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USING DIGRAPHS TO COMPUTE DETERMINANT, PERMANENT, AND DRAZIN INVERSE OF CIRCULANT MATRICES WITH TWO PARAMETERS

ANDRÉS M. ENCINAS, DANIEL A. JAUME, CRISTIAN PANELO, AND DENIS E. VIDELA

Abstract. This work presents closed formulas for the determinant, permanent, inverse, and Drazin inverse of circulant matrices with two non-zero coefficients.

1. Introduction

Circulant matrices appear in many applications, for example, to approximate the finite difference of elliptic equations with periodic boundary, and to approximate periodic functions with splines. Circulant matrices play an important role in coding theory and statistics. The standard reference for this topic is [9].

Among the main problems associated with circulant matrices are those of determining invertibility conditions and computing their Drazin inverse. These problems have been widely treated in the literature by using the primitive n-th root of unity and some polynomial associated with circulant matrices, see [9], [8], and [21]. There exist some classical and well-known results that enable us to solve almost everything we would raise about the inverse or Drazin inverse of circulant matrices. However, when dealing with specific families of circulant matrices, these classical results yield unwieldy formulas. Thus, it is interesting to seek alternative descriptions, and indeed, many papers have been devoted to this question. Direct computation for the inverse of some circulant matrices has been proposed in many works, see for example [11], [21], [25], [14], [8], and [17] (in chronological order).

Besides, the combinatorial structure of circulant matrices also has deserved attention. Graphs whose adjacency matrix is circulant, specially those with integral spectrum, have been studied in many works; see, for example, [22], [4], [12], [16], and [20].

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In this work, we delve into the combinatorial structure of circulating matrices with only two non-null generators, by considering the digraphs associated with this kind of matrices. Therefore, we complete a previous work of some of the authors, see [10], where only a specific class of these matrices was considered.

We use digraphs in the present work; for all the graph-theoretic notions not explicitly defined here, the reader is referred to [2].

We warn the reader that, as is usual in circulant matrices, our matrix indexes and permutations start at zero. Hence, permutations in this work are bijections over $\{0, \dots, n-1\}$, and if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

then the set of indexes of A is $\{0, 1\}$. So $A_{0\,0} = 1$ and $A_{1\,0} = 3$. This is also why in the present work $[n]$ denotes the set $\{0, \dots, n-1\}$ instead of $\{1, \dots, n\}$.

For permutations, we use the cyclic notation: $(0\ \ 2\ \ 1\ \ 4)(5\ \ 6)$ is the permutation (in row notation) $(2\ \ 4\ \ 1\ \ 3\ \ 0\ \ 6\ \ 5)$, see [18]. Given a permutation α of $[n]$, we denote by P_α the $n \times n$ matrix defined by $(P_\alpha)_{\alpha(j)\,j} = 1$ and 0 otherwise. The matrix P_α is known as the permutation matrix associated to α. It is well-known that $P_\alpha^{-1} = P_\alpha^T$. The assignation $\alpha \mapsto P_\alpha$ from the symmetric group $\mathbb{S}_n$ to the general lineal group $\mathrm{GL}(n)$ is a representation of $\mathbb{S}_n$, i.e., $P_{\alpha\beta} = P_\alpha P_\beta$ where product is composition. The cycle type of a permutation α is an expression of the form

$$(1^{m_1}, 2^{m_2}, \dots, n^{m_n}),$$

where m_k is the number of cycles of length k in α. It is well-known that the conjugacy classes of $\mathbb{S}_n$ are determined by the cycle type, see [19, p. 3]. Thus, α and β are conjugated (i.e., there exists a permutation σ such that $\sigma\alpha\sigma^{-1} = \beta$) if and only if α and β have the same cycle type.

We use the matrix associated to the permutation

$$\tau_n = (n-1\ \ \ n-2\ \cdots\ 2\ \ 1\ \ 0)$$

many times along this work, so instead of P_{τ_n} we just write P_n.

Notice that, for $k \in \mathbb{Z}$, we have that $P_n^0 = I_n$, $P_n^k = P_n^{k \bmod n} = P_{\tau_n^k}$, $\det(P_n) = (-1)^{n-1}$, and $\left(P_n^k\right)^{-1} = P_n^{n-k}$. Moreover, $\tau_n^k(i) = i - k \bmod n$.

A matrix $C = (c_{i\,j})$ is called *circulant* with parameters $c_0, c_1, \dots, c_{n-1}$ if

$$C = \begin{bmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_{n-1} & c_0 & \dots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \dots & c_0 \end{bmatrix} = \mathrm{Circ}(c_0, \dots, c_{n-1})$$

or, equivalently,

$$c_{i\,j} = c_{j-i \bmod n}.$$

We have that

$$\mathrm{Circ}(c_0, \dots, c_{n-1}) = c_0 I_n + \cdots + c_{n-1} P_n^{n-1}.$$

Let A be an $m \times n$ matrix and let $S \subseteq [m]$, $T \subseteq [n]$. The submatrix of A obtained by deleting the rows in S and the columns in T is denoted by $A(S|T)$.

The paper is organized as follows. In Section 2, we present our main idea about how to work with circulant matrices with just two non-zero parameters: untangling the associated digraphs. In Section 3, we explicitly find the matrices that untangle the digraphs associated with our matrices. In Section 4, we find explicit formulas for the determinant and permanent of circulant matrices with two non-zero parameters. In Section 5, we give an explicit formula for the inverse of non-singular circulant matrices with two non-zero coefficients. In Section 6, we give an explicit formula for the Drazin inverse of singular circulant matrices with two non-zero parameters. Finally, in Sections 7, 8, and 9 we generalize the previous results for block circulant matrices.

2. Untangling the skein: two key permutations

Let n, s_1, s_2, and s be non-negative integers such that $0 \leqslant s_1 < s_2 < n$ and $0 < s < n$. Let a, b be non-zero complex numbers. We are going to work with the following type of $n \times n$ circulant matrices:

$$aP_n^{s_1} + bP_n^{s_2}.$$

We call it a circulant matrix with two parameters.

There are $\binom{n}{2}$ forms of circulant matrices with two parameters. Since $aP_n^{s_1} + bP_n^{s_2} = P_n^{s_1}\left(aI_n + bP_n^{s_2-s_1}\right)$ and $aI_n + bP_n^{n-s_1} = \left(aI_n + bP_n^{s_1}\right)^T$, it looks like we only need to understand $(n-1)/2$ of them. As we will see later, it is enough with just one of them: $aI_n + bP_n$.

As usual in number theory, the greatest common divisor of two integers n and s is denoted by (n, s). We find the following notation useful:

$$n\backslash s := \tfrac{n}{(n,s)}.$$

We read it "n without s". Notice that $s\,(n\backslash s) = n\,(s\backslash n) = [n, s]$, where $[n, s]$ is the lowest common multiple of n and s. The remainder of the integer division of x by n is denoted by $x \bmod n$.

Let $A = (a_{i\,j})$ be a matrix of order n. It is usual to associate a digraph of order n to A, denoted by $D(A)$ (see [6]). The vertices of $D(A)$ are labeled by the integers $\{0, 1, \ldots, n-1\}$. If $a_{i\,j} \neq 0$, there is an arc from the vertex i to the vertex j of weight $a_{i\,j}$. When σ is a permutation, we just write $D(\sigma)$ instead of $D\left(P_\sigma\right)$, and we talk about digraphs and permutations associated.

In this section, we show that $aP_n^{s_1} + bP_n^{s_2}$ is associated with a digraph that has $(n, s_2 - s_1)$ main cycles of length $n\backslash(s_2 - s_1)$ each, and we give two permutations that allow us to untangle this digraph, in a sense that will be clear later.

The first untangle is given by the permutation associated with the matrix $P_n^{s_1}$, because $aP_n^{s_1} + bP_n^{s_2} = P_n^{s_1}\left(aI_n + bP_n^{s_2-s_1}\right)$. Therefore, we only need to study digraphs associated to matrices of the form $aI_n + bP_n^s$. In Figures 1 and 2 it can be seen how this permutation untangles these digraphs.

Notice that $D\left(aI_n + bP_n^s\right)$ has (n, s) connected components of order $n\backslash s$, and each of them has a main spanning cycle. Notice that for $u, v \in [n]$, in $D\left(aI_n + bP_n^s\right)$ we have a directed arc from u to v if and only if $v - u = s \bmod n$.

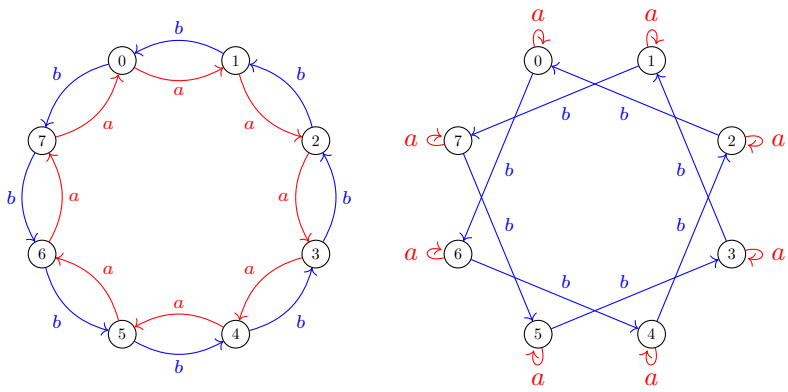


FIGURE 1. On the left $D\left(aP_8 + bP_8^7\right)$, and on the right $D\left(aI_8 + bP_8^6\right)$.

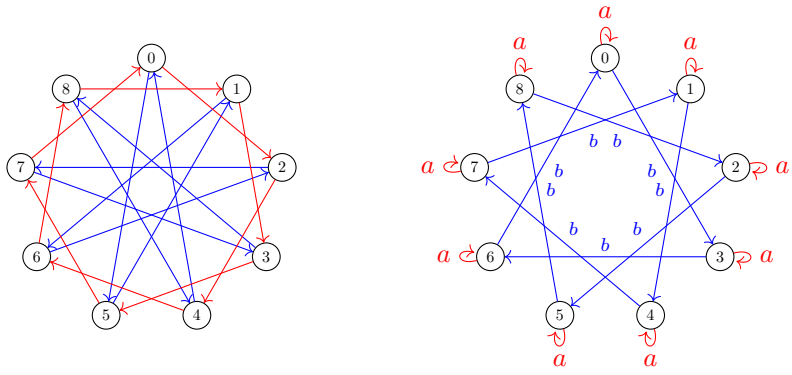


FIGURE 2. On the left $D\left(aP_9^2 + bP_9^5\right)$ (b–arcs in blue and a–arcs in red), and on the right $D\left(aI_9 + bP_9^3\right)$.

Given n and s positive integers such that $0 < s < n$, the (n, s)-canonical permutation is

$$\nu_{n,s} := \prod_{i=1}^{(n,s)} \nu_{n,s,i},$$

where

$$\nu_{n,s,i} := (i\,(n\backslash s)-1 \quad i\,(n\backslash s)-2 \;\cdots\; i\,(n\backslash s)-((n\backslash s)-1) \quad i\,(n\backslash s)-(n\backslash s)).$$

The permutation $\nu_{n,s}$ has (n, s) cycles of length $n\backslash s$ in the natural-cyclic-order. For instance, if we consider $n = 8$ and $s = 6$, we have that $\nu_{8,6} = (3\ \ 2\ \ 1\ \ 0)(7\ \ 6\ \ 5\ \ 4)$. Notice that

$$P_{\nu_{n,s}} = I_{(n,s)} \otimes P_{n\backslash s},$$

where $\otimes$ denotes the usual Kronecker product between matrices, see [23].

The permutations τ_n^s and $\nu_{n,s}$ have the same cycle type. Therefore, for each pair n, s with $0 < s < n$, there exists a permutation σ such that $\sigma\tau_n^s\sigma^{-1} = \nu_{n,s}$. The digraph $D(\nu_{8,6})$ can be seen in Figure 3. Notice that $D(\nu_{8,6})$ is an untangling version of $D\left(aI_8 + bP_8^6\right)$, up to loops. The untangling versions of digraphs of the form $D\left(aP_n^{s_1} + bP_n^{s_2}\right)$ are digraphs of the form $D\left(aI_n + bP_{\nu_{n,s_2-s_1}}\right)$, see Figures 4 and 5.

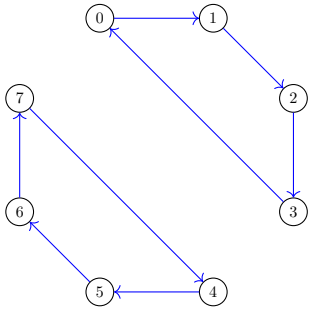
Let n, s_1, and s_2 be integers such that $0 < s_1 < s_2 < n$. If $n\backslash s_1 = n\backslash s_2$, then $\nu_{n,s_1} = \nu_{n,s_2}$, thus $D\left(\nu_{n,s_1}\right) = D\left(\nu_{n,s_2}\right)$.

The permutations $\tau_n^{s_1}$ and σ give us a block diagonal form of $aP_n^{s_1} + bP_n^{s_2}$:

$$P_\sigma P_9^7\left(aP_9^2 + bP_9^5\right)P_\sigma^T = aI_9 + bP_{\nu_{9,3}}$$

$$= \left[\begin{array}{ccc|ccc|ccc} a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & a & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & a \end{array}\right]$$

$$= I_3 \otimes (aI_3 + bP_3)\,.$$



i	$\varrho_{8,6}(i) \bmod (8,6)$	$\ell_{8,6}(i)$
0	0	0
1	0	3
2	0	2
3	0	1
4	1	0
5	1	3
6	1	2
7	1	1

FIGURE 3. $D(\nu_{8,6}) = D(\nu_{8,2})$. Note that $(8,6) = (8,2) = 2$ and $8\backslash 6 = 8\backslash 2 = 4$.

3. FINDING $\sigma_{n,s}$

In order to compute the Drazin (group) inverse of a matrix of the form $aP_n^{s_1} + bP_n^{s_2}$ (among other computations) we need to find one σ explicitly. This can be done if we find which vertices are together in the same connected component of $D(aI_n + bP_n^s)$. Since this digraph appears many times in our work, we just write $D_{n,s}(a,b)$ instead of $D\left(aI_n + bP_n^s\right)$.

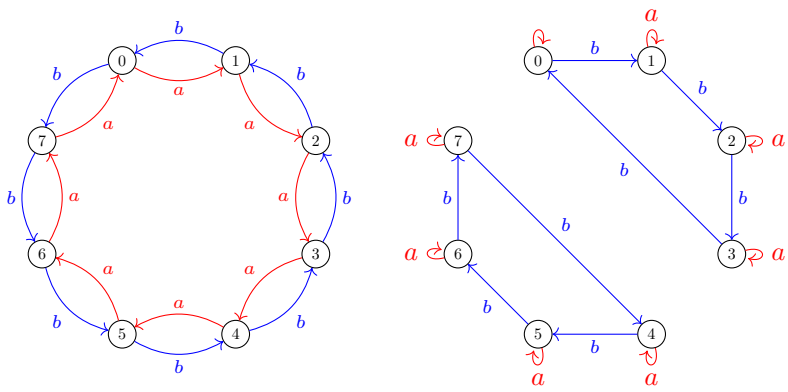


FIGURE 4. On the left $D\left(aP_8 + bP_8^7\right)$, and on the right its untangled version $D\left(aI_8 + bP_{\nu_{8,6}}\right)$.

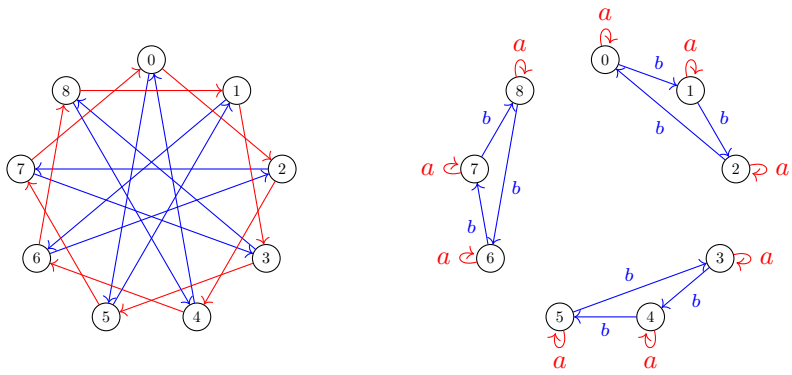


FIGURE 5. On the left $D\left(aP_9^2 + bP_9^5\right)$ (in blue b–arcs and in red a–arcs), and on the right its untangled version $D\left(aI_9 + bP_{\nu_{9,3}}\right)$.

Let n and s be non-negative integers such that $0 < s < n$. For each $i \in \mathbb{Z}$, we define

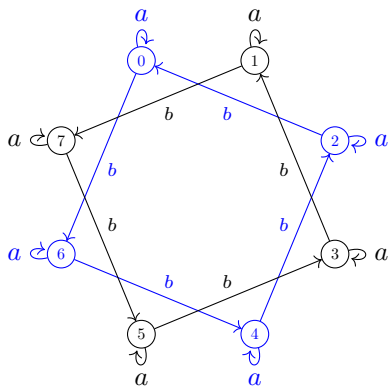
$$R(n, s, i) := \{i + k\, s \bmod n : k \in \mathbb{Z}\}.$$

This is the set of all reachable vertices from vertex i in $D_{n,s}(a, b)$. Notice that the following statements are all equivalent:

(1) $R(n, s, i_1) = R(n, s, i_2)$,
(2) $i_2 \bmod n \in R(n, s, i_1)$,
(3) $R(n, s, i_1) \cap R(n, s, i_2) \neq \varnothing$, and
(4) $i_1 - i_2 = 0 \bmod (n, s)$.

In Figure 6, we can see $R(8, 6, 0)$ and $R(8, 6, 1)$.

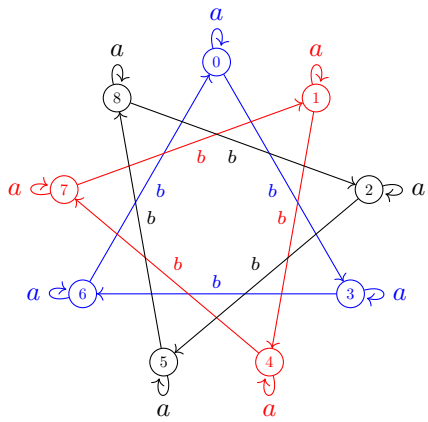
In order to obtain explicitly a σ, we introduce the following functions:



$R(8,6,0) = \{0,6,4,2\}$

$R(8,6,1) = \{1,7,5,3\}$

FIGURE 6. $D_{8,6}(a,b)$. Note that $(8,6) = 2$ and $8\backslash 6 = 4$.



i	$\mathrm{cycle}_{8,6}(i)$	$\mathrm{pos}_{8,6}(i)$
0	0	0
1	1	0
2	2	0
3	0	1
4	1	1
5	2	1
6	0	2
7	1	2
8	2	2

FIGURE 7. $D_{9,3}(a,b)$. Note that $(9,3) = 9\backslash 3 = 3$.

(1) $\varrho_{n,s} : [n] \longrightarrow [(n,s)]$,

$$\varrho_{n,s}(i) := \max\{k \in [(n,s)] : i \geqslant k\,(n\backslash s)\}.$$

(2) $\ell_{n,s} : [n] \longrightarrow [n\backslash s]$,

$$\ell_{n,s}(i) := i - \varrho_{n,s}(i)\,(n\backslash s) \bmod n\backslash s.$$

(3) $\mathrm{cycle}_{n,s} : [n] \longrightarrow [(n,s)]$; if $i \in R(n,s,h)$, then

$$\mathrm{cycle}_{n,s}(i) := h.$$

(4) $\mathrm{pos}_{n,s} : [n] \longrightarrow [n\backslash s]$; if $i \in R(n,s,h)$ and $i = t\,s + h$, then

$$\mathrm{pos}_{n,s}(i) = t.$$

The functions $\varrho_{n,s}$, and $\ell_{n,s}$ give us, essentially, the cycle and the position on the cycle of a vertex of $D(\nu_{n,s})$, respectively, while the functions $\mathrm{cycle}_{n,s}$ and $\mathrm{pos}_{n,s}$ give us the main cycle and the position on the main cycle in $D_{n,s}(a,b)$, respectively.

We embed these digraphs in the cylinder $[(n,s)]\times[n\backslash s]$. The following functions and their pullbacks are these embeddings:

(5) $J_{n,s} : [n] \longrightarrow [(n,s)] \times [n\backslash s]$,

$$J_{n,s}(i) = (\varrho_{n,s}(i), \ell_{n,s}(i)).$$

(6) $\widehat{J}_{n,s} : [(n,s)] \times [n\backslash s] \longrightarrow [n]$,

$$\widehat{J}_{n,s}(x,y) = y + x\,(n\backslash s).$$

(7) $F_{n,s} : [n] \longrightarrow [(n,s)] \times [n\backslash s]$,

$$F_{n,s}(i) = (\mathrm{cycle}_{n,s}(i), \mathrm{pos}_{n,s}(i)).$$

(8) $\widehat{F}_{n,s} : [(n,s)] \times [n\backslash s] \longrightarrow [n]$,

$$\hat{F}_{n,s}(x,y) = s\,y + x \bmod n.$$

The function $J_{n,s}$ embeds $D(\nu_{n,s})$ in $[(n,s)]\times[n\backslash s]$ "in the same way" that $F_{n,s}$ embeds $D_{n,s}(a,b)$ in $[(n,s)]\times[n\backslash s]$.

Let A be a set; we denote the identity map by $\mathrm{id}_A : A \to A$. If n and s are two integers such that $0 < s < n$, then the following identities are direct:

$$\begin{aligned}
\widehat{J}_{n,s} \circ J_{n,s} &= \mathrm{id}_{[n]},\\
J_{n,s} \circ \widehat{J}_{n,s} &= \mathrm{id}_{[(n,s)]\times[n\backslash s]},\\
\widehat{F}_{n,s} \circ F_{n,s} &= \mathrm{id}_{[n]},\\
F_{n,s} \circ \widehat{F}_{n,s} &= \mathrm{id}_{[(n,s)]\times[n\backslash s]}.
\end{aligned}$$

These expressions give us the next lemma.

Lemma 3.1. *Let n and s be integers such that $0 < s < n$. Then, the function $\sigma_{n,s} : [n] \longrightarrow [n]$, defined by $\sigma_{n,s} := \widehat{F}_{n,s} \circ J_{n,s}$, is a permutation of $[n]$ and $\sigma_{n,s}^{-1} = \widehat{J}_{n,s} \circ F_{n,s}$.*

The function $\mathrm{Shift}_{n,s} : [(n,s)] \times [n\backslash s] \longrightarrow [(n,s)] \times [n\backslash s]$, defined by

$$\mathrm{Shift}_{n,s}(x,y) := (x,\ y - 1 \bmod n\backslash s),$$

allows us to express τ_n^s in terms of $F_{n,s}$ and $\widehat{F}_{n,s}$, and $\nu_{n,s}$ in terms of $J_{n,s}$ and $\widehat{J}_{n,s}$.

Lemma 3.2. *Let n and s be integers such that $0 < s < n$. Then*

$$\begin{aligned}
\tau_n^s &= \widehat{F}_{n,s} \circ \mathrm{Shift}_{n,s} \circ F_{n,s},\\
\nu_{n,s} &= \widehat{J}_{n,s} \circ \mathrm{Shift}_{n,s} \circ J_{n,s}.
\end{aligned}$$

Corollary 3.3. *Let n and s be integers such that $0 < s < n$. Then*

$$\nu_{n,s} = \sigma_{n,s}^{-1} \circ \tau_{n,s} \circ \sigma_{n,s},$$
$$I_{(n,s)} \otimes P_{n\backslash s} = P_{\sigma_{n,s}}^T P_n^s P_{\sigma_{n,s}}.$$

Theorem 3.4. *Let n, s_1, and s_2 be integers such that $0 \leqslant s_1 < s_2 < n$, and let a, b be non-zero complex numbers. Then,*

$$aP_n^{s_1} + bP_n^{s_2} = P_n^{s_1} P_{\sigma_{n,s_2-s_1}} \left[I_{(n,s_2-s_1)} \otimes \left(aI_{n\backslash(s_2-s_1)} + bP_{n\backslash(s_2-s_1)} \right) \right] P_{\sigma_{n,s_2-s_1}}^T.$$

Proof. Let $s = s_2 - s_1$. Then, by Corollary 3.3,

$$\begin{aligned} P_{\sigma_{n,s}}^T P_n^{n-s_1} \left(aP_n^{s_1} + bP_n^{s_2} \right) P_{\sigma_{n,s}} &= aI_n + bP_{\sigma_{n,s}}^T P_n^s P_{\sigma_{n,s}} \\ &= aI_n + b\left(I_{(n,s)} \otimes P_{n\backslash s} \right) \\ &= I_{(n,s)} \otimes \left(aI_{n\backslash s} + bP_{n\backslash s} \right). \end{aligned}$$

This concludes the proof. □

The following theorem shows that we can untangle $\sum_{k=0}^{n\backslash s-1} a_k P_n^{ks}$, in the same way as in the case of P_n^s.

Theorem 3.5. *Let s, n be positive integers such that $s < n$ and let a_k be non-zero complex numbers for $k \in [n\backslash s]$. Then there exists a permutation $\sigma_{n,s}$ of $[n]$ such that*

$$P_{\sigma_{n,s}}^{-1} \left(\sum_{k=0}^{(n\backslash s)-1} a_k P_n^{s\,k} \right) P_{\sigma_{n,s}} = I_{(n,s)} \otimes \left(\sum_{k=0}^{(n\backslash s)-1} a_k P_{n\backslash s}^k \right).$$

Proof. Clearly $\nu_{n,s}$ has the same cycle type that τ_n^s. Then, there exists a permutation $\sigma_{n,s}$ such that

$$\sigma_{n,s}^{-1} \tau_n^s \sigma_{n,s} = \nu_{n,s}.$$

By taking into account that the application $P : \mathbb{S}_n \to \mathrm{GL}(n)$, where $\mathbb{S}_n$ is the symmetric group and $\mathrm{GL}(n)$ is the general lineal group, defined by $P(\sigma) = P_\sigma$ considered in the above section is a group homomorphism, we have that

$$P_{\sigma_{n,s}}^{-1} P_n^s P_{\sigma_{n,s}} = P_{\sigma_{n,s}^{-1} \tau_n^s \sigma_{n,s}} = P_{\nu_{n,s}} = I_{(n,s)} \otimes P_{n\backslash s}.$$

In the same way, if we consider the powers $P_n^{s\,k}$ with $k \in [n\backslash s]$, we obtain

$$P_{\sigma_{n,s}}^{-1} P_n^{k\,s} P_{\sigma_{n,s}} = (P_{\sigma_{n,s}}^{-1} P_n^s P_{\sigma_{n,s}})^k = P_{\sigma_{n,s}^{-1} \circ \tau_n^s \circ \sigma_{n,s}}^k = P_{\nu_{n,s}}^k.$$

By the property $(A \otimes B)(C \otimes D) = AC \otimes BD$ of the Kronecker product, we have that

$$P_{\sigma_{n,s}}^{-1} P_n^{k\,s} P_{\sigma_{n,s}} = P_{\nu_{n,s}}^k = (I_{(n,s)} \otimes P_{n\backslash s})^k = I_{(n,s)} \otimes P_{n\backslash s}^k$$

for all $k \in [n\backslash s]$. By taking into account that $A \otimes (B + C) = A \otimes B + A \otimes C$, we obtain

$$P_{\sigma_{n,s}}^{-1} \left(\sum_{k=0}^{(n\backslash s)-1} a_k P_n^{k\,s} \right) P_{\sigma_{n,s}} = \sum_{k=0}^{(n\backslash s)-1} a_k \left(I_{(n,s)} \otimes P_{n\backslash s}^k \right) = I_{(n,s)} \otimes \left(\sum_{k=0}^{n\backslash s-1} a_k P_{n\backslash s}^k \right),$$

as asserted. □

4. Determinant and permanent of $aP_n^{s_1} + bP_n^{s_2}$

A linear subdigraph L of a digraph D is a spanning subdigraph of D in which each vertex has indegree 1 and outdegree 1 (there is exactly one arc getting into each vertex, and there is exactly one arc (possibly the same) getting out of each vertex), see [6].

Theorem 4.1 ([6])**.** *Let $A = (a_{ij})$ be a square matrix of order n. Then*

$$\det(A) = \sum_{L \in \mathcal{L}(D(A))} (-1)^{n-c(L)} w(L)$$

and

$$\operatorname{perm} A = \sum_{L \in \mathcal{L}(D(A))} w(L),$$

where $\mathcal{L}(D(A))$ is the set of all linear subdigraphs of the digraph $D(A)$, $c(L)$ is the number of cycles contained in L, and $w(L)$ is the product of the weights of the edges of L.

Theorem 4.2. *Let n be a non-negative integer and let a, b be non-zero complex numbers. Then*

$$\det(aI_n + bP_n) = a^n - (-b)^n.$$

Proof. Notice that the digraph $D_{n,1}(a,b)$ has only two linear subdigraphs, the whole cycle and n loops. The result follows from Theorem 4.1. □

Corollary 4.3. *Let n, s_1, and s_2 be integers such that $0 \leqslant s_1 < s_2 < n$. Let a and b be non-zero complex numbers. Then*

$$\det(aP_n^{s_1} + bP_n^{s_2}) = (-1)^{(n-1)s_1} \left(a^{n\backslash(s_2-s_1)} - (-b)^{n\backslash(s_2-s_1)}\right)^{(n,s_2-s_1)}.$$

Hence, $aP_n^{s_1} + bP_n^{s_2}$ is singular if and only if $a^{n\backslash(s_2-s_1)} - (-b)^{n\backslash(s_2-s_1)} = 0$.

Notice that $a^{n\backslash(s_2-s_1)} - (-b)^{n\backslash(s_2-s_1)} = 0$ if and only if either $a = \pm b$ when $n\backslash(s_2 - s_1)$ is even or $a = -b$ when $n\backslash(s_2 - s_1)$ is odd.

Proof. Let $s = s_2 - s_1$. By Theorem 3.4,

$$\det(aP_n^{s_1} + bP_n^{s_2}) = \det(P_n^{s_1}) \det\left(I_{(n,s)} \otimes \left(aI_{n\backslash s} + bP_{n\backslash s}\right)\right).$$

Owing to $\det\left(P_n^i\right) = (-1)^{(n-1)\,i}$, the result follows from Theorem 4.2. □

Example 4.4. The matrix $aI_8 + bP_8^6$, whose associated digraph is in Figure 6, is singular if and only if $|a| = |b|$, and the same occurs with the matrix $aP_8^1 + bP_8^7$. The matrix $aI_9 + bP_9^3$, whose associated digraph is in Figure 7, is singular if and only if $a = -b$. The matrix $aP_9^2 + bP_9^5$ is also singular if and only if $a = -b$.

Corollary 4.5. *Let n, s_1, and s_2 be integers such that $0 \leqslant s_1 < s_2 < n$. Let a and b be non-zero complex numbers. Then*

$$\operatorname{perm}(aP_n^{s_1} + bP_n^{s_2}) = \left(a^{n\backslash(s_2-s_1)} + b^{n\backslash(s_2-s_1)}\right)^{(n,s_2-s_2)}.$$

Proof. The result follows from Theorem 4.1 and Corollary 4.3: if A is a square matrix and Q is a permutation matrix of the same order, then $\operatorname{perm} QA = \operatorname{perm} A$, see [15, Theorem 1.1]. □

5. Inverse of $aP_n^{s_1} + bP_n^{s_2}$

Theorem 5.1. *Let n be a non-negative integer and let a and b be non-zero complex numbers. If $aI_n + bP_n$ is non-singular, then*

$$(aI_n + bP_n)^{-1} = \frac{1}{a^n - (-b)^n} \sum_{i=0}^{n-1} (-1)^i b^i a^{n-1-i} P_n^i.$$

Proof. We just check that

$$(aI_n + bP_n) \left[\frac{1}{a^n - (-b)^n} \sum_{i=0}^{n-1} (-1)^i b^i a^{n-1-i} P_n^i \right] = I_n. \qquad \square$$

Example 5.2. Let a, b be non-zero complex numbers. If $a^4 - (-b)^4 \neq 0$, then

$$(aI_4 + bP_4)^{-1} = \frac{1}{a^4 - b^4} \operatorname{Circ}\left(a^3, -ba^2, b^2a, -b^3\right).$$

If $a^5 - (-b)^5 \neq 0$, then

$$(aI_5 + bP_5)^{-1} = \frac{1}{a^5 + b^5} \operatorname{Circ}\left(a^4, -ba^3, b^2a^2, -b^3a, b^4\right).$$

In order to obtain an explicit formula for the inverse of a non-singular circulant matrix of the form $aP_n^{s_1} + bP_n^{s_2}$, we define

$$\rho_{n,s}(i) = (-1)^{\operatorname{pos}_{n,s}(i)}\, \delta_{0,\operatorname{cycle}_{n,s}(i)}\, b^{\operatorname{pos}_{n,s}(i)}\, a^{(n\backslash s)-1-\operatorname{pos}_{n,s}(i)}, \tag{5.1}$$

where δ is the usual Kronecker delta. Notice that $\rho_{n,s}$ satisfies the following properties:

(1) For $n > 0$, $\rho_{n,1}(i) = (-1)^i b^i a^{n-i-1}$ for all $i = 0, \ldots, n-1$.
(2) For $0 < s < n$, $\rho_{n\backslash s,1}(i) = \rho_{n,s}(i\, s)$ for all $i = 0, \ldots, (n\backslash s) - 1$.

Corollary 5.3. *Let n, s_1, and s_2 be integers such that $0 \leqslant s_1 < s_2 < n$. Let a and b be non-zero complex numbers such that $a^{n\backslash(s_2-s_1)} - (-b)^{n\backslash(s_2-s_1)} \neq 0$. Then*

$$(aP_n^{s_1} + bP_n^{s_2})^{-1} = \frac{1}{a^{n\backslash(s_2-s_1)} - (-b)^{n\backslash(s_2-s_1)}} \sum_{i=0}^{n-1} \rho_{n,s_2-s_1}(i+s_1) P_n^i.$$

Proof. Let $s = s_2 - s_1$. By Theorem 3.4 we have that $(aP_n^{s_1} + bP_n^{s_2})^{-1}$ is equal to

$$P_{\sigma_{n,s}} \left[I_{(n,s)} \otimes \left(aI_{n\backslash s} + bP_{n\backslash s} \right)^{-1} \right] P_{\sigma_{n,s}}^T P_n^{n-s_1}.$$

Thus, by Theorem 5.1 and (5.1),

$$(aP_n^{s_1} + bP_n^{s_2})^{-1} = P_{\sigma_{n,s}} \left(\sum_{i=0}^{(n\backslash s)-1} I_{(n,s)} \otimes \frac{\rho_{n\backslash s,1}(i)}{a^{n\backslash s} - (-b)^{n\backslash s}} P_{n\backslash s}^i \right) P_{\sigma_{n,s}}^T P_n^{n-s_1}.$$

By Corollary 3.3 we have that

$$P_n^s = P_{\sigma_{n,s}} \left(I_{(n,s)} \otimes P_{n\backslash s} \right) P_{\sigma_{n,s}}^T \tag{5.2}$$

and

$$P_{\sigma_{n,s}} \left(I_{(n,s)} \otimes P_{n\backslash s}^i \right) P_{\sigma_{n,s}}^T = P_n^{i\,s}. \tag{5.3}$$

Then

$$\begin{aligned} (aP_n^{s_1} + bP_n^{s_2})^{-1} &= \sum_{i=0}^{(n\backslash s)-1} \frac{\rho_{n\backslash s,1}(i)}{a^{n\backslash s} - (-b)^{n\backslash s}} P_n^{(i\,s)-s_1} \\ &= \sum_{i=0}^{(n\backslash s)-1} \frac{\rho_{n,s}(i\,s)}{a^{n\backslash s} - (-b)^{n\backslash s}} P_n^{(i\,s)-s_1} \\ &= \sum_{j=0}^{n-1} \frac{\rho_{n,s}(j+s_1)}{a^{n\backslash s} - (-b)^{n\backslash s}} P_n^j. \end{aligned}$$

□

Example 5.4. Let a and b be two real numbers such $a^4 - (-b)^4 \neq 0$. We will compute $\left(aP_{12} + bP_{12}^4\right)^{-1}$. Since $4 - 1 = 3$ and $12\backslash 3 = 4$, by Theorems 3.4 and 5.1 we know that the inverse of $aI_{12} + bP_{12}^3$ is composed essentially by 3 blocks of

$$\operatorname{Circ}\left(a^3, -a^2b, ab^2, -b^3\right).$$

They are merged in a 12×12 matrix via P_{12} and $P_{\sigma_{12,3}}$. Since $s_1 = 1$, we have that $(12, 3) = 3$, so there are 3 major cycles of length 4 in $D\left(aI_{12} + bP_{12}^3\right)$:

$$\begin{array}{ccccccccc} 0 & \to & 3 & \to & 6 & \to & 9 & \to & 0, \\ 1 & \to & 4 & \to & 7 & \to & 10 & \to & 1, \\ 2 & \to & 5 & \to & 8 & \to & 11 & \to & 2. \end{array}$$

Now we compute the coefficients of the inverse:

i	$\operatorname{cycle}_{12,3}(i+1)$	$\delta_{0,\operatorname{cycle}_{12,3}(i+1)}$	$\operatorname{pos}_{12,3}(i+1)$	$\rho_{12,3}(i+1)$
0	1	0	0	0
1	2	0	0	0
2	0	1	1	$-a^2b$
3	1	0	1	0
4	2	0	1	0
5	0	1	2	ab^2
6	1	0	2	0
7	2	0	2	0
8	0	1	3	$-b^3$
9	1	0	3	0
10	2	0	3	0
11	0	1	0	a^3

Therefore,

$$\left(aP_{12}+bP_{12}^4\right)^{-1}=\frac{1}{a^4-b^4}\operatorname{Circ}\left(0,0,-ba^2,0,0,b^2a,0,0,-b^3,0,0,a^3\right).$$

6. Drazin inverse of $aP_n^{s_1}+bP_n^{s_2}$

Given a matrix A, the column space of A is denoted by $\operatorname{Rank}(A)$ and its dimension by $\operatorname{rank} A$. The null space of A is denoted by $\operatorname{Null}(A)$ and its dimension, called nullity, by $\operatorname{null}(A)$. The index of a square matrix A, denoted by $\operatorname{ind}(A)$, is the smallest non-negative integer k for which $\operatorname{Rank}(A^k)=\operatorname{Rank}(A^{k+1})$. It is well-known that a circulant matrix has index 0 or 1, see [9]. Let A be a matrix of index k. The Drazin inverse of A, denoted by A^D, is the unique matrix such that

(1) $AA^D=A^DA$,
(2) $A^{k+1}A^D=A^k$,
(3) $A^DAA^D=A^D$.

In [10], the following Bjerhammar-type condition for the Drazin inverse was proved. We find it handy for checking the Drazin conditions in combinatorial settings.

Theorem 6.1 ([10])**.** *Let A and D be square matrices of order n, with $\operatorname{ind}(A)=k$, such that $\operatorname{Null}\left(A^k\right)=\operatorname{Null}(D)$, and $AD=DA$. Then $A^{k+1}D=A^k$ if and only if $D^2A=D$.*

Theorem 6.2. *Let n, s_1, and s_2 be integers such that $0\leqslant s_1<s_2<n$. Let a and b be non-zero complex numbers such that $a^{n\backslash(s_2-s_1)}-(-b)^{n\backslash(s_2-s_1)}=0$. Then $\left(aP_n^{s_1}+bP_n^{s_2}\right)^D$ equals*

$$P_n^{n-s_1}P_{\sigma_{n,s_2-s_1}}\left[I_{(n,s_2-s_1)}\otimes\left(aI_{n\backslash(s_2-s_1)}+bP_{n\backslash(s_2-s_1)}\right)^D\right]P_{\sigma_{n,s_2-s_1}}^T.$$

Proof. The following two facts are well-known. If $A=XBX^{-1}$, then $A^D=XB^DX^{-1}$. If $AB=BA$, then $(AB)^D=B^DA^D=A^DB^D$. See [5] or [7].

In 1997, Wang proved that $(A\otimes B)^D=A^D\otimes B^D$ and $\operatorname{ind}(A\otimes B)=\max\{\operatorname{ind}(A),\operatorname{ind}(B)\}$, see [24, Theorem 2.2]. Let $s=s_2-s_1$. Then, by Theorem 3.4,

$$\begin{aligned}\left(aP_n^{s_1}+bP_n^{s_2}\right)^D&=P_n^{n-s_1}\left[P_{\sigma_{n,s}}\left[I_{(n,s)}\otimes\left(aI_{n\backslash s}+bP_{n\backslash s}\right)\right]P_{\sigma_{n,s}}^T\right]^D\\&=P_n^{n-s_1}P_{\sigma_{n,s}}\left[I_{(n,s)}\otimes\left(aI_{n\backslash s}+bP_{n\backslash s}\right)\right]^DP_{\sigma_{n,s}}^T\\&=P_n^{n-s_1}P_{\sigma_{n,s}}\left[I_{(n,s)}\otimes\left(aI_{n\backslash s}+bP_{n\backslash s}\right)^D\right]P_{\sigma_{n,s}}^T.\end{aligned}$$

□

Therefore, we just need to study the Drazin inverse of matrices of the form aI_n+bP_n.

By Corollary 4.3, we know that $aP_n^{s_1}+bP_n^{s_2}$ is singular if and only if $a^{n\backslash s}-(-b)^{n\backslash s}=0$, where $s=s_2-s_1$. If $n\backslash s$ is even, then $a^{n\backslash s}-(-b)^{n\backslash s}=0$ if and only if $|a|=|b|$. If $n\backslash s$ is odd, then $a^{n\backslash s}-(-b)^{n\backslash s}=0$ if and only if $a=-b$. Hence, matrices of the form $a\left(I_n-P_n\right)$ are always singular, but matrices of the

form $a\,(I_n + P_n)$ are singular if and only if n is even. All of these facts reduce the computation of the Drazin inverse of singular circulant matrices of the form $aP_n^{s_1} + bP_n^{s_2}$ to the computation of the Drazin inverse of just two matrices: $I_n - P_n$ and $I_{2n} + P_{2n}$.

6.1. **Drazin inverse of $I_n - P_n$.**

Theorem 6.3. *Let n be a non-negative integer. The Drazin inverse of $aI_n - aP_n$ is*

$$(I_n - P_n)^D := \frac{1}{2n} \sum_{i=0}^{n-1} (n - 2\,i - 1)\, P_n^i. \tag{6.1}$$

Example 6.4. Let a be a non-zero complex number. Thus, we have that

$$(aI_4 - aP_4)^D = \tfrac{1}{8a}\,\mathrm{Circ}\,(3, 1, -1, -3)$$

and

$$(aI_5 - aP_5)^D = \tfrac{1}{10a}\,\mathrm{Circ}\,(4, 2, 0, -2, -4)\,.$$

In order to prove this theorem, we will use some polynomial tools. The idea is to prove that both matrices in (6.1) have the same null space, and then use Theorem 6.1.

We denote by $\mathbb{1}_n$ the vector of all ones. The easy proof of the following lemma is left to the reader.

Lemma 6.5. *Let n be a non-negative integer. Then*

$$\mathbb{1}_n \in \mathrm{Null}\,(I_n - P_n) \cap \mathrm{Null}\left(\frac{1}{2n} \sum_{i=0}^{n-1} (n - 2\,i - 1)\, P_n^i\right).$$

Every circulant matrix $\mathrm{Circ}\,(c_0, \ldots, c_{n-1})$ has its associated polinomial $P_C(x) = \sum_{i=0}^{n-1} c_i x^i$. Ingleton proved the following proposition in 1955.

Proposition 6.6 ([13, Proposition 1.1])**.** *The rank of a circulant matrix C of order n is $n - d$, where d is the degree of the greatest common divisor of $x^n - 1$ and the associated polynomial of C.*

This means that in order to know the rank of some circulant matrix, it is enough to know how many n-th roots of the unit are roots of its associated polynomial.

Let us consider the polynomials

$$H_j^n(x) = \sum_{i=0}^{n-1} (n - 2\,(i + j) - 1)\; x^i \tag{6.2}$$

for $j = 0, \ldots, n-1$. Notice that $H_0^n(x)$ is the associated polynomial of the circulant matrix $\sum_{i=0}^{n-1} (n - 2\,i - 1)\, P_n^i$.

We denote by Ω_ℓ the set $\{x \in \mathbb{C} : x^\ell = 1\}$ of ℓ-th roots of the unit. Notice that

$$\Omega_n = \{\omega_n^k : k = 0, \ldots, n-1\},$$

where $\omega = \exp(\frac{2\pi i}{n})$, i.e., ω_n^k are exactly the roots of the polynomial $p(x) = x^n - 1$. Moreover, they are all simple roots of $p(x) = x^n - 1$, because they are all different and p has degree n. Moreover, if we define

$$\Phi_n(x) := \frac{x^n - 1}{x - 1} = \sum_{i=0}^{n-1} x^i,$$

then ω_n^k are all simple roots of Φ_n for any $k \neq 0$. In terms of derivatives, this means that

$$\Phi_n(\omega) = 0 \quad \text{and} \quad \Phi_n'(\omega) \neq 0$$

for all $\omega \in \Omega_n \smallsetminus \{1\}$.

Proposition 6.7. *Let n be a non-negative integer. Then*

$$H_j^n(\omega) \neq 0$$

for all $j = 0, \dots, n-1$ and $\omega \in \Omega_n \smallsetminus \{1\}$.

Proof. We have that

$$H_j^n(x) = \sum_{i=0}^{n-1} (n - 2\,(i+j) - 1)\; x^i = (n - 2\,j - 1) \sum_{i=0}^{n-1} x^i - 2 \sum_{i=0}^{n-1} i\, x^i.$$

Notice that $\sum_{i=0}^{n-1} i\, x^i = x\, \Phi_n'(x)$, and so we obtain

$$H_j^n(x) = (n - 2\,j - 1)\, \Phi_n(x) - 2\, x\, \Phi_n'(x). \tag{6.3}$$

Now, assume that $H_j^n(\omega) = 0$ for some $\omega \in \Omega_n \smallsetminus \{1\}$, thus (6.3) implies that

$$0 = (n - 2\,j - 1)\, \Phi_n(\omega) - 2\, \omega\, \Phi_n'(\omega).$$

Hence, we have that $\omega\, \Phi_n'(\omega) = 0$, since $\Phi_n(\omega) = 0$. But this cannot occur since $\omega \neq 0$ and ω are all simple roots of Φ_n. Therefore $H_j^n(\omega) \neq 0$ for all $j = 0, \dots, n-1$ and $\omega \in \Omega_n \smallsetminus \{1\}$, as asserted. □

We are ready to prove the main result of the subsection.

Proof of Theorem 6.3. Since the index of all singular circulant matrices is 1 and the rank of $I_n - P_n$ is clearly $n-1$, by Propositions 6.6 and 6.7 we have that

$$\operatorname{Null}(I_n - P_n) = \operatorname{Null}\left(\frac{1}{2\,n}\sum_{i=0}^{n-1}(n-2\,i-1)\,P_n^i\right).$$

By Theorem 6.1, we just need to check that

$$(I_n - P_n)^2\left(\frac{1}{2\,n}\sum_{i=0}^{n-1}(n-2\,i-1)\,P_n^i\right) = I_n - P_n,$$

which is left to the reader. □

In order to obtain an explicit formula for the Drazin inverse of circulant matrices of the form $P_n^{s_1} - P_n^{s_2}$ we define

$$\rho^*_{n,s}(i) := \delta_{0,\operatorname{cycle}_{n,s}(i)}\left(n\backslash s - 2\,\operatorname{pos}_{n,s}(i) - 1\right), \tag{6.4}$$

where δ is the usual Kronecker delta. In this case, ρ^* satisfies the following:

(1) For $n > 0$, $\rho^*_{n,1}(i) = (n-2i-1)$ for all $i = 0, \ldots, n-1$.
(2) For $0 < s < n$, $\rho^*_{n\backslash s,1}(i) = \rho^*_{n,s}(i\,s)$ for all $i = 0, \ldots, (n\backslash s) - 1$.

Corollary 6.8. *Let n, s_1, and s_2 be integers such that $0 \leqslant s_1 < s_2 < n$. Then*

$$(P_n^{s_1} - P_n^{s_2})^D = \frac{1}{2\,(n\backslash(s_2 - s_1))}\sum_{i=0}^{n-1}\rho^*_{n,s_2-s_1}(i+s_1)P_n^i.$$

Proof. Let $s = s_2 - s_1$. By Theorem 6.2 we have that $(P_n^{s_1} - P_n^{s_2})^D$ is equal to

$$P_n^{n-s_1}P_{\sigma_{n,s}}\left[I_{(n,s)} \otimes \left(I_{n\backslash s} - P_{n\backslash s}\right)^D\right]P_{\sigma_{n,s}}^T.$$

Thus, by Theorem 6.3 and (6.4),

$$(P_n^{s_1} - P_n^{s_2})^D = P_n^{n-s_1}P_{\sigma_{n,s}}\left(\sum_{i=0}^{(n\backslash s)-1} I_{(n,s)} \otimes \frac{\rho^*_{n\backslash s,1}(i)}{2\,(n\backslash s)}P_{n\backslash s}^i\right)P_{\sigma_{n,s}}^T.$$

Hence, by Corollary 3.3 and expressions (5.2) and (5.3),

$$\begin{aligned}(P_n^{s_1} - P_n^{s_2})^D &= \sum_{i=0}^{(n\backslash s)-1}\frac{\rho^*_{n\backslash s,1}(i)}{2\,(n\backslash s)}P_n^{(i\,s)-s_1}\\ &= \sum_{i=0}^{(n\backslash s)-1}\frac{\rho^*_{n,s}(i\,s)}{2\,(n\backslash s)}P_n^{(i\,s)-s_1}\\ &= \sum_{j=0}^{n-1}\frac{\rho^*_{n,s}(j+s_1)}{2\,(n\backslash s)}P_n^j.\end{aligned}$$

□

Example 6.9. We compute $\left(aP_{12} - aP_{12}^4\right)^D$. Since $12\backslash 3 = 4$, by Theorems 6.2 and 6.3 we know that the Drazin inverse is essentially 3 blocks of Circ $(3, 1, -1, -3)$ merged in a 12×12 matrix via P_{12} and $P_{\sigma_{12,3}}$.

i	$\mathrm{cycle}_{12,3}(i+1)$	$\delta_{0,\mathrm{cycle}_{12,3}(i+1)}$	$\mathrm{pos}_{12,3}(i+1)$	$\rho^*_{12,3}(i+1)$
0	1	0	0	0
1	2	0	0	0
2	0	1	1	1
3	1	0	1	0
4	2	0	1	0
5	0	1	2	-1
6	1	0	2	0
7	2	0	2	0
8	0	1	3	-3
9	1	0	3	0
10	2	0	3	0
11	0	1	0	3

Therefore, $\left(aP_{12} - aP_{12}^4\right)^D = \dfrac{1}{8\,a}\,\mathrm{Circ}\,(0, 0, 1, 0, 0, -1, 0, 0, -3, 0, 0, 3)$.

6.2. **Drazin inverse of $I_{2n} + P_{2n}$.**

Theorem 6.10. *Let n be a non-negative integer. The Drazin inverse of $I_{2n} + P_{2n}$ is*

$$(I_{2n} + P_{2n})^D = \frac{1}{4\,n} \sum_{i=0}^{2\,n-1} (-1)^i\,(2\,n - 2\,i - 1)\,P_{2n}^i. \tag{6.5}$$

The idea of the proof is the same as before, i.e., we want to show that both matrices in (6.5) have the same null space, and then use Theorem 6.1.

We define $\pm\mathbb{1}_n = ((-1)^i)_{i=0}^{2n-1} \in \mathbb{R}^{2n}$, a particular vector of ones and minus ones. The following lemma can be proved easily and its proof is left to the reader.

Lemma 6.11. *Let n be a non-negative integer. Then*

$$\pm\mathbb{1}_n \in \mathrm{Null}\,(I_{2n} + P_{2n}) \cap \mathrm{Null}\left(\frac{1}{4\,n} \sum_{i=0}^{2n-1} (-1)^i\,(2\,n - 2\,i - 1)\,P_{2n}^i\right).$$

Let us consider the polynomials

$$\widehat{H}_j^{2n}(x) = \sum_{i=0}^{2n-1} (-1)^{i+j}\,(2\,n - 2\,(i + j) - 1)\;x^i$$

for $j = 0, \ldots, 2n-1$. As before, notice that $\widehat{H}_0^{2n}(x)$ is the associated polynomial of the circulant matrix $\sum_{i=0}^{2n-1}(-1)^i\,(2\,n - 2\,i - 1)\,P_{2n}^i$. The following result ensures that its rank is $2n - 1$.

Proposition 6.12. *Let n be a non-negative integer. Then*

$$\widehat{H}_j^{2n}(\omega) \neq 0$$

for all $j = 0, \dots, 2n-1$ and $\omega \in \Omega_{2n} \setminus \{-1\}$.

Proof. On the one hand, the linear transformation $T(x) = -x$ for $x \in \mathbb{C}$ is a permutation on Ω_{2n}, i.e., we have that

$$\omega \text{ is a } (2n)\text{-th root of } 1 \iff -\omega \text{ is a } (2n)\text{-th root of } 1. \tag{6.6}$$

On the other hand, if $C_{i+j} = 2n - 2\,(i+j) - 1$ we have that

$$\widehat{H}_j^{2n}(x) = \sum_{i=0}^{2n-1} (-1)^{i+j} C_{i+j} x^i = (-1)^j \sum_{i=0}^{2n-1} C_{i+j} (-x)^i = (-1)^j H_j^{2n}(-x),$$

where H_j^{2n} is as in (6.2). By (6.6) and Proposition 6.7 we obtain

$$\widehat{H}_j^{2n}(\omega) = 0 \iff H_j^{2n}(-\omega) = 0 \iff \omega = -1.$$

So, if $\omega \neq -1$, then $\widehat{H}_j^{2n}(\omega) \neq 0$, as desired. □

We are ready to prove the main result of the subsection.

Proof of Theorem 6.10. The rank of $I_{2n} + P_{2n}$ is clearly $2n-1$. Hence, by Lemma 6.11 and Proposition 6.12,

$$\operatorname{Null}(I_{2n} + P_{2n}) = \operatorname{Null}\left(\frac{1}{4n} \sum_{i=0}^{2n-1} (-1)^i\,(2n - 2\,i - 1)\,P_{2n}^i \right).$$

By Theorem 6.1, it suffices to check that

$$(I_{2n} + P_{2n})^2 \left(\frac{1}{4n} \sum_{i=0}^{2n-1} (-1)^i\,(2n - 2\,i - 1)\,P_{2n}^i \right) = I_{2n} + P_{2n},$$

which is left to the reader. □

Corollary 6.13. *Let n, s_1, and s_2 be integers such that $0 \leqslant s_1 < s_2 < 2n$. Then*

$$(P_{2n}^{s_1} + P_{2n}^{s_2})^D = \frac{1}{2\,(2n \backslash (s_2 - s_1))} \sum_{i=0}^{2n-1} (-1)^{i+s_1} \rho^*_{2n, s_2 - s_1}(i + s_1) P_{2n}^i.$$

Example 6.14. Let us compute $\left(aP_{12} + aP_{12}^4\right)^D$. Since $12 \backslash 3 = 4$, by Theorems 6.2 and 6.3 we know that the Drazin inverse is essentially 3 blocks of $\operatorname{Circ}(3, -1, -1, 3)$ merged in a 12×12 matrix via P_{12} and $P_{\sigma_{12,3}}$. Since $4 - 1 = 3$, we have $12 \backslash 3 = 4$, and $(12, 3) = 3$. Therefore, we have

i	$\mathrm{cycle}_{12,3}(i+1)$	$\delta_{0,\mathrm{cycle}_{12,3}(i+1)}$	$\mathrm{pos}_{12,3}(i+1)$	$(-1)^{i+1}\rho^*_{12,3}(i+1)$
0	1	0	0	0
1	2	0	0	0
2	0	1	1	-1
3	1	0	1	0
4	2	0	1	0
5	0	1	2	-1
6	1	0	2	0
7	2	0	2	0
8	0	1	3	3
9	1	0	3	0
10	2	0	3	0
11	0	1	0	3

Therefore, $\left(aP_{12} + aP_{12}^4\right)^D = \dfrac{1}{8\,a}\,\mathrm{Circ}\,(0,0,-1,0,0,-1,0,0,3,0,0,3)$.

7. Block circulant matrices with two parameters

Let n, s_1, and s_2 be non-negative integers such that $0 \leqslant s_1 < s_2 \leqslant n-1$. Let A and B be two square matrices of order r. We will use the basic properties of the Kronecker product, see [23]. The matrices of the form $P_n^{s_1} \otimes A + P_n^{s_2} \otimes B$ we call block circulant matrices with two parameters.

For example, for the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}, \tag{7.1}$$

we have that $P_8 \otimes A + P_8^3 \otimes B$ is

$$\left[\begin{array}{cc|cc|cc|cc|cc|cc|cc|cc}
0 & 0 & 1 & 2 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 & -1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & -1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 3 & -1 \\
\hline
-1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\
\hline
0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
\hline
1 & 2 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

The following tells us that it is enough to study circulant matrices of the form $I_n \otimes A + P_n^s \otimes B$. We have that

$$P_n^{s_1} \otimes A + P_n^{s_2} \otimes B = P_n^{s_1} \otimes I_r \left(I_n \otimes A + P_n^{s_2 - s_1} \otimes B \right).$$

Thus, if $P_n^{s_1} \otimes A + P_n^{s_2} \otimes B$ is non-singular, then

$$(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B)^{-1} = (I_n \otimes A + P_n^{s_2 - s_1} \otimes B)^{-1} P_n^{n - s_1} \otimes I_r.$$

The next is a direct consequence of the following fact: $i - j = n - s \mod n$ if and only if $j - i = s \mod n$.

$$(I_n \otimes A + P_n^s \otimes B)^T = I_n \otimes A^T + P_n^{n-s} \otimes B^T.$$

Let $s = s_2 - s_1$. Note that by Theorem 3.4 we have that the matrix $I_{(n,s)} \otimes \left(I_{n \backslash s} \otimes A + P_{n \backslash s} \otimes B \right)$ is equal to

$$\left(P_{\sigma_{n,s}}^T \otimes I_r \right) \left(P_n^{n - s_1} \otimes I_r \right) \left(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B \right) \left(P_{\sigma_{n,s}} \otimes I_r \right). \tag{7.2}$$

For the matrices given in (7.1) we have that

$$\left(P_{\sigma_{8,2}}^T \otimes I_2 \right) \left(P_8^7 \otimes I_2 \right) \left(P_8 \otimes A + P_8^3 \otimes B \right) \left(P_{\sigma_{8,2}} \otimes I_2 \right)$$

$$= \left[\begin{array}{cccccccc|cccccccc}
\mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{-1} & \mathbf{3} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mathbf{3} & \mathbf{-1} & 0 & 0 & 0 & 0 & \mathbf{2} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{-1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} & \mathbf{3} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{3} & \mathbf{-1} & 0 & 0 & 0 & 0 & \mathbf{2} & \mathbf{1}
\end{array}\right].$$

8. Determinant of block circulant matrices with two parameters

From the Schur complement we have the following proposition.

Proposition 8.1. *Suppose E, F, G, and H are respectively $p \times p$, $p \times q$, $q \times p$, and $q \times q$ matrices, with H non-singular. If*

$$M = \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

then

$$\det(M) = \det(H)\det(E - FH^{-1}G).$$

Let k be a non-negative integer. We denote by $\vec{v}_k$ a $k \times 1$ vector and by $\vec{u}_k$ a $1 \times k$ vector given by

$$(\vec{v}_k)_{i\,0} = \begin{cases} 1 & \text{if } i = k-1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\vec{u}_k)_{0\,j} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the proof of the following lemma we use the notation given in [3]. Let K be an $m \times n$ matrix. If $S \subseteq [m]$ and $T \subseteq [n]$, then $K\,[S|T]$ will denote the submatrix of K determined by the rows corresponding to S and the columns corresponding to T.

Lemma 8.2. *Let n be a non-negative integer. Let A and B two non-zero square matrices or order r. If $AB = BA$ and A is non-singular, then*

$$\det(I_n \otimes A + P_n \otimes B) = \det(A)^n \det(I_r - (-A^{-1}B)^n).$$

Proof. For $1 \leqslant i \leqslant n-1$, we define the following sequences of matrices:

(1) $E_i := I_n\,[[n-i]|[n-i]] \otimes A + P_n\,[[n-i]|[n-i]] \otimes B$,
(2) $F_i := \vec{v}_{n-i} \otimes B$,
(3) $G_i := \vec{u}_{n-i} \otimes (-A)^{-i+1}B^i$ and
(4) $H_i := A$.

Note that

$$I_n \otimes A + P_n \otimes B = \begin{bmatrix} E_1 & F_1 \\ G_1 & H_1 \end{bmatrix} \quad \text{and} \quad E_{i-1} - F_{i-1}H_{i-1}^{-1}G_{i-1} = \begin{bmatrix} E_i & F_i \\ G_i & H_i \end{bmatrix}$$

for $2 \leqslant i \leqslant n-1$. Therefore, by Proposition 8.1,

$$\det(I_n \otimes A + P_n \otimes B) = \det(H_1)\det(E_1 - F_1H_1^{-1}G_1)$$

and

$$\det(E_{i-1} - F_{i-1}H_{i-1}^{-1}G_{i-1}) = \det(H_i)\det(E_i - F_iH_i^{-1}G_i)$$

for $2 \leqslant i \leqslant n-1$. Therefore

$$\begin{aligned}\det(I_n \otimes A + P_n \otimes B) &= \prod_{i=1}^{n-1} \det(H_i) \det(E_i - F_i H_i^{-1} G_i) \\ &= (\det(A))^{n-2} \det(E_{n-1} - F_{n-1} H_{n-1}^{-1} G_{n-1}) \\ &= \det(A)^n \det(I_r - (-A^{-1}B)^n). \end{aligned}$$ □

Given the matrix $P_n^{s_1} \otimes A + P_n^{s_2} \otimes B = (P_n^{s_1} \otimes I_r)(I_n \otimes A + P_n^{s_2 - s_1} \otimes B)$, note that

$$\det(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B) = \det(P_n^{s_1} \otimes I_r) \det(I_n \otimes A + P_n^{s_2 - s_1} \otimes B),$$

and $\det(P_n^{s_1} \otimes I_r) \det(I_n \otimes A + P_n^{s_2 - s_1} \otimes B)$ is equal to

$$\det(P_n^{s_1} \otimes I_r) \det((P_{\sigma_{n,s_2-s_1}}^T \otimes I_r)(I_n \otimes A + P_n^{s_2-s_1} \otimes B)(P_{\sigma_{n,s_2-s_1}} \otimes I_r)). \quad (8.1)$$

Theorem 8.3. *Let n, s_1, and s_2 be non-negative integers such that $0 \leqslant s_1 < s_2 \leqslant n-1$, and let A and B two non-zero square matrices or order r such that $AB = BA$. We have the following:*

(1) *if A is non-singular, then*

$$\det(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B) = \det(P_n^{s_1})^r \det(A)^n \left[\det\left(I_r - (-A^{-1}B)^{n \backslash (s_2 - s_1)}\right)\right]^{(n, s_2 - s_1)}.$$

(2) *If B is non-singular and A is singular, then*

$$\det(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B) = \det(P_n^{s_2})^r \det(B)^n \left[\det\left(I_r - (-B^{-1}A)^{n \backslash (n - s_2 + s_1)}\right)\right]^{(n, n - s_2 + s_1)}.$$

Proof. Let $s = s_2 - s_1$. Assume that A is non-singular. By (8.1) and Lemma 8.2 we have that

$$\begin{aligned}&\det(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B) \\ &\quad = \det(P_n^{s_1} \otimes I_r) \det(I_n \otimes A + P_n^s \otimes B) \\ &\quad = \det(P_n^{s_1} \otimes I_r) \det((P_{\sigma_{n,s}}^T \otimes I_r)(I_n \otimes A + P_n^s \otimes B)(P_{\sigma_{n,s}} \otimes I_r)) \\ &\quad = \det(P_n^{s_1} \otimes I_r) \det\left(I_{(n,s)} \otimes (I_{n \backslash s} \otimes A + P_{n \backslash s} \otimes B)\right) \\ &\quad = \det(P_n^{s_1} \otimes I_r) \left(\det(I_{n \backslash s} \otimes A + P_{n \backslash s} \otimes B)\right)^{(n,s)} \\ &\quad = \det(P_n^{s_1})^r \det(A)^n \left(\det\left(I_r - (-A^{-1}B)^{n \backslash s}\right)\right)^{(n,s)}. \end{aligned}$$

If B is non-singular and A is singular the proof is analogous to the previous proof. □

For example, for the matrices A and B given in (7.1), we have

$$\det\left(P_8 \otimes A + P_8^3 \otimes B\right) = (\det(P_8))^2 (\det(A))^8 \left(\det\left(I_2 - (-A^{-1}B)^4\right)\right)^2.$$

Since

$$I_2 - (-A^{-1}B)^4 = \frac{81}{20728}\begin{bmatrix} -1296 & -1295 \\ -1295 & -1296 \end{bmatrix}, \quad \det\left(I_2 - (-A^{-1}B)^4\right) = \frac{1}{2591},$$

and $\det(A) = -3$, we obtain $\det\left(P_8 \otimes A + P_8^3 \otimes B\right) = (-3)^8 \left(\frac{1}{2591}\right)^2$.

Corollary 8.4. *Let n, s_1, and s_2 be non-negative integers such that $0 \leqslant s_1 < s_2 \leqslant n-1$, and let A and B be two non-zero square matrices or order r such that $AB = BA$. We have that $\det(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B) = 0$ if and only if*

1. *A, B are singular, or*
2. *A is non-singular, but $I_r - (-A^{-1}B)^{(n\setminus(s_2-s_1))}$ is singular, or*
3. *A is singular, but B is non-singular and $I_r - (-B^{-1}A)^{(n\setminus(n-s_2+s_1))}$ is singular.*

9. Inverse of block circulant matrices with two parameters

Theorem 9.1. *Let n be a non-negative integer and let A and B be non-zero square matrices of order r, such that $AB = BA$. If $I_n \otimes A + P_n \otimes B$ and $A^n - (-B)^n$ are non-singular, then*

$$(I_n \otimes A + P_n \otimes B)^{-1} = \sum_{i=0}^{n-1} P_n^i \otimes \left[(-1)^i B^i A^{n-i-1} \left(A^n - (-B)^n\right)^{-1}\right]$$

Proof. We just check that

$$(I_n \otimes A + P_n \otimes B)\left[\sum_{i=0}^{n-1} P_n^i \otimes \left[(-1)^i B^i A^{n-i-1} \left(A^n - (-B)^n\right)^{-1}\right]\right] = I_n \otimes I_r. \quad \square$$

In order to obtain an explicit formula for the inverse of non-singular circulant matrices of the form $P_n^{s_1} \otimes A + P_n^{s_2} \otimes B$, we define

$$\Omega_{n,s}(i) = (-1)^{\mathrm{pos}_{n,s}(i)}\, \delta_{0,\mathrm{cycle}_{n,s}(i)}\, B^{\mathrm{pos}_{n,s}(i)}\, A^{n\setminus s-1-\mathrm{pos}_{n,s}(i)} \left[A^{n\setminus s} - (-B)^{n\setminus s}\right]^{-1},$$

assuming that $\left[A^{n\setminus s} - (-B)^{n\setminus s}\right]^{-1}$ exists. Notice that Ω satisfies the following properties:

1. For $n > 0$, $\Omega_{n,1}(i) = (-1)^i B^i A^{n-i-1}$ for all $i = 0, \ldots, n-1$.
2. For $0 < s < n$, $\Omega_{n\setminus s,1}(i) = \Omega_{n,s}(i\, s)$ for all $i = 0, \ldots, (n\setminus s) - 1$.

Theorem 9.2. *Let n, s_1, s_2, and r be non-negative integers such that $0 \leqslant s_1 < s_2 \leqslant n-1$. Let A and B be two square matrices of order r such that $AB = BA$. If $P_n^{s_1} \otimes A + P_n^{s_2} \otimes B$ and $A^{n\setminus s_2-s_1} - (-B)^{n\setminus s_2-s_1}$ are non-singular, then*

$$(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B)^{-1} = \sum_{i=0}^{n-1} P_n^i \otimes \Omega_{n,s_2-s_1}(i + s_1).$$

Proof. Let $s = s_2 - s_1$. By (7.2) we have that $(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B)^{-1}$ is equal to

$$\left(P_{\sigma_{n,s}} \otimes I_r\right) \left[I_{(n,s)} \otimes \left(I_{n\backslash s} \otimes A + P_{n\backslash s} \otimes B\right)\right]^{-1} \left(P_{\sigma_{n,s}}^T \otimes I_r\right) \left(P_n^{n-s_1} \otimes I_r\right).$$

Thus, by Theorem 9.1 and the definition of $\Omega_{n,s}$, $(P_n^{s_1} \otimes A + P_n^{s_2} \otimes B)^{-1}$ is

$$\left(P_{\sigma_{n,s}} \otimes I_r\right) \left(\sum_{i=0}^{(n\backslash s)-1} I_{(n,s)} \otimes P_{n\backslash s}^i \otimes \Omega_{n\backslash s,1}(i) \right) \left(P_{\sigma_{n,s}}^T \otimes I_r\right) \left(P_n^{n-s_1} \otimes I_r\right).$$

Then, by (5.2) and (5.3),

$$\begin{aligned} (P_n^{s_1} \otimes A + P_n^{s_2} \otimes B)^{-1} &= \sum_{i=0}^{(n\backslash s)-1} P_n^{(i\,s)-s_1} \otimes \Omega_{n\backslash s,1}(i) \\ &= \sum_{i=0}^{(n\backslash s)-1} P_n^{(i\,s)-s_1} \otimes \Omega_{n,s}(i\,s) \\ &= \sum_{j=0}^{n-1} P_n^j \otimes \Omega_{n,s}(j + s_1). \end{aligned}$$ □

For example, for the matrices given in (7.1) we have that $\left(P_8 \otimes A + P_8^3 \otimes B\right)^{-1}$ is equal to

$$P_8 \otimes \Omega_{8,2}(2) + P_8^3 \otimes \Omega_{8,2}(4) + P_8^5 \otimes \Omega_{8,2}(6) + P_8^7 \otimes \Omega_{8,2}(0),$$

where

$$\Omega_{8,2}(0) = \frac{1}{2519} \begin{bmatrix} 2357 & 2206 \\ 2206 & 2357 \end{bmatrix},$$

$$\Omega_{8,2}(2) = -\frac{1}{2519} \begin{bmatrix} 1823 & 1219 \\ 1219 & 1823 \end{bmatrix},$$

$$\Omega_{8,2}(4) = \frac{1}{2519} \begin{bmatrix} 202 & -194 \\ -194 & 202 \end{bmatrix},$$

and

$$\Omega_{8,2}(6) = -\frac{1}{2519} \begin{bmatrix} 5508 & -4156 \\ -4156 & 5508 \end{bmatrix}.$$

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Andrés M. Encinas
Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain
andres.marcos.encinas@upc.edu

Daniel A. Jaume
Departamento de Matemáticas. Facultad de Ciencias Físico-Matemáticas y Naturales, Universidad Nacional de San Luis, Instituto de Matemáticas Aplicadas de San Luis, IMASL-CONICET, San Luis, Argentina
djaume@unsl.edu.ar

Cristian Panelo ✉
Departamento de Matemáticas, Facultad de Ciencias Físico-Matemáticas y Naturales, Universidad Nacional de San Luis, San Luis, Argentina
crpanelo@unsl.edu.ar

Denis E. Videla
FaMAF - CIEM (CONICET), Universidad Nacional de Córdoba, Argentina
devidela@famaf.unc.edu.ar

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