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Deductive Systems of BCK-Algebras

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Abstract

In this paper we shall give some results on irreducible deductive systems in BCK-algebras and we shall prove that the set of all deductive systems of a BCK-algebra is a Heyting algebra. As a consequence of this result we shall show that the annihilator F^* of a deductive system F is the the pseudocomplement of F. These results are more general than that the similar results given by M. Kondo in [7].

Key words: BCK-algebras, deductive system, irreducible deductive system, Heyting algebras, annihilators.

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1 Introduction and preliminaries

In [7] it was shown that the set of all ideals (or deductive systems, in our terminology) of a BCK-algebra $\mathbf{A}$ is a pseudocomplement distributive lattice and that the annihilator F^* of a deductive system F of $\mathbf{A}$ is the pseudocomplement of F. Related results on annihilators in Hilbert algebras and Tarski algebras (or also called commutative Hilbert algebras [6] or Abbot's implication algebras) are given in [2] and [3]. On the other hand, it was shown in [9] that the set of deductive systems $Ds(\mathbf{A})$ of a BCK-algebra $\mathbf{A}$ is an infinitely distributive lattice, and thus it is a Heyting algebra. In this note we will give a description of this fact and we shall prove that the annihilator F^* of the deductive system F can be obtained as $F^* = F \Rightarrow \{1\}$, where $\Rightarrow$ is the Heyting implication defined in the lattice $Ds(\mathbf{A})$.

In the remaining part of this section we shall review some results on BCK-algebras. In section 2 we shall study the notion of irreducible deductive system. In particular, we shall give a generalization of a result given in [8] for BCK-algebras with supremum. In Section 3 we shall prove that the lattice of deductive system of a BCK-algebra is a Heyting algebra.

Definition 1 An algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $(2, 0)$ is a *BCK-algebra* if for all $a, b, c \in A$ the following conditions hold:

1. $a \to a = 1$,
2. $(a \to b) \to ((b \to c) \to (a \to c)) = 1$,
3. $a \to (b \to c) = b \to (a \to c)$,
4. $a \to (b \to a) = 1$
5. $a \to b = 1$ and $b \to a = 1$, implies $a = b$.

If $\mathbf{A}$ is a BCK-algebra and we define the binary relation $\leq$ on $\mathbf{A}$ by $a \leq b$ if and only if $a \to b = 1$, then $\leq$ is a partial order in $\mathbf{A}$.

Let us recall that in all BCK-algebras $\mathbf{A}$ the following properties are satisfied:

P1 $1 \to a = a$,

P2 $a \to ((a \to b) \to b) = 1$

P3 $a \to b \leq (c \to b) \to (c \to a)$,

P4 $a \to b = ((a \to b) \to b) \to b$,

P5 if $a \leq b$, then $c \to a \leq c \to b$ and $b \to c \leq a \to c$.

A BCK-algebra *with supremum*, or BCK$^{\vee}$-algebra is an algebra

$$\mathbf{A} = \langle A, \to, \vee, 1\rangle$$

where $\langle A, \to, 1\rangle$ is a BCK-algebra, $\langle A, \vee, 1\rangle$ is a join-semilattice, and $a \to b = 1$ if and only if $a \vee b = b$. For $a, b \in A$ we define inductively $a \to_n b$ as $a \to_0 b = b$ and $a \to_{n+1} b = a \to ((a \to_n b))$.

Let $\mathbf{A}$ be a BCK-algebra. A *deductive system* or *filter* of $\mathbf{A}$ is a nonempty subset F of A such that $1 \in F$, and for every $a, b \in A$, if a, $a \to b \in F$, then $b \in F$. It is clear that if F is a deductive system, $a \leq b$ and $a \in F$, then $b \in F$. The set of all deductive system of a BCK-algebra $\mathbf{A}$ is denoted by $Ds(\mathbf{A})$. The deductive system generated by a set $X \subseteq A$ is denoted by $\langle X\rangle$. Let us recall that

$$\langle X\rangle = \{a \in A : x_1 \to (\ldots(x_n \to a)\ldots) = 1 \text{ for some } x_1, \ldots, x_n \in X\}.$$

In particular, $\langle x\rangle = \{a \in A : x \to (\ldots(x \to a)\ldots) = x \to_n a = 1\}$.

Let $\mathbf{A}$ be a BCK-algebra. In [9] (see also [10]) it was proved that the structure $\langle Ds(\mathbf{A}), \vee, \wedge, \{1\}, A\rangle$ is a bounded (infinitely) distributive lattice where the operations $\wedge$ and $\vee$ are defined by:

$$F_1 \wedge F_2 = F_1 \cap F_2$$
$$F_1 \vee F_2 = \{a \in A : \exists(x, y) \in F_1 \times F_2;\ x \to (y \to a) = 1\}.$$

We note that

$$F \vee \langle a \rangle = \{c \in A : a \to_n c \in F \text{ for some } n \geq 0\}$$

for $F \in Ds(\mathbf{A})$ and $a \in A$. Indeed, let $c \in F \vee \langle a \rangle$. Then there exist $x \in F$ and $n \geq 0$ such that $x \to (y \to c) = 1$ and $a \to_n y = 1$. Since $x \to (y \to c) = 1 \in F$, $y \to c \in F$. So, $y \to c \leq (a \to_n y) \to (a \to_n c) = 1 \to (a \to_n c) = a \to_n c \in F$.

2 Irreducible deductive systems

In [8] the separation theorem for BCK$^{\vee}$-algebras was proved. In this section following the paper [1], we prove a separation theorem for any BCK-algebra.

Let $\mathbf{A}$ be a BCK-algebra. A deductive system F is *irreducible* if and only if for any $F_1, F_2 \in Ds(\mathbf{A})$ such that $F = F_1 \cap F_2$, we have $F = F_1$ or $F = F_2$. We denote by $X(\mathbf{A})$ the set of all irreducible deductive systems of a BCK-algebra $\mathbf{A}$.

Lemma 2 *Let $\mathbf{A}$ be a BCK-algebra. Let $F \in Ds(\mathbf{A})$. Then F is irreducible if and only if for every $a, b \notin F$ there exist $c \notin F$ and $n \geq 0$ such that $a \to_n c$, $b \to_n c \in F$.*

Proof $\Rightarrow$) Let $a, b \notin F$. Let us consider the deductive systems $F_a = \langle F \cup \{a\} \rangle = F \vee \langle a \rangle$ and $F_b = \langle F \cup \{b\} \rangle = F \vee \langle b \rangle$. Since $F \neq F_a$ and $F \neq F_b$, then by irreducibility of F we have $F \subset F_a \cap F_b$. It follows that there exists $c \in (F_a \cap F_b) - F$. Then $a \to_n c \in F$ and $b \to_m c \in F$ for some $n, m \geq 0$. If we assume that $n \geq m$, then by property P4 we have that $b \to_m c \leq b \to_n c$. So, $a \to_n c \in F$ and $b \to_n c \in F$.

$\Leftarrow$). Let $F_1, F_2 \in Ds(\mathbf{A})$ such that $F = F_1 \cap F_2$. Suppose that $F \neq F_1$ and $F \neq F_2$. Then there exist $a \in F_1 - F$ and $b \in F_2 - F$. So, by the assumption, there exists $c \notin F$ and $n \geq 0$ such that $a \to_n c \in F$ and $b \to_n c \in F$. As, a, $a \to_n c \in F_1$ and $F_1 \in Ds(\mathbf{A})$, then $c \in F_1$. Similarly, $c \in F_2$. Thus, $c \in F_1 \cap F_2 = F$, which is a contradiction. □

Let $\mathbf{A}$ be a BCK-algebra. A subset I of A is called an *ideal* of $\mathbf{A}$ if:

1. If $b \in I$ and $a \leq b$, then $a \in I$.
2. If $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$.

The set of all ideals of $\mathbf{A}$ will be denoted by $Id(\mathbf{A})$.

Theorem 3 *Let $\mathbf{A}$ be a BCK-algebra. Let $F \in Ds(\mathbf{A})$ and $I \in Id(\mathbf{A})$ such that $F \cap I = \emptyset$. Then there exists $P \in X(\mathbf{A})$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof Let us consider the following subset of $Ds(\mathbf{A})$:

$$\mathcal{F} = \{H \in Ds(\mathbf{A}) : F \subseteq H \text{ and } H \cap I = \emptyset\}.$$

Since $F \in \mathcal{F}$, then $\mathcal{F} \neq \emptyset$. It is clear that the union of a chain of elements of $\mathcal{F}$ is also in $\mathcal{F}$. So, by Zorn's lemma, there exists a maximal element P of $\mathcal{F}$. We prove that $P \in X(\mathbf{A})$. Let $a, b \notin P$ and let us consider the deductive systems $P_a = \langle P \cup \{a\}\rangle$ and $P_b = \langle P \cup \{b\}\rangle$. Clearly, $P \subset P_a \cap P_b$. Then, $P_a, P_b \notin \mathcal{F}$. Thus, $P_a \cap I \neq \emptyset$ and $P_a \cap I \neq \emptyset$. It follows that there exist $x, y \in I$ such that $a \rightarrow_n x \in P$ and $b \rightarrow_m y \in P$ for some $n, m \geq 0$. Suppose that $m \leq n$. Then $b \rightarrow_m y \leq b \rightarrow_n y \in P$. Since I is an ideal, there exists $c \in I$ such that $x \leq c$ and $y \leq c$. So, $a \rightarrow_n x \leq a \rightarrow_n c \in P$ and $b \rightarrow_n y \leq b \rightarrow_n c \in P$. Therefore, by Lemma 2, we conclude that $P \in X(\mathbf{A})$. □

Corollary 4 *Let* $\mathbf{A}$ *be a BCK-algebra. Let* $F \in Ds(\mathbf{A})$.

1. *For each* $a \notin F$ *there exists* $P \in X(\mathbf{A})$ *such that* $a \notin P$ *and* $F \subseteq P$.
2. $F = \bigcap \{P \in X(\mathbf{A}) : F \subseteq P\}$.

3 Annihilators

Let us recall that a *Heyting algebra* is an algebra $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the operation $\Rightarrow$ satisfies the condition: $a \wedge b \leq c$ if and only if $a \leq b \Rightarrow c$, for all $a, b, c \in A$. The *pseudocomplement* of an element $x \in A$ is the element $x^* = x \Rightarrow 0$.

Let $\mathbf{A}$ be a BCK-algebra. Let $a \in A$. Define the set $[a) = \{x \in A : a \leq x\}$. We note that in general the set $[a) \notin Ds(\mathbf{A})$.

For each pair $F, H \in Ds(\mathbf{A})$ let us define the subset $F \Rightarrow H$ of A as follows:

$$F \Rightarrow H = \{a \in A : [a) \cap F \subseteq H\}.$$

Theorem 5 *Let* $\mathbf{A}$ *be a BCK-algebra. Let* $F, H \in Ds(\mathbf{A})$. *Then*

1. $F \Rightarrow H \in Ds(\mathbf{A})$.
2. $F \Rightarrow H = \{x \in A : (x \rightarrow f) \rightarrow f \in H$ *for each* $f \in F\}$.
3. $\langle Fi(A), \vee, \wedge, \Rightarrow, \{1\}, A\rangle$ *is a Heyting algebra.*

Proof 1. Since, $[1) \cap F = \{1\} \subseteq H$, then $1 \in F \Rightarrow H$.

Let x, $x \rightarrow y \in F \Rightarrow H$. Then, $[x) \cap F \subseteq H$ and $[x \rightarrow y) \cap F \subseteq H$. Let $z \in [y) \cap F$. As, $y \leq z$, then by the property P5, $x \rightarrow y \leq x \rightarrow z$. By property P4., we have $x \rightarrow z \in F$. Thus,

$$x \rightarrow z \in [x \rightarrow y) \cap F.$$

On the other hand, as $x \leq (x \rightarrow z) \rightarrow z$ and $z \leq (x \rightarrow z) \rightarrow z$, we get $(x \rightarrow z) \rightarrow z \in [x) \cap F$. Therefore,

$$x \rightarrow z,\ (x \rightarrow z) \rightarrow z \in H,$$

and consequently $z \in H$. So, $F \Rightarrow H \in Ds(\mathbf{A})$.

2. We prove that

$$F \Rightarrow H \subseteq G = \{x \in A : (x \to f) \to f \in H \text{ for each } f \in F\}.$$

Let $x \in A$ such that $[x) \cap F \subseteq H$. Let $f \in F$. Since, $x \leq (x \to f) \to f$ and $f \leq (x \to f) \to f$, then $(x \to f) \to f \in [x) \cap F \subseteq H$. Thus, $x \in G$.

Let $x \in G$. Let $y \in A$ such that $x \leq y$ and $y \in F$. Since $(x \to y) \to y \in H$ and $x \to y = 1$, then $1 \to y = y \in H$. Thus, $x \in F \Rightarrow H$.

3. Let $F, H, K \in Ds(\mathbf{A})$. Then it is easy to check that

$$F \cap H \subseteq K \text{ if and only if } F \subseteq H \Rightarrow K.$$

Thus, $\langle Ds(\mathbf{A}), \vee, \wedge, \Rightarrow, \{1\}, A\rangle$ is a Heyting algebra. □

As a corollary we have the following result, first given by M. Kondo in [7].

Corollary 6 *Let* $\mathbf{A}$ *be a BCK-algebra. The annihilator of a deductive system* F *is the deductive system*

$$F^* = F \Rightarrow \{1\} = \{x \in A : [x) \cap F = \{1\}\}.$$

Proof It is immediate by the above theorem. □

For BCK$^{\vee}$-algebras we can give the following result which generalize a similar result given by M. Kondo in [7] for commutative BCK-algebras.

Proposition 7 *Let* $\mathbf{A}$ *be a BCK$^{\vee}$-algebra. Then for every* $F \in Ds(\mathbf{A})$

$$F^* = \{x \in A : x \vee f = 1 \textit{ for each } f \in F\}.$$

Proof Let $x \in A$ such that $x \vee f = 1$ for each $f \in F$. We prove that $[x) \cap F = \{1\}$. Let $a \in A$ such that $x \leq a$ and $a \in F$. Then $a = x \vee a = 1$. Thus, $x \in F^*$.

Let $x \in F^*$. Then $[x) \cap F = \{1\}$. Since $x \leq x \vee f$, $f \leq x \vee f$, for each $f \in F$, and as F is increasing, then $x \vee f \in [x) \cap F$. Thus, $x \vee f = 1$, for each $f \in F$. □

Now we prove that the annihilator of a subset X is the annihilator of the deductive system generated by X. This result was proved for Tarski algebras in [2].

Theorem 8 *Let* $\mathbf{A}$ *be a BCK$^{\vee}$-algebra. Then for every subset* X *of* A*, we have* $X^* = \langle X\rangle^*$.

Proof Since $X \subseteq \langle X\rangle$, then $\langle X\rangle^* \subseteq X^*$. Let $x \in X^*$. We prove that for every $a \in \langle X\rangle$, $x \vee a = 1$. Suppose that there exists $a \in \langle X\rangle$ such that $a \vee x \neq 1$. Then there exist $x_1, \ldots, x_k \in X$ such that

$$x_1 \to (x_2 \to \ldots (x_k \to a) \ldots) = 1.$$

As $x \in X^*$, $x \vee x_i = 1$ for every $x_i \in \{x_1, \ldots, x_k\}$. Since, $a \vee x \neq 1$, by Theorem 3 there exists an irreducible deductive system P such that $x \notin P$, $a \notin P$ and taking into account that $x \vee x_i = 1$, then $x_i \in P$ for every $x_i \in \{x_1, \ldots, x_k\}$. But since, $x_1 \to (x_2 \to \ldots (x_k \to a) \ldots) = 1 \in P$, then $a \in P$, which is a contradiction. Thus, $a \vee x = 1$ for every $a \in \langle X\rangle$ and consequently $x \in \langle X\rangle^*$. □

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