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The composite of irreducible morphisms in standard components

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ABSTRACT

In this work, we consider standard components of the Auslander–Reiten quiver with trivial valuation. We give a characterization of when there are n irreducible morphisms between modules in such a component with non-zero composite belonging to the $n+1$-th power of the radical. We prove that a necessary condition for their existence is that it has to be a non-zero cycle or a non-zero bypass in the component. For directed algebras, we prove that the composite of n irreducible morphisms between indecomposable modules belongs to a greater power of the radical, greater than n, if and only if it is zero.

The notions of irreducible morphisms and almost split sequences, introduced by Auslander and Reiten, play a fundamental role in the representation theory of algebras. There is an important connection between an irreducible morphism between indecomposable modules and the radical of the category mod$A$, and it is given by the fact that a morphism between indecomposable modules $f : X \to Y$ is irreducible if and only if it belongs to the radical $\Re(X, Y)$ and not to its square $\Re^2(X, Y)$. The study of the radical of the category mod$A$ gives a better understanding of the module category.

The purpose of this work is to study the composite of n irreducible morphisms between indecomposable modules. It is well known that such a composite belongs to $\Re^n$. It is not always true that it is not in $\Re^{n+1}$ (see, the examples in [8] and [9]).

In particular, we are interested in giving necessary and sufficient conditions for the composite of n irreducible morphisms between indecomposable modules to be a non-zero morphism in $\Re^{n+1}$. This problem has been solved for two irreducible morphisms over artin algebras (see [8]) and for n irreducible morphisms belonging to a special path, called left or right almost sectional paths over

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artin algebras (see [9]). Moreover, it has been also solved for three irreducible morphisms in the case of finite dimensional algebras over an algebraically closed field (see [10]).

In this work, we are going to consider A to be a finite dimensional algebra over an algebraically closed field k. In [6], the composite of n irreducible morphisms between modules over standard algebras has been studied, applying covering techniques in order to give a full solution to the problem. Here, we will present a generalization of those results for the composite of n irreducible morphisms between indecomposable modules belonging to a standard component of the Auslander–Reiten quiver Γ_A of mod A, with trivial valuation.

We generalize some useful results proven in [11] to translation quivers with length. The idea of the main proof in this paper is to use the fact that Γ admits a simply connected universal covering and therefore it is a component with length. Then we will use the universal covering functor to obtain our results.

Concerning the problem of the composite of n irreducible morphisms, we will prove the following result:

Theorem. *Let Γ be a standard component of Γ_A with trivial valuation and $X_i \in \Gamma$, for $1 \leqslant i \leqslant n+1$. Then the following conditions are equivalent*:

(a) *There are irreducible morphisms $h_i : X_i \to X_{i+1}$ such that $h_n \dots h_1 \neq 0$ and $h_n \dots h_1 \in \Re^{n+1}(X_1, X_{n+1})$.*
(b) *There are n irreducible morphisms $f_i : X_i \to X_{i+1}$ with zero composite and a morphism $\varphi = \epsilon_n \dots \epsilon_1 \neq 0$, where $\epsilon_i = f_i$ or $\epsilon_i \in \Re^2(X_i, X_{i+1})$.*

In the particular case of a directed algebra, we will prove that the composite of n irreducible morphisms between indecomposable modules belongs to a greater power of the radical, greater than n, if and only if it is zero.

This paper is organized as follows. In Section 1, we recall some notions and give notations needed throughout the paper. In Section 2, we discuss necessary and sufficient conditions for the existence of n irreducible morphisms between indecomposable modules in a standard component with trivial valuation, such that their composite is non-zero in $\Re^{n+1}$.

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1. Preliminaries

Throughout this section, we will give some notations, definitions and preliminaries notions for which we suggest the reader to see the general textbooks like [2,3,18]; for a detail account on coverings techniques we refer the reader to [4,5,13,16,17].

1.1. A *quiver* Γ is given by two sets Γ_0 and Γ_1 together with two maps $s, e : \Gamma_1 \to \Gamma_0$. The elements of Γ_0 are called *vertices* and the elements of Γ_1 are called *arrows*.

A quiver Γ is said to be *locally finite* if each vertex of Γ_0 is the starting and the ending point of at most finitely many arrows in Γ.

A pair (Γ, τ) is said to be a *translation quiver* provided Γ is a quiver without loops, no multiple arrows, and locally finite; and $\tau : \Gamma'_0 \to \Gamma''_0$ is a bijection whose domain Γ'_0 and codomain Γ''_0 are both subsets of Γ_0, and if for every $x \in \Gamma_0$ such that τx exists and for every $y \in x^- = \{y \in \Gamma_0 \mid \text{there exists an arrow } y \to x\}$, the number of arrows from y to x is equal to the number of arrows from τx to y.

1.2. A *k-category* A over an algebraically closed field is a category where for each pair of objects x, y in A, the set of morphisms $A(x, y)$ is a k-vector space and the composite of morphisms is k-bilinear.

Let (Q, I) be a connected locally finite bound quiver, in the sense of [4]. For each $x, y \in Q$ let $I(x, y) = e_x(kQ)e_y \cap I$. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$, (where $\lambda_i \in k^*$ and each w_i is a path from x to y) is *minimal* if $m \geqslant 2$ and for any non-empty proper subset $J \subset \{1, 2, \ldots, m\}$, we have $\sum_{j \in J} \lambda_i w_i \notin I(x, y)$. A walk in Q from x to y, is a sequence formed by arrows in Q and the formal inverses α^{-1} of the arrows $\alpha \in Q$. That is, it is a formal product $\alpha_1^{\varepsilon_1} \alpha_2^{\varepsilon_2} \ldots \alpha_t^{\varepsilon_t}$ where α_i are arrows in Q with $s(\alpha_{i+1}) = e(\alpha_i)$ or $e(\alpha_{i+1}) = e(\alpha_i)$ or $s(\alpha_{i+1}) = s(\alpha_i)$ or $e(\alpha_{i+1}) = s(\alpha_i)$, and $\varepsilon_i \in \{1, -1\}$ for all i, with source x and target y. We denote by e_x the trivial path at x.

On the set of walks of (Q, I) we define a *homotopy* relation as the smallest equivalence relation satisfying the following conditions:

(a) If $\alpha : X \to Y$ is an arrow, then $\alpha^{-1}\alpha \sim e_x$ and $\alpha\alpha^{-1} \sim e_y$.
(b) If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a minimal relation, then $w_i \sim w_j$ for all i, j.
(c) If $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions make sense.

1.3. Given a translation quiver (Γ, τ), the points of Γ_0 where τ (or τ^-) is not defined are called projective vertices (or injective vertices, respectively).

Consider a locally finite translation quiver Γ. The full subquiver of Γ given by a non-projective vertex x, the non-injective vertex τx and by the set $(\tau x)^+ = x^-$ is called *the mesh* starting at τx and ending at x. For each arrow $\alpha : y \to x$ with x non-projective, we denote by $\sigma\alpha$ the arrow $\tau x \to y$. The *mesh ideal* is the ideal I of the category $k\Gamma$, generated by the elements

$$\mu_x = \Sigma_\alpha \alpha(\sigma\alpha) \in k\Gamma(\tau x, x),$$

where x is not projective and α are the arrows of Γ ending at x. The *mesh category* of Γ is the quotient category $k(\Gamma) = k\Gamma/I$.

We observe that it is possible to define an equivalence relation over the paths of a translation quiver Γ, in the same way as we stated in (1.2) of this paper, setting $Q = \Gamma$ and setting I equal to the mesh ideal of $k\Gamma$. We call this equivalence relation *homotopy*, and it coincides with the one defined in [4, (1.2)].

Let $x \in \Gamma_0$ be arbitrary. The set $\pi_1(\Gamma, x)$ of equivalences classes $\bar{u}$ of closed paths u starting and ending at x has a group structure defined by the operation $\bar{u}.\bar{v} = \overline{u.v}$. Since Γ is connected then this group does not depend on the choice of x. We denote it $\pi_1(\Gamma, x)$ and call it the *fundamental group* of (Γ, x).

A translation quiver Γ is called *simply connected* if it is connected and $\pi_1(\Gamma, x) = 1$ for some $x \in \Gamma_0$.

We refer the reader to [4,16,17] for a detailed account on covering theory.

1.4. Throughout this paper A denotes a finite dimensional algebra over an algebraically closed field k. We denote by mod$A$ the category of all finitely generated left A-modules, and by ind$A$ the full subcategory of mod$A$ consisting of one representative of each isomorphism class of indecomposable A-module. We denote by Γ_A the Auslander–Reiten quiver of mod$A$, and by τ and τ^- the Auslander–Reiten translations DTr and TrD, respectively (see [3]).

1.5. The Auslander–Reiten quiver Γ_A, is a translation quiver with vertices the classes of isomorphisms of indecomposable A-modules. Denoting the vertex corresponding to an indecomposable module M by $[M]$, there is an arrow $[M] \to [N]$ between two vertices if and only if there is an irreducible morphism from M to N. For finite dimensional algebras over an algebraically closed field, it is known that an arrow $[M] \to [N]$ of Γ_A has *valuation* (a, a), that is, there is a minimal right almost split morphism $aM \oplus X \to N$ where M, N are indecomposable and M it is not a summand of X, and there is a minimal left almost split morphism $M \to aN \oplus Y$ where M, N are indecomposable and N is not a summand of Y. We say that the arrow has *trivial valuation* if $a = 1$.

A component Γ of Γ_A is said to be *a component with length* if parallel paths in Γ have the same length. In [11], we extended this notion to translation quivers. Observe that a component with length has no oriented cycles.

We say that a component Γ of Γ_A is *standard* if the full subcategory of mod$A$ generated by the modules of Γ is equivalent to the mesh category $k(\Gamma)$ of Γ (see [18]). This means, there is an isomorphism $\varphi : k(\Gamma) \to \text{ind}\,\Gamma$ such that

(a) $\varphi([M]) = M$, for each object $[M] \in k(\Gamma)$.
(b) For each arrow $\alpha \in k(\Gamma)$ $\varphi(\alpha) = f$ where f is an irreducible morphism in ind Γ, and
(c) If we consider the mesh ending at $[M]$

$$\begin{array}{ccccc} & \nearrow & [x_1] & \searrow & \\ [\tau M] & & \vdots & & [M] \\ & \searrow & [x_n] & \nearrow & \end{array}$$

then $\varphi(\text{mesh})$ is an almost split sequence ending at M in ind Γ.

Well-known examples of infinite standard components are the preprojective and preinjective components of a finite dimensional algebra over an algebraically closed field and the connecting components of tilted algebras. By [11], we know that if A is an algebra of finite representation type and Γ_A is with length, then Γ_A is standard.

From now on, we do not distinguish between the indecomposable A-modules and the corresponding vertices of Γ_A.

1.6. The *category of translation quivers* has as objects the translation quivers (Γ, τ), and as morphisms $F : (\Gamma', \tau') \to (\Gamma, \tau)$ the quiver-morphisms $F : \Gamma' \to \Gamma$ such that $F\tau' = \tau F$.

A morphism of translations quivers $F : (\Gamma', \tau') \to (\Gamma, \tau)$ is a *covering functor* if:

(i) for each point $x \in \Gamma'_0$ the induced applications $x^- \to (F(x))^-$ and $x^+ \to (F(x))^+$ are bijective; and
(ii) for each point $x \in \Gamma'_0$, τx and $\tau^{-1}x$ are defined if $\tau F(x)$ and $\tau^{-1}F(x)$ are respectively defined.

We fix a translation quiver (Γ, τ). The *category of coverings of the translation quiver* (Γ, τ) has as objects the pairs $((\Gamma', \tau'), F)$ such that $F : (\Gamma', \tau') \to (\Gamma, \tau)$ is a covering functor. A morphism $\varphi : ((\Gamma', \tau'), F) \to ((\Gamma'', \tau''), G)$ is a translation quiver morphism $\varphi : (\Gamma', \tau') \to (\Gamma'', \tau'')$ such that $G\varphi = F$. It is well known that this category has a universal object called *universal covering* of the translation quiver (Γ, τ) (see [4]).

Let Γ be a translation quiver and G a subgroup of the group of automorphisms $\mathrm{Aut}(\Gamma)$. The action of the group G in Γ is said to be *admissible* if none of the orbits of G in Γ_0 intersects a subquiver of the form $\{x\} \cup x^+$ or $\{x\} \cup x^-$ in more than one vertex.

By [4, 1.5], it is known that the fundamental group of Γ acts admissibly on the universal covering $\widetilde{\Gamma}$ of Γ, in such a way that Γ turns to be the orbit space of $\widetilde{\Gamma}$ under the action of the fundamental group, that is, there is a covering for Γ:

$$\pi : \widetilde{\Gamma} \to \big(\widetilde{\Gamma}/\pi_1(\Gamma)\big) \simeq \Gamma$$

We observe that $\widetilde{\Gamma}$ is simply connected. We refer to [4] and [13] for a detailed account on coverings.

1.7. Let $\widetilde{\Gamma}$ be the universal cover of Γ, defined as in [4, (1.2)]. Let $k(\widetilde{\Gamma})$ be the mesh category of $\widetilde{\Gamma}$, and ind Γ the full subcategory of ind$A$ whose objects are the indecomposable modules belonging to Γ.

Lemma 1.1. *There is a covering functor* $F : k(\widetilde{\Gamma}) \to \operatorname{ind}\Gamma$ *with fundamental group* $\Pi = \Pi_1(\Gamma, M)$ *for an arbitrary* $M \in \Gamma$.

Proof. Let $\pi : \widetilde{\Gamma} \to \Gamma$ be the universal cover of Γ. Consider the induced functor $F = k(\pi)$: $k(\widetilde{\Gamma}) \to k(\Gamma)$, where $k(\widetilde{\Gamma})$ and $k(\Gamma)$ are the mesh category of $\widetilde{\Gamma}$ and Γ, respectively. By [17, Proposition 2.2], F is a covering functor. $\square$

1.8. For $X, Y \in \operatorname{ind} A$, denote by $\Re_A(X, Y)$ the set of the morphisms $f : X \to Y$ which are not isomorphisms and by $\Re_A^\infty(X, Y)$ the intersection of all powers $\Re_A^i(X, Y)$, $i \geqslant 1$, of $\Re_A(X, Y)$.

We are also going to consider the radical of a k-category. We refer the reader to [14, p. 25], for a detailed account on this theory.

1.9. A component Γ of Γ_A is said to be a generalized standard component if $\Re^\infty(X, Y) = 0$, for each $X, Y \in \Gamma$.

1.10. Let Γ be a standard component of Γ_A and $\widetilde{\Gamma}$ the universal covering of Γ. Then the functor $F : k(\widetilde{\Gamma}) \to \operatorname{ind} \Gamma$, preserves irreducible morphisms (see [4, p. 337]), since it maps arrows into irreducible morphisms. Moreover, it was observed in [5, p. 27], that F also preserves the powers of the radical, that is, induces an isomorphism:

$$\bigsqcup_{F(Z)=F(Y)} \Re^n(X, Z)/\Re^{n+1}(X, Z) \simeq \Re^n\big(F(X), F(Y)\big)/\Re^{n+1}\big(F(X), F(Y)\big)$$

for $X, Y \in k(\widetilde{\Gamma})$, where $F(Z) = F(Y) \in k(\Gamma)$ (see [7]).

2. On the composite of irreducible morphisms

2.1. The aim of this section, is to look for necessary and sufficient conditions for the existence of n irreducible morphisms between indecomposable modules over standard components, such that their composite is a non-zero morphism in $\Re^{n+1}$. We will reduce the study of the component Γ to the study of a k-category, $k(\widetilde{\Gamma})$, passing from Γ to its universal cover $\widetilde{\Gamma}$. Moreover, since $\widetilde{\Gamma}$ is simply connected then by [4] it is a component with length, and we will get our results from those proven in [11], for translations quivers with length.

K. Bongartz and P. Gabriel considered the homotopy given by the mesh relations and defined simply connected translation quivers. They had implicitly proven that any component of a simply connected translation quiver is a component with length [4, proof of Proposition 1.6].

The notion of radical of $\operatorname{mod} A$ can be extended to k-categories and one can as well define the powers of the radical (see [4, p. 337]. Given a k-category, an irreducible morphism between objects in the k-category is a morphism f such that $f \in \Re \backslash \Re^2$.

We observe that Proposition 3.1 and Corollary 3.3 given in [11], can be stated in a more general context as follows:

Proposition 2.1. *Let Γ be a translation quiver with length and $X, Y \in k(\Gamma)$ such that $\ell(X, Y) = n$ with $n \geqslant 1$. Then*:

(a) $\Re^{n+1}(X, Y) = 0$.
(b) *If $g : X \to Y$ is a non-zero morphism in $k(\Gamma)$, then $g \in \Re^n(X, Y) \setminus \Re^{n+1}(X, Y)$.*
(c) $\Re^j(X, Y) = \Re^n(X, Y)$, *for each $j = 1, \ldots, n-1$.*

Proof. (a) Assume that there is a morphism $g \neq 0$ such that $g \in \Re^{n+1}(X, Y)$, with $X, Y \in k(\Gamma)$. We observe that Proposition 7.4 stated in [3] can be generalized to the mesh category of translation quivers. By [3, Proposition 7.4], there exist an integer $s \geqslant 1$, objects $B_1, B_2, \ldots, B_s$ in $k(\Gamma)$, morphisms $f_i \in \Re(X, B_i)$ and $g_i : B_i \to Y$ with each g_i a sum of composites of n irreducible morphism between objects such that $g = \Sigma_{i=1}^s g_i f_i$ with $g_i f_i \neq 0$. Moreover, each $f_i : X \to B_i$ can be written as $f_i = \Sigma_{k=1}^{r_i} \mu_{ik}$, where μ_{ik} is a composite of irreducible morphisms, for $k = 1, \ldots, r_i$. So the paths $g_i \mu_{ik} : X \to Y$ have length greater than n, contradicting that Γ is a component with length. Thus $\Re^{n+1}(X, Y) = 0$.

(b) If g is a non-zero morphism in $k(\Gamma)$ and $\ell(X, Y) = n$, then $g = \Sigma_{i=1}^s \lambda_i \overline{u_i}$, where each u_i is a path of length n from X to Y and $\overline{u_i}$ is the corresponding morphism in $k(\Gamma)$. Then, $g \in \Re^n(X, Y)$. Moreover, since g is a non-zero morphism, then by (a) we infer that $g \notin \Re^{n+1}(X, Y)$.

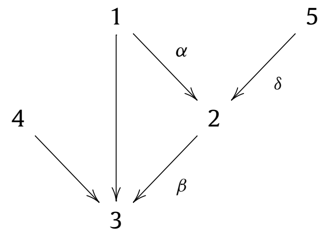
(c) Follows immediately from the fact that Γ is a component with length and $\ell(X, Y) = n$. $\square$

As an immediate application of the above result we get the following consequence:

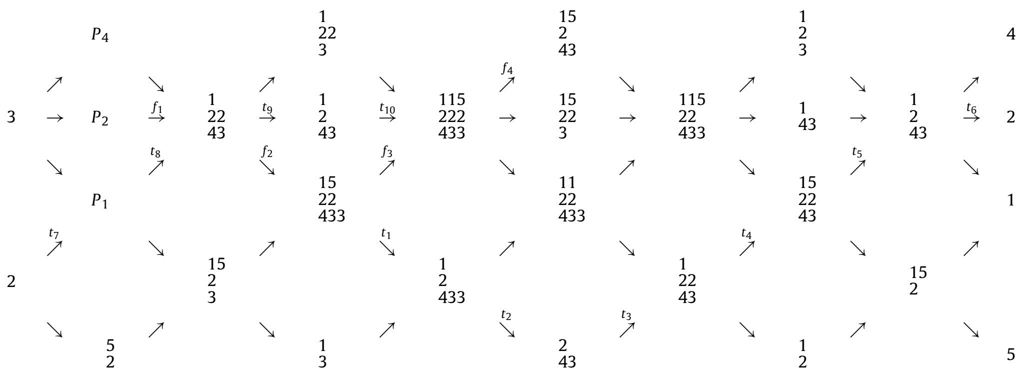
Corollary 2.2. *Let Γ be a translation quiver with length and $f_i : X_i \to X_{i+1}$ irreducible morphisms with $X_i \in k(\Gamma)$, for $i = 1, \dots, n$. Then $f_n \dots f_1 \in \Re^{n+1}$ if and only if $f_n \dots f_1 = 0$.*

2.2. Next, we will show an example of an algebra of finite representation type, having four irreducible morphisms with non-zero composite lying in $\Re^{13}$.

Example 2.3. Let A be the algebra of finite representation type over an algebraically closed field given by the quiver:



with the relations $\beta\alpha = 0$ and $\beta\delta = 0$. The Auslander–Reiten quiver is the following:



identifying the two vertices associated with the simple module S_2.

Consider $\mu = t_{10}t_9 \dots t_2t_1$. Then $\mu \in \Re^{10}$. Consider the irreducible morphisms $h_3 = f_3 + \mu$ and $h_i = f_i$ for $i = 1, 2, 4$. Observe that $f_4f_3f_2f_1 = 0$ and that $h_4h_3h_2h_1 = f_4\mu f_2f_1 \neq 0$. Moreover, $h_4h_3h_2h_1 \in \Re^{13}$. Hence we have found four irreducible morphisms such that their composite is non-zero and belongs to $\Re^{13}$.

2.3. For the convenience of the reader we state the following result, proven in [9, Lemma 3.4].

Lemma. *(See [9].) Let A be a finite dimensional algebra over an algebraically closed field k. Let Γ be a component of Γ_A with trivial valuation and $X_i \in \Gamma$ for $i = 1, \dots, n+1$. Then the following conditions are equivalent:*

(a) *There exist irreducible morphisms $f_i : X_i \to X_{i+1}$ with $f_n \cdots f_1 \notin \Re^{n+1}(X_1, X_{n+1})$.*
(b) *If $h_i : X_i \to X_{i+1}$, $1 \leqslant i \leqslant n$, are irreducible morphisms, then $h_n \dots h_1 \notin \Re^{n+1}(X_1, X_{n+1})$.*

It is proven in [15] that, if A is a finite dimensional algebra over an algebraically closed field, then any standard component of Γ_A is generalized standard. This fact is going to be essential for our results.

The next result will be fundamental in our further considerations.

Proposition 2.4. *Let A be a finite dimensional algebra over an algebraically closed field k. Let Γ be a standard component of Γ_A with trivial valuation and $X_i \in \Gamma$, for $i = 1, \dots, n+1$. Then the following conditions are equivalent*:

(a) *There exist irreducible morphisms $f_i : X_i \to X_{i+1}$, with $f_n \dots f_1 \in \Re^n(X_1, X_{n+1}) \backslash \Re^{n+1}(X_1, X_{n+1})$.*
(b) *For each set of irreducible morphisms $h_i : X_i \to X_{i+1}$, with $i = 1, \dots, n$, we have that $h_n \dots h_1 \neq 0$.*

Proof. Assume that (a) holds, and let $h_i : X_i \to X_{i+1}$ be irreducible morphisms for $i = 1, \dots, n$. Then, by Lemma [9] $h_n \dots h_1 \notin \Re^{n+1}(X_1, X_{n+1})$. Hence $h_n \dots h_1 \neq 0$.

(b) $\Rightarrow$ (a) The idea of this proof is to use that Γ has a simply connected universal covering $\widetilde{\Gamma}$. Since such a covering is with length, we can apply Proposition 2.1 to conclude that a non-zero composite of n irreducible morphisms in $k(\widetilde{\Gamma})$ does not belong to $\Re^{n+1}$. Then, using the Galois covering $k(\pi) : k(\widetilde{\Gamma}) \to \operatorname{ind} \Gamma$ together with 1.10, we obtain the result.

Let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1}$$

be a path of irreducible morphisms in $\operatorname{ind} \Gamma$. For each $i \in \{1, \dots, n\}$, there is a unique arrow $\alpha_i : X_i \to X_{i+1}$ in Γ such that $f_i = \lambda_i \overline{\alpha_i} + r_i$ where $\overline{\alpha_i}$ is the irreducible morphism associated with the arrow α_i, $r_i \in \Re^2(X_i, X_{i+1})$, and $\lambda_i \in k^*$.

Let

$$\widetilde{X}_1 \xrightarrow{\beta_1} \widetilde{X}_2 \xrightarrow{\beta_2} \widetilde{X}_3 \to \cdots \to \widetilde{X}_n \xrightarrow{\beta_n} \widetilde{X}_{n+1}$$

be a path in $\widetilde{\Gamma}$ lifting the path $\alpha_n \dots \alpha_1$ respect to $\pi : \widetilde{\Gamma} \to \Gamma$. If we denote by $\overline{\beta}_i$ the irreducible morphism in $k(\widetilde{\Gamma})$ corresponding to each arrow β_i in $\widetilde{\Gamma}$ then

$$F(\overline{\beta}_n \dots \overline{\beta}_1) = F(\overline{\beta_n \dots \beta_1}) = \overline{\alpha_n \dots \alpha_1} \neq 0$$

by hypothesis and therefore $\overline{\beta_n \dots \beta_1} \neq 0$.

On the other hand, by 2.1 we have that $\overline{\beta_n \dots \beta_1} \notin \Re^{n+1}(\widetilde{X}_1, \widetilde{X}_{n+1})$, so that $\overline{\alpha_n \dots \alpha_1} \notin \Re^{n+1}(X_1, X_{n+1})$ because of 1.10. Therefore, $f_n \dots f_1 = \lambda_n \dots \lambda_1 \overline{\alpha_n \dots \alpha_1} + \mu$ with $\mu \in \Re^{n+1}(\widetilde{X}_1, \widetilde{X}_{n+1})$. Then $f_n \dots f_1$ does not lie in $\Re^{n+1}(X_1, X_{n+1})$, proving the result. □

Note that the above result does not hold for artin algebras as we show in our next example.

Example 2.5. Consider the representation finite dimensional $\mathbb{R}$-algebra

$$A = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{R} \end{pmatrix}$$

where $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ the field of complex numbers. Let $P_1 = Ae_{11}$ and $P_2 = Ae_{22}$ be the indecomposable projective A-modules. The almost split sequence starting at S_2 can be written as

$$0 \to S_2 \xrightarrow{f} P_1 \xrightarrow{g} P_1/S_2 \to 0$$

where f is the irreducible morphism $f : S_2 \to P_1$ defined as

$$f \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

with $a \in \mathbb{R}$ and $g : P_1 \to P_1/S_2$ the canonical projection. Then $gf = 0$.

Now, if we consider the irreducible monomorphism $f' : S_2 \to P_1$ given by

$$f' \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ ai & 0 \end{pmatrix}$$

then the composite gf' is different from zero. The Auslander–Reiten quiver of $modA$:

$$\begin{array}{ccccccc} & & P_1 & & & & \tau^{-1}P_1 \\ & {}^{(2,1)}\nearrow & & \searrow^{(1,2)} & & {}^{(2,1)}\nearrow & \\ S_2 & & & & P_1/S_2 & & \end{array}$$

shows that Γ_A is a component with length. Then

$$gf' \in \Re^2(S_2, P_1/S_2) \backslash \Re^3(S_2, P_1/S_2)$$

proving that the above result does not hold in this case.

As an application of the above result, we get this useful corollary.

Corollary 2.6. *Let Γ be a standard component of Γ_A with trivial valuation. The composite of n irreducible epimorphisms between modules in Γ belongs to $\Re^n \backslash \Re^{n+1}$.*

Proof. Assume that there are n irreducible epimorphisms between modules in Γ such that their composite is in $\Re^{n+1}(X_1, X_{n+1})$. We observe that if $f : X \to Y$ is an irreducible epimorphism, then any other irreducible morphism from X to Y is also an epimorphism. By Proposition 2.4 and Lemma [9], there are irreducible epimorphisms $f_i : X_i \to X_{i+1}$ such that $f_n \ldots f_1 = 0$. Since f_1 is an epimorphism then $f_n \ldots f_2 = 0$. Iterating this argument we get that $f_n = 0$, a contradiction to the fact that f_n is an irreducible morphism. Then we prove that the composite of n irreducible epimorphisms between modules in Γ belongs to $\Re^n(X_1, X_{n+1}) \backslash \Re^{n+1}(X_1, X_{n+1})$, where $n \geqslant 2$. □

A dual statement holds for the composite of irreducible monomorphisms.

Now we are in the position to state and prove our main result.

Theorem 2.7. *Let A be a finite dimensional algebra over an algebraically closed field k. Let Γ be a standard component of Γ_A with trivial valuation and $X_i \in \Gamma$, for $i = 1, \ldots, n+1$. Then the following conditions are equivalent*:

(a) *There exist irreducible morphisms $h_i : X_i \to X_{i+1}$ with $h_n \ldots h_1 \neq 0$ and $h_n \ldots h_1 \in \Re^{n+1}(X_1, X_{n+1})$.*

(b) *There are a path $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots \to X_n \xrightarrow{f_n} X_{n+1}$ of irreducible morphisms with zero composite and a morphism $\varphi = \epsilon_n \ldots \epsilon_1 \neq 0$ where $\epsilon_i = f_i$ or $\epsilon_i \in \Re^2(X_i, X_{i+1})$.*

Proof. Consider irreducible morphisms $h_i : X_i \to X_{i+1}$ as in (a). By Lemma [9] and Proposition 2.4, there are irreducible morphisms $f_i : X_i \to X_{i+1}$ such that $f_n \ldots f_1 = 0$.

On the other hand, since Γ has trivial valuation then each irreducible morphism h_i is of the form $h_i = \alpha_i f_i + \mu_i$ with $\alpha_i \in k^*$ and $\mu_i \in \Re^2(X_i, X_{i+1})$.

We claim that $\mu_i \neq 0$ for some i. In fact, if $\mu_i = 0$ for each i, then $h_n \ldots h_1 = \alpha f_n \ldots f_1 = 0$ with $\alpha \in k$, contradicting the hypothesis. Therefore, $\Re^2(X_i, X_{i+1}) \neq 0$ for some i.

Now, we are going to prove that for each $i = 1, \ldots, n$ there exists a morphism $\varphi_i = \epsilon_i \ldots \epsilon_1 \in \Re^i(X_1, X_{i+1})$, where $\epsilon_i \in \Re^2(X_i, X_{i+1})$ or $\epsilon_i = f_i$, such that $h_n \ldots h_{i+1}\varphi_i \neq 0$ and $h_n \ldots h_{i+1}\varphi_i \in \Re^{n+1}(X_1, X_{n+1})$.

We will prove the result by induction on i. If $i = 1$, then

$$h_n \ldots h_1 = h_n \ldots h_2(\alpha_1 f_1 + \mu_1) = \alpha_1 h_n \ldots h_2 f_1 + h_n \ldots h_2 \mu_1 \neq 0$$

Therefore at least one summand is non-zero. If $\alpha_1 h_n \dots h_2 f_1 \neq 0$, then considering $\varphi_1 = f_1$ we get the result. Otherwise, $\varphi_1 = \mu_1 \in \Re^2(X_1, X_2)$ satisfies the result.

Assume that the result holds for $i = m$, that is, there exist a morphism $\varphi_m = \epsilon_m \dots \epsilon_1$, where $\epsilon_i \in \Re^2(X_i, X_{i+1})$ or $\epsilon_i = f_i$ for $i = 1, \dots, m$ satisfying that $h_n \dots h_{m+1}\varphi_m$ is a non-zero morphism in $\Re^{n+1}(X_1, X_{n+1})$. Since $h_{m+1} = \alpha_{m+1} f_{m+1} + \mu_{m+1}$ with $\alpha_{m+1} \in k^*$ and $\mu_{m+1} \in \Re^2(X_{m+1}, X_{m+2})$ then

$$h_n \dots (\alpha_{m+1} f_{m+1} + \mu_{m+1})\varphi_m = \alpha_{m+1} h_n \dots f_{m+1}\varphi_m + h_n \dots \mu_{m+1}\varphi_m \neq 0.$$

By the inductive hypothesis we know that φ_m is of the form $\varphi_m = \epsilon_m \dots \epsilon_1$ where $\epsilon_i \in \Re^2(X_i, X_{i+1})$ or $\epsilon_i = f_i$.

If $h_n \dots f_{m+1}\varphi_m$ is non-zero, then $\varphi_{m+1} = f_{m+1}\varphi_m$ satisfies the statement. If otherwise, $h_n \dots \mu_{m+1}\varphi_m \neq 0$, we define $\varphi_{m+1} = \mu_{m+1}\varphi_m$. Then $\mu_{m+1}\varphi_m$ belongs to $\Re^{m+3}(X_1, X_{m+2})$. Then we proved that there is a morphism $\varphi = \epsilon_n \dots \epsilon_1 \neq 0$, where $\epsilon_i = f_i$ or $\epsilon_i \in \Re^2(X_i, X_{i+1})$. Observe that for some $j = 1, \dots, n$ it follows that $\epsilon_j \in \Re^2(X_j, X_{j+1})$, since otherwise $\varphi = 0$ a contradiction.

Suppose that (b) holds. We know that there is a non-zero morphism $\varphi = \epsilon_n \dots \epsilon_1$, where $\epsilon_i = f_i$ or $\epsilon_i \in \Re^2(X_i, X_{i+1})$. We may assume that the number m of indices is such that $\epsilon_i \in \Re^2(X_i, X_{i+1})$ is minimum. We will prove (a), by induction on m.

If $m = 1$, let j be the unique index such that $\epsilon_j \in \Re^2(X_j, X_{j+1})$. Consider the irreducible morphisms $h_i = f_i$ and $h_j = f_j + \epsilon_j$. Then, since $f_n \dots f_1 = 0$ we have $h_n \dots h_1 = f_n \dots f_{j+1}\epsilon_j f_{j-1} \dots f_1$ and then $0 \neq h_n \dots h_1 \in \Re^{n+1}(X_1, X_{n+1})$. So the result holds in this case.

If $m = 2$, assume that $\epsilon_j \in \Re^2(X_j, X_{j+1})$ and $\epsilon_m \in \Re^2(X_m, X_{m+1})$ with $m \neq j$ and $1 \leqslant j, m \leqslant n$. Consider the irreducible morphisms $h_i = f_i$ if $i \neq j$, $i \neq m$ and $h_i = f_i + \epsilon_i$ for $i = j, m$. Then, $h_n \dots h_1 = f_n \dots f_{j+1}\epsilon_j f_{j-1} \dots f_{m+1}\epsilon_m f_{j-1} f_1$ is a non-zero morphism that belongs to $\Re^{n+1}(X_1, X_{n+1})$, since by the inductive hypothesis $f_n \dots f_1$, $f_n \dots f_{j+1}\epsilon_j f_{j-1} \dots f_1 = 0$ and $f_n \dots f_{m+1}\epsilon_m f_{m-1} \dots f_1 = 0$.

This argument can be iterated and considering a morphism φ with the minimum factors ϵ_i in $\Re^2(X_i, X_{i+1})$ such that $\varphi \neq 0$, we prove inductively the result for arbitrary $m > 1$, finding n irreducible morphisms with non-zero composite in $\Re^{n+1}(X_1, X_{n+1})$. □

In [12], Crawley-Boevey, Happel and Ringel, introduced the concept of bypass of an arrow for translations quivers. The notion of bypass below generalizes the one given by them.

2.4. Let $f : X \to Y$ be an irreducible morphism, where X and Y are indecomposable. A *bypass* of f is a path $X \xrightarrow{t_1} Y_1 \xrightarrow{t_2} Y_2 \to \cdots \to Y_n \xrightarrow{t_{n+1}} Y$ in $\operatorname{ind} A$ of length $n \geqslant 2$ where t_1 and t_{n+1} are irreducible morphisms, $X \not\simeq Y_n$ and $Y \not\simeq Y_1$. If all the morphisms t_i's are irreducible, then we say that the bypass belongs to Γ_A.

The following result holds for generalized standard components of Γ_A.

Lemma 2.8. *Let Γ be a generalized standard component of Γ_A and $\Re(X, Y) \setminus \Re^2(X, Y) \neq 0$ for $X, Y \in \Gamma$. If $\mu \neq 0$ in $\Re^2(X, Y)$, then there is a bypass of an irreducible morphism from X to Y, a cycle from X to X, or a cycle from Y to Y in* $\operatorname{mod} A$.

Proof. Consider a non-zero morphism $\mu \in \Re^2(X, Y)$. Since Γ is a generalized standard component, then $\mu = \Sigma_{i=1}^{t} u_i$ where each u_i is a non-zero composite of at least two irreducible morphisms between indecomposable modules. Without loss of generality we can consider the non-zero path u_1 and we write $u_1 = f_r \cdots f_1$ with $r \geqslant 2$ and

$$u_1 : X \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \to \cdots \to A_r \xrightarrow{f_r} Y.$$

If $A_1 \simeq Y$ or $A_r \simeq X$, then there is a cycle from Y to Y in Γ, or a cycle from X to X in Γ, respectively. Otherwise, if $A_1 \not\simeq Y$ and $A_r \not\simeq X$ then there is a bypass of an arrow from X to Y in Γ, proving the result. □

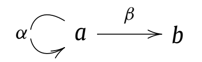
As an immediate consequence of the above lemma we have the following corollary.

Corollary 2.9. *Let Γ be a standard component of Γ_A. If there are irreducible morphisms $h_i : X_i \to X_{i+1}$ with $X_i \in \Gamma$, for $i = 1, \ldots, n+1$ with $0 \neq h_n \ldots h_1 \in \Re^{n+1}(X_1, X_{n+1})$, then there exist a bypass or a cycle in Γ_A passing through some X_i, for $i = 1, \ldots, n+1$.*

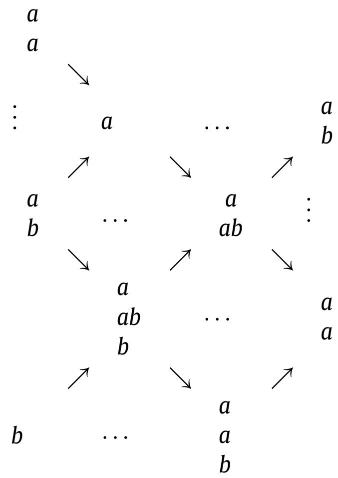
Proof. The result follows from Theorem 2.7 and the fact that Γ is a generalized standard component of Γ_A. □

Observe that in general the converse of Corollary 2.9 does not hold as we show in our next example (see [12, p. 527]).

Example 2.10. Consider the non-triangular algebra of finite representation type given by the quiver:



with the relation $\alpha^2 = 0$. The Auslander–Reiten quiver is the following, denoting the indecomposable modules by their Loewy series:



identifying the two vertices associated with $\begin{smallmatrix} a \\ a \end{smallmatrix}$ and the two associated with $\begin{smallmatrix} a \\ b \end{smallmatrix}$. Observe that there is a sectional cycle in Γ_A:

$$\begin{matrix} a \\ ab \\ b \end{matrix} \longrightarrow \begin{matrix} a \\ ab \end{matrix} \longrightarrow \begin{matrix} a \\ b \end{matrix} \longrightarrow \begin{matrix} a \\ ab \\ b \end{matrix}$$

and a non-sectional bypass in Γ_A:

$$\begin{matrix} a \\ ab \\ b \end{matrix} \longrightarrow \begin{matrix} a \\ a \\ b \end{matrix} \longrightarrow \begin{matrix} a \\ a \end{matrix} \longrightarrow a \longrightarrow \begin{matrix} a \\ ab \end{matrix}$$

of the arrow

$$\begin{matrix} a \\ ab \\ b \end{matrix} \longrightarrow \begin{matrix} a \\ ab \end{matrix} .$$

Then Γ_A is a component without length, with bypasses and cycles. Observe that any possible composition of irreducible morphisms belongs to a greater power of the radical if and only if it is zero.

Now, we are going to prove that if A is a directed algebra, then the composite of n irreducible morphisms belongs to $\Re^{n+1}$ if and only if it is zero. First, we will recall the definition of standard algebra given by Bongartz and Gabriel in [4] and the definition of a directed algebra given by C. Ringel in [18].

2.5. A finite dimensional algebra of finite representation type over an algebraically closed field k is a *standard algebra* if the category ind$A$ is equivalent to the mesh category $k(\Gamma_A)$ of Γ_A.

2.6. A finite dimensional algebra over an algebraically closed field k is a *directed algebra* if each indecomposable A-module is directed, that is, it does not belong to any cycle in mod A. In [18, Corollary 9′ p. 76], Ringel proved that a directed algebra is of finite representation type.

We observe that if A is a triangular algebra of finite representation type then, A is a standard algebra (see [5, p. 3]). In particular, a directed algebra is standard.

Crawley-Boevey, Happel, and Ringel and independently José Antonio de la Peña proved that there are no bypasses in the Auslander–Reiten quiver of a directed algebra (see [12,19]). In [1], a new proof of this fact is given.

As a consequence of the above results we have the following corollary:

Corollary 2.11. *Let A be a directed algebra. Let $h_1 : X_1 \to X_2, \ldots, h_n : X_n \to X_{n+1}$ be irreducible morphisms with $X_i \in \Gamma_A$ for $i = 1, \ldots, n+1$. Then $h_n \ldots h_1 \in \Re^{n+1}(X_1, X_{n+1})$ if and only if $h_n \ldots h_1 = 0$.*

Proof. Assume that there exist n irreducible morphisms between indecomposable modules h_i: $X_i \to X_{i+1}$ such that $h_n \ldots h_1 \neq 0$ in $\Re^{n+1}(X_1, X_{n+1})$. By [19], we know that there are no bypasses in a directed algebra. Then by Corollary 2.9 there exists a cycle passing through some X_i, for $i = 1, \ldots, n+1$. This contradicts that Γ_A is directed. Then, $h_n \ldots h_1 = 0$.

The converse is trivial. □

Remark 2.12. If Γ is a connecting component of a tilted algebra of type $Z\Delta$ with trivial valuation, where Δ has no bypass, then $h_n \ldots h_1 \in \Re^{n+1}(X_1, X_{n+1})$ if and only if $h_n \ldots h_1 = 0$ (see [1]).

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