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CFT thermal 2-point function*

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This is a short review of our recent work[1] with some new original results. We numerically calculate the conformal two point correlation function of operators with arbitrary scale dimension up to order T^{2d} in the low temperature expansion. We analytically compute its large scale dimension limit up to the same order.

1. Introduction

In d-dimensional Euclidean Conformal Field Theories (CFT), the 2-point correlation function (propagator) of the operator $\mathcal{O}(x)$ of anomalous dimension Δ is constrained by conformality to be

$$G_2(x) \propto \frac{1}{|x|^{2\Delta}} \quad , \quad |x|^2 = \vec{x}^2 + t^2 \quad , \tag{1}$$

with the Fourier transform

$$G_2(k) \propto k^{2\Delta - d} \quad , \quad k^2 = \vec{k}^2 + \omega^2 \quad . \tag{2}$$

The splitting $x^\mu \to \vec{x}$, $t(= x^0)$ or $k^\mu \to \vec{k}$, $\omega(= k^0)$ is not necessary here, but is necessary for temperature $T \neq 0$.

A generic CFT is usually nonperturbative so it is not obvious how to introduce the temperature. But if we assume that such a CFT has a gravitational dual, then we could use the AdS/CFT correspondence. The temperature in the CFT corresponds to the Hawking temperature of the horizon in the bulk, which comes as the black hole solution or, in our case, the black brane solution (large mass black hole limit) in the anti de Sitter (AdS) space.

We will be interested in the small temperature corrections, $T \ll k, T \ll \omega$.[1] So far these corrections were estimated only in the limit of $\vec{k} = 0$ for the absorption

*Talk given by B.B. on September 2023 in Beograd cross section by black branes[2] and in the large Δ limit without[3–6] or with a chemical potential.[7] It turns out that the opposite limit, large T corrections, is much simpler.

This paper is a short review of[1] (sections 2-5) but it has on top of it some original contributions (sections 6 and Appendix A).

2. The equation

We will get the CFT thermal propagator from the AdS/CFT correspondence, i.e. from the solution of the equation of motion for a black brane in Euclidean AdS_{d+1} (we measure in units of the AdS scale, $L = 1$):

$$ds^2 = g_{ab}dx^a dx^b = \frac{1}{z^2}\left(\frac{dz^2}{f(z)} + f(z)dt^2 + d\vec{x}^2\right) \quad , \tag{3}$$

$$f(z) = 1 - \left(\frac{z}{z_h}\right)^d \quad . \tag{4}$$

Here $z \to 0$ is the boundary, while $z \to z_h$ is the horizon. The Hawking temperature is $T = \frac{d}{4\pi z_h}$. Small temperature thus means large z_h.

Put now a scalar field into this non-dynamical background

$$S_{bulk} = \frac{1}{2}\int d^{d+1}x\sqrt{det\ g_{ab}}\left(\partial_a\phi\, g^{ab}\, \partial_b\phi + m^2\phi^2\right) \quad . \tag{5}$$

According to the AdS/CFT dictionary, the operator in the boundary CFT dual to ϕ has anomalous dimension

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2} = \frac{d}{2} + \nu \quad , \tag{6}$$

where we will throughout assume Δ and ν non-integer.

The perturbation $\xi(z) = \xi(z, k)$ is defined as

$$\phi(x, z) = \int \frac{d^d k}{(2\pi)^d} e^{i\omega t + i\vec{k}\cdot\vec{x}}\, \xi(z) \tag{7}$$

and must satisfy

$$f(z)z^2\, \xi''(z) - (d - f(z))z\, \xi'(z) - \left(z^2\left(k^2 + \omega^2\frac{1 - f(z)}{f(z)}\right) + \Delta(\Delta - d)\right)\xi(z) = 0 \quad . \tag{8}$$

The solution must satisfy the following boundary condition on the horizon

$$z \to z_h \ :\ \xi(z) < \infty \tag{9}$$

According to AdS/CFT the solution close to the boundary gives the propagator

$$z \to 0 \ :\ \xi(z) \propto z^{d-\Delta} + \underbrace{G_2(\omega, k)}_{\text{our hero}}\, z^\Delta \quad . \tag{10}$$

This amounts essentially to solve the usual *connection problem*: given the independent solutions $\xi_a^{1,2}(z)$ known expanded around point a

$$\xi_a^{1,2}(z) = (z - z_a)^{\alpha_{1,2}} \sum_{n=0}^{\infty} c_n^{1,2} (z - z_a)^n \quad , \tag{11}$$

how are they related to two other independent solutions $\xi_b^{1,2}(z)$ expanded around point b ?

$$\xi_b^{1,2}(z) = (z - z_b)^{\beta_{1,2}} \sum_{n=0}^{\infty} d_n^{1,2} (z - z_b)^n \tag{12}$$

In our case: $a = z_h$ (horizon) and $b = 0$ (boundary). In other words, the connection problem is to find q_{ij}, $i, j, = 1, 2$:

$$\xi^1_{z_h}(z) = q_{11}\ \xi^1_0(z) + q_{12}\ \xi^2_0(z) \tag{13}$$

$$\xi^2_{z_h}(z) = q_{21}\ \xi^1_0(z) + q_{22}\ \xi^2_0(z) \tag{14}$$

If $\xi^1_{z_h}(z)$ is well behaved for $z \to z_h$, and

$$\xi^1_0(z) \xrightarrow{z\to 0} A\ z^{d-\Delta} \tag{15}$$

$$\xi^2_0(z) \xrightarrow{z\to 0} B\ z^{\Delta} \tag{16}$$

$$\to \quad G_2(\omega, k) = \frac{B\ q_{12}}{A\ q_{11}}$$

This problem has been recently solved analytically for our case[8] based on a general solution of the connection problem for the Heun equation.[9] The propagator is related to

- the known Nekrasov-Shatashvili function NS (sums over instantons in $\mathcal{N} = 2$ SQCD, gauge SU(2) with $F = 4$ quarks with masses m_i)

  $$NS = \sum_{n=1}^{\infty} c_n(a, m_i) t^n$$

  where c_n is calculable but complicated for large n (sums over partitions) and $t \propto \exp(-1/g^2)$ is the instanton parameter;
- the parameter a is got from the Matone relation (between the parameters of the model), schematically

  $$p(a, \omega/T, k/T, m_i, \Delta) = t\ \partial_t NS(t)$$

The limit in our case to be taken is a special $m_i \to \infty$ one. The reason for the problem is exactly this $m_i \to \infty$ limit. Does it mean that quarks need to be integrated out, $F = 4 \to F = 0$? Usually[10] this is done by taking $\Lambda_F \to 0$ with

$$\Lambda_{F-1} = m_F^p \Lambda_F^{1-p} \quad . \tag{17}$$

but in our case

$$t \propto \Lambda_F^{p'} = 1/2 \tag{18}$$

is fixed and so we do not end up in the theory with lower F.

So the exact solution is

- very nice for expansion in large T/k, T/ω for the black hole case;
- but very bad (or impossible) for expansion in small T/k, T/ω for the black brane case (F written as infinite expansion in positive powers of ω/T).

In fact all instantons (powers of t count number of instantons) contribute, the gauge coupling is large. Essentially it boils down to infinite sums very difficult to evaluate. So this solution is not useful for our program, perturbative expansion in positive powers of T/k

Now, which are the parameters of the problem? We have 4 parameters:

(1) d, we will consider mostly $d = 2$ or $d = 4$
(2) T/ω (first dimensionless ratio)
(3) T/k (second dimensionless ratio)
We will be interested in expansion in positive powers of T/ω and T/k (small temperature)
(4) Δ, for large $\Delta \gg 1$ one can approximate the propagator as the exponent of the geodesic length between the two points in spacetime. Solutions have been found for $d = 4$ to order T^8 [4]

$$G_2(\eta, x) = \frac{1}{|x|^{2\Delta}} \left(1 + \frac{\Delta\pi^4 T^4}{120} C_2^{(1)}(\eta)|x|^4 + \frac{\Delta^2\pi^8 T^8}{28800} \left(C_4^{(1)}(\eta) + C_2^{(1)}(\eta) + C_0^{(1)}(\eta)\right)|x|^8 + \mathcal{O}(T^{12})\right) \tag{19}$$

where $|x|^2 = t^2 + \vec{x}^2$, $C_n^{(1)} \ldots$ are Gegenbauer polynomials, and $\eta = t/|x|$.
Our goal here is to calculate these coefficients but for general values of Δ and d.

We will solve our equation (8) by expanding in small $1/z_h = 4\,\pi\,T/d$

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} \left(\frac{z}{z_h}\right)^{nd} \quad , \quad \frac{1}{f(z)^2} = \sum_{n=0}^{\infty} (n+1) \left(\frac{z}{z_h}\right)^{nd} \tag{20}$$

and solve perturbatively (8) at fixed power in $1/z_h$:

$$\xi(z) = \xi_0(z) + \frac{\xi_1(z)}{{z_h}^d} + \frac{\xi_2(z)}{{z_h}^{2d}} + \ldots$$

The boundary conditions are always

$$z \to \infty \quad : \quad |\xi(z)| < \infty \tag{21}$$

$$z \to 0 \quad : \quad \xi(z) = z^{d-\Delta}\left(1 + \mathcal{O}(z)\right) + G_2(\omega, k) z^{\Delta}\left(1 + \mathcal{O}(z)\right) \tag{22}$$

and

$$G_2(\omega, k) = G_2^{(0)}(k)\ (1 + g_1(\omega/k) + g_2(\omega/k) + \ldots) \tag{23}$$

For sake of clearness, we present the computations slightly different from;[1] for more details we refer the reader to this reference and section 6 of the present paper.

3. CFT: order T^0

The equation to solve is (8) at $1/{z_h}^0$:

$$z^2\, \xi_0''(z) - (d-1)z\, \xi_0'(z) - \left((kz)^2 + \Delta(\Delta - d)\right) \xi_0(z) = 0 \tag{24}$$

The solution is ($\nu = \Delta - d/2$)

$$\xi_0(z) = z^{d/2}\ K_\nu(kz) \tag{25}$$

For small z

$$\xi_0(z) \propto z^{d/2} \left((kz)^{-\nu} + (kz)^{\nu}\right) \propto z^{d/2-\nu} + G_2(k) z^{d/2+\nu} \tag{26}$$

and from here we get

$$G_2^{(0)}(k) = k^{2\nu} \tag{27}$$

4. First correction to CFT: order T^d

The equation at the next order is

$$z^2\, \xi_1''(z) - (d-1)z\, \xi_1'(z) - \left((kz)^2 + \Delta(\Delta - d)\right) \xi_1(z) = \frac{1}{z_h^d} \hat{\mathcal{O}} \xi_0(z) \tag{28}$$

with the solution

$$\begin{aligned} \xi_1(z) \quad &= \quad z^{d/2} I_\nu(z) \times \int_\infty^z dz' \left(\left(k^2 + \omega^2\right) z'^{d+1} + \left(\nu^2 - d^2/4\right) z'^{d-1} \right) K_\nu(kz')^2 \\ &\xrightarrow{z\to 0} z^{d/2-\nu} \int_\infty^0 dz' \,(\text{integrand above}) \end{aligned} \tag{29}$$

The correction to the propagator results

$$g_1(\omega/k) \equiv \frac{\sqrt{\pi}}{4(kz_h)^d} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+3}{2}\right)} \frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma\left(1+\nu\right)\Gamma\left(1-\nu\right)} \nu \left(d \left(\frac{\omega}{k}\right)^2 - 1 \right) \tag{30}$$

which for $d = 4$ becomes

$$g_1(\omega/k) = \left(\frac{\pi T}{k}\right)^4 \frac{2}{15} \nu(\nu^2 - 1)(\nu^2 - 4) \left(4\, \frac{\omega^2}{k^2} - 1 \right) \tag{31}$$

Transforming to x-space we get

$$g_1(x) = \Delta \frac{\pi^4\, T^4\, |x|^4}{120} \left(4\, \frac{t^2}{|x|^2} - 1 \right) \tag{32}$$

This result coincides exactly with the large Δ result. As we will see, this is simply a coincidence, which will not be true anymore at the next order.

5. Second correction to CFT: order T^{2d}

The integrals here cannot be done analytically. We do them numerically. But we can check the known expressions in $d = 2$ where the results are known at all orders.[11] At T^4 it looks

$$g_2(\omega/k) = \left(\frac{2\pi T}{k}\right)^4 \frac{1}{90}\,\Delta(\Delta-1)(\Delta-2)\,\big(9\,\Delta\,(\Delta-2) - 12 + (5\,\Delta+2)\,(\Delta-3)\,(\Delta-4)\,(1-2\,(\omega/k)^2)^2\big) \tag{33}$$

Schematically

$$\text{second correction} \propto \gamma_0 + \gamma_2(\omega/k)^2 + \gamma_4(\omega/k)^4$$

where our $\gamma_{0,2,4}$ (single dots) are compared with the full result[11] on fig. 1.

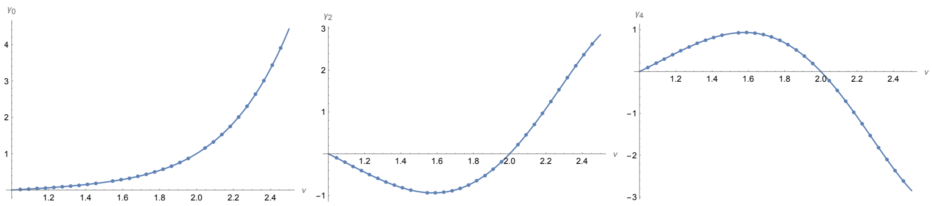

Fig. 1.

We do now the same thing for $d = 4$ (here we do not know the T^8 exact result) with the result on fig. 2. This is our prediction.

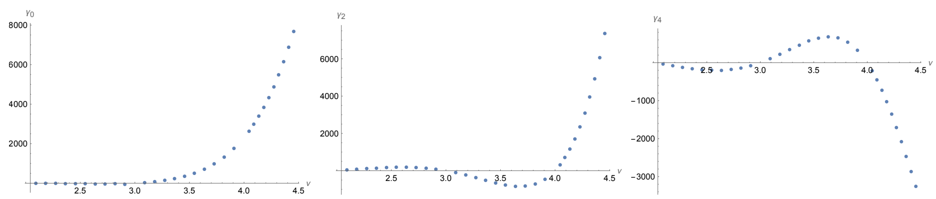

Fig. 2.

We compare our result (discrete points) with the large ν result[5] (full curve) on fig. 3. The large ν result works very badly for low ν. This is not that surprising but here we checked it.

The method we presented here is suited for low ν computations but is quite bad for large but finite ν: the two methods are thus complementary. However, we tried to analytically take the strict limit $\nu \to \infty$. The result for an arbitrary dimension

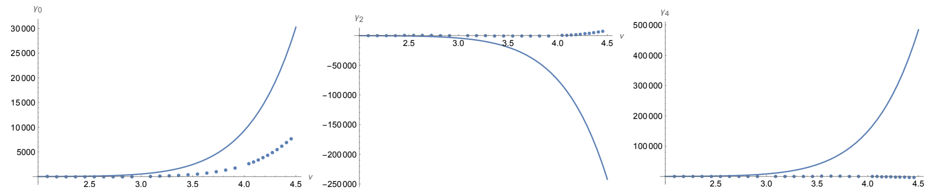

Fig. 3.

d is

$$g_2^{\text{large }\nu}(\omega/k) = \frac{2^{2d-5}\,\Gamma(d/2)^4}{(d+1)^2\,\Gamma(d)^2}\,\nu^{2d+2}\left(d\left(\frac{\omega}{k}\right)^2 - 1\right)^2 \quad . \tag{34}$$

This result coincides with that presented in[4] for $d = 4$ and with the large ν limit of (33) for $d = 2$. The key reason why the extreme large limit is doable is that one does not need to do it numerically, because analytical approximation of Bessel functions at large ν can be used. The proof of (34) is given in Appendix A. We notice that (34) confirms the exponentiation property of the propagator at large Δ.[4]

6. The $D1 - D5$ system

The method used in[1] to get the low temperature expansion of the progator in a CFT is indeed applicable outside the context of the AdS/CFT duality. All one needs is

- to have a parameter τ present analytically in the equation whose limit $\tau = 0$ is well-defined (in our case, the ratio temperature/frequency);
- to know the exact solutions in the limit $\tau = 0$.

As an exercise which shows the validity of the method in another context, we consider the system of $D1-D5$ branes in type IIB string theory,[13] whose background metric, after appropriate compactification, gives rise to the five-dimensional black hole geometry

$$ds^2 = g(r)^{-\frac{2}{3}}\,f(r)\,dt^2 + g(r)^{\frac{1}{3}}\left(\frac{dr^2}{f(r)} + r^2\,d\Omega_3^2\right) \tag{35}$$

where

$$g(r) = \left(1 + \left(\frac{r_1}{r}\right)^2\right)\left(1 + \left(\frac{r_5}{r}\right)^2\right) \qquad ; \qquad f(r) = 1 - \left(\frac{r_h}{r}\right)^2 \quad . \tag{36}$$

The constants r_1 and r_5 are related to the charges of the system, and we work in the dilute gas approximation where $r_h \ll r_1, r_5$. The position of the horizon r_h is associated to the Hawking temperature by $T = \frac{r_h}{2\,\pi\,L^2}$, where we have introduced the scale $L \equiv \sqrt{r_1\,r_5}$.

We would like to study the absorption of free massless scalars by this black hole, using our method and the exact result obtained in,[14] as well as the low temperature expansion of such result computed in[2] with the Langer-Olver expansion. If we write: $\Phi(t,r,\Omega) = e^{-i\,\omega\,t}\, Y_{\nu-1}(\Omega)\,\phi(r)$, where $Y_{\nu-1}(\Omega)$ are generic harmonics in S^3 with laplacian eigenvalue equal to ν^2-1, the radial equation to solve is

$$\frac{f(r)}{r^3}\,\partial_r\left(r^3\,f(r)\,\partial_r\phi(r)\right) + \left(\omega^2\,g(r) - (\nu^2-1)\,\frac{f(r)}{r^2}\right)\phi(r) = 0 \tag{37}$$

In the low energy limit $\omega\, r_i \ll 1, i = 1,5$, we can solve (37) at good approximation as follows. First, we consider the far region $r \gg r_1, r_5$ (then $f(r) = g(r) \simeq 1$). The temperature-independent general solution is given in terms of Hankel functions[a] by

$$\phi^{(far)}(\omega\, r) = \alpha_{ref}\,\frac{H^{(1)}_\nu(\omega\, r)}{\omega\, r} + \alpha_{inc}\,\frac{H^{(2)}_\nu(\omega\, r)}{\omega\, r} \tag{40}$$

Second, we consider the near region $\omega\, r \ll 1$. By introducing the variable z and $h(z)$ according to

$$\phi^{(near)}(r) \equiv \sqrt{\frac{z}{f(r)}}\; h(z) \qquad ; \qquad z \equiv \frac{L}{r} \quad . \tag{41}$$

equation (37) can be recast into

$$\left(\frac{d^2}{dz^2} - U_T(z)\right) h(z) = 0 \qquad , \tag{42}$$

where

$$U_T(z) = \frac{\nu^2+1}{z^2\, f(L/z)} - \frac{(\omega\, L)^2}{f(L/z)^2} - \frac{1}{z^2\, f(L/z)^2} - \frac{1}{4\, z^2} \quad . \tag{43}$$

We note that $f(L/r) = 1 - z^2/{z_h}^2$, where $z_h = L/r_h$. The temperature is

$$T = \frac{1}{2\,\pi\, L\, z_h} \quad . \tag{44}$$

Even if equation (37) had no exact solution, it is clear that in the low energy limit there exists a region $r_1, r_5 \ll r \ll 1/\omega$ where we can match (40) and (41). Now, to study absorption by the black hole, we must impose ingoing boundary conditions at the horizon to (41), i.e.

$$\phi^{(near)}(r) \quad \overset{z\to\infty}{\sim} \quad e^{+i\,\omega\, L\, z} \tag{45}$$

[a]The Hankel functions are convenient in our context, due to the simple asymptotics as right and left waves for large argument

$$H^{(1,2)}_\nu(x) \quad \overset{x\to\infty}{\sim} \quad \sqrt{\frac{2}{\pi\, x}}\; e^{\pm i\,(x-(\nu+\frac{1}{2})\,\frac{\pi}{2})} \quad . \tag{38}$$

For low argument the behavior is

$$H^{(1,2)}_\nu(x) \quad \overset{x\to 0}{\longrightarrow} \quad \pm\frac{\Gamma(\nu)}{i\,\pi}\left(\frac{x}{2}\right)^{-\nu} + \cdots + \left(\frac{1}{\Gamma(1+\nu)} \pm \cos(\pi\,\nu)\,\frac{\Gamma(-\nu)}{i\,\pi}\right)\left(\frac{x}{2}\right)^{\nu} + \ldots \tag{39}$$

Admitting that (42) has no exact solution for finite temperature, we try our method to get an expansion Then we start with the lower order, zero temperature case. The solution to (42) obeying (45) is

$$h^{(0)}(z) = \alpha_{abs} \sqrt{\frac{\pi}{2}\,\omega\,L\,z}\; H_\nu^{(1)}(\omega\,L\,z) \quad . \tag{46}$$

By matching this near solution for $\omega\,L\,z \ll 1$ with the far solution (40) for $\omega\,r \ll 1$ we get

$$\begin{aligned} \frac{\alpha_{ref}}{\alpha_{abs}} &= -\frac{i}{\sqrt{\pi}} \left(\Gamma(\nu)\,\Gamma(1+\nu) \left(\frac{\omega\,L}{2}\right)^{\frac{3}{2}-2\,\nu} + (\nu \leftrightarrow -\nu) \right) \\ \frac{\alpha_{inc}}{\alpha_{abs}} &= -\frac{i}{\sqrt{\pi}}\, e^{-i\,\pi\,\nu} \left(e^{i\,\pi\,\nu}\,\Gamma(\nu)\,\Gamma(1+\nu) \left(\frac{\omega\,L}{2}\right)^{\frac{3}{2}-2\,\nu} + (\nu \leftrightarrow -\nu) \right) \end{aligned} \tag{47}$$

Now, the conserved flux $j = j(r)$

$$j(r) \equiv |\Im\left(\phi(r)^*\,r^3\,f(r)\,\partial_r\phi(r)\right)| \quad , \quad \partial_r j(r) = 0 \tag{48}$$

can be computed at large z from (41) and (46)

$$j = j_{abs} = \omega\,L^3\,|\alpha_{abs}|^2 \tag{49}$$

On the other hand, the fluxes associated to the incident and reflected waves in (40) can be computed at large r to give

$$j_{inc} = \frac{2}{\pi\,\omega^2}\,|\alpha_{inc}|^2 \quad ; \quad j_{ref} = \frac{2}{\pi\,\omega^2}\,|\alpha_{ref}|^2 \tag{50}$$

Conservation of the flux (48) implies that $j_{inc} = j_{ref} + j_{abs}$, as can be checked by using (47). In particular, the probability of absorption (transmission coefficient in the quantum mechanics language) is

$$\tau^{(0)} \equiv \frac{j_{abs}}{j_{inc}} = \frac{\pi}{2}\,(\omega\,L)^3\,\frac{|\alpha_{abs}|^2}{|\alpha_{inc}|^2} \overset{\omega\,L\ll 1}{\simeq} \frac{4\,\pi^2}{\Gamma(\nu)^2\,\Gamma(1+\nu)^2} \left(\frac{\omega\,L}{2}\right)^{4\,\nu} \tag{51}$$

which coincides with equation (4.6) of.[2]

To get the low temperature corrections, we follow[1] and rewrite (42) as a first order system for a basis of solutions $h_\mp(z)$ obeying ingoing (the relevant one for us, see (45)) and outgoing boundary conditions at the horizon

$$\frac{d}{dz}\mathbf{w}\,(h_+, h_-; z) = \mathbf{A}(z)\;\mathbf{w}\,(h_+, h_-; z) \quad . \tag{52}$$

where $\mathbf{w}(f, g; z)$ is the wronskian matrix of the functions $f(z), g(z)$. The potential $\mathbf{A}(z)$ can be split into a zero-temperature and a temperature-dependent part as

$$\mathbf{A}(z) = \mathbf{A}^{(0)}(z) + \mathbf{A}_T(z) \quad . \tag{53}$$

The $T = 0$ matrix is [b]

$$\mathbf{A}^{(0)}(z) = \begin{pmatrix} 0 & 1 \\ (\omega\, L)^2 \left(-1 + \frac{\nu^2 - \frac{1}{4}}{q^2}\right) & 0 \end{pmatrix} \tag{54}$$

where we have introduced the variable $q \equiv \omega\, L\, z$, while that the temperature-dependent part can be expanded in powers of $1/{q_h}^2 = \left(\frac{2\,\pi\, T}{\omega}\right)^2$ as

$$\mathbf{A_T(z)} = (\omega\, L)^2\; u_T(q) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad u_T(q) = \sum_{m \in \mathbb{N}^+} \frac{u^{(m)}(q)}{(q_h)^{2\,m}} \quad , \tag{55}$$

where

$$u^{(m)}(q) = (\nu^2 - m)\, q^{2\,m-2} - (1+m)\, q^{2\,m} \quad . \tag{56}$$

At this point, we observe that the problem posed by equations (52)-(56) is exactly that studied in[1] if we:

- identify the ν here with $\nu = \Delta - d/2$ there;
- put $d = 2$;
- replace k there with $i\,\omega\, L$;
- put spatial momentum $\vec{q} = \vec{0}$ there.

Furthermore, the smoothness boundary conditions considered in[1] are naturely mapped to the ingoing boundary conditions considered here. Therefore, we can use the results in[1] replacing parameters according to the mapping expresed above. For example, if we expand the transmision coefficient

$$\tau^{(T)} = \tau^{(0)} \left(1 + \frac{\Delta\tau_1}{{q_h}^2} + \dots\right) \quad , \tag{57}$$

from eq. 30 the first correction results

$$\Delta\tau_1 = -\frac{\nu}{3}\,(\nu^2 - 1) \quad , \tag{58}$$

which coincides with equation (4.5) of.[2]

7. Conclusions

- Thermal CFT propagators has been known so far in expansion of T/k, T/ω only for $\Delta \to \infty$
- the recent exact solution of the problem is not appropriate for such expansion
- we performed such a calculation analytically to order T^d and numerically to order T^{2d} for small values of Δ finding agreement with the exactly known all order $d = 2$ result

[b] We point out a misprint in equation (2.17) of,[1] where the factor $(\omega\, L)^2$ is erroneously extended to the (12)-element of the matrix (54).

- we predict numerically the behaviour for $d = 4$ at order T^{2d} and find that the large ν approximation works badly for low ν
- we study the D1-D5 system and are able to reproduce correctly the known result from the literature[2]
- we prove in Appendix A our eq. 34, which was only conjectured in.[1]

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Appendix A. Proof of the large Δ limit.

In this appendix we will make heavy use of definitions introduced in section 4 and appendix A of reference.[1]

To get the large Δ result, we will need the large order expansions of the Bessel functions,[12]

$$\begin{aligned} I_\nu(\nu z) &\overset{\nu\to+\infty}{\sim} \frac{e^{+\nu\,\eta(z)}}{(2\pi\nu)^{\frac{1}{2}}\,(1+z^2)^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{U_k(p(z))}{\nu^k} \\ K_\nu(\nu z) &\overset{\nu\to+\infty}{\sim} \left(\frac{\pi}{2\nu}\right)^{\frac{1}{2}} \frac{e^{-\nu\,\eta(z)}}{(1+z^2)^{\frac{1}{4}}} \sum_{k=0}^{\infty} (-)^k \frac{U_k(p(z))}{\nu^k} \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} \eta(z) &= (1+z^2)^{\frac{1}{2}} + \ln\frac{z}{1+(1+z^2)^{\frac{1}{2}}} = h(z) + \ln z \\ p(z) &= (1+z^2)^{-\frac{1}{2}} \end{aligned} \tag{A.2}$$

We note that, with z running from 0 to ∞,

- $p(z) \in (0,1]$ is a monotonic decreasing and bounded positive function;
- $\eta(z) \in (-\infty,+\infty)$ is monotonic increasing, having a zero $\eta(z_0) = 0$ at $z_0 \sim 0.662743$;
- $h(z)$ is a is monotonic increasing and positive function, with: $h(z) \overset{z\to 0}{\longrightarrow} h(0) + \frac{z^2}{4}$, $h(0) = 1 - \ln 2 = 0.306853$, $h(z_0) = -\ln z_0 = 0.411368$, and $h(z) \overset{z\to\infty}{\longrightarrow} z$.

Furthermore, the coefficients $U_k(x)$ are polynomials in x of degree $3k$. This guarantees that $U_k(p(z))$ is bounded for any z, and then in the large ν limit we can retain safely the leading terms in (A.1), that is, if we work at leading order we only need to know that $U_0(x) = 1$ (for higher orders see[12] and references therein).

It is convenient to start from

$$\begin{aligned} g_{rs}(q) \equiv \frac{1}{2}\,\alpha_r\,\alpha_s + \frac{\alpha_r}{8\,\nu}\,(F_s(q) - 2\,\nu\,\alpha_s) + \frac{\alpha_s}{8\,\nu}\,(F_r(q) - 2\,\nu\,\alpha_r) \\ + \frac{1}{4\,\nu}\int_\infty^q dz\ \left(W(F_r, G_s; z) + W(F_s, G_r; z)\right) \end{aligned} \tag{A.3}$$

that after an integration by parts can be written in the form

$$\begin{aligned} g_{rs}(q) - \frac{1}{2}\,\alpha_r\,\alpha_s = \frac{\alpha_r}{8\,\nu}\,(F_s(q) - 2\,\nu\,\alpha_s) + \frac{1}{4\,\nu}\,F_r(q)\,G_s(q) \\ + \frac{1}{2}\int_{\frac{q}{\nu}}^\infty dz\ G_s(\nu\,z)\,F_r'(\nu\,z) + (r \leftrightarrow s) \end{aligned} \tag{A.4}$$

From the low q limit

$$g_{rs}(q) \quad \xrightarrow{q\to 0} \quad g_{rs} + \frac{1}{4\,\nu}\left(\frac{\gamma_r^{(-)}}{r^2} + \frac{\gamma_s^{(-)}}{s^2}\right)\frac{q^{-2\,\nu+r+s}}{-2\,\nu + r + s}\,(1 + o(q)) \tag{A.5}$$

we see that the terms in the first line of (A.4) give no contribution to $\hat{g}_{rs}$ [c], so we remain with

$$g_{rs} - \frac{1}{2}\,\alpha_r\,\alpha_s = \frac{1}{2}\int_{\frac{q}{\nu}}^\infty dz\ G_s(\nu\,z)\,F_r'(\nu\,z)\Bigg|_{\hat{g}_{rs}} + (r \leftrightarrow s) \tag{A.6}$$

Now, because we are interested in the large Δ limit, we need the large ν limit of the integrand. Since now on we assume this leading order limit, throwing away any correction of order $1/\nu$ and beyond.

It is convenient to introduce the functions

$$\begin{aligned} L_r(\nu; z) &\equiv -\int_z^\infty dx\,\tfrac{x^{r-1}}{\sqrt{1+x^2}}\,e^{-2\,\nu\,\eta(x)} \quad \longrightarrow \quad L_r'(\nu; z) = \tfrac{z^{r-1}}{\sqrt{1+z^2}}\,e^{-2\,\nu\,\eta(z)} \\ S_r(z) &\equiv \tfrac{z^r}{r}\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{r}{2}; 1 + \tfrac{r}{2}; -z^2\right) + \tfrac{\alpha_r}{\nu^{r-1}} \quad \longrightarrow \quad S_r'(z) = \tfrac{z^{r-1}}{\sqrt{1+z^2}} \end{aligned} \tag{A.7}$$

We note that $S_r(z)$ is a monotonic increasing and well behaved function for finite z, and independent of ν in the large ν limit with behaviors

$$S_r(z) \quad \xrightarrow{z\to 0} \quad \frac{\alpha_r}{\nu^{r-1}} \qquad ; \qquad S_r(z) \quad \xrightarrow{z\to\infty} \quad \frac{z^{r-1}}{r-1} \tag{A.8}$$

[c] What is more, in view of (A.5), the terms of the form $q^{-2\,\nu+s}$, $q^{-2\,\nu+r}$ and $q^{-4\,\nu+r+s}$ that appear in the first line can not exist on the r.h.s. of (A.4); they must cancel with terms coming from the second line.

On the other hand, $L_r(\nu;z)$ has the limits (see (A.11))

$$\begin{aligned} L_r(\nu;z) &\overset{z\to 0}{\longrightarrow} -\frac{1}{\sin(\pi\nu)}\left(\frac{\alpha_r}{\nu^{r-1}}+\frac{\gamma_r^{(-)}}{2\,r\,\nu^{2\nu}}\,z^{-2\nu+r}\right) \\ L_r(\nu;z) &\overset{z\to\infty}{\longrightarrow} -\frac{z^{r-2}}{2\,\nu}\,e^{-2\,\nu\,z} \end{aligned} \tag{A.9}$$

By using (A.1), it is straight to get,

$$\begin{aligned} F_r'(q)\big|_{q=\nu\,z} &\overset{\nu\to+\infty}{\sim} -2\,\nu^{r-1}\,\sin(\pi\nu)\,L_r'(\nu;z)\,\,(1+o(1/\nu)) \\ G_r'(q)\big|_{q=\nu\,z} &\overset{\nu\to+\infty}{\sim} \frac{\nu^{r-2}}{2}\left(\frac{z^{r-1}}{(1+z^2)^{\frac{1}{2}}}+\sin(\pi\nu)\,L_r'(z)\right)\,(1+o(1/\nu)) \end{aligned} \tag{A.10}$$

By integrating we get,

$$\begin{aligned} F_r(\nu\,z) &\overset{\nu\to+\infty}{\sim} -2\,\nu^{r}\,\sin(\pi\nu)\,L_r(\nu;z)\,\,(1+o(1/\nu)) \\ G_r(\nu\,z) &\overset{\nu\to+\infty}{\sim} \frac{\nu^{r-1}}{2}\,\,(S_r(z)+\sin(\pi\nu)\,L_r(\nu;z))\,\,(1+o(1/\nu)) \end{aligned} \tag{A.11}$$

where the integration constants were fixed by imposing that

$$F_r(\nu\,z) \overset{z\to\infty}{\longrightarrow} 0 \quad ; \quad G_r(\nu\,z) \overset{z\to 0}{\longrightarrow} -\frac{\gamma_r^{(-)}}{4\,\nu\,r}\,(\nu\,z)^{-2\,\nu+r} \tag{A.12}$$

hold, and we have used the definition

$$\frac{\alpha_r}{\nu^{r-1}} \overset{\nu\to+\infty}{\sim} \sin(\pi\nu)\int_0^\infty dz\,z^{r-1}\,\frac{e^{-2\,\nu\,\eta(z)}}{\sqrt{1+z^2}}\bigg|_{substr.} = -\frac{1}{r\,\sqrt{\pi}}\,\Gamma\left(1+\frac{r}{2}\right)\,\Gamma\left(\frac{1}{2}-\frac{r}{2}\right) \tag{A.13}$$

By inserting (A.10) and (A.11) in (A.6), and taking into account (A.12), we obtain

$$\begin{aligned} \nu^{-r-s+2}\left(g_{rs}-\frac{1}{2}\,\alpha_r\,\alpha_s\right) &= -\frac{\sin(\pi\nu)}{2}\int_{\frac{q}{\nu}}^\infty\left(\frac{z^s}{s}\,{}_2F_1\left(\frac{1}{2},\frac{s}{2};1+\frac{s}{2};-z^2\right)\right. \\ &+\left.\frac{\alpha_s}{\nu^{s-1}}+\sin(\pi\nu)\,L_s(\nu;z)\right)\,L_r'(\nu;z)\bigg|_{g_{rs}}+(r\leftrightarrow s) \\ &= -\frac{\sin(\pi\nu)}{2}\int_{\frac{q}{\nu}}^\infty\left(\frac{z^s}{s}\,{}_2F_1\left(\frac{1}{2},\frac{s}{2};1+\frac{s}{2};-z^2\right)\,L_r'(\nu;z)+(r\leftrightarrow s)\right)\bigg|_{g_{rs}} \\ &-\frac{1}{2}\left(\frac{\alpha_r}{\nu^{r-1}}+\sin(\pi\nu)\,L_r(\nu;z)\right)\left(\frac{\alpha_s}{\nu^{s-1}}+\sin(\pi\nu)\,L_s(\nu;z)\right)\bigg|_{\frac{q}{\nu}\,;\,g_{rs}}^\infty \\ &= -\frac{\sin(\pi\nu)}{2}\left(\frac{1}{s}J_{rs}\left(\nu;\frac{q}{\nu}\right)+\frac{1}{r}J_{sr}\left(\nu;\frac{q}{\nu}\right)\right)\bigg|_{g_{rs}} \end{aligned} \tag{A.14}$$

where in the last line we used (A.9), and we defined

$$\begin{aligned} J_{rs}(\nu;z) &= \int_z^\infty dx\,x^{-2\,\nu+r+s-1}\,f_s(\nu;x) \\ f_s(\nu;x) &\equiv {}_2F_1\left(\frac{1}{2},\frac{s}{2};1+\frac{s}{2};-z^2\right)\frac{e^{-2\,\nu\,h(x)}}{\sqrt{1+x^2}} \end{aligned} \tag{A.15}$$

We are going to show that

$$\lim_{\nu\to\infty} J_{rs}\left(\nu; \frac{q}{\nu}\right)\Big|_{g_{rs}} = 0 \tag{A.16}$$

To this end, with the help of the formula obtained by integrating by parts succesively $m+1$ times,

$$\int_{z_1}^{z_2} dx\, x^{a-1}\, f(x) = \sum_{k=0}^{m} \frac{(-)^k}{(a)_{k+1}}\, x^{a+k}\, f^{(k)}(x)\Bigg|_{z_1}^{z_2} + \frac{(-)^{m+1}}{(a)_{m+1}} \int_{z_1}^{z_2} dx\, x^{a+m}\, f^{(m+1)}(x) \tag{A.17}$$

and valid for arbitrary $f(x)$ and constants (a, z_1, z_2), we can decompose $J_{rs}(\nu; z)$ as a sum of finite and divergent terms in $z = 0$ as follows:

$$J_{rs}(\nu; z) = J^{div}_{rs}(\nu; z) + J^{fin}_{rs}(\nu; z) \tag{A.18}$$

where

$$\begin{aligned} J^{div}_{rs}(\nu; z) &= -\sum_{k=0}^{[2\,\nu]-r-s} \frac{(-)^k}{(-2\,\nu + r + s)_{k+1}}\, z^{-2\,\nu+r+s+k}\, f^{(k)}_s(\nu; z) \\ J^{fin}_{rs}(\nu; z) &= \frac{(-)^{[2\,\nu]-r-s+1}}{(-2\,\nu + r + s)_{[2\,\nu]-r-s+1}} \int_z^{\infty} dx\, x^{-2\,\nu+[2\,\nu]}\, f^{([2\,\nu]-r-s+1)}_s(\nu; x) \end{aligned} \tag{A.19}$$

and the Pochhammer symbols relevant are

$$(-2\,\nu + r)_{k+1} = (-)^{k+1}\, \frac{\Gamma\,(2\,\nu - r + 1)}{\Gamma\,(2\,\nu - r - k)} \tag{A.20}$$

The derivatives of $f_s(\nu; z)$ have the form

$$f^{(k)}_s(\nu; z) = \frac{f_s(\nu; z)}{\left(1+\sqrt{1+z^2}\right)^k\, (1+z^2)^k}\, P_k(\nu; z) \tag{A.21}$$

where $P_k(\nu; z)$ are polinomials of degree k in ν, with coefficients well-behaved functions of z for finite z. We will need only the highest term

$$P_k(\nu; z) = \left(-2\,z\,(1+z^2)\right)^k\, \nu^k\, \left(1 + o\left(\frac{1}{\nu}\right)\right) \tag{A.22}$$

So, at the leading order we are working, we can approximate (A.21) by

$$f^{(k)}_s(\nu; z) = f_s(\nu; z)\, \left(-\frac{2\,\nu\, z}{1+\sqrt{1+z^2}}\right)^k \tag{A.23}$$

With (A.20) and (A.23) we write (A.19) as

$$\begin{aligned} J^{div}_{rs}(\nu; z) &= -f^{(k)}_s(\nu; z)\, \frac{z^{-2\,\nu+r+s}}{2\,\nu} \sum_{k=0}^{[2\,\nu]-r-s} \frac{\left(1-\sqrt{1+z^2}\right)^k}{\prod_{l=0}^{k}\left(1-\frac{l+r+s}{2\,\nu}\right)} \\ J^{fin}_{rs}(\nu; z) &= (-)^{[2\,\nu]-r-s}\, \frac{\Gamma(\epsilon_{2\,\nu})}{\sqrt{2\,\pi}}\, (2\,\nu)^{\frac{1}{2}-\epsilon_{2\,\nu}} \end{aligned}$$

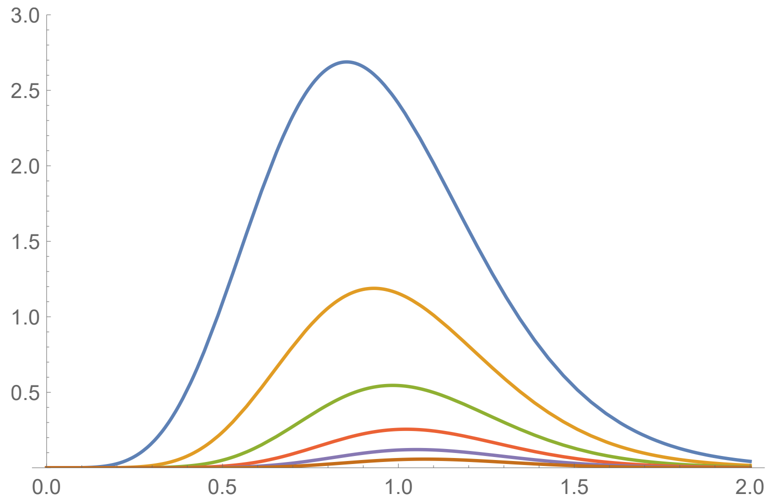

Fig. 4. The integrand of $J^{fin}_{rs}(\nu;0)$ in (A.24) (including the factor $(2\,\nu)^{\frac{1}{2}-\epsilon_{2\,\nu}}$) is plotted as a function of z, for $r=4, s=2$ and $\nu=5$ (*blue*), $\ldots, 10$ (*orange*).

$$\times \int_z^\infty dx\; x^{r+s-1}\left(\sqrt{1+x^2}-1\right)^{1-\epsilon_{2\,\nu}-r-s}\, {}_2F_1\left(\frac{1}{2},\frac{s}{2};1+\frac{s}{2};-x^2\right)\frac{e^{-2\,\nu\,\bar{h}(x)}}{\sqrt{1+x^2}} \tag{A.24}$$

where $\epsilon_{2\,\nu} \equiv 2\,\nu - [2\,\nu] \in (0,1)$, and we have introduced

$$\bar{h}(x) \equiv \sqrt{1+x^2} - 1 - \ln x \tag{A.25}$$

Now, it is clear from (A.24) that the divergent part does not contribute to $\hat{g}_{rs}$

$$J^{div}_{rs}\left(\nu;\frac{q}{\nu}\right)\Big|_{\hat{g}_{rs}} = 0 \tag{A.26}$$

Then the contribution is just given by

$$J_{rs}\left(\nu;\frac{q}{\nu}\right)\Big|_{\hat{g}_{rs}} = J^{fin}_{rs}(\nu;0) \tag{A.27}$$

But, since $\bar{h}(x) > 0$ for any $x > 0$, it follows that the integrand in (A.24) at any fixed x goes to zero for very large ν, and then

$$J^{fin}_{rs}(\nu;0) \quad \stackrel{\nu\to\infty}{\longrightarrow} \quad 0 \tag{A.28}$$

This fact can be confirmed numerically by considering the plots of the integrand presented in figure 4, where it is clear that the area under the curve tends to zero for increasing values of ν.

A somewhat more direct proof is a straightforward saddle point analysis of the integrand. Assuming that the saddle point has the expansion

$$x_m(\nu) = x_m^{(0)} + \frac{x_m^{(1)}}{\nu} + \ldots \quad , \tag{A.29}$$

the leading order minimizes $\bar{h}(x)$, $\bar{h}'(x_m^{(0)}) = 0 \to x_m^{(0)} \sim 1.272$, and then we get

$$J_{rs}^{fin}(\nu;0) \sim \nu^{-\frac{1}{2}}\, e^{-2\,\bar{h}\left(x_m^{(0)}\right)\nu} \quad \overset{\nu\to\infty}{\longrightarrow} \quad 0 \tag{A.30}$$

From (A.27) and (A.28) the assertion (A.16) is proved. From (A.14), this means that

$$g_{rs} \quad \overset{\nu\to\infty}{\longrightarrow} \quad \nu^{r+s-2}\,\frac{1}{2}\,\frac{\alpha_r}{\nu^{r-1}}\,\frac{\alpha_s}{\nu^{s-1}} \tag{A.31}$$

where $\frac{\alpha_r}{\nu^{r-1}}$ is given in (A.13). This is the key relation; by using it in equation (4.4) of,[1] (34) follows.

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