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Quasi-Modal Lattices

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Abstract. We introduce the class of bounded distributive lattices with two operators, Δ and ∇, the first between the lattice and the set of its ideals, and the second between the lattice and the set of its filters. The results presented can be understood as a generalization of the results obtained by S. Celani.

1. Introduction

In [8] Priestley found a topological duality for bounded distributive lattices. This duality has been extended to diverse algebraic structures whose reducts are distributive lattices. As examples, we can cite the topological duality for distributive lattices with operators (see [3, 5–7], etc.). In this context, it is well known that if in a bounded distributive lattice $\mathbf{A}$ we consider its associated Priestley space

$$\widetilde{\mathcal{T}}_{X(\mathbf{A})} = \langle X(\mathbf{A}), \subseteq, \mathcal{T}_{X(\mathbf{A})}\rangle$$

then $\mathbf{A}$ will be isomorphic to the lattice of the clopen increasing of the topology $\mathcal{T}_{X(\mathbf{A})}$ (i.e. $\mathbf{A} \cong \mathbf{D}(X(\mathbf{A}))$). If the structure being considered is a modal lattice $\mathbf{A} = \langle A, \vee, \wedge, \Box, \Diamond, 0, 1\rangle$, i.e., a bounded distributive lattice with two unary modal operators, $\Box$ (preserving $\wedge$ and 1) and $\Diamond$ (preserving $\vee$ and 0), then the operator $\Box$ can be interpreted as a function

$$f\colon A \to D(X(\mathbf{A}))$$

such that $f(a \wedge b) = f(a) \cap f(b)$ and $f(1) = X(\mathbf{A})$. The operator $\Diamond$ can be interpreted as a function

$$g\colon A \to D(X(\mathbf{A}))$$

such that $g(a \vee b) = g(a) \cup g(b)$ and $g(0) = \emptyset$. In other words, the unary modal operators can be understood as functions that assign to each element of the lattice

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a clopen increasing in the topology $\mathcal{T}_{X(\mathbf{A})}$ and at the same time comply with the expressed conditions. We can now ask a series of questions: for example, what type of operators can be obtained if, for each $a \in A$, $f(a)$ were only open and increasing and $g(a)$ closed and increasing in $\mathcal{T}_{X(\mathbf{A})}$? For the first case we will bear in mind that in $\widetilde{\mathcal{T}}_{X(\mathbf{A})}$ the lattice of the ideals of $\mathbf{A}$ is isomorphic to the lattice of the increasing open of $\mathcal{T}_{X(\mathbf{A})}$. Thus, the new operator obtained will be of the form $\Delta\colon A \to \mathrm{Id}(\mathbf{A})$ such that $\Delta(a \wedge b) = \Delta a \cap \Delta b$ and $\Delta 1 = A$. If $g(a)$ is closed and increasing we will bear in mind the fact that in $\widetilde{\mathcal{T}}_{X(\mathbf{A})}$ the lattice of the filters of $\mathbf{A}$ is anti-isomorphic to the lattice of the increasing closed of $\mathcal{T}_{X(\mathbf{A})}$. Consequently, the new operator obtained will be of the form $\nabla\colon A \to \mathrm{Fi}(\mathbf{A})$ such that $\nabla(a \vee b) = \nabla a \cap \nabla b$ and $\nabla 0 = A$. This gives rise to what we term as quasi-modal lattices, and their study is the main purpose of the present paper. Some results of this paper are an extension from the work done by S. Celani in [2]. We note that quasi-modal lattices are not algebras according to the standard terminology of universal algebra. The expression "quasi-modal" is due to the fact that the functions Δ and ∇ are not modal unary operations like those usually defined in a modal lattice, but in some way behave as such.

In Section 2 we basically review some notions of the Priestley duality and the representation of the bounded distributive lattices (see [8–10]). In the following section, we define the quasi-modal lattices and we show, among other things, that the class of these new structures is a generalization of the modal lattices (see [5, 6] and [7]). We also establish a notion of homomorphism between quasi-modal lattices and give a representation theorem. Next we define the descriptive quasi-modal spaces and prove that they are dually equivalent to quasi-modal lattices. The concept of quasi-modal sublattices is introduced and characterized in Section 4. In the last section, we define and characterize the congruences in the quasi-modal lattices. Finally, as an application of the congruences, we introduce the notion of simple and subdirectly irreducible quasi-modal lattices and give a characterization in terms of the dual space. For this characterization we take into account the ideas of A. Petrovich [7].

2. Preliminaries

A *totally order-disconnected* topological space is a triple $\langle X, \leqslant, \mathcal{T}_X \rangle$, where $\langle X, \leqslant \rangle$ is a poset, $\langle X, \mathcal{T}_X \rangle$ is a topological space and given $x, y \in X$ such that $x \nleqslant y$, there is a clopen increasing set U such that $x \in U$ and $y \notin U$. A *Priestley space* is a compact totally order-disconnected topological space. If $\langle X, \leqslant, \mathcal{T}_X \rangle$ is a Priestley space the set of all clopen increasing sets of $\mathcal{T}_X$ is denoted by $D(X)$, and it is well known that $\mathbf{D}(X) = \langle D(X), \cup, \cap, \emptyset, X \rangle$ is a bounded distributive lattice.

If $\mathbf{A} = \langle A, \vee, \wedge, 0, 1 \rangle$ is a bounded distributive lattice, we denote the set of all prime filters of $\mathbf{A}$ by $X(\mathbf{A})$ and the families of all ideals and filters of $\mathbf{A}$ by $\mathrm{Id}(\mathbf{A})$ and $\mathrm{Fi}(\mathbf{A})$, respectively. Given a bounded distributive lattice $\mathbf{A}$ it is known that $\langle \mathcal{P}_i(X(\mathbf{A})), \cup, \cap, \emptyset, X(\mathbf{A}) \rangle$, where $\mathcal{P}_i(X(\mathbf{A}))$ denotes the family of all $\subseteq$-

increasing subsets of $X(\mathbf{A})$, is a bounded distributive lattice and that the function $\beta\colon A \to \mathcal{P}_i(X(\mathbf{A}))$ given by $\beta(a) = \{P \in X(\mathbf{A}) : a \in A\}$ is a one-to-one lattice homomorphism, i.e. $\mathbf{A} \cong \beta(\mathbf{A})$. Moreover, the structure $\langle X(\mathbf{A}), \subseteq, \mathcal{T}_{X(\mathbf{A})}\rangle$ where the topology $\mathcal{T}_{X(\mathbf{A})}$ has the set

$$\beta(A) \cup \{X(\mathbf{A}) - \beta(a) : \beta(a) \in \beta(A)\}$$

as a subbase is a Priestley space and $\mathbf{A} \cong \mathbf{D}(X(\mathbf{A}))$. Also, the map $\varphi\colon \mathrm{Id}(\mathbf{A}) \to \mathcal{O}_i(X(\mathbf{A}))$ given by

$$\varphi(I) = \{P \in X(\mathbf{A}) : P \cap I \neq \emptyset\},$$

where $\mathcal{O}_i(X(\mathbf{A}))$ denotes the set of all open increasing subsets of $X(\mathbf{A})$, is a lattice isomorphism. Similarly, the function $\psi\colon \mathrm{Fi}(\mathbf{A}) \to C_i(X(\mathbf{A}))$ given by

$$\psi(F) = \{P \in X(\mathbf{A}) : F \subseteq P\},$$

where $C_i(X(\mathbf{A}))$ denotes the set of all closed increasing subsets of $X(\mathbf{A})$, is a lattice anti-isomorphism. Such functions can be expressed in terms of β as follows: $\varphi(I) = \bigcup_{a\in I} \beta(a)$ for each $I \in \mathrm{Id}(\mathbf{A})$ and $\psi(F) = \bigcap_{a\in F} \beta(a)$ for each $F \in \mathrm{Fi}(\mathbf{A})$.

In addition, if $\langle X, \leqslant, \mathcal{T}_X\rangle$ is a Priestley space then the map $\varepsilon_X\colon X \to X(D(X))$ defined by $\varepsilon_X(x) = \{U \in D(X) : x \in U\}$ is a homeomorphism and order-isomorphism. Moreover, we will say that $R \subseteq X \times X$ is a *lattice preorder* if is reflexive, transitive and verifies the following condition: $\forall x, y \in X$ if $(x, y) \notin R$, $\exists U \in D(X) : y \in U, x \notin U$ and $R^{-1}(U) \subseteq U$.

Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be bounded distributive lattices and let $h\colon A_1 \to A_2$ be a lattice homomorphism. Then the application $\mathcal{F}(h)\colon X(\mathbf{A}_2) \to X(\mathbf{A}_1)$ given by $\mathcal{F}(h)(P) = h^{-1}(P)$, for each $P \in X(\mathbf{A}_2)$, is a continuous and monotonic function. Let $\langle X, \leqslant, \mathcal{T}_X\rangle$ and $\langle Y, \leqslant, \mathcal{T}_Y\rangle$ be two Priestley spaces. If $f\colon X \to Y$ is a continuous and monotonic function, then the application $\mathcal{A}(f)\colon D(Y) \to D(X)$ given by $\mathcal{A}(f)(U) = f^{-1}(U)$, for each $U \in D(Y)$, is a lattice homomorphism.

Let $\langle X, \leqslant\rangle$ be a poset and let Y be a subset of X. The set of maximal (minimal) elements of Y will be denoted by $\max Y$ ($\min Y$).

Let $\langle X, \leqslant, \mathcal{T}_X\rangle$ be a Priestley space and Y a closed subset of X. Then it is known (see [10]) that for each $x \in Y$ there is $z \in \max Y$ ($\min Y$) such that $x \leqslant z$ ($z \leqslant x$). In particular, if $Y \neq \emptyset$ then $\max Y \neq \emptyset$ and $\min Y \neq \emptyset$.

Given a topological space $\langle X, \mathcal{T}_X\rangle$ and $Y \subseteq X$, we will use the notation $\mathrm{Cl}(Y)$ to express the closure of Y. Also, the set of all closed subset of X will be denoted by $C(X)$.

Let $\mathbf{A}$ be a bounded distributive lattice. The filter (ideal) generated by a subset $H \subseteq A$ will be denoted by $[H)$ or $F(H)$ ($(H]$ or $I(H)$). When $H = \{a\}$ we will write $[a)$ or $F(a)$ ($(a]$ or $I(a)$) instead of $[\{a\})$ or $F(\{a\})$ ($(\{a\}]$ or $I(\{a\})$), respectively.

If X is an arbitrary set and R a binary relation defined on X, then for any $x \in X$, $R(x)$ will denote the image of x by R. More precisely, $R(x) = \{y \in X : (x, y)$

$\in R\}$. While dom(R) denotes the domain of R, i.e., dom$(R) = \{x \in X : \exists y \in X$ such that $(x, y) \in R\}$. The identity or diagonal relation i_X on X is the set $\{(x, x) : x \in X\}$.

Finally, if Y is a subset of a set X then Y^c will denote the set-theoretical complement of Y, i.e. $Y^c = X - Y$.

3. Quasi-Modal Lattices

DEFINITION 1. A *quasi-modal lattice*, or *qm-lattice* for short, is a structure $\mathbf{A} = \langle A, \wedge, \vee, \Delta, \nabla, 0, 1\rangle$, where $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and Δ, ∇ are two functions

$$\Delta\colon A \to \mathrm{Id}(\mathbf{A}); \qquad \nabla\colon A \to \mathrm{Fi}(\mathbf{A})$$

that verify the following conditions:

(1) $\Delta(a \wedge b) = \Delta a \cap \Delta b$,
(2) $\Delta 1 = A$,
(3) $\nabla(a \vee b) = \nabla a \cap \nabla b$,
(4) $\nabla 0 = A$.

The class of qm-lattices is denoted by $\mathcal{QML}$.

EXAMPLE 1. Let $\mathbf{A}$ be a bounded distributive lattice and X a subset of A. If we define the functions $\Delta\colon A \to \mathrm{Id}(\mathbf{A})$ and $\nabla\colon A \to \mathrm{Fi}(\mathbf{A})$ by

$$\Delta a = I(X \cup \{a\}) \quad \text{and} \quad \nabla a = F(X \cup \{a\})$$

for any $a \in A$, then the structure $\langle A, \wedge, \vee, \Delta, \nabla, 0, 1\rangle$, is a quasi-modal lattice.

EXAMPLE 2. Let ω be the set of natural numbers. Next, we define the following correspondences

$$\Delta(X) \mapsto \begin{cases} \{Y \subseteq \omega : Y \subseteq X \text{ and } Y \text{ is finite}\} & \text{if } X \neq \omega, \\ \mathcal{P}(\omega) & \text{otherwise} \end{cases}$$

and

$$\nabla(X) \mapsto \begin{cases} \{Y \subseteq \omega : X \subseteq Y \text{ and } Y \text{ is cofinite}\} & \text{if } X \neq \emptyset, \\ \mathcal{P}(\omega) & \text{otherwise.} \end{cases}$$

It is easy to check that $\Delta(X)$ is an ideal of $\mathcal{P}(\omega)$ and that $\nabla(X)$ is a filter of $\mathcal{P}(\omega)$, for any $X \subseteq \omega$. We can see that Δ and ∇ verify the conditions of Definition 1. So, the structure

$$\langle \mathcal{P}(\omega), \cup, \cap, \Delta, \nabla, \emptyset, \omega\rangle$$

is a qm-lattice.

The first example shows that it is always possible to define a qm-lattice from an arbitrary bounded distributive lattice.

We see the relation between quasi-modal lattices and modal lattices.

Let $\mathbf{A} \in \mathcal{QML}$ with the property that for each $a \in A$, Δa is a principal ideal and ∇a is a principal filter. If we define the functions, $\Box\colon A \to A$ and $\Diamond\colon A \to A$ by

$$\Box a = \text{the } x \text{ such that } \Delta a = I(x) \quad \text{and} \quad \Diamond a = \text{the } z \text{ such that } \nabla a = F(z)$$

then the structure $\mathbf{A} = \langle A, \vee, \wedge, \Box, \Diamond, 0, 1\rangle$ is a modal lattice. In other words, the following conditions hold:

M1 $\Box(a \wedge b) = \Box a \wedge \Box b$,
M2 $\Box 1 = 1$,
M3 $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,
M4 $\Diamond 0 = 0$.

Moreover, given a modal lattice $\mathbf{A} = \langle A, \vee, \wedge, \Box, \Diamond, 0, 1\rangle$, if for each $a \in A$ we define

$$\Delta a = I(\Box a) \quad \text{and} \quad \nabla a = F(\Diamond a)$$

then $\mathbf{A} = \langle A, \vee, \wedge, \Delta, \nabla, 0, 1\rangle$ is a qm-lattice.

Now, we define two operators which will be useful in the development of the representation. Next, we show some properties that these operators satisfy.

Let $\mathbf{A} \in \mathcal{QML}$. For each $P \in X(\mathbf{A})$ we define

$$\Delta^{-1}(P) = \{a \in A : \Delta a \cap P \neq \emptyset\}.$$

Dually, we define

$$\nabla^{-1}(P) = \{a \in A : \nabla a \subseteq P\}.$$

LEMMA 2. *Let* $\mathbf{A}$ *be a qm-lattice. Then for each* $P \in X(\mathbf{A})$

(1) $\Delta^{-1}(P) \in \mathrm{Fi}(\mathbf{A})$,
(2) $(\nabla^{-1}(P))^c \in \mathrm{Id}(\mathbf{A})$.

Proof. (1) Since $\Delta 1 = A$, $\Delta 1 \cap P \neq \emptyset$, i.e. $1 \in \Delta^{-1}(P)$. Let x, y be such that $x \leqslant y$ and $x \in \Delta^{-1}(P)$. Thus $\Delta x \subseteq \Delta y$ and $\Delta x \cap P \neq \emptyset$. So, $y \in \Delta^{-1}(P)$ since $\Delta y \cap P \neq \emptyset$. To prove the other condition, we consider $x, y \in \Delta^{-1}(P)$. So, $\Delta x \cap P \neq \emptyset$ and $\Delta y \cap P \neq \emptyset$. Then there are elements $a, b \in A$ such that $a \in \Delta x \cap P$ and $b \in \Delta y \cap P$. Since $\Delta x, \Delta y \in \mathrm{Id}(\mathbf{A})$, we have that $a \wedge b \in \Delta x \cap \Delta y$. As P is a filter, we have that $a \wedge b \in P$. Thus $a \wedge b \in \Delta x \cap \Delta y \cap P$. Then $\Delta(x \wedge y) \cap P \neq \emptyset$. Therefore $\Delta^{-1}(P) \in \mathrm{Fi}(\mathbf{A})$.

(2) Since $\nabla 0 = A$ and $P \subsetneq A$, $\nabla 0 \nsubseteq P$, i.e. $0 \in (\nabla^{-1}(P))^c$. Let $x, y \in (\nabla^{-1}(P))^c$. Thus $\nabla x \nsubseteq P$ and $\nabla y \nsubseteq P$, hence $\nabla x \cap P^c \neq \emptyset$ and $\nabla y \cap P^c \neq \emptyset$. Then there are elements $a, b \in A$ such that $a \in \nabla x \cap P^c$ and $b \in \nabla y \cap P^c$.

Suppose that $a \vee b \notin P^c$, $a \vee b \in P$. Since P is prime filter, $a \in P$ or $b \in P$, which is a contradiction. Now suppose that $a \vee b \notin \nabla x \cap \nabla y$, hence $a \vee b \notin \nabla x$ or $a \vee b \notin \nabla y$. As $\nabla x \in \mathrm{Fi}(\mathbf{A})$ and $a \leqslant a \vee b$, if $a \vee b \notin \nabla x$, then $a \notin \nabla x$, which is also a contradiction. Similarly we can prove that $b \notin \nabla y$, which is a contradiction. Thus $a \vee b \in \nabla x \cap \nabla y \cap P^c = \nabla(x \vee y) \cap P^c$, hence $\nabla(x \vee y) \cap P^c \neq \emptyset$. To prove the other condition, we consider $x, y \in A$ such that $x \leqslant y$ and $y \in (\nabla^{-1}(P))^c$. It is easy to check that $x \in (\nabla^{-1}(P))^c$. □

Moreover, it is easy to check that Δ^{-1} and ∇^{-1} are monotonous on $X(\mathbf{A})$, i.e. for any $P, Q \in X(\mathbf{A})$ if $P \subseteq Q$ then $\Delta^{-1}(P) \subseteq \Delta^{-1}(Q)$ and $\nabla^{-1}(P) \subseteq \nabla^{-1}(Q)$.

3.1. REPRESENTATION

In this subsection we show a representation theorem for quasi-modal lattices in terms of quasi-modal lattices of sets, using the well-known representation theorem for bounded distributive lattices.

DEFINITION 3. Let $\mathcal{F} = \langle X, \leqslant, R_1, R_2, D\rangle$ be a relational structure, where $\langle X, \leqslant\rangle$ is a poset and R_1, R_2 are two binary relations defined on X. We say that $\mathcal{F}$ is a *quasi-modal space*, or *qm-space* for short, if

(1) D is a distributive sublattice of $\langle \mathcal{P}_i(X), \cup, \cap, \emptyset, X\rangle$ with $\emptyset, X \in D$,
(2) $\leqslant \circ R_1 \subseteq R_1$,
(3) $\leqslant^{-1} \circ R_2 \subseteq R_2$,
(4) In the topological space $(X, \mathcal{T}_X)$ defined by taking the family

$$D \cup \{(X - U) : U \in D\}$$

as a subbase, for each $U \in D$

$$\Delta_{R_1}(U) = \{x \in X : R_1(x) \subseteq U\} \quad \text{is open and } \leqslant\text{-increasing}$$

and

$$\nabla_{R_2}(U) = \{x \in X : R_2(x) \cap U \neq \emptyset\} \quad \text{is closed and } \leqslant\text{-increasing.}$$

Now, given a qm-space $\mathcal{F} = \langle X, \leqslant, R_1, R_2, D\rangle$ we can see that for every $U \in D$ the set

$$\overline{\Delta}_{R_1}(U) = \{V \in D : V \subseteq \Delta_{R_1}(U)\}$$

is an ideal of D, and the set

$$\overline{\nabla}_{R_2}(U) = \{V \in D : \nabla_{R_2}(U) \subseteq V\}$$

is a filter of D.

LEMMA 4. *Let* $\mathcal{F} = \langle X, \leqslant, R_1, R_2, D\rangle$ *be a qm-space. Then*

$$\mathcal{A}(\mathcal{F}) = \langle D, \cup, \cap, \overline{\Delta}_{R_1}, \overline{\nabla}_{R_2}, \emptyset, X\rangle$$

is a qm-lattice.

Proof. Since for any $U, V \in D$, $\Delta_{R_1}(U) \cap \Delta_{R_1}(V) = \Delta_{R_1}(U \cap V)$ and $\Delta_{R_1}(X) = X$, we conclude that $\overline{\Delta}_{R_1}(U \cap V) = \overline{\Delta}_{R_1}(U) \cap \overline{\Delta}_{R_1}(V)$ and $\overline{\Delta}_{R_1}(X) = D$. Moreover, for any $U, V \in D$, $\nabla_{R_2}(U) \cup \nabla_{R_2}(V) = \nabla_{R_2}(U \cup V)$ and $\nabla_{R_2}(\emptyset) = \emptyset$. Therefore, we conclude that $\overline{\nabla}_{R_2}(U \cup V) = \overline{\nabla}_{R_2}(U) \cup \overline{\nabla}_{R_2}(V)$ and $\overline{\nabla}_{R_2}(\emptyset) = D$. □

Now, given $\mathbf{A} \in \mathcal{QML}$ we consider the set $X(\mathbf{A})$ of the prime filters of the lattice reduct of $\mathbf{A}$ and define two binary relations R_Δ and R_∇ on $X(\mathbf{A})$. We will prove that $\mathcal{F}(\mathbf{A}) = \langle X(\mathbf{A}), \subseteq, R_\Delta, R_\nabla, \beta(A)\rangle$ is a qm-space. In this case for each $a \in A$

$$\Delta_{R_\Delta}(\beta(a)) = \{P \in X(\mathbf{A}) : R_\Delta(P) \subseteq \beta(a)\}$$

and

$$\nabla_{R_\nabla}(\beta(a)) = \{P \in X(\mathbf{A}) : R_\nabla(P) \cap \beta(a) \neq \emptyset\}.$$

Moreover, using the above lemma, we will have the qm-lattice $\mathcal{A}(\mathcal{F}(\mathbf{A}))$ and will say that $\mathcal{F}(\mathbf{A})$ and $\mathcal{A}(\mathcal{F}(\mathbf{A}))$ are the qm-space associated with $\mathbf{A}$, and the qm-lattice associated with $\mathbf{A}$ respectively.

Let $\mathbf{A} \in \mathcal{QML}$. We define two binary relations, R_Δ and R_∇ on $X(\mathbf{A})$ by,

$$(P, Q) \in R_\Delta \Leftrightarrow \Delta^{-1}(P) \subseteq Q \quad \text{and} \quad (P, Q) \in R_\nabla \Leftrightarrow Q \subseteq \nabla^{-1}(P).$$

LEMMA 5. *Let* $\mathbf{A} \in \mathcal{QML}$. *Then*

(1) $\subseteq \circ R_\Delta \subseteq R_\Delta$,
(2) $\subseteq^{-1} \circ R_\nabla \subseteq R_\nabla$.

Proof. (1) Let $(P, Q) \in \subseteq \circ R_\Delta$, hence there is $T \in X(\mathbf{A})$ such that $P \subseteq T$ and $(T, Q) \in R_\Delta$. Thus, we have that $P \subseteq T$ and $\Delta^{-1}(T) \subseteq Q$. Therefore $\Delta^{-1}(P) \subseteq Q$, since $\Delta^{-1}(P) \subseteq \Delta^{-1}(T)$. Thus $(P, Q) \in R_\Delta$.

(2) Let $(P, Q) \in \subseteq^{-1} \circ R_\nabla$, hence there is $T \in X(\mathbf{A})$ such that $T \subseteq P$ and $Q \subseteq \nabla^{-1}(T)$. Then $Q \subseteq \nabla^{-1}(P)$, since $\nabla^{-1}(T) \subseteq \nabla^{-1}(P)$. Thus $(P, Q) \in R_\nabla$. □

We give an auxiliary result.

LEMMA 6. *Let* $\mathbf{A} \in \mathcal{QML}$. *Let* $a \in A$ *and* $P \in X(\mathbf{A})$. *Then*

(1) $a \in \Delta^{-1}(P) \Leftrightarrow \forall Q \in X(\mathbf{A})$ (*if* $\Delta^{-1}(P) \subseteq Q$ *then* $a \in Q$),
(2) $a \in \nabla^{-1}(P) \Leftrightarrow \exists Q \in X(\mathbf{A})$ ($Q \subseteq \nabla^{-1}(P)$ *and* $a \in Q$).

Proof. (1) The implication from left to right is immediate. Let us assume that there is $a \in A$, such that $a \notin \Delta^{-1}(P)$. Let $(a] \subseteq A$ be the ideal generated by the element a. Suppose that

$$(a] \cap \Delta^{-1}(P) \neq \emptyset.$$

Let $x \in (a] \cap \Delta^{-1}(P)$. Thus, $x \leqslant a$ and $\Delta x \cap P \neq \emptyset$. Since $x \wedge a = x$, $\Delta x \subseteq \Delta a$. Thus, $\Delta a \cap P \neq \emptyset$, i.e. $a \in \Delta^{-1}(P)$, which is a contradiction. So, $(a] \cap \Delta^{-1}(P) = \emptyset$. Since $\Delta^{-1}(P)$ is a filter and $(a]$ is an ideal, by Birkhoff–Stone's theorem, there is $Q \in X(\mathbf{A})$ such that $\Delta^{-1}(P) \subseteq Q$ and $Q \cap (a] = \emptyset$. Then, there is $Q \in X(\mathbf{A})$ such that $\Delta^{-1}(P) \subseteq Q$ and $a \notin Q$.

(2) Let $a \in \nabla^{-1}(P)$, i.e. $a \notin (\nabla^{-1}(P))^c$. Let $[a) \subseteq A$ be the filter generated by the element a. Suppose that

$$(\nabla^{-1}(P))^c \cap [a) \neq \emptyset.$$

Then, there is $b \in (\nabla^{-1}(P))^c \cap [a)$. Thus, $b \notin \nabla^{-1}(P)$ and $a \leqslant b$. It follows that $\nabla b \nsubseteq P$ and $\nabla b \subseteq \nabla a$. Then $\nabla a \nsubseteq P$, which is a contradiction. So, $(\nabla^{-1}(P))^c \cap [a) = \emptyset$. Since $(\nabla^{-1}(P))^c$ is an ideal, then by Birkhoff–Stone's theorem there is $Q \in X(\mathbf{A})$ such that $[a) \subseteq Q$ and $Q \cap (\nabla^{-1}(P))^c = \emptyset$. So, $Q \subseteq \nabla^{-1}(P)$ and $a \in Q$. In the other direction the proof is immediate. □

In order to see that $\Delta_{R_\Delta}(\beta(a))$ is open and increasing, and that $\nabla_{R_\nabla}(\beta(a))$ is closed and increasing, we will use the duality between ideals and open increasing sets, and the duality between filters and closed increasing sets respectively, which hold in Priestley spaces.

LEMMA 7. *Let* $\mathbf{A} \in \mathcal{QML}$. *Then, for each* $a \in A$

(1) $\Delta_{R_\Delta}(\beta(a)) = \varphi(\Delta a)$,
(2) $\nabla_{R_\nabla}(\beta(a)) = \psi(\nabla a)$.

Proof. (1) Let $P \in \Delta_{R_\Delta}(\beta(a))$, then by definition we have $R_\Delta(P) \subseteq \beta(a)$. Thus, if $(P, Q) \in R_\Delta$, then $Q \in \beta(a)$, i.e. $a \in Q$. Using the definition of R_Δ, we can say that if $Q \in X(\mathbf{A})$ is such that $\Delta^{-1}(P) \subseteq Q$, then $a \in Q$. But by Lemma 6, $a \in \Delta^{-1}(P)$, i.e. $\Delta a \cap P \neq \emptyset$. So, $P \in \varphi(\Delta a)$. Now, let $P \in \varphi(\Delta a)$. Then, $\Delta a \cap P \neq \emptyset$, i.e. $a \in \Delta^{-1}(P)$. Let $(P, Q) \in R_\Delta$, then $\Delta^{-1}(P) \subseteq Q$. Thus $a \in Q$, i.e. $Q \in \beta(a)$. So $R_\Delta(P) \subseteq \beta(a)$, i.e. $P \in \Delta_{R_\Delta}(\beta(a))$.

(2) Let $P \in \nabla_{R_\nabla}(\beta(a))$. Then $R_\nabla(P) \cap \beta(a) \neq \emptyset$, i.e. there is $Q \in R_\nabla(P) \cap \beta(a)$. Thus we have that $Q \subseteq \nabla^{-1}(P)$ and $a \in Q$. Using Lemma 6 we can see that $a \in \nabla^{-1}(P)$, i.e. $\nabla a \subseteq P$. So $P \in \psi(\nabla a)$. Now, let $P \in \psi(\nabla a)$, then $\nabla a \subseteq P$, i.e. $a \in \nabla^{-1}(P)$. Suppose that $R_\nabla(P) \cap \beta(a) = \emptyset$. Thus, if $Q \in X(\mathbf{A})$ is such that $Q \subseteq \nabla^{-1}(P)$, then $Q \notin \beta(a)$. So $a \notin Q$, but by Lemma 6 this is a contradiction. □

The preceding result, in a way, allows us to identify the operators Δ_{R_R} and ∇_{R_∇} with Δ and ∇ respectively.

LEMMA 8. *Let* $\mathbf{A} \in \mathcal{QML}$. *Then* $\mathcal{F}(\mathbf{A}) = \langle X(\mathbf{A}), \subseteq, R_\Delta, R_\nabla, \beta(A)\rangle$ *is a qm-space.*

Proof. It follows from Lemma 5, Lemma 7 and Definition 3. □

LEMMA 9. *Let* $\mathbf{A} \in \mathcal{QML}$. *Then*

$$\mathcal{A}(\mathcal{F}(\mathbf{A})) = \langle \beta(A), \cup, \cap, \overline{\Delta}_{R_\Delta}, \overline{\nabla}_{R_\nabla}, \emptyset, X(\mathbf{A})\rangle$$

is a qm-lattice.

Proof. It follows from Lemma 4 and the above lemma. □

We introduce the notion of homomorphism between two qm-lattices.

DEFINITION 10. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two qm-lattices. A function $h\colon A_1 \to A_2$ is a *homomorphism of qm-lattices*, or *qm-homomorphism* for short, if

(1) h is a homomorphism of lattices,
(2) for any $a \in A_1$, $I(h(\Delta_1 a)) = \Delta_2(h(a))$,
(3) for any $a \in A_1$, $F(h(\nabla_1 a)) = \nabla_2(h(a))$.

If h is a lattice isomorphism then we say that h is a *qm-isomorphism*, and we write $\mathbf{A}_1 \cong_q \mathbf{A}_2$.

THEOREM 11 (of Representation). *Let* $\mathbf{A} \in \mathcal{QML}$. *Then* $\mathbf{A}$ *and* $\mathcal{A}(\mathcal{F}(\mathbf{A}))$ *are qm-isomorphics.*

Proof. By the representation theorem for bounded distributive lattices, we know that $\beta\colon A \to \beta(A)$ is an isomorphism. Consequently $I(\beta(\Delta a) = \beta(\Delta a)$ and $F(\beta(\nabla a)) = \beta(\nabla a)$. Therefore, we have only to prove that $\beta(\Delta a) = \overline{\Delta}_{R_\Delta}(\beta(a))$ and that $\beta(\nabla a) = \overline{\nabla}_{R_\nabla}(\beta(a))$. These equalities are obtained using Lemma 7. So, $\mathbf{A} \cong_q \mathcal{A}(\mathcal{F}(\mathbf{A}))$. □

3.2. DUALITY

We will develop a duality for qm-lattices by means of Priestley spaces with two binary relations to deal with Δ and ∇.

DEFINITION 12. A *descriptive quasi-modal space*, or *descriptive qm-space* for short, is a quasi-modal space $\mathcal{F} = \langle X, \leqslant, R_1, R_2, D\rangle$, such that

(1) $\langle X, \leqslant, \mathcal{T}_X\rangle$ is a Priestley space, where the set $D \cup \{(X - U) : U \in D\}$ is a subbase for $\mathcal{T}_X$,
(2) $R_1(x)$ is closed and increasing, for any $x \in X$,
(3) $R_2(x)$ is closed and decreasing for any $x \in X$.

LEMMA 13. *Let* $\mathbf{A}$ *be a qm-lattice. Then*

$$\mathcal{F}(\mathbf{A}) = \langle X(\mathbf{A}), \subseteq, R_\Delta, R_\nabla, \beta(A)\rangle$$

is a descriptive qm-space.

Proof. We prove only (2) and (3) of the previous definition, since by Lemma 8 we already know that $\mathcal{F}(\mathbf{A}) = \langle X(\mathbf{A}), \subseteq, R_\Delta, R_\nabla, \beta(A)\rangle$ is qm-space. Moreover it is well known that $\langle X(\mathbf{A}), \subseteq, \mathcal{T}_{X(\mathbf{A})}\rangle$ is Priestley space, where the set $\beta(A) \cup \beta(A)^c$ is subbase for the topology $\mathcal{T}_{X(\mathbf{A})}$.

(2) Let $P \in X(\mathbf{A})$, hence $R_\Delta(P) = \{Q \in X(\mathbf{A}) : \Delta^{-1}P \subseteq Q\} = \psi(\Delta^{-1}P)$. So $R_\Delta(P)$ is closed and increasing.

(3) Let $P \in X(\mathbf{A})$, hence $R_\nabla(P) = \{Q \in X(\mathbf{A}) : Q \cap (\nabla^{-1}P)^c = \emptyset\}$. Thus $R_\nabla(P) = (\varphi(\nabla^{-1}P)^c)^c$ since $(\nabla^{-1}P)^c$ is an ideal. So $R_\nabla(P)$ is closed and decreasing. □

LEMMA 14. *Let* $\mathcal{F} = \langle X, \leqslant, R_1, R_2, D\rangle$ *be a qm-space, such that the ordered topological space* $\langle X, \leqslant, \mathcal{T}_X\rangle$ *defined by the subbase*

$$D \cup \{(X - U) : U \in D\}$$

is a Priestley space. Then the following conditions are equivalent:

(1) *For all* $x \in X$, $R_1(x)$ *is a closed and increasing subset of* X.
(2) *For all* $x, y \in X$, $(x, y) \in R_1$ *iff* $(\varepsilon(x), \varepsilon(y)) \in R_{\overline{\Delta}_{R_1}}$.

Proof. (1) $\Rightarrow$ (2) Let $(x, y) \in R_1$. We consider $U \in \overline{\Delta}_{R_1}^{-1}(\varepsilon(x))$. Thus, using the definition of $\overline{\Delta}_{R_1}^{-1}$ we infer that there exists $V \in D$ such that $V \in \overline{\Delta}_{R_1}(U)$ and $x \in V$. Hence $V \subseteq \Delta_{R_1}(U)$. Therefore $R_1(x) \subseteq U$. So $U \in \varepsilon(y)$. Now we prove the other direction. Let $(\varepsilon(x), \varepsilon(y)) \in R_{\overline{\Delta}_{R_1}}$ and suppose that $y \notin R_1(x)$. Since by hypothesis $R_1(x)$ is closed and increasing, and moreover $\langle X, \leqslant, \mathcal{T}_X\rangle$ is a Priestley space, there is a family of clopen and increasing $\{U_j\}_{j \in J}$ such that $R_1(x) = \bigcap_{j \in J} U_j$. This implies that there is U_k with $k \in J$ such that $R_1(x) \subseteq U_k$ and $U_k \notin \varepsilon(y)$. Under these conditions, it is easy to check that $(\varepsilon(x), \varepsilon(y)) \notin R_{\overline{\Delta}_{R_1}}$, which is a contradiction.

(2) $\Rightarrow$ (1) Let $x, y \in X$ be such that $y \in \mathrm{Cl}(R_1(x))$ and suppose that $y \notin R_1(x)$. Then $(\varepsilon(x), \varepsilon(y)) \notin R_{\overline{\Delta}_{R_1}}$. Thus, we can say that there exists $U \in D$ such that $\overline{\Delta}_{R_1}(U) \cap \varepsilon(x) \neq \emptyset$ and $U \notin \varepsilon(y)$. Equivalently, there are $U, V \in D$ such that $V \subseteq \Delta_{R_1}(U)$, $x \in V$ and $y \notin U$. Then, $R_1(x) \subseteq U$ and $y \notin U$. So, $y \notin \mathrm{Cl}(R_1(x))$ which is a contradiction. In order to show that $R_1(x)$ is increasing, we consider $y \leqslant z$ with $y \in R_1(x)$. As ε is an order isomorphism, we conclude that $\varepsilon(y) \subseteq \varepsilon(z)$. Moreover $\overline{\Delta}_{R_1}^{-1}(\varepsilon(x)) \subseteq \varepsilon(y)$, since $(x, y) \in R_1$. It follows that $(\varepsilon(x), \varepsilon(z)) \in R_{\overline{\Delta}_{R_1}}$. So, $z \in R_1(x)$. □

The lemma below can be proved in a similar way.

LEMMA 15. *Let $\mathcal{F} = \langle X, \leqslant, R_1, R_2, D\rangle$ be a qm-space, such that the ordered topological space $\langle X, \leqslant, \mathcal{T}_X\rangle$ defined by the subbase*

$$D \cup \{(X - U) : U \in D\}$$

is a Priestley space. Then the following conditions are equivalent:

(1) *For all $x \in X$, $R_2(x)$ is a closed and decreasing subset of X.*
(2) *For all $x, y \in X$, $(x, y) \in R_2$ iff $(\varepsilon(x), \varepsilon(y)) \in R_{\nabla_{R_2}}$.*

In order to obtain a full duality between quasi-modal lattices and quasi-modal spaces we need to define the notion of morphism between two qm-spaces.

DEFINITION 16. Let $\mathcal{F}_1 = \langle X_1, \leqslant_1, R_1, S_1, D_1\rangle$ and $\mathcal{F}_2 = \langle X_2, \leqslant_2, R_2, S_2, D_2\rangle$ be two qm-spaces. A function $f\colon X_1 \to X_2$ is a *quasi-modal morphism*, or *qm-morphism* for short, provided the following implications hold:

(1) if $(x, y) \in R_1$, then $(f(x), f(y)) \in R_2$,
(2) if $(x, y) \in S_1$, then $(f(x), f(y)) \in S_2$,
(3) if $(f(x), y) \in R_2$, then there is $z \in X_1$ such that $(x, z) \in R_1$ and $f(z) \leqslant_2 y$,
(4) if $(f(x), y) \in S_2$, then there is $z \in X_1$ such that $(x, z) \in S_1$ and $y \leqslant_2 f(z)$,
(5) for any $U \in D_2$, $f^{-1}(U) \in D_1$.

THEOREM 17. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two qm-lattices. A lattice homomorphism $h\colon A_1 \to A_2$ is a qm-homomorphism if and only if the map $\mathcal{F}(h)\colon X(\mathbf{A}_2) \to X(\mathbf{A}_1)$ defined by $\mathcal{F}(h)(P) = h^{-1}(P)$ for each $P \in X(\mathbf{A}_2)$, is a qm-morphism.*

Proof. ($\Rightarrow$) (1) Let $P, Q \in X(\mathbf{A}_2)$ be such that $(P, Q) \in R_{\Delta_2}$. Thus $\Delta_2^{-1}(P) \subseteq Q$. Let $a \in \Delta_1^{-1}(h^{-1}(P))$, hence $\Delta_1 a \cap h^{-1}(P) \neq \emptyset$. This implies that $I(h(\Delta_1 a)) \cap P \neq \emptyset$. Since h is a qm-homomorphism, we have that $\Delta_2 h(a) \cap P \neq \emptyset$. Thus, $h(a) \in \Delta_2^{-1}(P)$. So, $a \in h^{-1}(Q)$.

(2) Let $P, Q \in X(\mathbf{A}_2)$ be such that $(P, Q) \in R_{\nabla_2}$. Thus, $Q \subseteq \nabla_2^{-1}(P)$. Let $a \in h^{-1}(Q)$, hence $h(a) \in \nabla_2^{-1}(P)$. Thus, $\nabla_2 h(a) \subseteq P$ and since h is a qm-homomorphism we have that $F(h(\nabla_1 a)) \subseteq P$. It follows that $h(\nabla_1 a) \subseteq P$, i.e., $\nabla_1 a \subseteq h^{-1}(P)$. So, $a \in \nabla_1^{-1}(h^{-1}(P))$.

(3) Let $P \in X(\mathbf{A}_2)$ and $Q \in X(\mathbf{A}_1)$ be, such that $\Delta_1^{-1}(h^{-1}(P)) \subseteq Q$. We prove that $\Delta_2^{-1}(P) \cap h(Q^c) = \emptyset$. Suppose the opposite. Then there is an element x such that $x \leqslant h(q)$ for some $q \notin Q$ and $\Delta_2 x \cap P \neq \emptyset$. By the monotony of Δ_2, we conclude that $\Delta_2 x \subseteq \Delta_2 h(q)$. Thus $\Delta_2 h(q) \cap P \neq \emptyset$. As h is a qm-homomorphism, $I(h(\Delta_1 q)) \cap P \neq \emptyset$. Let $y \in I(h(\Delta_1 q)) \cap P$. Then there are $y_1, \ldots, y_n \in h(\Delta_1 q)$ such that $y \leqslant y_1 \vee \cdots \vee y_n$. As P is prime filter, there is $y_i \in h(\Delta_1 q$ for some $1 \leqslant i \leqslant n$, such that $y_i \in P$. So, $h(\Delta_1 q) \cap P \neq \emptyset$. This implies that $\Delta_1 q \cap h^{-1}(P) \neq \emptyset$. Thus $q \in \Delta_1^{-1}(h^{-1}(P))$. So $q \in Q$, which is a contradiction. Thus, by Birkhoff–Stone's theorem there is a prime filter $Z \in X(\mathbf{A}_2)$ such that, $\Delta_2^{-1}(P) \subseteq Z$ and $Z \cap h(Q^c) = \emptyset$. So, $(P, Z) \in R_{\Delta_2}$ and $h^{-1}(Z) \subseteq Q$.

(4) Let $P \in X(\mathbf{A}_2)$ and $Q \in X(\mathbf{A}_1)$ be such that $Q \subseteq \nabla_1^{-1}(h^{-1}(P))$. We show that $h(Q) \cap (\nabla_2^{-1}(P))^c = \emptyset$. Suppose the opposite. So, there is $q \in Q$ such that

$h(q) \notin \nabla_2^{-1}(P)$, i.e. $\nabla_2 h(q) \nsubseteq P$. As h is a qm-homomorphism, $F(h(\nabla_1 q)) \nsubseteq P$. Thus, there is $x \in F(h(\nabla_1 q))$ and $x \notin P$. Consequently, there are $y_1, \ldots, y_n \in \nabla_1 q$, such that $h(y_1) \wedge \cdots \wedge h(y_n) \leqslant x$. Also, since h is a homomorphism and $\nabla_1 q$ is filter, there is $y \in \nabla_1 q$ such that $h(y) \leqslant x$, where $y = y_1 \wedge \cdots \wedge y_n$. Since P is $\leqslant$-increasing, we obtain that $h(y) \notin P$. Moreover, $\nabla_1^{-1}(h^{-1}(P)) = \{t \in X_1 : \nabla_1 t \subseteq h^{-1}(P)\}$, and by hypothesis Q is included in the above set. So, for any $q \in Q$ it follows that $y \notin \nabla_1 q$, which is a contradiction. Thus, by Birkhoff–Stone's theorem there is a prime filter $Z \in X(\mathbf{A})$ such that $h(Q) \subseteq Z$ and $Z \cap (\nabla_2^{-1}(P))^c = \emptyset$. So, $Q \subseteq h^{-1}(Z)$ and $Z \subseteq \nabla_2^{-1}(P)$. □

We give two lemmas that will be needed in some of the proofs below. The first one corresponds to Lemma 13 in [2] and its proof is analogous. The second can be considered as its dual.

LEMMA 18. *Let $\mathcal{F}_1 = \langle X_1, \leqslant_1, R_1, S_1, D_1\rangle$ and $\mathcal{F}_2 = \langle X_2, \leqslant_2, R_2, S_2, D_2\rangle$ be two descriptive qm-spaces. Let $f\colon X_1 \to X_2$ be a function such that for each $U \in D_2$, $f^{-1}(U) \in D_1$. The following conditions are equivalent, for any $U \in D_2$.*

(1) $I(f^{-1}(\overline{\Delta}_{R_2} U)) = \overline{\Delta}_{R_1}(f^{-1}(U))$,
(2) $\Delta_{R_1}(f^{-1}(U)) = f^{-1}(\Delta_{R_2}(U))$.

LEMMA 19. *Let $\mathcal{F}_1 = \langle X_1, \leqslant_1, R_1, S_1, D_1\rangle$ and $\mathcal{F}_2 = \langle X_2, \leqslant_2, R_2, S_2, D_2\rangle$ be two descriptive qm-spaces. Let $f\colon X_1 \to X_2$ be a function such that, for each $U \in D_2$, $f^{-1}(U) \in D_1$. The following conditions are equivalent, for any $U \in D_2$.*

(1) $F(f^{-1}(\overline{\nabla}_{S_2} U)) = \overline{\nabla}_{S_1}(f^{-1}(U))$,
(2) $\nabla_{S_1}(f^{-1}(U)) = f^{-1}(\nabla_{S_2}(U))$.

Proof. (1) $\Rightarrow$ (2) We show that $f^{-1}(\nabla_{S_2}(U)) \subseteq \nabla_{S_1}(f^{-1}(U))$. Suppose the opposite. Thus, there is x such that $f(x) \in \nabla_{S_2}(U)$ and $x \notin \nabla_{S_1}(f^{-1}(U))$. Since $\nabla_{S_1}(f^{-1}(U))$ is a closed and $\leqslant_1$-increasing set and $\langle X_1, \leqslant_1, \mathcal{T}_{X_1}\rangle$ is a Priestley space, hence there is a family of clopen and $\leqslant_1$-increasing subsets $\mathcal{G} = \{V_i \in D_1 : i \in I\}$ such that

$$\nabla_{S_1}(f^{-1}(U)) = \bigcap_{i \in I} V_i.$$

Therefore, there is $V_j \in \mathcal{G}$ such that $\nabla_{S_1}(f^{-1}(U)) \subseteq V_j$ and $x \notin V_j$. Using the definition of $\overline{\nabla}_{S_1}$ and hypothesis 1, $V_j \in F(f^{-1}(\overline{\nabla}_{S_2} U))$. So, there is $Z \in f^{-1}(\overline{\nabla}_{S_2} U)$ such that $Z \subseteq V_j$. As $f^{-1}(\overline{\nabla}_{S_2}(U)) = \{f^{-1}(W) : W \in \overline{\nabla}_{S_2}(U)\}$, $Z = f^{-1}(W)$ for some $W \in D_2$ such that $\nabla_{S_2}(U) \subseteq W$. Since $x \notin V_j$, $f(x) \notin W$. Hence $f(x) \notin \nabla_{S_2}(U)$, which is a contradiction.

(2) $\Rightarrow$ (1) We show that $F(f^{-1}(\overline{\nabla}_{S_2} U)) \subseteq \overline{\nabla}_{S_1}(f^{-1}(U))$. Assume that $V \in F(f^{-1}(\overline{\nabla}_{S_2}(U)))$. Then there is $Z \in D_1$ such that $Z \subseteq V$ and $Z \in f^{-1}(\overline{\nabla}_{S_2} U)$. Thus, there is $W \in D_2$ such that $Z = f^{-1}(W)$ and $\nabla_{S_2}(U) \subseteq W$. Then, $f^{-1}(\nabla_{S_2}(U)) \subseteq f^{-1}(W)$. Using hypothesis (2), $\nabla_{S_1}(f^{-1}(U)) \subseteq f^{-1}(W)$. So, $\nabla_{S_1}(f^{-1}(U)) \subseteq V$ and consequently $V \in \overline{\nabla}_{S_1}(f^{-1}(U))$. □

THEOREM 20. *Let* $\mathcal{F}_1 = \langle X_1, \leqslant_1, R_1, S_1, D_1\rangle$ *and* $\mathcal{F}_2 = \langle X_2, \leqslant_2, R_2, S_2, D_2\rangle$ *be two descriptive qm-spaces. Then* $f\colon X_1 \to X_2$ *is a qm-morphism if only if the function* $\mathcal{A}(f)\colon D_2 \to D_1$ *given by* $\mathcal{A}(f)(U) = f^{-1}(U)$ *for each* $U \in D_2$ *is a qm-homomorphism.*

Proof. Let us assume that f is a qm-morphism and prove that $\mathcal{A}(f)$ is qm-homomorphism. Let $U \in D_2$. According to Definition 10 and using the two above lemmas, it is sufficient to show that $\Delta_{R_1}(f^{-1}(U)) = f^{-1}(\Delta_{R_2}(U))$ and $\nabla_{S_1}(f^{-1}(U)) = f^{-1}(\nabla_{S_2}(U))$. We prove only the second equality, since the other is contained in Theorem 14 of [2]. Let $x \in \nabla_{S_1}(f^{-1}(U))$, hence $S_1(x) \cap f^{-1}(U) \neq \emptyset$. Thus, there is $z \in X_1$ such that $(x, z) \in S_1$ and $f(z) \in U$. Since, by hypothesis, f is a qm-morphism we have that $(f(x), f(z)) \in S_2$. Hence $S_2(f(x)) \cap U \neq \emptyset$, i.e. $f(x) \in \nabla_{S_2}(U)$. So $\nabla_{S_1}(f^{-1}(U)) \subseteq f^{-1}(\nabla_{S_2}(U))$. We prove the other inclusion. Let $f(x) \in \nabla_{S_2}(U)$. Hence $S_2(f(x)) \cap U \neq \emptyset$. Thus there is $y \in X_2$ such that $(f(x), y) \in S_2$ and $y \in U$. Since f is a qm-morphism, there is $z \in X_1$ such that $(x, z) \in S_1$ and $y \leqslant f(z)$. As U is $\leqslant$-increasing, we have $z \in f^{-1}(U)$. So, $S_1(x) \cap f^{-1}(U) \neq \emptyset$, i.e. $x \in \nabla_{S_1}(f^{-1}(U))$.

To prove the other implication, assume that $\mathcal{A}(f)$ is a qm-homomorphism. Let $x, y \in X_1$ such that $(x, y) \in S_1$. Suppose that $(f(x), f(y)) \notin S_2$. Since $S_2(f(x))$ is closed and decreasing and X_2 is a Priestley space, there is a clopen decreasing set V such that $S_2(f(x)) \subseteq V$ and $y \notin f^{-1}(V)$. Let $U = V^c$ be, thus U belong to D_2. Hence $S_1(x) \cap f^{-1}(U) \neq \emptyset$, because $y \in f^{-1}(U)$ and by hypothesis $y \in S_1(x)$. It follows that $x \in \nabla_{S_1}(f^{-1}(U))$ hence, using Lemma 19, $f(x) \in \nabla_2(U)$. So $S_2(f(x)) \cap U \neq \emptyset$, which is a contradiction.

(4) Let $(f(x), y) \in S_2$, and suppose that for any $z \in S_1(x)$, $y \nleqslant f(z)$. As $\langle X_1, \leqslant, D_1\rangle$ is a Priestley space, then for each $z \in S_1(x)$ there is $U_z \in D_1$ such that $y \in U_z$ and $z \in f^{-1}(U_z^c)$. Let $V_z = U_z^c$ for each $z \in S_1(x)$, hence V_z is a clopen decreasing set. Thus

$$S_1(x) \subseteq \bigcup_{z \in S_1(x)} f^{-1}(V_z).$$

Moreover $S_1(x)$ is compact, since $S_1(x)$ is closed and X_1 is compact. Therefore we have that there are $V_{z_1}, \ldots, V_{z_n}$ with $z_i \in S_1(x)$ and $1 \leqslant i \leqslant n$, such that

$$S_1(x) \subseteq f^{-1}(V_{z_1} \cup \cdots \cup V_{z_n}).$$

Let $V = V_{z_1} \cup \cdots \cup V_{z_n}$. It is easy to check that V is clopen decreasing. Consider $U = V^c$, hence $U \in D_2$. We conclude that $S_1(x) \cap f^{-1}(U) = \emptyset$. It follows that $x \notin \nabla_1(f^{-1}(U))$. Consequently, by Lemma 19 we can see that $f(x) \notin \nabla_{S_2}(U)$. Thus $S_2(f(x)) \cap U = \emptyset$. So $y \notin U$, i.e., $y \in V$ which is a contradiction. □

By the above results and the Priestley duality for bounded distributive lattices we have a duality between the class of quasi-modal lattices with qm-homomorphisms and the descriptive qm-spaces with qm-morphisms.

4. Quasi-Modal Sublattices

In this section we define and characterize the quasi-modal sublattices. In the characterization we use the known lattice preorder relation associated to the Priestley space $X(\mathbf{A})$ (see [4]).

Throughout this section, we will frequently work with sublattices of a given lattice. In order to avoid any confusion, if **A** and **B** are two qm-lattices then we use the symbols Δ_A and ∇_A to denote the corresponding operations of **A**; similarly, we use Δ_B and ∇_B. Also, the ideal generated in **A** (filter generated) by some $X \subseteq A$, will be denoted by $I_A(X)$, $(F_A(X))$. If **A** is a bounded distributive lattice and **B** is a bounded sublattice of **A** then we will assume that the maximum and minimum of **A** and **B** are the same (0 and 1).

DEFINITION 21. Let **A** and **B** be two qm-lattices. We shall say that the structure $\mathbf{B} = \langle B, \vee, \wedge, \Delta_B, \nabla_B, 0, 1\rangle$ is a *quasi-modal sublattice* of **A**, or *qm-sublattice* for short, if **B** is a bounded sublattice of **A**, and for any $a \in B$,

$$I_A(\Delta_B(a)) = \Delta_A(a) \quad \text{and} \quad F_A(\nabla_B(a)) = \nabla_A(a).$$

PROPOSITION 22. *Let* **A** *be a bounded distributive lattice and let* **B** *be a bounded sublattice of* **A**. *Let H be an ideal of* **B** *and let J be an ideal of* **A**. *Let F be a filter of* **B** *and let G be a filter of* **A**. *Then*

(1) *if $J = I_A(H)$ then $H = J \cap B$,*
(2) *if $G = F_A(F)$ then $F = G \cap B$.*

Proof. (2) In order to prove that $F = G \cap B$ we need to show that $F_A(G \cap B) = F_A(F)$, which is easy to check. Let $x \in G \cap B$. Then $x \in F_A(G \cap B) = F_A(F)$. Therefore, there is $f \in F$ such that $f \leqslant x$. So $x \in F$. The proof of (1) is similar. □

LEMMA 23. *Let* **A** *be a qm-lattice. Let* **B** *be a bounded sublattice of* **A**. *Then the following conditions are equivalent*

(1) *There are two quasi-modal operators, $\Delta_B\colon B \to \mathrm{Id}(\mathbf{B})$ and $\nabla_B\colon B \to \mathrm{Fi}(\mathbf{B})$, such that $\mathbf{B} = \langle B, \vee, \wedge, \Delta_B, \nabla_B, 0, 1\rangle$ is a qm-sublattice of* **A**.
(2) *For any $a \in B$*

$$I_A(\Delta_A(a) \cap B) = \Delta_A(a) \quad \textit{and} \quad F_A(\nabla_A(a) \cap B) = \nabla_A(a).$$

Proof. Suppose that (1) holds. Thus $F_A(\nabla_B(a)) = \nabla_A(a)$, for any $a \in B$. Using Proposition 22, we have that $\nabla_B(a) = \nabla_A(a) \cap B$. So $F_A(\nabla_A(a) \cap B) = F_A(\nabla_B(a)) = \nabla_A(a)$.

Assume (2). We define $\nabla_B(a) = \nabla_A(a) \cap B$, for each $a \in B$. We can see that $\nabla_B(a)$ is a filter of **B**. So $F_A(\nabla_B(a)) = F_A(\nabla_A(a) \cap B) = \nabla_A(a)$. □

Now we present a characterization for the quasi-modal sublattices. For this development, given a quasi-modal lattice $\mathbf{A}$ we consider the relation S_B defined by

$$S_B = \{(P, Q) \in X(\mathbf{A}) \times X(\mathbf{A}) : Q \cap B \subseteq P\},$$

where B is a subset of A. It is well known that S_B is a lattice preorder in the Priestley space $\langle X(\mathbf{A}), \subseteq, \mathcal{T}_{X(\mathbf{A})}\rangle$ (see [4]).

THEOREM 24. *Let* $\mathbf{A} \in \mathcal{QML}$ *and* $\mathbf{B}$ *be a bounded sublattice of* $\mathbf{A}$. *The following conditions are equivalent*

(1) $\mathbf{B}$ *is a qm-sublattice,*
(2) $S_B^{-1} \circ R_{\Delta_A} \subseteq R_{\Delta_A} \circ S_B^{-1}$ *and* $S_B \circ R_{\nabla_A} \subseteq R_{\nabla_A} \circ S_B$.

Proof. (1) $\Rightarrow$ (2) We show that $S_B^{-1} \circ R_{\Delta_A} \subseteq R_{\Delta_A} \circ S_B^{-1}$. Let $P, D, Q \in X(\mathbf{A})$ such that $P \cap B \subseteq D$ and $\Delta_A^{-1}(D) \subseteq Q$. We see that

$$\Delta_A^{-1}(P) \cap (Q^c \cap B] = \emptyset.$$

Suppose the opposite. Hence there is $p \in \Delta_A^{-1}(P)$ such that $p \leqslant q$, for some $q \in Q^c \cap B$. Therefore $\Delta_A(p) \cap P \neq \emptyset$, $\Delta_A(p) \subseteq \Delta_A(q)$ and by the above lemma $\Delta_A(q) = I_A(\Delta_A(q) \cap B)$. Thus $I_A(\Delta_A(q) \cap B) \cap P \neq \emptyset$. Consequently there is $x \in P$ and $x \in \Delta_A(q) \cap B$. So $x \in P \cap B \subseteq D$. Moreover $x \in \Delta_A(q) \cap D$, which implies that $\Delta_A(q) \cap D \neq \emptyset$, i.e., $q \in \Delta_A^{-1}(D)$. Using the hypothesis we conclude that $q \in Q$, which is a contradiction. Next, there is $Z \in X(\mathbf{A})$ such that $\Delta_A^{-1}(P) \subseteq Z$ and $Z \cap B \subseteq Q$. In other words, $(P, Q) \in R_{\Delta_A} \circ S_B^{-1}$.

Now we check that $S_B \circ R_{\nabla_A} \subseteq R_{\nabla_A} \circ S_B$. Let $P, D, Q \in X(\mathbf{A})$ be such that $D \cap B \subseteq P$ and $Q \subseteq \nabla_A^{-1}(D)$. We prove that

$$F(Q \cap B) \cap (\nabla_A^{-1}(P))^c = \emptyset.$$

Suppose the opposite. Then there is $p \notin \nabla_A^{-1}(P)$ such that $q \leqslant p$ with $q \in Q \cap B$. Therefore $\nabla_A(p) \subseteq \nabla_A(q)$, $\nabla_A(p) \nsubseteq P$ and $\nabla_A(q) \subseteq D$. Hence $\nabla_A(q) \nsubseteq P$. Since $q \in B$ we have that $F(\nabla_A(q) \cap B) = \nabla_A(q)$. Consequently $F(\nabla_A(q) \cap B) \nsubseteq P$. Thus there is $x \in \nabla_A(q) \cap B$ and $x \notin P$, which is a contradiction since $\nabla_A(q) \subseteq D$ and $D \cap B \subseteq P$. So, there is $Z \in X(\mathbf{A})$ such that $Z \subseteq \nabla_A^{-1}(P)$ and $Q \cap B \subseteq Z$.

(2) $\Rightarrow$ (1) First we show that

$$I_A(\Delta_A(a) \cap B) = \Delta_A(a),$$

for any $a \in B$. Suppose that $\Delta_A(a) \nsubseteq I(\Delta_A(a) \cap B)$, for some $a \in B$. Thus there is $x \in \Delta_A(a)$ such that $x \nleqslant y$ for every $y \in \Delta_A(a) \cap B$. It is easy to check that $[[x) \cap B) \cap \Delta_A(a) = \emptyset$. Then there is $D \in X(\mathbf{A})$ such that $[x) \cap B \subseteq D$ and $a \notin \Delta_A^{-1}(D)$. Thus, there exists $Q \in X(\mathbf{A})$ such that $(D, Q) \in R_{\Delta_A}$ and $a \notin Q$. Now we see that

$$[x) \cap (B \cap D^c] = \emptyset.$$

If this is not the case, there is $p \in B \cap D^c$ such that $x \leqslant p$. So $p \in [x) \cap B \subseteq D$, which is a contradiction. Consequently there is $P \in X(\mathbf{A})$ such that $x \in P$ and $B \cap P \subseteq D$. We can conclude that $(P, Q) \in S_B^{-1} \circ R_{\Delta_A}$. Therefore, by hypothesis there is $Z \in X(\mathbf{A})$ such that $\Delta_A^{-1}(P) \subseteq Z$ and $Z \cap B \subseteq Q$. It is clear that $\Delta_A(a) \cap P \neq \emptyset$, because x belong to both sets. So $a \in Z \cap B$, and this implies that $a \in Q$, which is a contradiction. The other inclusion is immediate.

Now we check that

$$F_A(\nabla_A(a) \cap B) = \nabla_A(a),$$

for any $a \in B$. Suppose that $\nabla_A(a) \nsubseteq F(\nabla_A(a) \cap B)$, for some $a \in B$. Thus, there is $x \in \nabla_A(a)$ such that $y \nleqslant x$ for every $y \in \nabla_A(a) \cap B$. We can see that $((x] \cap B] \cap \nabla_A(a) = \emptyset$. If we suppose the opposite, there exists $y \in \nabla_A(a)$ such that $y \leqslant x$ and $y \in B$. So, $y \leqslant x$ and $y \in \nabla_A(a) \cap B$ which is a contradiction. Therefore, there is $D \in X(\mathbf{A})$ such that $\nabla_A(a) \subseteq D$ and $((x] \cap B] \cap D = \emptyset$. So, $a \in \nabla_A^{-1}(D)$. Using Lemma 6 we can ensure that there is $Q \in X(\mathbf{A})$ such that $(D, Q) \in R_{\nabla_A}$ and $a \in Q$. It is easy to prove that

$$(x] \cap F(B \cap D) = \emptyset.$$

Then, there is $P \in X(\mathbf{A})$ such that $x \notin P$ and $B \cap D \subseteq P$. Next, we have that $(P, Q) \in S_B \circ R_{\nabla_A}$. Thus, by hypothesis there is $Z \in X(\mathbf{A})$ such that $Z \subseteq \nabla_A^{-1}(P)$ and $Q \cap B \subseteq Z$. Since $a \in Q \cap B$, $a \in \nabla_A^{-1}(P)$. So $\nabla_A(a) \subseteq P$, and consequently $x \notin \nabla_A(a)$ since $x \notin P$, which is a contradiction. □

5. Quasi-Modal Congruences

In this section we introduce the concept of quasi-modal congruence, or qm-congruence for short, in a qm-lattice. Given a qm-lattice $\mathbf{A}$ and $\theta \subseteq A \times A$ a lattice congruence, our main goal will be to establish a compatibility property for Δ and ∇.

In the study of the qm-congruences we will take into account that, if $\mathbf{A}$ is a bounded distributive lattice and $\theta \subseteq A \times A$ is a lattice congruence then θ can be expressed in terms of closed subsets of the topology $\mathcal{T}_{X(\mathbf{A})}$. More precisely, every lattice congruence $\theta \subseteq A \times A$ has associated a closed subset Y of $X(\mathbf{A})$ such that $\theta = \theta(Y)$, where $\theta(Y)$ is defined by:

$$(a, b) \in \theta(Y) \Leftrightarrow \beta(a) \cap Y = \beta(b) \cap Y.$$

DEFINITION 25. Let $\mathbf{A}$ be a qm-lattice, and $\theta \subseteq A \times A$ a lattice congruence. Let $a, b \in A$. We shall say that $\langle \Delta a, \Delta b \rangle \in \theta$ if the following conditions hold:

(1) $\forall x \in \Delta a\ \exists y \in \Delta b : (x, y) \in \theta$,
(2) $\forall y \in \Delta b\ \exists x \in \Delta a : (x, y) \in \theta$.

DEFINITION 26. Let $\mathbf{A}$ be a qm-lattice, and $\theta \subseteq A \times A$ a lattice congruence. Let $a, b \in A$. We shall say that $\langle \nabla a, \nabla b \rangle \in \theta$ if the following conditions hold:

(1) $\forall x \in \nabla a \, \exists y \in \nabla b : (x, y) \in \theta$,
(2) $\forall y \in \nabla b \, \exists x \in \nabla a : (x, y) \in \theta$.

We will say that a lattice congruence θ is a *qm-congruence* if for every $(a, b) \in \theta$, it holds that $\langle \Delta a, \Delta b \rangle, \langle \nabla a, \nabla b \rangle \in \theta$.

LEMMA 27. *Let $\langle A, \wedge, \vee, \Delta, \nabla, 0, 1 \rangle$ be a qm-lattice. Let Y be a closed subset of $X(\mathbf{A})$ such that $(a, b) \in \theta(Y)$. Then $\varphi(\Delta a) \cap Y = \varphi(\Delta b) \cap Y$ iff the following conditions hold*

(1) $\forall x \in \Delta a \, \exists y \in \Delta b : (x, y) \in \theta(Y)$,
(2) $\forall y \in \Delta b \, \exists x \in \Delta a : (x, y) \in \theta(Y)$.

Proof. Suppose that $\varphi(\Delta a) \cap Y = \varphi(\Delta b) \cap Y$. We prove (1). Let $x \in \Delta a$. Thus

$$\beta(x) \cap Y \subseteq \bigcup_{x \in \Delta a} (\beta(x) \cap Y) = \varphi(\Delta a) \cap Y$$
$$= \varphi(\Delta b) \cap Y = \bigcup_{y \in \Delta b} (\beta(y) \cap Y) \subseteq \bigcup_{y \in \Delta b} \beta(y).$$

Since $\beta(x) \cap Y$ is a closed subset of $X(\mathbf{A})$, then by compactness there are $y_1, \ldots, y_n \in \Delta b$ such that $\beta(x) \cap Y \subseteq \beta(y_1) \cup \cdots \cup \beta(y_n)$. So $\beta(x) \cap Y \subseteq \beta(z) \cap Y$, where $z = y_1 \vee \cdots \vee y_n$. Hence $\beta(x) \cap Y \cap \beta(z) = \beta(x \wedge z) \cap Y = \beta(x) \cap Y$. Let $y = x \wedge z$. Then $y \in \Delta b$, because $y \leqslant z$ and $z \in \Delta b$ which is an ideal. In other words, there is $y \in \Delta b$ such that $\beta(x) \cap Y = \beta(y) \cap Y$. We can prove (2) in a similar manner.

Now suppose that (1) and (2) hold. Let $P \in \varphi(\Delta a) \cap Y$, i.e., $\Delta a \cap P \neq \emptyset$ and $P \in Y$. Let $x \in \Delta a \cap P$. By (1), there is $y \in \Delta b$ such that $(x, y) \in \theta(Y)$. Hence $\beta(x) \cap Y = \beta(y) \cap Y$ and since $x \in P$, we have that $P \in \beta(y) \cap Y$. So $\Delta b \cap P \neq \emptyset$; thus $P \in \varphi(\Delta b) \cap Y$. We can prove the other inclusion analogously. □

LEMMA 28. *Let $\langle A, \wedge, \vee, \Delta, \nabla, 0, 1 \rangle$ be a qm-lattice. Let Y be a closed subset of $X(\mathbf{A})$ such that $\langle a, b \rangle \in \theta(Y)$. Then $\psi(\nabla a) \cap Y = \psi(\nabla b) \cap Y$ iff the following conditions hold*

(1) $\forall x \in \nabla a \, \exists y \in \nabla b : (x, y) \in \theta(Y)$,
(2) $\forall y \in \nabla b \, \exists x \in \nabla a : (x, y) \in \theta(Y)$.

Proof. Suppose that $\psi(\nabla a) \cap Y = \psi(\nabla b) \cap Y$. We prove (1). Let $x \in \nabla a$. Hence $\bigcap_{x \in \nabla a} \beta(x) \cap Y \subseteq \beta(x) \cap Y$. So by hypothesis $\bigcap_{y \in \nabla b} \beta(y) \cap Y \subseteq \beta(x) \cap Y$. Thus we have that

$$\beta(x)^c \subseteq (\beta(x) \cap Y)^c \subseteq \left(\bigcap_{y \in \nabla b} \beta(y) \cap Y \right)^c = \bigcup_{y \in \nabla b} (\beta(y) \cap Y)^c.$$

By compactness there are $y_1, \dots, y_n \in \nabla b$ such that $\beta(x)^c \subseteq \beta(y_1)^c \cup \cdots \cup \beta(y_n)^c \cup Y^c$. Hence $\beta(x)^c \subseteq (\beta(z) \cap Y)^c$ where $z = y_1 \wedge \cdots \wedge y_n$. This implies that $\beta(z) \cap Y \subseteq \beta(x) \cap Y$. As in previous lemma, we can conclude that $\beta(x) \cap Y = \beta(y) \cap Y$ with $y = z \vee x$. As $z \in \nabla b$ and $z \leqslant y$ it follows that $y \in \nabla b$, because ∇b is filter. (2) can be proved analogously.

Now suppose that (1) and (2) hold. Let $P \in \psi(\nabla a) \cap Y$, i.e., $\nabla a \subseteq P$ and $P \in Y$. Suppose that $\nabla b \nsubseteq P$. Hence there is $y \in \nabla b$ such that $y \notin P$. So $y \notin \nabla a$. By hypothesis (2), there is $x \in \nabla a$ such that $(x, y) \in \theta(Y)$, i.e., $\beta(x) \cap Y = \beta(y) \cap Y$. We can see that $P \notin \beta(y)$. Hence $x \notin P$, which is a contradiction since $x \in \nabla a \subseteq P$. We can show the other inclusion analogously. □

THEOREM 29. *Let* **A** *be a qm-lattice,* Y *a closed subset of* $X(\mathbf{A})$ *and let* $\theta(Y)$ *be its associated lattice congruence. Then,* $\theta(Y)$ *is a qm-congruence iff for every* $(a, b) \in \theta(Y)$ *the following conditions hold*

(1) $\varphi(\Delta a) \cap Y = \varphi(\Delta b) \cap Y$,
(2) $\psi(\nabla a) \cap Y = \psi(\nabla b) \cap Y$.

Proof. It follows from the two previous lemmas. □

Given a lattice **A** and a lattice congruence $\theta \subseteq A \times A$, it is well known that the quotient algebra

$$\mathbf{A}/\theta = \langle A/\theta, \vee, \wedge, 0_\theta, 1_\theta \rangle$$

is a bounded distributive lattice, where $A/\theta = \{x_\theta : x \in A\}$ and x_θ denotes the equivalence class of x, and also that the function $q\colon A \to A/\theta$ given by $q(a) = a_\theta$, is a lattice homomorphism. If $\mathbf{A} \in \mathcal{QML}$ and $\theta \subseteq A \times A$ is a qm-congruence and for each $a \in A$ we consider the sets

$$\Delta_\theta a_\theta = I(\{x_\theta : x \in \Delta a\}) \quad \text{and} \quad \nabla_\theta a_\theta = F(\{x_\theta : x \in \nabla a\}),$$

then it is easy to see that $\mathbf{A}/\theta = \langle A/\theta, \vee, \wedge, \Delta_\theta, \nabla_\theta, 0_\theta, 1_\theta \rangle$ is a qm-lattice and q is a qm-homomorphism.

Moreover, given a lattice **A**, it is known that the structure

$$\mathbf{Con}\,\mathbf{A} = \langle \mathrm{Con}\,\mathbf{A}, \vee, \cap, i_A, A \times A \rangle$$

is a lattice, where Con **A** denotes the family of all lattice congruences on **A**. Now, if $\mathbf{A} \in \mathcal{QML}$ then

$$\mathbf{Con}_q\,\mathbf{A} = \langle \mathrm{Con}_q\,\mathbf{A}, \vee, \cap, i_A, A \times A \rangle$$

is a sublattice of **Con A**, where $\mathrm{Con}_q\,\mathbf{A}$ denotes the set of all qm-congruences on **A**.

We will give a characterization of the qm-congruences in terms of certain subsets of the Priestley space $\langle X(\mathbf{A}), \subseteq, \mathcal{T}_{X(\mathbf{A})} \rangle$.

DEFINITION 30. Let $\langle X, \leqslant, R_1, R_2, D\rangle$ a qm-space. A subset $Y \subseteq X$ is called

(1) R_1-*saturated*, if $\min R_1(x) \subseteq Y$ for each $x \in Y$,
(2) R_2-*saturated*, if $\max R_2(x) \subseteq Y$ for each $x \in Y$.

The subsets that are at the same time R_1-saturated and R_2-saturated of X will be called $R_{1,2}$-*saturated*, and their family will be denoted by $S_{R_{1,2}}(X)$. In accordance with the above definition, it is easy to check that the intersection and union of any subfamily of $S_{R_{1,2}}(X)$ is a $R_{1,2}$-saturated set. Moreover, since X and $\emptyset$ are $R_{1,2}$-saturated sets, the family $S_{R_{1,2}}(X)$ is a complete sublattice of $\mathcal{P}(X)$. Also, the structure $\mathbf{C}_{R_{1,2}}(X) = \langle C_{R_{1,2}}(X), \cup, \cap, \emptyset, X\rangle$ forms a complete sublattice of $\mathbf{C}(X) = \langle C(X), \cup, \cap, \emptyset, X\rangle$, where $C_{R_{1,2}}(X) = C(X) \cap S_{R_{1,2}}(X)$ (the family of the $R_{1,2}$-saturated and closed subsets).

LEMMA 31. *Let* $\mathbf{A}$ *be a qm-lattice and* $\theta \subseteq A \times A$ *a lattice congruence. Let* $Y \subseteq X(\mathbf{A})$ *be the closed set associated with* θ. *The following conditions are equivalent*

(1) $\langle \Delta a, \Delta b\rangle \in \theta(Y)$, *for each* $(a, b) \in \theta(Y)$,
(2) Y *is* R_Δ-*saturated.*

Proof. (1) $\Rightarrow$ (2) Let $P \in Y$ and $Q \in \min R_\Delta(P)$. Suppose that $Q \notin Y$. As Y is closed, there are $a, b \in A$ such that $Y \subseteq \beta(a)^c \cup \beta(b)$ and $Q \notin \beta(a)^c \cup \beta(b)$. This implies that $(a, a \wedge b) \in \theta(Y)$, $a \in Q$ and $b \notin Q$. We prove that

$$\Delta^{-1}(P) \cap I(Q^c \cup \{a\}) \neq \emptyset.$$

Suppose the opposite. Hence there is $D \in X(\mathbf{A})$ such that $\Delta^{-1}(P) \subseteq D$, $D \subseteq Q$ and $a \notin D$. Thus $D = Q$ since $D \in R_\Delta(P)$ and Q is minimal. So $a \notin Q$, which is a contradiction. Therefore there is $p \in \Delta^{-1}(P)$ such that $p \leqslant a \vee q$, with $q \notin Q$. Consequently $\Delta p \subseteq \Delta(a \vee q)$ and $\Delta(a \vee q) \cap P \neq \emptyset$. Since $(a, a \wedge b) \in \theta(Y)$ and $\theta(Y)$ is congruence,

$$(a \vee q, (a \wedge b) \vee q) \in \theta(Y).$$

Using hypothesis (1), we have that

$$\beta(a \vee q) \cap Y = \beta((a \wedge b) \vee q) \cap Y.$$

Considering this equality, and also that $P \in Y$ and $\Delta(a \vee q) \cap P \neq \emptyset$, we can conclude that $\Delta((a \wedge b) \vee q) \cap P \neq \emptyset$. Thus $(a \wedge b) \vee q \in \Delta^{-1}(P)$, and since $(P, Q) \in R_\Delta$ we have $(a \wedge b) \vee q \in Q$. So $a \wedge b \in Q$, because $Q \in X(\mathbf{A})$ and $q \notin Q$. Therefore $b \in Q$, which is a contradiction.

(2) $\Rightarrow$ (1) Let $(a, b) \in \theta(Y)$. We show only that $\varphi(\Delta a) \cap Y \subseteq \varphi(\Delta b) \cap Y$. Suppose the opposite, i.e., there is $P \in X(\mathbf{A})$ such that $\Delta a \cap P \neq \emptyset$, $P \in Y$ and $\Delta b \cap P = \emptyset$. Thus $b \notin \Delta^{-1}(P)$. Therefore there is $Q \in X(\mathbf{A})$ such that $\Delta^{-1}(P) \subseteq Q$ and $b \notin Q$. Thus $Q \in R_\Delta(P)$ and as $R_\Delta(P)$ is closed, $\min R_\Delta(P) \neq \emptyset$. Hence there is $D \in X(\mathbf{A})$ such that $D \in R_\Delta(P)$ and $D \subseteq Q$. So $b \notin D$. As Y is

R_Δ-saturated by hypothesis, we have that $D \in Y$. Moreover $a \in \Delta^{-1}(P)$ since $\Delta a \cap P \neq \emptyset$, and as $D \in R_\Delta(P)$ we conclude that $a \in D$. Thus $D \in \beta(a) \cap Y = \beta(b) \cap Y$. So $b \in D$, which is a contradiction. □

LEMMA 32. *Let* $\mathbf{A}$ *be a qm-lattice and* $\theta \subseteq A \times A$ *a lattice congruence. Let* $Y \subseteq X(\mathbf{A})$ *be the closed set associated with* θ. *The following conditions are equivalent*

(1) $\langle \nabla a, \nabla b \rangle \in \theta(Y)$, *for each* $(a, b) \in \theta(Y)$,
(2) Y *is* R_∇*-saturated.*

Proof. (1) $\Rightarrow$ (2) Let $P \in Y$ and $Q \in \max R_\nabla(P)$. Suppose that $Q \notin Y$. As in the proof of the implication (1) $\Rightarrow$ (2) of the above lemma, we can conclude that there are $a, b \in A$ such that $(a, a \wedge b) \in \theta(Y)$, $a \in Q$ and $b \notin Q$. It is also easy to see that $[Q \cup \{b\}) \cap (\nabla^{-1}(P))^c \neq \emptyset$. Thus there is $q \in Q$ such that $q \wedge b \leqslant x$ and $x \in (\nabla^{-1}(P))^c$. Hence $q \wedge b \notin \nabla^{-1}(P)$ since $(\nabla^{-1}(P))^c$ is ideal. This implies that $\nabla(q \wedge b) \nsubseteq P$. It is clear that $(a \wedge q, (a \wedge b) \wedge q) \in \theta(Y)$. Using the hypothesis we have that

$$\psi(\nabla(a \wedge q)) \cap Y = \psi(\nabla((a \wedge b) \wedge q)) \cap Y.$$

Moreover $\nabla(a \wedge q) \subseteq P$, since $a, q \in Q \subseteq \nabla^{-1}(P)$. Therefore $P \in \psi(\nabla(a \wedge q)) \cap Y$ and consequently $\nabla((a \wedge b) \wedge q) \subseteq P$. Moreover, $\nabla(q \wedge b) \subseteq \nabla((a \wedge b) \wedge q)$ because $(a \wedge b) \wedge q \leqslant b \wedge q$. So $\nabla(q \wedge b) \subseteq P$ which is a contradiction.

(2) $\Rightarrow$ (1) Let $(a, b) \in \theta(Y)$. We show that $\psi(\nabla a) \cap Y \subseteq \psi(\nabla b) \cap Y$. Suppose the opposite, i.e., there is $P \in X(\mathbf{A})$ such that $\nabla a \subseteq P$, $P \in Y$ and $\nabla b \nsubseteq P$. So $b \notin \nabla^{-1}(P)$. It is easy to check that $(\nabla^{-1}(P))^c \cap [a) = \emptyset$. Thus there exists $Q \in X(\mathbf{A})$ such that $Q \subseteq \nabla^{-1}(P)$ and $a \in Q$. Since $R_\nabla(P)$ is closed and $Q \in R_\nabla(P)$, $\max R_\nabla(P) \neq \emptyset$. Therefore there is $D \in X(\mathbf{A})$ such that $D \in R_\nabla(P)$ and $Q \subseteq D$. So $a \in D$, and since Y is R_∇-saturated $D \in Y$. Consequently $D \in \beta(a) \cap Y = \beta(b) \cap Y$. Hence $b \in D \subseteq \nabla^{-1}(P)$, which is a contradiction. We can prove the other inclusion in a similar manner. □

COROLLARY 33. *Let* $\mathbf{A}$ *be a qm-lattice and* $\theta \subseteq A \times A$ *a lattice congruence. Let* $Y \subseteq X(\mathbf{A})$ *be the closed associated with* θ. *Then* $\theta(Y)$ *is a qm-congruence iff* Y *is* $R_{\Delta,\nabla}$*-saturated.*

Proof. It follows immediately from the two lemmas above. □

COROLLARY 34. *Let* $\mathbf{A}$ *be a qm-lattice. Then the correspondence* $Y \mapsto \theta(Y)$ *establishes an anti-isomorphism between* $\mathbf{C}_{\Delta,\nabla}(X(\mathbf{A}))$ *and* $\mathbf{Con}_q\, \mathbf{A}$.

5.1. SIMPLE AND SUBDIRECTLY IRREDUCIBLE QM-LATTICES

In this subsection, we introduce the concepts of Simple and Subdirectly Irreducible quasi-modal lattices. For its characterization, we rely on the characterization of the simple and subdirectly irreducible algebras given by A. Petrovich in [7].

DEFINITION 35. Let $\mathbf{A} \in \mathcal{QML}$. We will say that

(1) $\mathbf{A}$ is *simple* if and only if the lattice of the qm-congruences has only two elements.
(2) $\mathbf{A}$ is *subdirectly irreducible* if and only if there exists a minimum nontrivial qm-congruence θ in $\mathbf{A}$.

According to the above definition and Corollary 34 we can ensure that a qm-lattice $\mathbf{A}$ is simple iff $C_{\Delta,\nabla}(X(\mathbf{A})) = \{\emptyset, X(\mathbf{A})\}$.

Let $\langle X, \leqslant, R_1, R_2, D\rangle$ be a qm-space. The set

$$\mathcal{T}_{R_{1,2}} = \{X - Y : Y \in C_{R_{1,2}}(X)\}$$

defines a topology on X, whose closed sets are the elements of $C_{R_{1,2}}(X)$. Let $S \in \mathcal{P}(X)$, we denote with $\mathrm{Cl}_{R_{1,2}}(S)$ the closure of S with respect to the topology $\mathcal{T}_{R_{1,2}}$. In addition, a set S is called $R_{1,2}$*-closed* ($R_{1,2}$*-dense*) if it is closed (dense) with respect to the topology $\mathcal{T}_{R_{1,2}}$.

THEOREM 36. *Let* $\mathbf{A} \in \mathcal{QML}$. *Then*

(1) $\mathbf{A}$ *is simple iff either* $\mathrm{dom}(R_\Delta \cup R_\nabla) = X(\mathbf{A})$, $\min R_\Delta(P) \cup \max R_\nabla(P)$ *is* $R_{\Delta,\nabla}$*-dense in* $X(\mathbf{A})$ *for each* $P \in X(\mathbf{A})$, *or* $\mathrm{dom}(R_\Delta \cup R_\nabla) = \emptyset$ *and* $X(\mathbf{A})$ *is a singleton.*
(2) $\mathbf{A}$ *is subdirectly irreducible but not simple iff only one of the following conditions holds:*
   (i) *The set*

   $$\{P \in X(\mathbf{A}) : \min R_\Delta(P) \cup \max R_\nabla(P) \text{ is } R_{\Delta,\nabla}\text{-dense in } X(\mathbf{A})\}$$

   is a non-empty open subset of $X(\mathbf{A})$ *different from* $X(\mathbf{A})$.
   (ii) *There is* $P \notin \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_\Delta(P) \cup \max R_\nabla(P))$ *such that*

   $$\{P\} \cup \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_\Delta(P) \cup \max R_\nabla(P)) = X(\mathbf{A})$$

   and $P \in \mathrm{dom}(R_\Delta \cup R_\nabla)$.

Proof. (1) ($\Rightarrow$) Let $\mathbf{A}$ be a simple qm-lattice. Suppose that $\mathrm{dom}(R_\Delta \cup R_\nabla) = X(\mathbf{A})$. Let $P \in X(\mathbf{A})$. Therefore $R_\Delta(P) \neq \emptyset$ or $R_\nabla(P) \neq \emptyset$. This implies that $\min R_\Delta(P) \neq \emptyset$ or $\max R_\nabla(P) \neq \emptyset$. So

$$\mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_\Delta(P) \cup \max R_\nabla(P)) = X(\mathbf{A}),$$

because $\mathbf{A}$ is simple. Now we suppose that $P \in X(\mathbf{A}) - \mathrm{dom}(R_\Delta \cup R_\nabla)$. Thus $R_\Delta(P) = \emptyset$ and $R_\nabla(P) = \emptyset$. Then $\{P\}$ is $R_{\Delta\nabla}$-closed. Since $\mathbf{A}$ is simple, we have $\{P\} = X(\mathbf{A})$ and consequently $\mathrm{dom}(R_\Delta \cup R_\nabla) = \emptyset$.

We prove the other implication. Let us assume that $\mathrm{dom}(R_\Delta \cup R_\nabla) = X(\mathbf{A})$ and $\min R_\Delta(P) \cup \max R_\nabla(P)$ is $R_{\Delta,\nabla}$-dense in $X(\mathbf{A})$ for every $P \in X(\mathbf{A})$. Let Y be a non-empty $R_{\Delta\nabla}$-closed subset of $X(\mathbf{A})$. Then, for every $P \in Y$, $\min R_{\Delta}(P) \subseteq Y$ and $\max R_{\nabla}(P) \subseteq Y$. Thus, since Y is $R_{\Delta,\nabla}$-closed,

$$\mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P)) \subseteq Y.$$

Therefore, $Y = X(\mathbf{A})$. If $\mathrm{dom}(R_{\Delta} \cup R_{\nabla}) = \emptyset$ and $X(\mathbf{A})$ is a singleton it is clear that the sets $\emptyset$ and $X(\mathbf{A})$ are the only $R_{\Delta\nabla}$-closed sets.

(2) We can see that conditions (i) and (ii) are incompatible. Suppose the opposite. Then, there are P, Z in $X(\mathbf{A})$ such that $\min R_{\Delta}(P) \cup \max R_{\nabla}(P)$ is $R_{\Delta\nabla}$-dense in $X(\mathbf{A})$,

$$Z \notin \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(Z) \cup \max R_{\nabla}(Z))$$

and

$$\{Z\} \cup \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(Z) \cup \max R_{\nabla}(Z)) = X(\mathbf{A}).$$

It is clear that $P \neq Z$, and that $P \in \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(Z) \cup \max R_{\nabla}(Z))$. As the set $\mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(Z) \cup \max R_{\nabla}(Z))$ is $R_{\Delta,\nabla}$-saturated we have that

$$\min R_{\Delta}(P) \cup \max R_{\nabla}(P) \subseteq \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(Z) \cup \max R_{\nabla}(Z)),$$

which is a contradiction since by hypothesis $\min R_{\Delta}(P) \cup \max R_{\nabla}(P)$ is $R_{\Delta,\nabla}$-dense and besides $\mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(Z) \cup \max R_{\nabla}(Z)) \neq X(\mathbf{A})$.

Suppose that $\mathbf{A}$ is subdirectly irreducible but not simple. Let Y be the greatest element of $C_{R_{\Delta,\nabla}}(X(\mathbf{A})) - \{X(\mathbf{A})\}$. We define the set

$$T = \{P \in X(\mathbf{A}) : \min R_{\Delta}(P) \cup \max R_{\nabla}(P) \text{ is not } R_{\Delta,\nabla}\text{-dense in } X(\mathbf{A})\}.$$

Since Y is $R_{\Delta,\nabla}$-closed and different from $X(\mathbf{A})$ we have $Y \subseteq T$. Suppose that $Y = T$. Clearly $X(\mathbf{A}) - Y$ is a non-empty open subset of $X(\mathbf{A})$, and in accordance with the definition of T we obtain (i). Now, suppose that $Y \subsetneq T$. Let $P \in T - Y$. Then $\mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P))$ is a $R_{\Delta,\nabla}$-saturated subset of $X(\mathbf{A})$, and different from $X(\mathbf{A})$. Thus

$$\mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P)) \subseteq Y.$$

It is clear that the set

$$\{P\} \cup \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P))$$

is a closed subset of $X(\mathbf{A})$ and $R_{\Delta,\nabla}$-saturated. Since $P \notin Y$ we may conclude that $\{P\} \cup \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P)) = X(\mathbf{A})$, and thus $Y = \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P))$. So, we obtain (ii).

Let us prove the reciprocal. Suppose (i). Let T be the set defined previously. We can see that T is different from $X(\mathbf{A})$, because otherwise $\{P \in X(\mathbf{A}) : \min R_{\Delta}(P) \cup \max R_{\nabla}(P)$ is $R_{\Delta,\nabla}$-dense in $X(\mathbf{A})\}$ would be empty, against the hypothesis (i). Also it is clear that $\mathbf{A}$ is not simple, and T is closed.

We show that T is $R_{\Delta,\nabla}$-saturated. Let $P \in T$ and suppose $Z \in \min R_{\Delta}(P) \cup \max R_{\nabla}(P)$. So $Z \in \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P))$ and consequently $\min R_{\Delta}(Z) \cup \max R_{\nabla}(Z)$ is not $R_{\Delta,\nabla}$-dense. Therefore $T \in C_{R_{\Delta,\nabla}}(X(\mathbf{A}))$. We see now that T is the greatest element of $C_{R_{\Delta,\nabla}}(X(\mathbf{A})) - \{X(\mathbf{A})\}$. Let $Y \in C_{R_{\Delta,\nabla}}(X(\mathbf{A})) - \{X(\mathbf{A})\}$. Let $P \in Y$. Then $\min R_{\Delta}(P) \cup \max R_{\nabla}(P) \subseteq Y$ and thus

$$\mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P)) \subseteq Y.$$

Since $Y \neq X(\mathbf{A})$ we have that $P \in T$, i.e. $Y \subseteq T$. Now suppose (ii). It is clear that $\mathbf{A}$ is not simple. We consider the set

$$Z = \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P)).$$

So, Z is a $R_{\Delta,\nabla}$-closed set different from $X(\mathbf{A})$, since P is the element considered in (ii). Let $Y \in C_{R_{\Delta,\nabla}}(X(\mathbf{A})) - \{X(\mathbf{A})\}$ and let $Q \in Y$. We assume that $Q = P$, therefore since $Y \in C_{R_{\Delta,\nabla}}(X(\mathbf{A}))$ we have $X(\mathbf{A}) = \{P\} \cup \mathrm{Cl}_{R_{\Delta,\nabla}}(\min R_{\Delta}(P) \cup \max R_{\nabla}(P)) \subseteq Y$, which is a contradiction. So, $Y \subseteq Z$. □

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