T]$.

First, we solve an analogous problem in a bounded domain.

Lemma 2.1. *Let $U \subset \mathbb{R}^d$ a bounded smooth domain, let $\tilde{T} < T$ and let φ_U be defined as in Remark 2.1. Then the problem*

$$\begin{cases} Lu - u_t = g(u, x, t) & \text{in } U \times (0, \tilde{T}) \\ u(x, 0) = u_0(x) & \text{in } U \times \{0\} \\ u(x, t) = \varphi_U(x, t) & \text{in } \partial U \times (0, \tilde{T}) \end{cases} \tag{2.1}$$

admits at least one solution u with $0 \le u(x, t) \le \beta(x, t)$ for $x \in U$, $0 \le t \le \tilde{T}$.

Proof. Set $\lambda > 0$ such that the function $g(u, x, t) - \lambda u$ is non-increasing on u for $0 \le u \le \max_{x \in \partial U} \beta(x, \tilde{T})$. Set $u^0 = 0$ and $V = U \times (0, \tilde{T})$. By standard results, we may define $u^{n+1} \in W_p^{2,1}(V)$ as the unique solution of the problem

$$\begin{cases} Lu^{n+1} - u_t^{n+1} - \lambda u^{n+1} = g(u^n, x, t) - \lambda u^n & \text{in } U \times (0, \tilde{T}) \\ u^{n+1}(x, 0) = u_0(x) & \text{in } U \times \{0\} \\ u^{n+1}(x, t) = \varphi_U(x, t) & \text{in } \partial U \times (0, \tilde{T}). \end{cases} \tag{2.2}$$

We claim that

$$0 \le u^n(x,t) \le u^{n+1}(x,t) \le \beta(x,t) \quad \forall\ (x,t) \in \overline{U} \times [0,\tilde{T}], \forall\ n \in \mathbb{N}_0.$$

Indeed, by the maximum principle it follows that $u^1 \ge 0$; moreover,

$$Lu^1 - u^1_t - \lambda u^1 = g(0,x,t) \ge g(\beta,x,t) - \lambda\beta \ge L\beta - \beta_t - \lambda\beta$$

and hence $u^1 \le \beta$. Inductively,

$$\begin{aligned} Lu^{n+1} - u^{n+1}_t - \lambda u^{n+1} &= g(u^n,x,t) - \lambda u^n \le g(u^{n-1},x,t) - \lambda u^{n-1} \\ &= Lu^n - u^n_t - \lambda u^n. \end{aligned}$$

Thus $u^{n+1} \ge u^n$. In the same way as before it follows that $u^{n+1} \le \beta$.

We define

$$u(x,t) = \lim_{n\to\infty} u^n(x,t).$$

By the standard L^p-estimates [7, Thm 7.17],

$$\|D^2(u^n - u^m)\|_{L^p(V)} + \|(u^n - u^m)_t\|_{L^p(V)} \le c\left(\|L(u^n - u^m) - (u^n - u^m)_t\|_{L^p(V)} + \|u^n - u^m\|_{L^p(V)}\right).$$

By construction,

$$L(u^n - u^m) - (u^n - u^m)_t = g(u^{n-1},x,t) - g(u^{m-1},x,t) - \lambda(u^{n-1} - u^{m-1}).$$

As g is continuous, and using that $0 \le u^n \le \beta$, by dominated convergence it follows that $\{u^n\}$ is a Cauchy sequence in $W^{2,1}_p(V)$. Hence $u^n \to u$ in the $W^{2,1}_p$-norm, and then u is a strong solution. Moreover, by the Morrey imbedding and Schauder estimates, it follows that u is a classical solution. □

Proof of Theorem 1.1. We approximate the domain Ω by an non-decreasing sequence $(\Omega_N)_{N\in\mathbb{N}}$ of bounded smooth subdomains of Ω, which can be chosen in such a way that $\partial\Omega$ is also the union of the non-decreasing sequence $\Omega_N \cap \Omega$.

Then, define u^N as a solution of the problem

$$\begin{cases} Lu - u_t = g(u,x,t) & \text{in } \Omega_N \times \left(0, T - \dfrac{1}{N}\right) \\ u(x,0) = u_0(x) & \text{in } \Omega_N \times \{0\} \\ u(x,t) = h(x,t) & \text{in } \partial\Omega_N \times \left(0, T - \dfrac{1}{N}\right) \end{cases} \tag{2.3}$$

such that $0 \le u^N \le \beta$ in $\Omega_N \times (0, T - \frac{1}{N})$. Define $V_N = \Omega_N \times (0, T - \frac{1}{N})$ and choose $p > d$. For $M > N$, we have that

$$\|D^2(u^M)\|_{L^p(V_N)} + \|(u^M)_t\|_{L^p(V_N)} \le c\left(\|Lu^M - (u^M)_t\|_{L^p(V_N)} + \|u^M\|_{L^p(V_N)}\right) \le c\left(\|g(u^M,\cdot)\|_{L^p(V_N)} + \|\beta\|_{L^p(V_N)}\right) \le C$$

for some constant C depending only on N. By Morrey imbedding, there exists a subsequence that converges uniformly on $\overline{V}_N$. Using a standard diagonal argument, we may extract a subsequence (still denoted $\{u^M\}$) such that u^M converges uniformly to some function u over compact subsets of $\Omega \times (0,T)$. For $V = U \times (0,\tilde{T})$, $U \subset\subset \Omega$, $\tilde{T} < T$, taking M, N large enough we have that

$$\|D^2(u^N - u^M)\|_{L^p(V)} + \|(u^N - u^M)_t\|_{L^p(V)} \le c\left(\|L(u^N - u^M) - (u^N - u^M)_t\|_{L^p(V)} + \|u^N - u^M\|_{L^p(V)}\right).$$

By construction,

$$L(u^N - u^M) - (u^N - u^M)_t = g(u^{M-1},x,t) - g(u^{M-1},x,t) - \lambda(u^{N-1} - u^{M-1}).$$

As before, using that g is continuous, and that $0 \le u^N \le \beta$, by dominated convergence it follows that $\{u^N\}$ is a Cauchy sequence in $W^{2,1}_p(V)$. Hence $u^N \to u$ in the $W^{2,1}_p$-norm, and then u is a classical solution in V. It follows that u satisfies the equation on $\Omega \times (0,T)$. Furthermore, it is clear that $u(x,0) = u_0(x)$. For $M > N$ we have that $u_M(x,t) = u_N(x,t)$ for $x \in \partial\Omega \cap \partial\Omega_N$, $t \in (0, T - \frac{1}{N})$. Thus, it follows that u satisfies the boundary condition $u(x,t) = h(x,t)$ on $\partial\Omega \times [0,T)$. □

3. A maximum principle for problem (1.1)

In this section we give a proof of Theorem 1.2. For $\varepsilon > 0$ set

$$v(x,t) = u(x,t) - \varepsilon\beta(x,t).$$

As β is non-decreasing in t,

$$v(x,t) \le Ke^{A|x|^2} - \varepsilon T^{-\frac{d}{2}} e^{\frac{\theta}{T}|x|^2}.$$

Choosing $\theta > TA$, we conclude that

$$\lim_{|x|\to\infty}\left[\sup_{0\leq t\leq\tilde{T}} v(x,t)\right] = -\infty.$$

We may choose R large enough such that

$$v(x,t) \leq M := \sup_{\partial' V} u(x,t)$$

for $x \in \Omega$ with $|x| \geq R$ and $0 \leq t \leq \tilde{T}$. On the other hand,

$$Lv - v_t - g(v,x,t) \geq Lu - u_t - g(u,x,t) - \varepsilon(L\beta - \beta_t) \geq 0.$$

As in the proof of Theorem 1.1, we approximate the domain Ω by an non-decreasing sequence $(\Omega_N)_{N\in\mathbb{N}}$ of bounded smooth sub-domains of Ω, which can be chosen in such a way that $\partial\Omega$ is also the union of the non-decreasing sequence $\Omega_N \cap \Omega$. For N large enough, we may assume that if $x \in \partial\Omega_N$ then $x \in \partial\Omega$ or $|x| \geq R$.

Hence, by the classical maximum principle for bounded domains,

$$v(x,t) \leq \sup_{\partial'\Omega_N} v(x,t) \leq M.$$

Letting $\varepsilon \to 0$, we conclude that $u(x,t) \leq M$ for $0 \leq t \leq \tilde{T}$.

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