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From iterated tilted algebras to cluster-tilted algebras ☆

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Abstract

In this paper the relationship between iterated tilted algebras and cluster-tilted algebras and relation extensions is studied. In the Dynkin case, it is shown that the relationship is very strong and combinatorial.

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1. Introduction and results

Cluster algebras were conceived around 2000 by Fomin and Zelevinsky, see [19], where they axiomatized a kind of combinatorics which was rapidly recognized to have been present before in

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different areas. Such a connection was established in the seminal paper [10] to the representation theory of finite-dimensional algebras, where the authors introduced the concept of *cluster category* $\mathcal{C}$, defined as orbit category of the bounded derived category $\mathrm{D}^{\mathrm{b}}(H)$ of a finite-dimensional hereditary algebra H over a field k. They established the connection in the special case when k is algebraically closed and H is of finite representation type, that is, the quiver of H is the disjoint union of Dynkin diagrams. The case A_n was also considered in [15]. It is remarkable that in the setting of cluster algebras the concept of finite type also exists naturally and that it is given by the Cartan–Killing classification, see [20]. The connection between cluster algebras and cluster categories was deepened by various authors and expanded over the original limit of finite type to hereditary finite-dimensional algebras (over an algebraically closed field) in general, see for example [14,16].

We assume throughout the whole article that the base field k is algebraically closed. The connection established thus far shows that to each hereditary algebra H, a cluster algebra $\mathcal{A}$ can be associated in such a way that its cluster variables (respectively clusters) correspond precisely to the indecomposable rigid objects, that is, objects T with $\mathrm{Hom}_{\mathcal{C}}(T, T[1]) = 0$ where $[1]$ is the shift induced by the shift in $\mathrm{D}^{\mathrm{b}}(H)$ (respectively *cluster-tilting objects*, see Section 2.8) of the cluster category $\mathcal{C}$. This turned the attention to *cluster-tilted algebras*, that is, endomorphism algebras of cluster-tilting objects of $\mathcal{C}$, see [12,13]. Buan, Marsh and Reiten showed in [13] that the quivers of the cluster-tilted algebras arising from a given cluster category are exactly the quivers corresponding to the exchange matrices of the associated cluster algebra. Moreover, they showed that for each cluster-tilting object $T = T' \oplus T_i$ with indecomposable summands T_i there exists precisely one indecomposable object $T_i' \not\simeq T_i$ such that $T' \oplus T_i'$ is again a cluster-tilting object and that this procedure corresponds in natural way to the mutation of the associated seeds.

In [2] the authors studied the relationship between tilted algebras $\mathrm{End}_H(M)$ for tilting H-modules M, and cluster-tilted algebras $\mathrm{End}_{\mathcal{C}}(T)$ for cluster-tilting objects T in $\mathcal{C}$. For this they introduced the concept of *relation extension* of an algebra B with $\mathrm{gldim}\, B \leqslant 2$ and defined it to be the algebra $\mathcal{R}(B) = B \ltimes \mathrm{Ext}^2_B(\mathrm{D}B, B)$, where $\mathrm{D}B$ is the dual of B, that is, the injective cogenerator $\mathrm{Hom}_k(B, k)$ of the module category $\mathrm{mod}\, B$. They proved that an algebra C is a cluster-tilted algebra if and only if it is the relation extension of some tilted algebra B. This result has an analogy with a well-known theorem about the relation between trivial extensions $T(A) = A \ltimes D(A)$ of Artin algebras A and tilted algebras, due to Hughes and Waschbüsch [22]. They prove that $T(A)$ is of finite representation type if and only if there exists a tilted algebra B of Dynkin type such that $T(A) \simeq T(B)$. This connection was extended to iterated tilted algebras by Assem, Happel and Roldán [4], who proved that a trivial extension $T(A)$ is of finite representation type if and only if A is an iterated tilted algebra of Dynkin type. Keeping these results in mind, we want to further extend the mentioned connection between cluster-tilted algebras and tilted algebras to iterated tilted algebras. It turns out that it is possible to do so, but one needs to restrict to iterated tilted algebras of global dimension at most two. The following is one of our main results.

Theorem 1.1. *If B is an iterated tilted algebra of* $\mathrm{gldim}\, B \leqslant 2$ *then there exists a cluster-tilted algebra C which is a split extension of B. More precisely, if $B = \mathrm{End}_{\mathrm{D}^{\mathrm{b}}(H)}(T)$ with H a hereditary algebra and T is a tilting complex in $\mathrm{D}^{\mathrm{b}}(H)$ then $C = \mathrm{End}_{\mathcal{C}(H)}(T)$ is a cluster-tilted algebra and there exists a sequence of algebra homomorphisms*

$$B \to C \xrightarrow{\pi} \mathcal{R}(B) \to B$$

whose composition is the identity map. Moreover, the kernel of π is contained in $\mathrm{rad}^2\, C$. *In particular C and $\mathcal{R}(B)$ have the same quivers and are both split extensions of B.*

The last assertion, relating the quivers of C and $\mathcal{R}(B)$, was also proven independently by Amiot in [1, 4.17] with different techniques.

To achieve the result we introduce a mechanism of obtaining a new iterated tilted algebra $\rho(B)$ with $\operatorname{gldim}\rho(B) \leqslant 2$, from a given one B with $\operatorname{gldim} B \leqslant 2$. We shall call the new algebra $\rho(B)$ the *rolling* of B. The key result in our proof is the following.

Theorem 1.2. *Let B be an iterated tilted algebra of type Q with* $\operatorname{gldim} B \leqslant 2$ *then for sufficiently large h the algebra $\rho^h(B)$ is tilted of type Q.*

We then focus on the finite type, where much more precise information is available on the combinatorial structure of the quiver and relations of a cluster-tilted algebra, see [12]. To do this we need the notion of admissible cut of a quiver Q, introduced in [17] (see also [18]), and define it to be a subset Δ of the arrows such that each oriented chordless cycle of Q contains precisely one element of Δ. Then for an algebra B, given as the quotient of a path algebra kQ_B by an admissible ideal I_B, we define the *quotient* of B *by an admissible cut* Δ to be $kQ_B/\langle I_B \cup \Delta\rangle$.

The following shows that the relationship between cluster-tilted algebras and iterated tilted algebras of the same type is strong and combinatorial.

Theorem 1.3. *An algebra B with* $\operatorname{gldim} B \leqslant 2$ *is iterated tilted of Dynkin type Q if and only if it is the quotient of a cluster-tilted algebra of type Q by an admissible cut.*

Moreover, we characterize the iterated tilted algebras B with $\operatorname{gldim} B \leqslant 2$ for which the relation extension $\mathcal{R}(B)$ is isomorphic to the corresponding cluster-tilted algebra $C(B)$, see Proposition 4.22.

Results along these lines were proven in [17] and [18] for admissible cuts of trivial extensions. In her PhD thesis E. Fernández showed that they are a very useful tool in the study of classification problems. In this way, she classified all trivial extensions of finite representation type, and gave a method to get all iterated tilted algebras of Dynkin type obtaining, under a unified approach, results proven with diverse techniques by other authors. Though in a different context, we consider that the results in this paper can be applied in a similar way to obtain analogous classification results for cluster-tilted algebras and also provide a new insight on tilted an iterated tilted algebras to study their quivers and relations.

2. Basic definitions and notations

2.1. Quivers and path algebras

A *quiver* is a directed graph, that is, a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 the set of arrows and $s, t : Q_1 \to Q_0$ are the maps which assign to each arrow α its *source* $s(\alpha)$ and its *target* $t(\alpha)$. We usually write $\alpha : s(\alpha) \to t(\alpha)$ to express this.

A subquiver Q' of a quiver Q is called a *chordless* (or *minimal*) *cycle* if Q' is full, connected and in every vertex of Q' exactly two arrows of Q' incide (starting or stopping there). In case exactly one arrow stops and the other starts the cycle is called *oriented*.

A *path* is a tuple $\gamma = (y|\alpha_r, \alpha_{r-1}, \dots, \alpha_1|x)$ of vertices $x, y \in Q_0$ and arrows $\alpha_1, \dots, \alpha_r \in Q_1$ with $x = y$ if $r = 0$ and $s(\alpha_1) = x$, $t(\alpha_r) = y$, $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \dots, r-1$ if $r > 0$. The number r is called the *length* of γ and the functions t, s are naturally extended by setting $s(\gamma) = x$ and $t(\gamma) = y$. We usually abbreviate $(y|\alpha_r, \alpha_{r-1}, \dots, \alpha_1|x)$ by $\alpha_r\alpha_{r-1}\cdots\alpha_1$ and $(x\,\|\,x)$ by e_x.

For a field k and a quiver Q, let kQ be the *path algebra* of Q: the underlying k-vector space has the set of all paths as basis and the multiplication is induced linearly by the concatenation of paths, that is, if $\delta = \beta_s\cdots\beta_1$ and $\gamma = \alpha_r\cdots\alpha_1$ then $\delta\gamma$ is defined as

$$\delta\gamma = \beta_s\cdots\beta_1\alpha_r\cdots\alpha_1$$

if $s(\beta_1) = t(\alpha_r)$ and $\delta\gamma = 0$ otherwise. The ideal of kQ generated by all paths of positive length is called *radical* and will be denoted by $\operatorname{rad} kQ$.

If the field k is algebraically closed, then each finite-dimensional algebra A is Morita-equivalent to the quotient of a path algebra by an *admissible* ideal I, that is, I is contained in $\operatorname{rad}^2 kQ$ and the quotient kQ/I is finite-dimensional. If, moreover, A is basic then $A \simeq kQ/I$, and the pair (Q, I) is called a *presentation* for A. If Q, Q' are two quivers and $I \subset kQ$, $I' \subset kQ'$ two ideals then we call (Q', I') an *extension of* (Q, I) if $Q_0 \subseteq Q'_0$, $Q_1 \subseteq Q'_1$ and $I \subseteq I'$.

2.2. Split extensions

We say that the algebra A is a *split extension* of the algebra B by the ideal M of A if there exists a split surjective algebra morphism $\pi : A \to B$ whose kernel M is a nilpotent ideal. This means that there exists a short exact sequence of k-vector spaces

$$0 \to M \xrightarrow{l} A \xrightarrow{\pi} B \to O$$

such that there exists an algebra morphism $\sigma : B \to A$ with $\pi\sigma = 1_B$. In particular σ identifies B with a subalgebra of A. Note that $M \subseteq \operatorname{rad} A$ since M is a nilpotent ideal.

Let B be a finite-dimensional algebra and consider a B–B-bimodule M. The *trivial extension* $B \ltimes M$ is the algebra whose underlying k-vector space is $B \times M$ with multiplication $(b, m) \cdot (b', m') = (bb', bm' + mb')$. When $\operatorname{gldim} B \leqslant 2$, the trivial extension $\mathcal{R}(B) = B \ltimes \operatorname{Ext}^2_B(\mathrm{D}B, B)$ is called the *relation extension* of B, see [2].

2.3. Quadratic forms

For an algebra B of finite global dimension, we denote by $\operatorname{mod} B$ the category of finitely generated (or equivalently finite-dimensional) left B-modules. Furthermore, we denote by $\mathrm{K}_\circ(B)$ the associated *Grothendieck group*, that is, the free abelian group on the isomorphism classes of objects of $\operatorname{mod} B$ modulo the subgroup generated by $\{E - X - Y \mid 0 \to X \to E \to Y \to 0$ is exact$\}$. The class of a B-module X shall be denoted by $[X]$. Notice that $\mathrm{K}_\circ(B) \simeq \mathbb{Z}^n$ where n is the number of isomorphism classes of simple B-modules. We denote by $\chi_B : \mathrm{K}_\circ(B) \to \mathbb{Z}$ the *homological form* (or *Euler form*) of B, that is, χ_B is the quadratic form associated to the bilinear form defined by

$$\big([X], [Y]\big) = \sum_{i=0}^{\infty} (-1)^i \dim \operatorname{Ext}^i_B(X, Y)$$

for $X, Y \in \operatorname{mod} B$.

We denote by q_B the *geometrical form* (or *Tits form*), defined by the “truncated” bilinear form defined for the classes of the simple modules S_i by

$$\langle [S_h], [S_j] \rangle = \sum_{i=0}^{2} (-1)^i \dim \mathrm{Ext}_B^i(S_h, S_j).$$

Remark 2.1. If $\operatorname{gldim} B \leqslant 2$ then $\chi_B = q_B$.

2.4. Algebras which are simply connected

An algebra A with connected quiver Q with no oriented cycles is called *simply connected* if for each presentation (Q, I) of A the fundamental group $\pi(Q, I)$ is trivial, for precise definitions we refer to [8] and [28].

A full subquiver Q' of Q is called *convex* if for any two paths γ, δ with $t(\gamma) = s(\delta)$ and $s(\gamma), t(\delta) \in Q'_0$ then $t(\gamma) \in Q'_0$. An algebra $A = kQ/I$ is called *strongly simply connected* if for every full and convex subquiver Q' of Q the induced algebra $kQ'/(kQ' \cap I)$ is simply connected.

Remark 2.2. By [28, Def. 2.2] and [8, 2.9], if A is of finite representation type then A is simply connected if and only if it is strongly simply connected.

2.5. Tilted and iterated tilted algebras

Let A be a finite-dimensional k-algebra. We recall that a module $M \in \operatorname{mod} A$ is called *tilting module* if M has projective dimension at most one, $\mathrm{Ext}_A^1(M, M) = 0$ and the decomposition of M into indecomposables contains precisely n pairwise non-isomorphic summands, where n is the number of pairwise non-isomorphic simple A-modules, or equivalently the number of vertices of the quiver of A.

If H is a hereditary algebra and M a tilting H-module then $\mathrm{End}_H^{\mathrm{op}}(M)$ is called a *tilted algebra*. Since the opposite of a tilted algebra is again a tilted algebra we often prefer to look at the endomorphism algebras themselves instead of their opposites. An algebra B is called an *iterated tilted algebra of type Q* if there exists a sequence of algebras $A_1, A_2, \ldots, A_t$ such that A_1 is hereditary with quiver Q, $A_t = B$ and for each $i = 1, \ldots, t-1$ we have $A_{i+1} \simeq \mathrm{End}_{A_i}(M_i)$ for some tilting A_i-module M_i or $A_i \simeq \mathrm{End}_{A_{i+1}}(N_i)$ for some tilting A_{i+1}-module N_i.

2.6. Structure of the derived category over a hereditary algebra

Throughout the rest of the article H denotes a finite-dimensional hereditary algebra over an algebraically closed field k. We denote by $\mathrm{D^b}(H)$ the bounded derived category of finitely generated H-modules, see [21] for generalities on derived categories. Since H is hereditary, each indecomposable object of $\mathrm{D^b}(H)$ is isomorphic to a complex concentrated in one degree. We shall identify the objects in $\operatorname{mod} H$ with the complexes concentrated in degree zero.

Recall that in $\mathrm{D^b}(H)$ *Serre duality* holds, that is, for any objects X and Y of $\mathrm{D^b}(H)$, we have

$$\mathrm{Hom}_{\mathrm{D^b}(H)}(X, \tau Y) = \mathrm{D}\,\mathrm{Hom}(Y, X[1]),$$

where τ denotes the Auslander–Reiten translation and [1] the suspension in $\mathrm{D^b}(H)$. The autoequivalence $F = \tau^{-1} \circ [1]$ will play a crucial role in the rest of the paper.

If the quiver Q of H is Dynkin then the Auslander–Reiten quiver Γ of $\mathrm{D^b}(H)$ consists of a single transjective component isomorphic to the translation quiver $\mathbb{Z}Q$, see [21, Ch. 1, Cor. 5.6]. In particular, the arrows induce a partial order in the vertices of Γ, that is, if $L \to M$ is an arrow in Γ then we write $L < M$. Moreover if there exists a path from L to M then all paths have the same length $d(L, M)$ and we set $d(L, M) = 0$ if there is no path at all.

In case Q is Dynkin, a set of representatives $\Sigma_1, \dots, \Sigma_n$ of the τ-orbits of Γ is called *section* if $\Sigma_1, \dots, \Sigma_n$ induce a connected subquiver of Γ. Here n is the number of vertices in the quiver Q.

If the quiver Q of H is not Dynkin then the structure of the Auslander–Reiten quiver Γ of $\mathrm{D^b}(H)$ is completely different. Denote by $\mathcal{P}$ (respectively $\mathcal{I}$) the preprojective (respectively preinjective) component of the Auslander–Reiten quiver of H and by $\mathcal{R}$ the full subcategory of $\operatorname{mod} H$ given by the regular components. For each $r \in \mathbb{Z}$ the regular part $\mathcal{R}$ gives rise to $\mathcal{R}[r]$, given by the complexes $X \in \mathrm{D^b}(H)$ concentrated in degree r with $X_r \in \mathcal{R}$. Moreover, for each $r \in \mathbb{Z}$ there is a transjective component $\mathcal{I}[r-1] \vee \mathcal{P}[r]$ of Γ which we shall denote by $\mathcal{R}[r - \frac{1}{2}]$, and each component of Γ is contained in $\mathcal{R}[r]$ for some half-integer r. The notation has the advantage that the different parts are ordered in the sense that $\operatorname{Hom}(\mathcal{R}[a], \mathcal{R}[b]) = 0$ for any two half-integers $a > b$. Also note that $\operatorname{Hom}(\mathcal{R}[a], \mathcal{R}[b]) = 0$ if $a < b - 1$.

2.7. *Tilting complexes*

An object T of $\mathrm{D^b}(H)$ is called *tilting complex* if $\operatorname{Hom}(T, T[i]) = 0$ for each $i \neq 0$ and if the only object X for which $\operatorname{Hom}(T, X[i]) = 0$ for all i is the zero object. It follows from [26, Cor. 3.3 and Lem. 3.5] that T is a tilting complex if and only if $\operatorname{Hom}_{\mathrm{D^b}(H)}(T, T[i]) = 0$ for all $i \neq 0$ and T has exactly n non-isomorphic indecomposable summands, where n is the number of simple H-modules (up to isomorphism).

Note that by [21, Cor. 5.5 of Ch. 4] and [27], an algebra A is iterated tilted of type Q if and only if A is isomorphic to the endomorphism algebra of a tilting complex T in $\mathrm{D^b}(kQ)$ (equivalently if and only if there exists an equivalence of triangulated categories $\mathrm{D^b}(A) \simeq \mathrm{D^b}(kQ)$).

2.8. *The cluster category*

Let H be a hereditary algebra. Then the orbit category $\mathcal{C} = \mathrm{D^b}(H)/F^{\mathbb{Z}}$ is called *cluster category of H*, see [10]. By construction the objects of $\mathcal{C}$ are the objects of $\mathrm{D^b}(H)$ and the morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathrm{D^b}(H)}\big(X, F^i Y\big)$$

with the natural composition, see [24], where it is also shown that $\mathcal{C}$ is a triangulated category.

An object T of $\mathcal{C}$ is a *cluster-tilting object* if $\operatorname{Hom}(T, T[1]) = 0$ and if T is decomposed into indecomposables $T = \bigoplus_{i=1}^n T_i$ then there are precisely n pairwise non-isomorphic summands, where n is the number of simple H-modules.

3. Iterated tilted algebras of global dimension two

3.1. Generalities on tilting complexes

If T is a tilting complex in $\mathrm{D^b}(H)$ (see Section 2.7) and $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$ then we have an equivalence of categories $G : \mathrm{D^b}(H) \to \mathrm{D^b}(B)$ derived from $\mathrm{Hom}(T, -)$ such that $G(T) = B$ and $G(\tau T[1]) = \mathrm{D}B$. For any direct summand X of T we write

$$\mathbf{P}_{X,T} = G(X) = \mathrm{Hom}_{\mathrm{D^b}(H)}(T, X) \quad \text{and} \quad \mathbf{I}_{X,T} = G\big(\tau X[1]\big).$$

Moreover, if GX and GY are B-modules, for two objects X and Y of $\mathrm{D^b}(H)$, then $\mathrm{Ext}^i_B(GX, GY) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(X, Y[i])$ for all $i \in \mathbb{Z}$.

Lemma 3.1. *Let T be a tilting complex in $\mathrm{D^b}(H)$ such that $\mathrm{gldim}\, B \leqslant 2$, where $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$. Then $\mathrm{Hom}_{\mathrm{D^b}(H)}(T, F^{-1}T) = 0$ and $\mathrm{Hom}_{\mathrm{D^b}(H)}(T, F^{-2}T) = 0$.*

Proof. By Serre duality and the fact that T is a tilting complex we have $\mathrm{Hom}_{\mathrm{D^b}(H)}(T, F^{-1}T) = \mathrm{Hom}_{\mathrm{D^b}(H)}(T[1], \tau T) = \mathrm{D}\,\mathrm{Hom}_{\mathrm{D^b}(H)}(T, T[2]) = 0$. Also $\mathrm{Hom}_{\mathrm{D^b}(H)}(T, F^{-2}T) = \mathrm{Hom}_{\mathrm{D^b}(H)}(T[3], \tau^2 T[1]) = \mathrm{D}\,\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau T[1], T[4]) = \mathrm{Ext}^4_B(\mathrm{D}B, B) = 0$ again by Serre duality and $\mathrm{gldim}\, B \leqslant 2$. $\square$

If T is a tilting complex then we have as in [2] that $\mathrm{Ext}^2_B(\mathrm{D}B, B) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(\tau T[1], T[2]) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}T, T) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(T, FT)$ with the natural structure of B–B-bimodules.

3.2. The rolling of tilting complexes

We are now going to define a procedure which is important in the forthcoming. It defines for each tilting complex T a new complex $\rho(T)$ such that $T \simeq \rho(T)$ in the cluster category $\mathcal{C}$. Since the structure of the derived category $\mathrm{D^b}(H)$ is substantially different whether the quiver Q of H is Dynkin or not, we have to distinguish these two cases in the construction.

Let first Q be a Dynkin quiver and T a tilting complex of $\mathrm{D^b}(kQ)$. Since $T = \bigoplus_{i=1}^n T_i$ has only finitely many summands we can easily find a section $\Sigma = \{\Sigma_1, \dots, \Sigma_n\}$ such that $T \leqslant \Sigma$, that is, $T_i \leqslant \Sigma_j$ for all i and j. If Σ_j is maximal in Σ and $\Sigma_j \notin \{T_1, \dots, T_n\}$ then $\Sigma' = \Sigma \setminus \{\Sigma_j\} \cup \{\tau\Sigma_j\}$ is also a section satisfying $T \leqslant \Sigma'$. After finitely many steps we get a section $\Sigma(T)$ such that $T \leqslant \Sigma(T)$ and all maximal elements in $\Sigma(T)$ belong to $\mathrm{add}\, T$. Notice that the section $\Sigma(T)$ is uniquely defined by T.

Definition 3.2 *(Rolling of tilting complex, the Dynkin case).* With the previous notations, let X be the sum of those summands of T which belong to $\Sigma(T)$ and T' a complement of X in T. Then define the *rolling* of T to be $\rho(T) = T' \oplus F^{-1}X$.

Now consider the case where Q is not Dynkin. Recall from Section 2.6 that $\mathrm{D^b}(kQ)$ is composed by the parts $\mathcal{R}[r]$ for $r \in \mathbb{Z}/2$ where $\mathcal{R}[r]$ denotes the regular (respectively transjective) part if r is an integer (respectively not an integer). Now, write $T = \bigoplus_{a \in \mathbb{Z}/2} T_{\mathcal{R}[a]}$, where $T_{\mathcal{R}[a]} \in \mathcal{R}[a]$.

Definition 3.3 *(Rolling of tilting complex, the non-Dynkin case).* With the previous notation let m be the largest half-integer such that $T_{\mathcal{R}[m]}$ is non-zero. Then define $X = T_{\mathcal{R}[m]}$ and T' to be the complement of X in T. Define the *rolling* of T to be $\rho(T) = T' \oplus F^{-1}X$.

Remark 3.4. If $T = T' \oplus X$ is a tilting complex in $\mathrm{D^b}(H)$ and $\rho(T) = T' \oplus F^{-1}X$ then we have $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$.

Definition 3.5 *(Rolling of iterated tilted algebras).* Let B be an iterated tilted algebra. Then define $\rho(B)$ to be the endomorphism algebra $\mathrm{End}_{\mathrm{D^b}(H)}(\rho(T))$, where H is a hereditary algebra with $\mathrm{D^b}(B) \simeq \mathrm{D^b}(H)$ and T a tilting complex in $\mathrm{D^b}(H)$ with $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$.

Notice that $\rho(B)$ does not depend on the choice of H or T. In fact, if T and $\widehat{T}$ are tilting complexes in $\mathrm{D^b}(H)$ such that $\mathrm{End}_{\mathrm{D^b}(H)}(T) \simeq \mathrm{End}_{\mathrm{D^b}(H)}(\widehat{T})$ then there is an equivalence of categories $G : \mathrm{D^b}(H) \to \mathrm{D^b}(H)$ with $G(T) = \widehat{T}$, and G preserves the partial order in $\mathrm{D^b}(H)$. Thus in the Dynkin case $G(\Sigma(T)) \simeq \Sigma(\widehat{T})$, and the sum X of the maximal elements in $\Sigma(T)$ corresponds under G to the sum $\widehat{X}$ of the maximal elements in $\Sigma(\widehat{T})$. Thus $\rho(T)$ and $G(\rho(T)) \simeq \rho(\widehat{T})$ have isomorphic endomorphism rings. The argument in the non-Dynkin case is similar.

3.3. *Characterization when $\rho(T)$ is again a tilting complex*

The following results provide necessary and sufficient conditions for the rolling $\rho(T)$ to be a tilting complex again.

Lemma 3.6. *Let $T = T' \oplus X$ be a tilting complex in $\mathrm{D^b}(H)$ such that $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$ and let $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$. Then $\overline{T} = T' \oplus F^{-1}X$ is a tilting complex if and only if $\mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}X, T'[j]) = 0$ for all $j \neq 0$ if and only if $\mathrm{Ext}^j_B(\mathbf{I}_{X,T}, \mathbf{P}_{T',T}) = 0$ for each $j \neq 2$.*

Proof. Observe that $\mathrm{Hom}_{\mathrm{D^b}(H)}(T', F^{-1}X[j]) = \mathrm{Hom}_{\mathrm{D^b}(H)}(T', \tau X[j-1]) = \mathrm{D}\,\mathrm{Hom}_{\mathrm{D^b}(H)}(X[j-1], T'[1]) = \mathrm{D}\,\mathrm{Hom}_{\mathrm{D^b}(H)}(X[j-2], T') = 0$ for all j (for $j \neq 2$ since T is a tilting complex and for $j = 2$ by hypothesis). Therefore $\overline{T}$ is a tilting complex if and only if $\mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}X, T'[j]) = 0$ for all $j \neq 0$, that is, if and only if $\mathrm{Ext}^j_B(\mathbf{I}_{X,T}, \mathbf{P}_{T',T}) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X[1], T'[j]) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}X, T'[j-2])$ equals zero for all $j \neq 2$. □

We can strengthen the former result under an additional hypothesis on the global dimension of B.

Lemma 3.7. *Let $T = T' \oplus X$ be a tilting complex in $\mathrm{D^b}(H)$ such that $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$ and let $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$. If $\mathrm{gldim}\, B \leqslant 2$, then $\widetilde{T} = T' \oplus F^{-1}X$ is a tilting complex in $\mathrm{D^b}(H)$ if and only if $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T'[k]) = 0$ for $k = 0, -1$.*

Proof. We have $\mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}X, T'[i]) = \mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T'[i+1]) = \mathrm{Ext}^{i+2}_B(\mathbf{I}_{X,T}, \mathbf{P}_{T',T})$, which equals zero for all $i \neq 0, -1, -2$.

By Lemma 3.6 the complex $T' \oplus F^{-1}X$ is a tilting complex if and only if $\mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}X, T'[i]) = 0$ for $i = -1, -2$. Hence the result follows. □

Lemma 3.8. *Let Q be a Dynkin quiver and T a tilting complex in* $\mathrm{D^b}(H)$. *Then* $\rho(T) < \tau(\Sigma(T))$.

Proof. As usual, let $T = T' \oplus X$ with $\rho(T) = T' \oplus F^{-1}X$ and $\Sigma = \Sigma(T)$. Let $\Sigma_1 \in \Sigma$, and let Σ_2 be a maximal element in Σ such that $\Sigma_1 \leqslant \Sigma_2$. That is, $\mathrm{Hom}_{\mathrm{D^b}(H)}(\Sigma_1, \Sigma_2) \neq 0$, so $\mathrm{Ext}^1_{\mathrm{D^b}(H)}(\Sigma_2, \tau(\Sigma_1)) \neq 0$ by the Serre duality. Our choice of Σ implies that $\Sigma_2 \in \operatorname{add} T$, so that $\tau\Sigma_1 \notin \operatorname{add} T$ because T is a tilting complex in $\mathrm{D^b}(H)$. Thus no summand of T is in $\tau\Sigma$ and therefore $T' < \tau\Sigma$, since $T' < \Sigma$ by the definition of T'.

Since $\operatorname{add} X \subseteq \Sigma$, then $F^{-1}(X) < \tau\Sigma$, ending the proof of the lemma. □

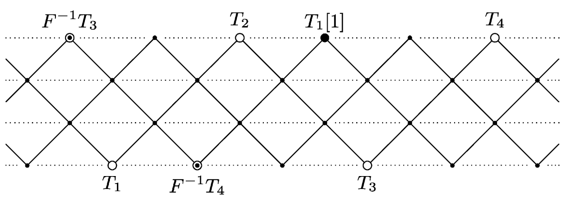
Proposition 3.9. *Let T be a tilting complex in* $\mathrm{D^b}(H)$ *such that* $\operatorname{gldim}\mathrm{End}_{\mathrm{D^b}(H)}(T) \leqslant 2$. *Then $\rho(T)$ is again a tilting complex.*

Proof. Again, let $T = T' \oplus X$ and $\rho(T) = T' \oplus F^{-1}X$. First consider the case when Q is a Dynkin quiver and let $\Sigma = \Sigma(T)$. By the lemma we know that $T' < \tau\Sigma$. We also get $T' < \tau\Sigma[1]$ because $\tau\Sigma < \tau\Sigma[1]$. Since the summands of X are in Σ, it follows that $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T') = 0$ and $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T'[-1]) = 0$. We conclude from Lemma 3.7 that $\rho(T)$ is a tilting complex.

Now consider the case where the quiver Q is not Dynkin and let $H = kQ$. As in Definition 3.3, let m be the largest half-integer such that $T_{\mathcal{R}[m]} \neq 0$. Hence we have $T = T' \oplus X$ and $\rho(T) = T' \oplus F^{-1}X$ where $X = T_{\mathcal{R}[m]}$. Then clearly we have $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$ and $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X, T'[k]) = 0$ for $k = 0, -1$ since τX belongs to $\mathcal{R}[m]$ and $T'[k]$ to $\bigvee_{i>0} \mathcal{R}[m - \frac{i}{2}]$. We conclude again by Lemma 3.7 that $\rho(T)$ is a tilting complex. □

Remark 3.10. The following example shows that the hypothesis on the global dimension of the endomorphism algebra is necessary.

Let $Q = \mathbb{A}_4$ and $T = \bigoplus_{i=1}^4 T_i$ the tilting complex in $\mathrm{D^b}(H)$ whose relative positions of the indecomposable summands T_i are as indicated in the following picture.



Then $B = \mathrm{End}_{\mathrm{D^b}(kQ)}(T)$ has global dimension 3. By Definition 3.2, the section $\Sigma(T)$ is precisely the section containing T_3 and T_4 and therefore $X = T_3 \oplus T_4$. Then $\rho(T)$ is not a tilting complex since $\mathrm{Hom}_{\mathrm{D^b}(kQ)}(F^{-1}T_4, T_1[1]) \neq 0$.

3.4. Global dimension two is preserved

The next result is fundamental in order for the iteration to work properly.

Proposition 3.11. *Let B be an iterated tilted algebra. If* $\operatorname{gldim} B \leqslant 2$ *then* $\operatorname{gldim} \rho(B) \leqslant 2$.

Proof. Let H be a hereditary algebra and $T = T' \oplus X$ a tilting complex in $\mathrm{D^b}(H)$ such that $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$ and $\rho(T) = T' \oplus F^{-1}X$. Then we have $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$ by Remark 3.4 and by Proposition 3.9 the complex $\rho(T)$ is a tilting complex in $\mathrm{D^b}(H)$. To shorten notations we set $\widetilde{T} = \rho(T)$ and $\widetilde{B} = \rho(B)$. We shall prove that $\mathrm{Ext}^j_{\widetilde{B}}(\mathrm{D}\widetilde{B}, \widetilde{B}) = 0$ for all $j \geqslant 3$. Since $\widetilde{T}$ is a tilting complex, we can show this by proving that $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau\widetilde{T}[1], \widetilde{T}[j])$ is zero for $j \geqslant 3$.

First note that

$$\mathrm{Hom}_{\mathrm{D^b}(H)}\big(\tau T[1], T[i]\big) = 0 \quad \text{for all } i \neq 0, 1, 2, \tag{3.1}$$

since $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau T[1], T[i]) \simeq \mathrm{Ext}^i_B(\mathrm{D}B, B)$.

Therefore $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau F^{-1}X[1], F^{-1}X[j]) = \mathrm{Hom}_{\mathrm{D^b}(H)}(\tau X[1], X[j]) = 0$ for $j \geqslant 3$ and $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau T'[1], T'[j]) = 0$ for $j \geqslant 3$. Also, $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau T'[1], F^{-1}X[j]) = \mathrm{Hom}_{\mathrm{D^b}(H)}(T'[1], X[j-1])$, which is zero for all $j \neq 2$ since T is a tilting complex.

Hence, it remains to see that $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau^2 X, T'[j]) = 0$ for $j \geqslant 3$. The minimal projective resolution of $\mathbf{I}_{X,T}$ in $\mathrm{mod}\, B$

$$0 \to P_2 \to P_1 \to P_0 \xrightarrow{\varphi} \mathbf{I}_{X,T} \to 0$$

gives rise to two exact triangles $\Delta_a : K \to P_0 \to \mathbf{I}_{X,T} \to K[1]$ and $\Delta_b : P_2 \to P_1 \to K \to P_2[1]$, where K denotes the kernel of φ.

To both triangles apply first the inverse of the equivalence $G : \mathrm{D^b}(H) \to \mathrm{D^b}(B)$ and then τ, to obtain exact triangles of the form $S \to \tau T_0 \to \tau^2 X[1] \to S[1]$ and $\tau T_2 \to \tau T_1 \to S \to \tau T_2[1]$ with $S = \tau G^{-1}(K)$ and some $T_0, T_1, T_2 \in \mathrm{add}\, T$. To these triangles apply the homological functor $\mathrm{Hom}_{\mathrm{D^b}(H)}(-, T'[j])$ to get exact sequences

$$\big(\tau T_0[1], T'[j]\big) \to \big(S[1], T'[j]\big) \to \big(\tau^2 X[1], T'[j]\big) \to \big(\tau T_0, T'[j]\big), \tag{3.2}$$

$$\big(\tau T_2[2], T'[j]\big) \to \big(S[1], T'[j]\big) \to \big(\tau T_1[1], T'[j]\big), \tag{3.3}$$

where we abbreviated $(Y, Z) = \mathrm{Hom}_{\mathrm{D^b}(H)}(Y, Z)$. By (3.1), the end terms of both sequences (3.2) and (3.3) are zero for $j > 3$ and hence we get $\mathrm{Hom}_{\mathrm{D^b}(H)}(\tau^2 X[1], T'[j]) \simeq \mathrm{Hom}_{\mathrm{D^b}(H)}(S[1], T'[j]) = 0$ for $j > 3$, which is what we wanted to prove. □

3.5. Iterated rolling

We now study the iteration of rolling. Fix a quiver Q, set $H = kQ$. Now start from a given tilting complex T with endomorphism algebra B with $\mathrm{gldim}\, B \leqslant 2$. By Proposition 3.9 the complex $\rho(T)$ is again a tilting complex and by Proposition 3.11 the endomorphism algebra $\rho(B) = \mathrm{End}_{\mathrm{D^b}(H)}(\rho(T))$ satisfies $\mathrm{gldim}\, \rho(B) \leqslant 2$. Iterating we get a sequence of tilting complexes $\rho^h(T)$ with endomorphism algebras $\rho^h(B)$. We will show that for sufficiently large h the algebra $\rho^h(B)$ is tilted.

For this we need some preliminary result in case where Q is Dynkin. Recall from Section 2.6 that for Q Dynkin, $d(Y, Z)$ denotes the length of the paths in the Auslander–Reiten quiver Γ of $\mathrm{D^b}(kQ)$ from Y to Z.

Let $\rho^h(T) = \bigoplus_{i=1}^n T_i^{(h)}$ be the decomposition into indecomposables and define the natural number

$$m_h(i) = \sum_{j=1}^n d\big(T_i^{(h)}, T_j^{(h)}\big).$$

The following definition will be helpful to simplify the arguments.

Definition 3.12. Let Q be a Dynkin quiver. For each section Σ we denote by $H(\Sigma)$ the hereditary algebra which has as injectives (concentrated in degree zero) the objects in Σ. That is, we can define $H(\Sigma)[0] = \bigoplus_{i=1}^n \tau^{-1}\Sigma_i[-1]$. Notice that Q and the quiver of $H(\Sigma)$ coincide up to the orientation of the arrows.

Now, for each for each $h \geqslant 0$ and each section Σ define the set

$$G_h(\Sigma) = \big\{ i \bigm| T_i^{(h)} \notin \operatorname{mod} H(\Sigma)[0] \big\}$$

and the natural number

$$n_h(\Sigma) = \sum_{i \in G_h(\Sigma)} m_h(i).$$

Notice that $n_h(\Sigma) = 0$ if and only if $\rho^h(T) \in \operatorname{mod} H(\Sigma)[0]$. Finally, let $\Sigma^{(h)} = \Sigma(\rho^h(T))$ be the section uniquely defined by $\rho^h(T)$ as in Section 3.2.

Lemma 3.13. *If $n_h(\Sigma^{(h)}) > 0$ then $n_{h+1}(\Sigma^{(h+1)}) < n_h(\Sigma^{(h)})$ and if $n_h(\Sigma^{(h)}) = 0$ then $n_{h+1}(\Sigma^{(h+1)}) = 0$.*

Proof. First suppose that $n_h(\Sigma^{(h)}) > 0$. If $\rho^h(T) = T' \oplus X$ and $\rho^{h+1}(T) = T' \oplus F^{-1}X$ then for $\Sigma' = \tau^2\Sigma^{(h)}$ we have $F^{-1}X \in \operatorname{mod} H(\Sigma')[0]$ and $d(Y, F^{-1}X_i) < d(Y, X_i)$ for all indecomposable summands Y of T', X_i of X. Consequently $n_{h+1}(\Sigma') < n_h(\Sigma^{(h)})$ and since clearly $n_{h+1}(\Sigma^{(h+1)}) \leqslant n_{h+1}(\Sigma')$ the claim follows.

If $n_h(\Sigma^{(h)}) = 0$ then with the same argument as above we have $F^{-1}X \in \operatorname{mod} H(\Sigma')[0]$ if $\Sigma' = \tau^2\Sigma^{(h)}$. Thus we see $\rho^{h+1}(T)$ belongs to $\operatorname{mod} H(\Sigma')[0]$ and consequently $n_{h+1}(\Sigma^{(h+1)}) = 0$. □

We are now in a position to prove Theorem 1.2, stated in the introduction.

Theorem 1.2. *Let B be an iterated tilted algebra of type Q with* gldim $B \leqslant 2$ *then for sufficiently large h the algebra $\rho^h(B)$ is tilted of type Q.*

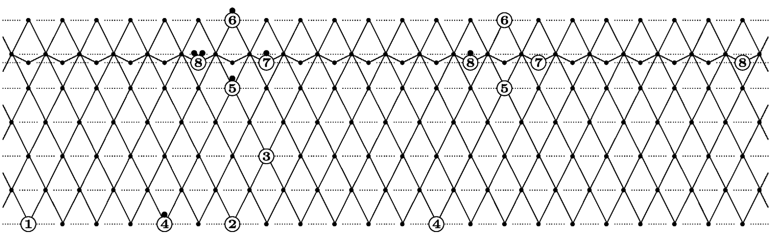
Proof. Let $H = kQ$ and T be a tilting complex in $\mathrm{D^b}(H)$ such that $B = \operatorname{End}_{\mathrm{D^b}(H)}(T)$. We have to show that for sufficiently large h there exists a hereditary algebra H' (which depends on h) with $\rho^h(T) \in \operatorname{mod} H'[0]$.

In case Q is Dynkin this follows directly from Lemma 3.13. In case that Q is not Dynkin we write $T = \bigoplus_{i=d}^s T_{\mathcal{R}[i/2]}$ for some integers $d \leqslant s$. Then by definition $\rho(T)$ belongs to $\bigcup_{i=d}^{s-1} \mathcal{R}[i/2] \cup \mathcal{R}[\frac{s}{2} - 1]$. By iterating, we get that for sufficiently large h the complex $\rho^h(T)$

belongs to $\mathcal{R}[p] \cup \mathcal{R}[p+\frac{1}{2}]$ for some half-integer p. If p is an integer then let $\Sigma_1, \ldots, \Sigma_n$ be any section of $\mathcal{R}[p+\frac{1}{2}]$ such that $T_i \leqslant \Sigma_j$ for each j and each indecomposable summand of $T_{\mathcal{R}[p+1/2]}$. Then $H' = H(\Sigma)$ is a hereditary algebra for which $\rho^h(T) \in \operatorname{mod} H'[0]$. If p is a half-integer then choose a section Σ in $\mathcal{R}[p]$ such that $\Sigma \leqslant T$ and define H' to be the hereditary algebra having its projectives in Σ. Again we have $\rho^h(T) \in \operatorname{mod} H'[0]$. □

We illustrate the former result by an example.

Example 3.14. Let Q be a quiver of type $\mathbb{D}_8$ with some orientation and $H = kQ$. In the following picture the Auslander–Reiten quiver Γ of the derived category $\mathrm{D^b}(H)$ is indicated; the arrows are going from left to right and are drawn as lines to simplify the picture. The indecomposable summand T_i of the tilting complex $T = \bigoplus_{i=1}^{8} T_i$ has been indicated by the number i inside a circle, that is, the symbol ⓘ. Furthermore, $F^{-1}T_i$, respectively $F^{-2}T_i$, has been indicated by the symbol ⓘ with one dot, respectively ⓘ with two dots.



We then have

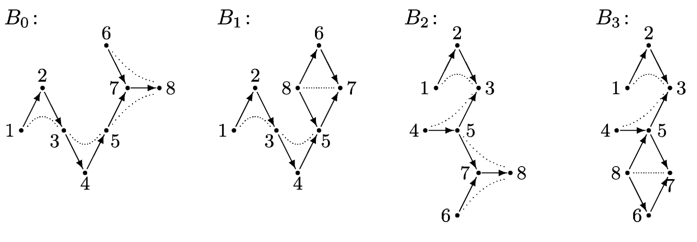
$$T = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6 \oplus T_7 \oplus T_8,$$

$$\rho(T) = T_1 \oplus T_2 \oplus T_3 \oplus T_4 \oplus T_5 \oplus T_6 \oplus T_7 \oplus F^{-1}T_8,$$

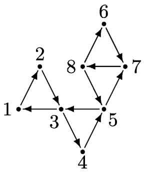
$$\rho^2(T) = T_1 \oplus T_2 \oplus T_3 \oplus F^{-1}T_4 \oplus F^{-1}T_5 \oplus F^{-1}T_6 \oplus F^{-1}T_7 \oplus F^{-1}T_8,$$

$$\rho^3(T) = T_1 \oplus T_2 \oplus T_3 \oplus F^{-1}T_4 \oplus F^{-1}T_5 \oplus F^{-1}T_6 \oplus F^{-1}T_7 \oplus F^{-2}T_8.$$

Define $B_h = \operatorname{End}_{\mathrm{D^b}(H)}(\rho^h(T))$. The following picture shows $B_h = kQ_h/I_h$ for $h = 0, 1, 2, 3$ by a presentation. As usual, relations are indicated by dotted lines.



Note that B_3 is tilted. By the above result all algebras B_i for $i > 3$ are also tilted. By calculating the further tilting complexes $\rho^h(T)$ for $h = 4, \ldots, 8$ one verifies that $B_h \simeq B_{h+3}$ for $h \geqslant 5$. Observe that in this example all relation extensions $\mathcal{R}(\rho^h(B))$ have isomorphic quivers as shown in the following picture. This is no coincidence and will be shown in Section 3.6 below.



The next result shows the importance of iterated tilted algebras with global dimension less or equal than two. It has been obtained independently by Osamu Iyama in [23, Thm. 1.22] and also by Claire Amiot in [1, 4.10] using different techniques.

Corollary 3.15. *Let H be a hereditary algebra. If T is a titling complex in $\mathrm{D}^{\mathrm{b}}(H)$ such that* $\operatorname{gldim} B \leqslant 2$, *where $B = \operatorname{End}_{\mathrm{D}^{\mathrm{b}}(H)}(T)$ then T is a cluster-tilting object in the cluster category $\mathcal{C}$ and $C = \operatorname{End}_{\mathcal{C}}(T)$ is a cluster-tilted algebra.*

Proof. By Theorem 1.2 there exists a number h such that $\rho^h(B)$ is a tilted algebra. By [10, Thm. 3.3], the object $\rho^h(T)$ defines a cluster-tilting object in $\mathcal{C}$ and $C' = \operatorname{End}_{\mathcal{C}}(\rho^h(T))$ is a cluster-tilted algebra. Since T and $\rho^h(T)$ define isomorphic objects in $\mathcal{C}$ the result follows. □

3.6. Behavior of the relation extensions under rolling

Notice that for any object T of $\mathrm{D}^{\mathrm{b}}(H)$, the endomorphism algebra

$$\operatorname{End}_{\mathcal{C}}(T) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(H)}\big(T, F^i T\big)$$

is naturally $\mathbb{Z}$-graded and contains $B = \operatorname{End}_{\mathrm{D}^{\mathrm{b}}(H)}(T)$ as a subalgebra. Recall from Section 3.1 that if T is a tilting complex then we have canonically that $\operatorname{Ext}^2_B(\mathrm{D}B, B) \simeq \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(H)}(T, FT)$ with the natural structure of B–B-bimodules. Therefore we get a canonical projection $\pi(B)\colon \operatorname{End}_{\mathcal{C}}(T) \to \mathcal{R}(B)$ of vector spaces and it was proven in [2, Lem. 3.3] that $\pi(B)$ is in fact an algebra isomorphism when T is a stalk complex concentrated in degree zero. However, in general $\pi(B)$ will not be an algebra homomorphism. Observe that if $\operatorname{gldim} B \leqslant 2$ then $\mathcal{R}(B) \simeq \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(H)}(T, T) \oplus \operatorname{Hom}_{\mathrm{D}^{\mathrm{b}}(H)}(T, FT)$. The next result is straightforward.

Lemma 3.16. *If $\pi(B)$ is an algebra homomorphism then the sequence of homomorphisms of algebras*

$$B \xrightarrow{j} \operatorname{End}_{\mathcal{C}}(T) \xrightarrow{\pi(B)} \mathcal{R}(B) \xrightarrow{p} B, \tag{3.4}$$

is the identity map, where j and p are the canonical inclusion and projection maps respectively. In particular $\operatorname{End}_{\mathcal{C}}(T)$ *is a split extension of B. Moreover, the canonical graded inclusion $\delta(B)\colon \mathcal{R}(B) \to \operatorname{End}_{\mathcal{C}}(T)$ is a homomorphism of B–B-bimodules and satisfies $\pi(B)\delta(B) =$* $\mathrm{id}_{\mathcal{R}(B)}$.

The next result shows that the relation extensions are closely related under rolling.

Proposition 3.17. *Let T be a tilting complex in $\mathrm{D}^{\mathrm{b}}(H)$ such that its endomorphism algebra B satisfies* $\operatorname{gldim} B \leqslant 2$. *Let $\widetilde{T} = \rho(T)$ and $\widetilde{B} = \rho(B)$. Then there exists a canonical algebra homomorphism $\Theta\colon \mathcal{R}(\widetilde{B}) \to \mathcal{R}(B)$ which is surjective and whose kernel is contained in* $\operatorname{rad}^2 \mathcal{R}(\widetilde{B})$.

Furthermore, Θ and the canonical isomorphism $\Psi : \mathrm{End}_{\mathcal{C}}(\widetilde{T}) \to \mathrm{End}_{\mathcal{C}}(T)$ commute with the projections, that is, $\Theta\pi(\widetilde{B}) = \pi(B)\Psi$. Moreover, if $\pi(\widetilde{B})$ is an algebra homomorphism then also $\pi(B)$ is an algebra homomorphism.

Proof. The canonical isomorphism $\Psi : \mathrm{End}_{\mathcal{C}}(\widetilde{T}) \to \mathrm{End}_{\mathcal{C}}(T)$ is given by the direct sum of the following bijective maps

$$\mathrm{id} : \mathrm{End}_{\mathcal{C}}(T') \to \mathrm{End}_{\mathcal{C}}(T'), \qquad \sigma^{-1} : \mathrm{Hom}_{\mathcal{C}}(T', F^{-1}X) \to \mathrm{Hom}_{\mathcal{C}}(T', X),$$
$$\sigma F : \mathrm{Hom}_{\mathcal{C}}(F^{-1}X, T') \to \mathrm{Hom}_{\mathcal{C}}(X, T'), \qquad F : \mathrm{End}_{\mathcal{C}}(F^{-1}X) \to \mathrm{End}_{\mathcal{C}}(X),$$

where σ denotes the shift in the $\mathbb{Z}$-graduation, that is

$$\sigma : \bigoplus_{i\in\mathbb{Z}} (Y, F^i Z) \to \bigoplus_{i\in\mathbb{Z}} (Y, F^{i+1} Z), \qquad (f_i)_{i\in\mathbb{Z}} \mapsto (f_{i+1})_{i\in\mathbb{Z}},$$

where we abbreviated again $(Y, Z) = \mathrm{Hom}_{\mathrm{D^b}(H)}(Y, Z)$, as we shall do also in the forthcoming. Now, $\Theta : \mathcal{R}(\widetilde{B}) \to \mathcal{R}(B)$ is defined by the following four maps.

$$\mathrm{id} : (T', T') \oplus (T', FT') \to (T', T') \oplus (T', FT'), \tag{3.5}$$

$$\begin{bmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{bmatrix} : (T', F^{-1}X) \oplus (T', X) \to (T', X) \oplus (T', FX), \tag{3.6}$$

$$\begin{bmatrix} 0 & 0 \\ F & 0 \end{bmatrix} : (F^{-1}X, T') \oplus (F^{-1}X, FT') \to (X, T') \oplus (X, FT'), \tag{3.7}$$

$$F : (F^{-1}X, F^{-1}X) \oplus (F^{-1}X, X) \to (X, X) \oplus (X, FX). \tag{3.8}$$

Since by hypothesis $\mathrm{Hom}_{\mathrm{D^b}(H)}(T', FX) = 0$, respectively $\mathrm{Hom}_{\mathrm{D^b}(H)}(X, T') = 0$, the maps in (3.6), respectively (3.7) are surjective. Therefore the map Θ is surjective and $\Theta\pi(\widetilde{B}) = \pi(B)\Psi$.

Now, the kernel of Θ is clearly $\mathrm{Hom}_{\mathrm{D^b}(H)}(T', F^{-1}X) \oplus \mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}X, FT')$, but by Lemma 3.1 the first summand is zero. We have $\mathrm{Hom}_{\mathrm{D^b}(H)}(F^{-1}X, FT') = \mathrm{Hom}_{\mathrm{D^b}(H)}(\tau(F^{-1}X)[1], T'[2]) \simeq \mathrm{Ext}^2_{\widetilde{B}}(\mathbf{I}_{F^{-1}X,\widetilde{T}}, \mathbf{P}_{T',\widetilde{T}})$ since $\widetilde{T}$ is a tilting complex. We will show that the last term is contained in the radical of $\mathrm{Ext}^2_{\widetilde{B}}(\mathrm{D}\widetilde{B}, \widetilde{B})$. By [2, §2.4] we have $\mathrm{top}\,\mathrm{Ext}^2_{\widetilde{B}}(\mathrm{D}\widetilde{B}, \widetilde{B}) = \mathrm{Ext}^2_{\widetilde{B}}(\mathrm{soc}\,\mathrm{D}\widetilde{B}, \mathrm{top}\,\widetilde{B})$. Hence it suffices to prove that $\mathrm{Ext}^2_{\widetilde{B}}(S_i, S_j) = 0$ for all indecomposable simples S_i, respectively S_j, which are direct summands of $\mathrm{soc}\,\mathbf{I}_{F^{-1}X,\widetilde{T}}$, respectively $\mathrm{top}\,\mathbf{P}_{T',\widetilde{T}}$.

Suppose the contrary, that is, there exist such summands S_i and S_j with $\mathrm{Ext}^2_{\widetilde{B}}(S_i, S_j) \neq 0$. Let $0 \to Q_2 \to Q_1 \to P_i \to S_i$ be the projective resolution in $\mathrm{mod}\,\widetilde{B}$ of S_i and $\varphi : Q_2 \to S_j$ some morphism defining a non-zero element of $\mathrm{Ext}^2_{\widetilde{B}}(S_i, S_j)$. This shows that some direct summand of Q_2 is isomorphic to P_j and hence we get a sequence

$$P_j \to Q' \to P_i$$

of non-zero maps between indecomposable projective $\widetilde{B}$-modules. One of these non-zero morphisms then must map from a summand of $\mathbf{P}_{F^{-1}X,\widetilde{T}}$ to a summand of $\mathbf{P}_{T',\widetilde{T}}$. This contradicts the fact that $\mathrm{Hom}_{\widetilde{B}}(\mathbf{P}_{F^{-1}X,\widetilde{T}}, \mathbf{P}_{T',\widetilde{T}}) = \mathrm{Hom}_{\mathrm{D^b}(H)}(X, T')$ equals zero.

It remains to see that if $\pi(\widetilde{B})$ is an algebra homomorphism then also $\pi(B)$ is an algebra homomorphism. That is, we suppose that for all $j \neq 0, 1$ and all morphisms

$$\widetilde{T} \xrightarrow{f} F^j\widetilde{T} \xrightarrow{g} \widetilde{T} \oplus F\widetilde{T} \tag{3.9}$$

the composition gf is zero and have to show that for all $h \neq 0, 1$ and all morphisms $T \xrightarrow{f'} F^hT \xrightarrow{g'} T \oplus FT$ the composition $g'f'$ is zero. For this we consider 16 different combinations: for $A, B \in \{T', X\}$ and $C \in \{T', X, FT', FX\}$, we consider the compositions

$$A \xrightarrow{f'} F^hB \xrightarrow{g'} C \tag{3.10}$$

for $h \neq 0, 1$. For some of the combinations, the proof that $g'f' = 0$ is straightforward using (3.9), as for instance if $A = B = T'$ and $C = T', X, FT'$. Also, by hypothesis there is nothing to show if (A, C) equals (X, T') or (T', FX). The remaining combinations are then divided in two cases:

(a) $A = T'$, $B = X$ and $C \in \{T', X, FT'\}$;
(b) $A = X$, $B \in \{T', X\}$ and $C \in \{X, FT', FX\}$.

Let $j = h - 1$. In case (a) observe that by (3.9) the composition (3.10) holds for all $h \geqslant 3$ and all $h < 0$. In case (b), apply F^{-1} to (3.10), in order to see that again the composition is zero if $h \geqslant 3$ or $h < 0$. So it only remains to consider the case where $h = 2$. In any case $g' = 0$ by Lemma 3.1. This finishes the proof of the proposition. □

We prove now Theorem 1.1, stated in the introduction. See also [1, 4.17] for a different proof of the last assertion of the theorem, relating the quivers of C and $\mathcal{R}(B)$.

Theorem 1.1. *If B is an iterated tilted algebra of* gldim $B \leqslant 2$ *then there exists a cluster-tilted algebra C which is a split extension of B. More precisely, if $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$ with H a hereditary algebra and T is a tilting complex in* $\mathrm{D^b}(H)$ *then $C = \mathrm{End}_{\mathcal{C}(H)}(T)$ is a cluster-tilted algebra and there exists a sequence of algebra homomorphisms*

$$B \to C \xrightarrow{\pi} \mathcal{R}(B) \to B$$

whose composition is the identity map. Moreover, the kernel of π is contained in $\mathrm{rad}^2 C$. *In particular C and $\mathcal{R}(B)$ have the same quivers and are both split extensions of B.*

Proof. Let H be a hereditary algebra with quiver Q and T be a tilting complex in $\mathrm{D^b}(H)$ such that $B = \mathrm{End}_{\mathrm{D^b}(H)}(T)$.

We already know from Theorem 1.2 that for sufficiently large h the algebra $\rho^h(B)$ is tilted of type Q and $C = \mathrm{End}_{\mathcal{C}}(\rho^h(T))$ is cluster-tilted. It follows now from [2, Thm. 3.4] that $\pi(\rho^h(B)) : C \to \mathcal{R}(\rho^h(B))$ is an isomorphism. Hence by Proposition 3.17 we get inductively for $i = h-1, h-2, \ldots, 1$ that the projection $\pi(\rho^i(B))$ is an algebra homomorphism and $\Theta_i : \mathcal{R}(\rho^i(B)) \to \mathcal{R}(\rho^{i-1}(B))$ is surjective with kernel contained in $\mathrm{rad}^2\,\mathcal{R}(\rho^i(B))$. Therefore the same holds for the composition $\Theta_1\Theta_2\cdots\Theta_h$. Thus $\pi(B): C \to \mathcal{R}(B)$ has the same property, because it is obtained from $\Theta_1\Theta_2\cdots\Theta_h$ by composing with isomorphisms, as follows by repeated application of Proposition 3.17 and using that $\pi(\rho^h(B))$ is an isomorphism. In particular $\mathcal{R}(B)$ has the same quiver (up to isomorphism) as C. By Lemma 3.16 the composition of the morphisms $B \to \operatorname{End}_{\mathcal{C}}(T) \xrightarrow{\pi(B)} \mathcal{R}(B) \to B$ is the identity map and consequently both algebras C and $\mathcal{R}(B)$ are split extensions of B. $\square$

Definition 3.18. For an iterated tilted algebra B with $\operatorname{gldim} B \leqslant 2$ choose a hereditary algebra H and tilting complex T in $\mathrm{D^b}(H)$ with $B = \operatorname{End}_{\mathrm{D^b}(H)}(T)$. We then define $C(B)$ to be the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}}(T)$.

We notice that $C(B) \simeq C(\rho(B))$ because $\rho(T) \simeq T$ in the cluster category $\mathcal{C}$, so $C(B) \simeq \mathcal{R}(\rho^h(B))$ for any h such that $\rho^h(B)$ is tilted. Such h always exists, by Theorem 1.2, and $\rho(B)$ does not depend on the choices of H and T, as observed after Definition 3.5. It follows that also $C(B)$ is uniquely defined up to isomorphism independently of the choices of H and T.

Proposition 3.19. *For each iterated tilted algebra B with* $\operatorname{gldim} B \leqslant 2$ *there are presentations of the algebras B, $\mathcal{R}(B)$ and $C(B)$ in which $\{\alpha_1, \dots, \alpha_r\}$ are the arrows of B and $\{\alpha_1, \dots, \alpha_r, \eta_1, \dots, \eta_s\}$ are the arrows of $\mathcal{R}(B)$ and of $C(B)$. Then* $\operatorname{Ker} \pi(B) = \langle \eta_1, \dots, \eta_s \rangle^2$.

Proof. To get the desired presentations one can take suitable basis of $\operatorname{rad} B / \operatorname{rad}^2 B$ and of $\operatorname{Ext}^2_B(\mathrm{D}B, B)/\operatorname{rad}\operatorname{Ext}^2_B(\mathrm{D}B, B)$. The map $\delta(B): \mathcal{R}(B) \to C(B)$ induces the identity on B and satisfies $\delta(B)(\eta_i) = \eta_i$. To simplify the notation, we shall write π and δ instead of $\pi(B)$ and $\delta(B)$, respectively.

It follows from the definition of the multiplication in $\mathcal{R}(B)$ that $\langle \eta_1, \dots, \eta_s \rangle^2 \subseteq \operatorname{Ker} \pi$. By Theorem 1.1 we have $\operatorname{Ker} \pi \subseteq \langle \alpha_1, \dots, \alpha_r, \eta_1, \dots, \eta_s \rangle^2$.

Let $z \in \mathcal{R}(B)$, say

$$z = a + \sum_{i=1}^{s} b_i \eta_i c_i + h$$

with $a, b_i, c_i \in B$ and $h \in \langle \eta_1, \dots, \eta_s \rangle^2$. Assume now that $z \in \operatorname{Ker} \pi(B)$. Then, by the above, $h \in \operatorname{Ker} \pi$ and consequently

$$y := z - h = a + \sum_{i=1}^{s} b_i \eta_i c_i$$

belongs to $\operatorname{Ker} \pi$ and also to $\operatorname{Hom}_{\mathrm{D^b}(H)}(T, T) \oplus \operatorname{Hom}_{\mathrm{D^b}(H)}(T, FT)$, a space to which the map π restricts as the identity. Hence $\pi(y) = 0$ implies $y = 0$ and consequently $z \in \langle \eta_1, \dots, \eta_s \rangle^2$. $\square$

We observe that though the algebras $C(B)$ and $\mathcal{R}(B)$ have the same quiver, they are in general not isomorphic, not even in the Dynkin case, as will be shown in Remark 4.21.

Remark 3.20. Theorem 1.1 together with [13, Thm. 2.13] and the classification given in [5] can be used as criteria for discarding an algebra of being iterated tilted, see Remark 4.13 for an example.

Corollary 3.21. *If B is an iterated tilted algebra of Dynkin type with* $\operatorname{gldim} B \leqslant 2$ *then* $\mathcal{R}(B)$ *is of finite representation type.*

Proof. By [13, Cor. 2.4] the cluster-tilted algebra $C(B)$ is of finite representation type. Hence so is $\mathcal{R}(B)$ being a quotient of $C(B)$. □

4. Admissible cuts of cluster-tilted algebras of Dynkin type

4.1. Cluster-tilted algebras of Dynkin type

We now want to give a more combinatorial description of the relationship between an iterated tilted algebra B with $\operatorname{gldim} B \leqslant 2$, its relation extension $\mathcal{R}(B)$ and the corresponding cluster-tilted algebra $C(B)$ in the case where these algebras are of finite representation type.

Recall from [11] that the quivers of the cluster-tilted algebras arising from a given cluster category are exactly the quivers corresponding to the exchange matrices of the associated cluster algebra. The following result follows therefore from [20, Thm. 1.8 and Lem. 7.5].

Proposition 4.1. *Each chordless cycle in the quiver Q_C of a cluster-tilted algebra C of Dynkin type is oriented.*

Also the following result, proven in [11, Prop. 1.4] will be useful.

Theorem 4.2. *Let C be a cluster-tilted algebra and e an idempotent of C. Then C/CeC is again a cluster-tilted algebra.*

4.2. Relations for cluster-tilted algebras of Dynkin type

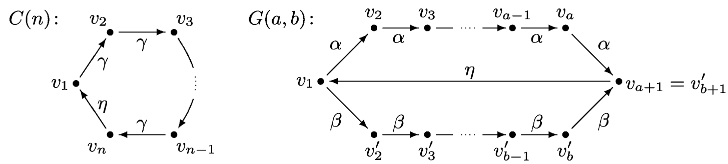
We will need the description of the relations for cluster-tilted algebras of Dynkin type given in [11] (see also [15] for the A_n case). We start by recalling that if there is an arrow from i to j, a path from j to i is called *shortest* if it contains no proper subpath which is a cycle and if the full subquiver generated by the path and the arrow contains no further arrows. A relation ρ is called *minimal* if whenever $\rho = \sum_i \beta_i \rho_i \gamma_i$ where ρ_i is a relation for every i, then β_i and γ_i are scalars for some index i (see [11]).

The following definition will simplify the language.

Definition 4.3 *(Parallel and antiparallel paths).* An arrow α is called *parallel* (respectively *antiparallel*) to a relation (or a path or an arrow) ρ if $s(\alpha) = s(\rho)$ and $t(\alpha) = t(\rho)$ (respectively $s(\alpha) = t(\rho)$ and $t(\alpha) = s(\rho)$).

The following description is an immediate consequence of [11, Thm. 4.1].

Theorem 4.4. *Let $C = kQ_C/I_C$ be a cluster-tilted algebra of Dynkin type. Then, in Q_C for each arrow η there exist at most two shortest antiparallel paths to η. If there is at least one and Σ_η denotes the full subquiver of Q_C given by the vertices of η and the antiparallel paths, then the quiver Σ_η is isomorphic to $C(n)$ (for some n) or to $G(a,b)$ (for some a, b), as shown in the following picture.*



The ideal I_C is generated by minimal zero relations and minimal commutativity relations, and each of them is antiparallel to exactly one arrow. If an arrow η is antiparallel to the minimal zero relation ρ, then $\Sigma_\eta \simeq C(n)$ and $\rho = \gamma^{n-1}$. If η is antiparallel to the minimal commutativity relation $\rho_1 = \rho_2$, then $\Sigma_\eta \simeq G(a,b)$ and $\rho_1 = \alpha^a \neq 0$, $\rho_2 = \beta^b \neq 0$.

Hence each arrow in an oriented cycle is antiparallel to precisely one minimal relation (up to scalars), and the relations obtained this way form a minimal set of generators of I_C.

Lemma 4.5. *Let C be a cluster-tilted algebra of Dynkin type with quiver Q. Then for each arrow α there is no other shortest path than α which is parallel to α in Q.*

Proof. Assume otherwise, that is, there exists a path γ parallel to α which is different in Q. Since C is of finite representation type, γ cannot be an arrow. Let $\gamma = \gamma_t\gamma_{t-1}\cdots\gamma_1$ be as follows.

$$x_0 \xrightarrow{\gamma_1} x_1 \to \cdots \to x_{t-2} \xrightarrow{\gamma_{t-1}} x_{t-1} \xrightarrow{\gamma_t} x_t.$$

By Proposition 4.1, the cycle $\alpha\gamma$ is not chordless. Let $m \geqslant 0$ be minimal such that there exists an arrow between x_m and x_s for some $s > m+1$. Then let M with $m+1 < M \leqslant t$ be maximal such that there exists an arrow δ between x_m and x_M. Then the arrows

$$\alpha, \gamma_t, \ldots, \gamma_{M+1}, \delta, \gamma_m, \ldots, \gamma_1$$

form a non-oriented cycle which by construction is chordless, in contradiction to Proposition 4.1. □

4.3. Definition of admissible cut

We are now ready to give the combinatorial description of how the iterated tilted algebras B with $\operatorname{gldim} B \leqslant 2$ can be obtained from a cluster-tilted algebra C. For this we introduce the following concept.

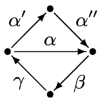
Definition 4.6 *(Admissible cut).* A subset of the set of arrows Q_1 of a quiver Q is called *admissible cut* of Q if it contains exactly one arrow of each oriented chordless cycle in Q.

Definition 4.7 *(Quotient by an admissible cut).* Let $C = kQ_C/I$ be an algebra given by a quiver Q_C and an admissible ideal I. A *quotient of C by an admissible cut* (or *an admissible cut of C*) is an algebra of the form $kQ_C/\langle I \cup \Delta\rangle$ where Δ is an admissible cut of Q_C.

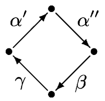
This is, B is an admissible cut of C if B is the algebra obtained by deleting in Q_C the arrows of an admissible cut Δ and considering the induced relations.

Remark 4.8. The definition is not independent of the presentation of B, that is, for two ideals I_1 and I_2 such that $kQ/I_1 \simeq kQ/I_2$ the same cut may give non-isomorphic quotients $kQ/\langle I_1 \cup \Delta\rangle \not\simeq kQ/\langle I_2 \cup \Delta\rangle$, as shows the following example. Let Q be the quiver as given in the following picture.

Q:



$Q_{B_1} = Q_{B_2}$:



Furthermore, let $I_1 = \langle \beta\alpha, \gamma\beta\rangle$ and $I_2 = \langle \beta(\alpha - \alpha''\alpha'), \gamma\beta\rangle$. Then the quotients kQ/I_1 and kQ/I_2 are isomorphic. Furthermore $\Delta = \{\alpha\}$ is an admissible cut but the quotients $B_1 = kQ/\langle I_1 \cup \Delta\rangle$ and $B_2 = kQ/\langle I_2 \cup \Delta\rangle$ are non-isomorphic since $\langle I_1 \cup \Delta\rangle = \langle \alpha, \gamma\beta\rangle$ whereas $I_2 = \langle \alpha, \beta\alpha''\alpha', \gamma\beta\rangle$, that is, B_2 is a proper quotient of B_1.

However, an admissible cut of a cluster-tilted algebra C of Dynkin type is independent of the presentation of C. This follows from the next lemma, and the fact that any such algebra C is schurian, that is, $\dim_k e_y C e_x \leqslant 1$ for any pair of vertices $x, y \in Q_C$. See [11, Lem. 1.8].

Lemma 4.9. *If C is a schurian algebra and Δ an admissible cut of the quiver Q of C then the quotient of C by Δ is independent of the presentation of C.*

Proof. Let $f : kQ/I \to kQ/J$ be an isomorphism. By composing, if necessary, with the isomorphism of kQ induced by an isomorphism of the quiver Q, we may assume that $f(e_x) = e_x$, for each $x \in Q_0$.

Since C is schurian, $\dim_k e_y(kQ/J)e_x \leqslant 1$ for each $x, y \in Q_0$. So for each arrow α we have that $f(\alpha) = \lambda_\alpha \alpha$ for some non-zero $\lambda_\alpha \in k$. Thus if Δ is an admissible cut of Q then Δ and $f(\Delta)$ generate the same ideal in kQ/J, and therefore the map $KQ/(I \cup \Delta) \to kQ/(J \cup \Delta)$ induced by f is an isomorphism. □

Notice that the example given in Remark 4.8 also shows that it is possible that the quiver Q_{B_1} of a quotient of an algebra C by an admissible cut may have oriented chordless cycles. However, this cannot happen in case where C is a cluster-tilted algebra of Dynkin type.

Lemma 4.10. *Let C be a cluster-tilted algebra of Dynkin type and Δ an admissible cut of the quiver Q_C of C. Then for any presentation $C = kQ_C/I$, the quiver Q_B of the quotient $B = kQ_C/\langle I \cup \Delta\rangle$ has no oriented chordless cycle.*

Proof. Suppose the contrary, namely that in Q_B there exists an oriented chordless cycle, given by a path

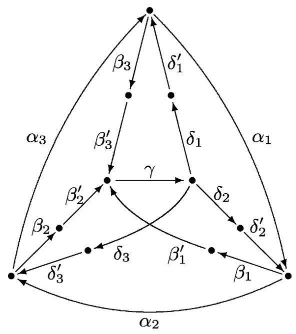
$$\gamma : x_0 \xrightarrow{\gamma_1} x_1 \to \cdots \to x_{t-2} \xrightarrow{\gamma_{t-1}} x_{t-1} \xrightarrow{\gamma_t} x_t = x_0.$$

Then γ cannot be chordless in Q_C by the definition of admissible cut. Thus there exists an arrow α between x_r and x_s for some $s > r + 1$. After renumbering the vertices x_i and the arrows γ_i we can assume without loss of generality that $\alpha : x_0 \to x_s$ for some s with $1 < s < t$. This contradicts Lemma 4.5. □

4.4. Existence of admissible cuts

We start by the observation that there exist quivers which do not admit an admissible cut.

Example 4.11. Let Q be the following quiver.



The only chordless cycles in Q are given by the paths

$$\alpha_3\alpha_2\alpha_1, \qquad \delta_i'\delta_i\gamma\beta_i'\beta_i\alpha_i, \qquad \delta_{i+1}'\delta_{i+1}\gamma\beta_i'\beta_i$$

for $i = 1, 2, 3$ where the indices have to be taken modulo 3.

Suppose that there exists an admissible cut Δ in Q. Then one (and only one) of the arrows α_i has to belong to Δ. Because of the cyclic symmetry (by interchanging the indices cyclically modulo 3) we can without loss of generality assume that α_1 belongs to Δ. Since $\delta_1'\delta_1\gamma\beta_1'\beta_1\alpha_1$ is a chordless cycle we have $\delta_1', \delta_1, \gamma, \beta_1', \beta_1 \notin \Delta$. Since $\delta_1'\delta_1\gamma\beta_3'\beta_3$ (respectively $\delta_2'\delta_2\gamma\beta_1'\beta_1$) is a chordless cycle, one (and only one) of the arrows β_3 or β_3' (respectively δ_2 or δ_2') must also belong to Δ. We can assume the two arrows are β_3 and δ_2 since the argument for any other choice is completely similar.

Let C be the set of chordless cycles which contain an arrow from $\Delta' = \{\alpha_1, \beta_3, \delta_2\}$. Observe that C contains all chordless cycles except $\delta_3'\delta_3\gamma\beta_2'\beta_2$ and that each arrow of Q occurs in one of the cycles in C. Hence on one hand the admissible cut Δ must contain another arrow from $\delta_3'\delta_3\gamma\beta_2'\beta_2$ and on the other hand Δ cannot contain any more since otherwise one of the cycles of C would contain two arrows from Δ, a contradiction. This proves that Q does not admit an admissible cut.

The following result shows in particular that the quiver of any cluster-tilted algebra of Dynkin type admits an admissible cut.

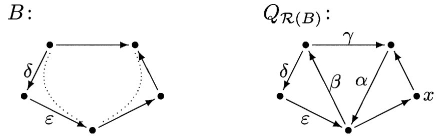
Proposition 4.12. *Let B be an iterated tilted algebra of Dynkin type with* $\operatorname{gldim} B \leqslant 2$. *Then B is the quotient by an admissible cut of the corresponding cluster-tilted algebra $C(B)$.*

Proof. Suppose that B is not the quotient by an admissible cut of $C = C(B)$. Then there exists a chordless cycle γ in the quiver Q_C of C which contains at least two arrows which do not belong to the quiver Q_B of B. Denote by $\gamma_L\gamma_{L-1}\cdots\gamma_1$ the path obtained by passing along the cycle starting from some vertex $s(\gamma_1)$ of γ and let Φ be the set of indices such that $\{\gamma_j \mid j \in \Phi\}$ are the arrows which do not belong to Q_B.

Write $B = kQ_B/I_B$ and $C = kQ_C/I_C$. Now, by Theorem 1.1 we have $\mathcal{R}(B) = C/J$ for some ideal J of C with $J \subseteq \operatorname{rad}^2 C$ and the arrows of Q_C coincide with the arrows of $Q_{\mathcal{R}(B)}$. For each $j \in \Phi$ the arrow γ_j corresponds to a generating relation ρ_j since $\mathcal{R}(B)$ is the relation extension of B.

Observe that $\delta = \gamma_{j-1}\gamma_{j-2}\cdots\gamma_1\gamma_L\cdots\gamma_{j+1}$ is a shortest path in Q_C which is antiparallel to γ_j and δ is not contained in the path algebra Q_B since by hypothesis Φ consists of at least two elements. By Theorem 4.4, there are at most two shortest paths in Q_C which are antiparallel to γ_j and therefore there exists precisely one path δ' in Q_B which is antiparallel to γ_j in Q_C. Consequently $\rho_j = \delta'$ is a zero relation. Hence the smallest full subquiver of Q_C containing δ and δ' is isomorphic to $G(a,b)$, defined as in Section 4.1. Since $C(B)$ is a split extension of B, by [3, 2.3] it follows that the ideal I_B is contained in I_C. Thus $\delta' = 0$ in C in contradiction to Theorem 4.4. □

Remark 4.13. By Theorem 1.1, for B an iterated tilted algebra with $\operatorname{gldim} B \leqslant 2$, the algebras $C(B)$ and $\mathcal{R}(B)$ have the same quiver, and therefore if B is of Dynkin type Q then B is the quotient of $\mathcal{R}(B)$ by an admissible cut. This however is not true in general, as shows the following example. Let $B = kQ_B/I_B$ be the algebra presented on the left-hand side in the following picture. Then the quiver of $\mathcal{R}(B)$ is as depicted on the right-hand.



Clearly $\Delta = \{\beta, \alpha\}$ is not an admissible cut of $Q_{\mathcal{R}(B)}$ since the cycle given by the path $\gamma\beta\alpha$ contains two arrows from Δ. However B is not an iterated tilted algebra as shows the following argument. Suppose that B is iterated tilted of type Q. Then by Theorem 1.1 the algebras $C(B)$ and $\mathcal{R}(B)$ have the same quiver and both are split extensions of B. In particular, since $\varepsilon\delta = 0$ in B we have also $\varepsilon\delta = 0$ in $C(B)$. For the ideal $J = C(B)e_xC(B)$, the quotient $C' = C(B)/J$ is again a cluster-tilted algebra by Theorem 4.2. By [9, Thm. 2.3] there is a unique cluster-tilted algebra with quiver $Q_{C'}$ and that algebra is known to be of Dynkin type $\mathbb{D}_4$. This contradicts the description of the relations in [12], see Section 4.1, where $\varepsilon\delta \neq 0$. Thus B is not iterated tilted.

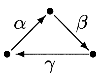
Remark 4.14. (a) Each iterated tilted algebra B with $\operatorname{gldim} B \leqslant 2$ is the quotient of $\mathcal{R}(B)$ by $\operatorname{Ext}^2_B(DB, B)$, which is generated by a set Δ of arrows corresponding to relations of B. It is unknown to the authors whether each such algebra B is the quotient by an admissible cut of $\mathcal{R}(B)$ by Δ.

(b) It follows from Theorem 1.1 that the algebras $\mathcal{R}(B)$ and $C = C(B)$ are split extensions of B, have isomorphic quivers, and their presentations can be chosen so that $I_B \subseteq I_{C(B)} \subseteq I_{\mathcal{R}(B)}$. We observe that for such presentations and with Δ as in (a), B is an admissible cut of $\mathcal{R}(B)$ by Δ if and only if B is an admissible cut of $C(B)$ by Δ. In fact, by (a) we know that B is an admissible cut of the relation extension $\mathcal{R}(B)$ by Δ if and only if the set Δ is an admissible cut of the quiver $Q_{\mathcal{R}(B)}$. Since the algebras $\mathcal{R}(B)$ and $C = C(B)$ have isomorphic quivers to prove the assertion we only need to prove that if Δ is an admissible cut of Q_C, then $B \simeq kQ_C/\langle I_C \cup \Delta\rangle$. Let J be the ideal of kQ_C such that $B \simeq kQ_C/J$. By the above we have $J \supseteq \langle I_C \cup \Delta\rangle$ and it remains to show that $J \subseteq \langle I_C \cup \Delta\rangle$. So let ρ be a relation of kQ_C which belongs to J. Then $\rho = \sum_{i=1}^t \lambda_i \rho_i$ for some non-zero scalars λ_i and some parallel paths $\rho_i = \rho_{i,N_i}\rho_{i,N_i-1}\cdots\rho_{i,1}$.

If some $\rho_{i,j} \in \Delta$ then $\rho' = \rho - \lambda_i \rho_i \in J$ and by induction over the number of summands we can assume that $\rho' \in \langle I_C \cup \Delta \rangle$. Hence it remains to consider the case where no summand of ρ contains an arrow of Δ, that is, ρ can be considered as element of kQ_B. Since $\rho \in J$, then $\rho = 0$ in $kQ_C/J \simeq B$, that is, $\rho \in I_B$. Since $I_B \subseteq I_C$ it follows that $\rho \in I_C \subseteq \langle I_C \cup \Delta \rangle$, as desired.

(c) It is interesting to notice that the fact that both $\mathcal{R}(B)$ and C are split extensions of B is essential for the preceding statement to hold. Let C, D be algebras such that D is a quotient of C inducing an isomorphism of quivers $Q_D = Q_C$. Clearly the sets of arrows which are admissible cuts for the quivers of the two algebras are the same. However, if an algebra B is an admissible cut of D, then it is not always true that B is also an admissible cut of C, as the following simple example shows.

Let Q be the quiver



$C = kQ/\langle \gamma\beta\alpha \rangle$ and $D = C/\langle \beta\alpha \rangle$. Then $B = D/\langle \gamma \rangle \simeq C/\langle \gamma, \beta\alpha \rangle$ is an admissible cut of D, but is not an admissible cut of C. Observe that C is not a split extension of B since $I_B \neq 0$.

4.5. Admissible cuts and antiparallel relations

We now start the investigation on quotients of cluster-tilted algebras by admissible cuts by the following basic fact.

Proposition 4.15. *Let B be a quotient by an admissible cut of a cluster-tilted algebra C of Dynkin type. Write $B = kQ_B/I_B$ where Q_B is the quiver of B and I_B is an admissible ideal generated by the minimal set of minimal relations $\{\rho_i \mid i = 1, \dots, t\}$. Then C is a split extension of B by an ideal $M = \langle \alpha_1, \alpha_2, \dots, \alpha_t \rangle$, generated by arrows such that α_i is antiparallel to ρ_i for each $i = 1, \dots, t$.*

Proof. Let $\Gamma = \{\alpha_1, \dots, \alpha_t\}$ be an admissible cut of Q_C such that $B = C/\langle \Gamma \rangle$. Notice that for each subquiver $\Sigma \simeq G(a,b)$ of Q_C either $\eta \in \Gamma$ or $\alpha : v_i \to v_{i+1}$ and $\beta : v'_j \to v'_{j+1}$ belong both to Γ (for some i, j). This shows that in each minimal relation $\sigma = \sum_{j=1}^{N} c_j \sigma_j$ (where σ_j are parallel paths and $c_j \neq 0$ coefficients) defining the ideal I_C we have that if $\sigma_j \in \langle \Gamma \rangle$ for some j then $\sigma_j \in \langle \Gamma \rangle$ for all j and consequently $\sigma \in \langle \Gamma \rangle$. Hence by [3, Thm. 2.5] we know that C is the split extension of B by the ideal $\langle \Gamma \rangle$. $\square$

Remark 4.16. By the above the arrows in the admissible cut Γ are in one-to-one correspondence to the relations defining I_B, with each arrow antiparallel to the corresponding relation. Hence if $\operatorname{gldim} B \leqslant 2$ then the quiver of C is precisely the quiver of B with arrows added antiparallel to the relations in I_B. Thus, according to the description of the quiver of the relation extension given in [2, Thm. 2.6], if $\operatorname{gldim} B \leqslant 2$ the quiver of C coincides with the quiver of $\mathcal{R}(B)$.

4.6. Strongly simple connectedness

We refer to Section 2.4 and the references cited there for the definition of simple connectedness and strongly simple connectedness of algebras.

Lemma 4.17. *Let B be a quotient by an admissible cut Γ of a cluster-tilted algebra C of Dynkin type. Then B is a strongly simply connected algebra.*

Proof. We know from Proposition 4.1 that each chordless cycle in Q_C is oriented and from Lemma 4.10 each chordless cycle in Q_B is non-oriented. We now proceed in steps.

(1) *Each chordless cycle in Q_B is obtained from a subquiver of Q_C which is isomorphic to $G(a,b)$ (for some a and b) by removing the arrow corresponding to η.*

Indeed, let $\Sigma: v_1 — v_2 — \cdots — v_t — v_1$ be a chordless cycle in Q_B. Then Σ is non-oriented and by 4.1 it cannot be chordless in Q_C. So there exists a chord $v_i — v_j$ for some $i \not\equiv j \pm 1 \pmod t$. We can assume that $i = 1$ and that $\Sigma_1: v_1 — v_2 — \cdots — v_j — v_1$ is a chordless cycle in Q_C and therefore oriented. If we assume that $\Sigma_2: v_1 — v_j — v_{j+1} — \cdots v_t — v_1$ is not a chordless cycle in Q_C then there exists a chord $\eta_2: v_l — v_h$ for some $j \leqslant l < h - 1 \leqslant t$ (where $v_{t+1} := v_1$) and if we take $l \geqslant j$ minimal and $h \leqslant t + 1$ maximal then $\Sigma': v_1 \overset{\eta_1}{—} v_j — \cdots — v_l \overset{\eta_2}{—} v_h — \cdots — v_t — v_1$ is a chordless (and therefore oriented) cycle in Q_C with two arrows η_1 and η_2 belonging to the admissible cut, a contradiction. This shows that Σ_2 is also oriented and therefore (1) holds.

(2) *The quiver Q_B is directed, that is, it does not contain an oriented cycle.*

Assume by contradiction that an oriented cycle Σ exists in Q_B and suppose that Σ is minimal with respect to the number of vertices. By (1) the cycle Σ is not chordless in Q_B. This chord divides Σ into two smaller cycles, one of them necessarily is oriented, in contradiction to the minimality of Σ.

(3) *The algebra is strongly simply connected.*

Using (1) and (2) it is easy to see that the (Q_B, I_B) is its own universal cover, in the sense of [25]. Therefore by [25, Thm. 4.2] the algebra B is simply connected. Since C is of Dynkin type, then by [11, Prop. 1.2] algebra C and hence B is of finite representation type and therefore by Remark 2.2 the algebra B is strongly simply connected. □

4.7. Behavior of the quadratic form

For a definition of the quadratic forms χ_B and q_B associated to an algebra B we refer to Section 2.3 and the references cited there.

Proposition 4.18. *Let B be a quotient by an admissible cut of a cluster-tilted algebra C of Dynkin type such that* gldim $B \leqslant 2$. *Then q_B is positive definite.*

Proof. Since C is mutation equivalent to a Dynkin diagram, we know by [7] that the quiver Q_C admits a positive definite quasi-Cartan companion A_C. By Remark 2.1 it suffices thus to show that the quasi-Cartan matrix A defined by the homological from χ_B is equivalent to A_C.

Let $\Gamma = \{\alpha_1, \ldots, \alpha_t\}$ be an admissible cut of Q_C such that $B = C/\langle \Gamma \rangle$. It follows from Proposition 4.15 that

$$q_B(x) = \sum_{i=1}^{n} x_i^2 - \sum_{\alpha \in (Q_B)_1} x_{s(\alpha)} x_{t(\alpha)} + \sum_{\gamma \in \Gamma} x_{s(\gamma)} x_{t(\gamma)}.$$

Therefore, the quasi-Cartan matrix A defined by $q_B(x) = \frac{1}{2} x^{\top} A x$ satisfies the property that $|A_{ij}|$ equals the number of arrows or relations (in either direction) in B between the vertices i and j or equivalently the number of arrows (in either direction) in Q_C. This shows that A is quasi-Cartan companion of Q_C. Since Γ is an admissible cut in Q_C, in each oriented cycle of Q_C there is precisely one arrow $i \to j$ for which $A_{ij} = 1$ and for all other arrows $i \to j$ in the same cycle we have $A_{ij} = -1$. Therefore A satisfies the sign condition in [7, Prop. 1.4] and by [7, Prop. 1.5] the two matrices A and A_C are equivalent. □

4.8. Main result on admissible cuts

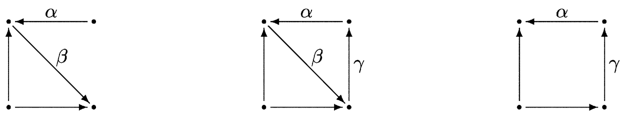
We have now gathered sufficient information on admissible cuts to be able to prove the main result on admissible cuts for cluster-tilted algebras of Dynkin type.

Theorem 4.19. *Let B be a quotient by an admissible cut of a cluster-tilted algebra C of Dynkin type Q. If* $\operatorname{gldim} B \leqslant 2$ *then B is iterated tilted of Dynkin type Q.*

Proof. By Proposition 4.18 the geometric form q_B of B is positive definite and by Lemma 4.17 the algebra B is strongly simply connected. It follows thus from [6] that B is iterated tilted of Dynkin type. □

Remark 4.20. The following example shows that this result cannot be extended to cluster-tilted of type $\widetilde{\mathbb{A}}_n$. Let $B = kQ_B/I_B$ where Q_B is as depicted below on the left-hand side and I_B is generated by the relation $\beta\alpha$. We indicated this below the quiver Q_B. In the middle column the quiver and relations of $\mathcal{R}(B)$ are shown. Observe that $\{\beta\}$ is an admissible cut of the quiver of $\mathcal{R}(B)$. Finally on the right-hand side you can see the quotient of $\mathcal{R}(B)$ by the admissible cut $\{\beta\}$.

B: $\mathcal{R}(B)$: $\mathcal{R}(B)/\langle\beta\rangle$:



$\beta\alpha = 0$ $\beta\alpha = \gamma\beta = \alpha\gamma = 0$ $\alpha\gamma = 0$

Notice that B is a tilted algebra of type $\widetilde{\mathbb{A}}_3$ and that $\operatorname{gldim} B \leqslant 2$ and hence $C = \mathcal{R}(B)$ is a cluster-tilted algebra of type $\widetilde{\mathbb{A}}_3$, but the quotient $B' = \mathcal{R}(B)/\langle\beta\rangle$ is not iterated tilted of any type as shows the following argument. Assume that B' is an iterated tilted algebra. Then the quiver of $\mathcal{R}(B')$ is isomorphic to $Q_{\mathcal{R}(B)} = Q_C$. But by [9, Thm. 2.3] there is a unique cluster-tilted algebra with quiver Q_C and consequently by Theorem 1.1 the algebra B' is iterated tilted of type $\widetilde{\mathbb{A}}_3$. But this contradicts the description in [5] of iterated tilted algebras of type $\widetilde{\mathbb{A}}_n$, where it is shown that in a non-oriented cycle there must be as many relations in clockwise orientation as there are relations in counter-clockwise orientation.

We prove now the main result of this section.

Theorem 1.3. *An algebra B with* $\operatorname{gldim} B \leqslant 2$ *is iterated tilted of Dynkin type Q if and only if it is the quotient of a cluster-tilted algebra of type Q by an admissible cut.*

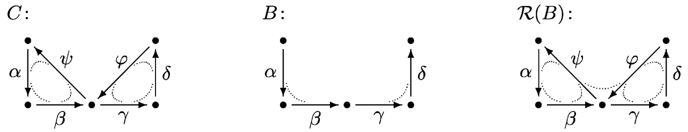
Proof. If C is a cluster-tilted algebra of Dynkin type Q then by Theorem 4.19, each quotient B of C by an admissible cut is an iterated tilted algebra with $\operatorname{gldim} B \leqslant 2$.

Conversely, if B is an iterated tilted algebra of Dynkin type Q with $\operatorname{gldim} B \leqslant 2$ then by Proposition 4.12 the algebra B is a quotient of the cluster-tilted algebra $C(B)$ by an admissible cut. □

4.9. Characterization when $\mathcal{R}(B) \simeq C(B)$

We now want to study the relationship between a cluster-tilted algebra C, a quotient B of C by an admissible cut and its relation extension $\mathcal{R}(B)$.

Remark 4.21. The following example shows that in general C is not the relation extension of B. To abbreviate notation we indicated by dotted arcs where the composition of two consecutive arrows is zero.



On the left-hand side the cluster-tilted algebra C is depicted. Then $\Gamma = \{\varphi, \psi\}$ is an admissible cut and $B = C/\langle \Gamma \rangle$ is as shown in the middle. On the right-hand side we see the relation extension $\mathcal{R}(B)$. Note that here we have $\psi\varphi = 0$ whereas in C this composition is non-zero.

We describe now when $C(B) \simeq \mathcal{R}(B)$ for an iterated tilted algebra B such that $\operatorname{gldim} B \leqslant 2$. We know by Theorem 1.1 that there is an exact sequence of algebra homomorphisms $B \to C(B) \xrightarrow{\pi} \mathcal{R}(B) \to B$ whose composition is the identity map. Moreover, the kernel of π is contained in $\operatorname{rad}^2 C$. Thus, we may assume that the presentations of $C(B)$ and $\mathcal{R}(B)$ extend the presentation of B, see Section 2.1, and denote by $\eta_1, \ldots, \eta_n$ the arrows of $Q_{C(B)} = Q_{\mathcal{R}(B)}$ which are not arrows of B. Then by Proposition 3.19, we have $\operatorname{Ker} \pi = \langle \eta_1, \ldots, \eta_n \rangle^2_{C(B)}$, where the subscript indicates that $\langle \eta_1, \ldots, \eta_n \rangle$ has to be considered as ideal of $C(B)$.

Proposition 4.22. *Let B be an iterated tilted algebra such that* $\operatorname{gldim} B \leqslant 2$ *and let $\eta_1, \ldots, \eta_n$ be as above. Then the following conditions are equivalent.*

(a) $\mathcal{R}(B) \simeq C(B)$.
(b) $\langle \eta_1, \ldots, \eta_n \rangle^2_{C(B)} = 0$.
(c) $\eta_i \mu \eta_j = 0$ *in $C(B)$ for any path $\mu \in kQ_B$ and for all $1 \leqslant i, j \leqslant n$.*

If we assume moreover that B is of Dynkin type then (a), (b) *and* (c) *are equivalent to the following condition.*

(d) *Let $\rho_1 : a \to i$, $\rho_2 : j \to b$ be minimal relations in B such that there is a non-zero path $\mu : a \to b$ in kQ_B. Then for $h = 1$ or $h = 2$ the following holds: there are paths μ_1, μ_2 such that $\mu = \mu_2\mu_1$, an arrow α_h and, in case ρ_h is not a zero relation then there exists a path γ_h not involving α_h (set $\gamma_h = 0$ otherwise) such that $\rho_1 = \alpha_1\mu_1 - \gamma_1$ or $\rho_2 = \mu_2\alpha_2 - \gamma_2$ respectively. Furthermore, ρ_h is the only minimal relation involving α_h.*

Proof. Since $\operatorname{Ker}\pi = \langle \eta_1, \dots, \eta_n \rangle^2_{C(B)}$, the equivalence of (a) and (b) follows from the fact that $\langle \eta_1, \dots, \eta_n \rangle^2_{\mathcal{R}(B)} = 0$. The equivalence of (b) and (c) is straightforward, so we only need to prove that (c) and (d) are equivalent in the Dynkin case.

Thus we assume from now on that B is of Dynkin type. Then $\{\eta_1, \dots, \eta_n\}$ is an admissible cut of $C(B)$, by Proposition 4.12.

First assume that (c) holds, and consider ρ_1, μ and ρ_2 in kQ_B as in (d). Then each relation ρ_i corresponds to an arrow η_{k_i} of $Q_{\mathcal{R}(B)} = Q_{C(B)}$. We may assume that $k_h = h$ and by (c) we have that $\eta_2\mu\eta_1 = 0$ in $C(B)$. If this relation is minimal we know from Theorem 4.4 that there exists an arrow α of $Q_{C(B)}$ so that $\alpha\eta_2\mu\eta_1$ is a chordless oriented cycle, contradicting that $\{\eta_1, \dots, \eta_n\}$ is an admissible cut of $C(B)$. Therefore the relation $\eta_2\mu\eta_1 = 0$ in $C(B)$ is not minimal, and hence there are paths μ_1, μ_2 such that $\mu = \mu_2\mu_1$ and either $\mu_1\eta_1$ or $\eta_2\mu_2$ is a minimal zero relation in $C(B)$. In the first case, by Theorem 4.4, there is an arrow α_1 such that $\mu_1\eta_1\alpha_1$ is an oriented chordless cycle in $C(B)$, and α_1 is not contained in any other chordless cycle in $C(B)$. Then $\alpha_1\mu_1$ is a shortest path antiparallel to η_1 and the statement follows from Theorem 4.4 using that ρ_1 is the relation antiparallel to η_1. The case when $\eta_2\mu_2$ is a minimal zero relation can be handled in a similar way, so (d) holds.

Now assume that (d) holds and consider a path $\eta_s\mu\eta_r$ in $kQ_{C(B)}$ with $\mu \in kQ_B$. Consider the minimal relations ρ_1, ρ_2 in I_B antiparallel to η_r, η_s respectively and let h and α_h, μ_1, μ_2, γ_h be as in (d). If $h = 1$, that is $\rho_1 = \alpha_1\mu_1 - \gamma_1$ in kQ_B, then η_r is antiparallel to $\alpha_1\mu_1$, since η_r is antiparallel to ρ_1. Then $\alpha_1\mu_1\eta_r$ is a chordless cycle in $C(B)$ and from the description of the relations in Theorem 4.4 we obtain that $\mu_1\eta_r = 0$, since α_1 is involved in a unique minimal relation of $C(B)$. Thus $\eta_s\mu\eta_r = \eta_s\mu_2\mu_1\eta_r = 0$ in this case. The same argument applies in the other case, proving (c). $\square$

When the iterated tilted algebra B is given by its quiver and relations and is of Dynkin type then (d) provides an easy way to determine if $\mathcal{R}(B)$ and $C(B)$ are isomorphic. For example, if two minimal relations of B are consecutive then (d) is not satisfied. In fact, if $\rho_1 : a \to i$ and $\rho_2 : j \to a$, then $\mu : a \to a$ is the trivial path e_a, and so are any μ_1 and μ_2 such that $\mu = \mu_1\mu_2$. Then (d) is not satisfied because $\rho_1, \rho_2 \in \operatorname{rad}^2 B$. Using this one readily verifies that $\mathcal{R}(B) \not\simeq C(B)$ for the algebra B in Remark 4.21, since $\delta\rho$ and $\beta\alpha$ are consecutive relations.

On the other hand, in the algebra B_2 of Example 3.14 we consider $a = 5$ and $b = 3$ in (d). We have relations $\rho_1 : 1 \to 3$, $\rho_2 : 5 \to 8$, and the arrow $\mu : 5 \to 3$. Since μ is neither an arrow of ρ_1 nor an arrow of ρ_2 then (d) is not satisfied.

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