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Exploring the complexity boundary between coloring and list-coloring

Flavia Bonomo · Guillermo Durán · Javier Marenco

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Abstract Many classes of graphs where the vertex coloring problem is polynomially solvable are known, the most prominent being the class of perfect graphs. However, the list-coloring problem is NP-complete for many subclasses of perfect graphs. In this work we explore the complexity boundary between vertex coloring and list-coloring on such subclasses of perfect graphs where the former admits polynomial-time algorithms but the latter is NP-complete. Our goal is to analyze the computational complexity of coloring problems lying "between" (from a computational complexity viewpoint) these two problems: precoloring extension, μ-coloring, and (γ, μ)-coloring.

Keywords Coloring · Computational complexity · List-coloring

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F. Bonomo (✉)
CONICET and Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina
e-mail: fbonomo@dc.uba.ar

G. Durán
Departamento de Ingeniería Industrial, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile
e-mail: gduran@dii.uchile.cl

G. Durán
CONICET and Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Buenos Aires, Argentina

J. Marenco
Instituto de Ciencias, Universidad Nacional de General Sarmiento, Buenos Aires, Argentina
e-mail: jmarenco@ungs.edu.ar

1 Introduction

A *coloring* of a graph $G = (V, E)$ is a function $f : V \to \mathbb{N}$ such that $f(v) \neq f(w)$ whenever $vw \in E$. A *k-coloring* is a coloring f such that $f(v) \leq k$ for every $v \in V$. The *vertex coloring problem* takes as input a graph G and a natural number k, and consists in deciding whether G is k-colorable or not. This well-known problem is a basic model for frequency assignment and resource allocation problems.

In order to take into account particular constraints arising in practical settings, more elaborate models of vertex coloring have been defined in the literature. One of such generalized models is the *list-coloring problem*, which considers a prespecified set of available colors for each vertex. Given a graph G and a finite list $L(v) \subseteq \mathbb{N}$ for each vertex $v \in V$, the list-coloring problem asks for a *list-coloring* of G, i.e., a coloring f such that $f(v) \in L(v)$ for every $v \in V$.

Many classes of graphs where the vertex coloring problem is polynomially solvable are known, the most prominent being the class of perfect graphs (Grötschel et al. 1981). Meanwhile, the list-coloring problem is NP-complete for general perfect graphs, and is also NP-complete for many subclasses of perfect graphs, including split graphs (Jansen and Scheffler 1997), interval graphs (Biro et al. 1992; Marx 2006), and bipartite graphs (Jansen and Scheffler 1997). However, using dynamic programming techniques this problem can be solved in polynomial time for a well known subclass of bipartite graphs: trees (Jansen and Scheffler 1997). Another class of graphs where list-coloring can be polynomially solved is the class of complete graphs: we can reduce this problem to maximum matching on bipartite graphs, a known polynomial problem. Combining these two ideas, list-coloring can be solved in polynomial time for block graphs (Jansen 1997).

We are interested in the complexity boundary between vertex coloring and list-coloring. Our goal is to analyze the computational complexity of coloring problems lying “between” (from a computational complexity viewpoint) these two problems.

We consider some particular cases of the list-coloring problem. The *precoloring extension* (PrExt) problem takes as input a graph $G = (V, E)$, a subset $W \subseteq V$, a coloring f' of W, and a natural number k, and consists in deciding whether G admits a k-coloring f such that $f(v) = f'(v)$ for every $v \in W$ or not (Biro et al. 1992). In other words, a prespecified vertex subset is colored beforehand, and our task is to extend this partial coloring to a valid k-coloring of the whole graph. This is a typical case of a completion problem. Many efficiently-solvable combinatorial problems have a more difficult general solution by the imposition of a partial one (we refer to Easton et al. 2000 for a review about some completion problems).

Given a graph G and a function $\mu : V \to \mathbb{N}$, G is *μ-colorable* if there exists a coloring f of G such that $f(v) \leq \mu(v)$ for every $v \in V$ (Bonomo and Cecowski 2005). This model arises in the context of classroom allocation to courses, where each course must be assigned a classroom which is large enough so it fits the students taking the course. We define here a new variation of this problem. Given a graph G and functions $\gamma, \mu : V \to \mathbb{N}$ such that $\gamma(v) \leq \mu(v)$ for every $v \in V$, we say that G is *(γ, μ)-colorable* if there exists a coloring f of G such that $\gamma(v) \leq f(v) \leq \mu(v)$ for every $v \in V$.

The classical vertex coloring problem is clearly a special case of μ-coloring and precoloring extension, which in turn are special cases of (γ, μ)-coloring. Furthermore, (γ, μ)-coloring is a particular case of list-coloring. These observations imply that all the problems in this hierarchy are polynomially solvable in those graph classes where list-coloring is polynomial and, on the other hand, all the problems are NP-complete in those graph classes where vertex coloring is NP-complete. Furthermore, list-coloring can be polynomially reduced to precoloring extension in a straightforward way. To this end, attach precolored vertices of degree 1 to each vertex in order to reduce the available colors from which it can be colored, creating the desired lists. But note that this reduction, unlike the previous ones, does not preserve the graph. In particular, many graph classes are not closed under this kind of operations. List-coloring can be polynomially reduced to μ-coloring in a similar way, but again this reduction does not preserve the graph structure.

It is interesting to note that the list-coloring problem can be polynomially reduced to the (γ, μ)-coloring problem while preserving the original graph, if the list of colors can be renamed in such a way that each list is an interval of colors. This renaming is possible if and only if there exists a row permutation of the $0-1$ color-vertex matrix such that the ones in each column of the resulting matrix are consecutive (Hell 2006). This property is known as the consecutive ones property and can be checked in linear time (Booth and Lueker 1976).

In this work, we are interested in the computational complexity of these problems over graph classes where vertex coloring is polynomially solvable and the complexity of list-coloring is NP-complete. In Sect. 2, we show some known complexity results about these coloring problems.

In Sect. 3, we prove new complexity results about precoloring extension, μ-coloring, (γ, μ)-coloring, and list-coloring in some subclasses of perfect graphs and line graphs of complete graphs. As a consequence of our results, we prove that, unless $P = NP$, μ-coloring and precoloring extension are strictly more difficult than vertex coloring. On the other hand, we show that list-coloring is strictly more difficult than (γ, μ)-coloring, and (γ, μ)-coloring is strictly more difficult than precoloring extension.

In Sect. 4, some general theorems are stated showing polynomial-time reductions from list-coloring to the other problems. These reductions involve changes in the graph, but are closed within some graph classes. They can be used, therefore, to prove that the problems studied here are polynomially equivalent in those classes. Finally, Sect. 5 presents a table reviewing the complexity situation of these problems in the classes of graphs we analyzed.

An extended abstract of a preliminary version of this work appears in Bonomo et al. (2006).

2 Known results

Most of the graph classes considered in this paper are subclasses of perfect graphs. A graph G is *perfect* when the chromatic number is equal to the cardinality of a maximum complete subgraph for every induced subgraph of G.

A graph is an *interval graph* if it is the intersection graph of a set of intervals over the real line. A *unit interval graph* is the intersection graph of a set of intervals of length one. Since interval graphs are perfect, vertex coloring over interval and unit interval graphs is polynomially solvable. On the other hand, precoloring extension over unit interval graphs is NP-complete (Marx 2006), implying that (γ, μ)-coloring and list-coloring are NP-complete over this class and over interval graphs.

A *split graph* is a graph whose vertex set can be partitioned into a complete graph K and an independent set I. A split graph is said to be *complete* if its edge set includes all possible edges between K and I. It is trivial to color a split graph in polynomial time, and it is a known result that precoloring extension is also solvable in polynomial time on split graphs (Hujter and Tuza 1996), whereas list-coloring is known to be NP-complete even over complete split graphs (Jansen and Scheffler 1997).

A *bipartite graph* is a graph whose vertex set can be partitioned into two independent sets V_1 and V_2. A bipartite graph is said to be *complete* if its edge set includes all possible edges between V_1 and V_2. Again, the vertex coloring problem over bipartite graphs is trivial, whereas precoloring extension (Hujter and Tuza 1993) and μ-coloring (Bonomo and Cecowski 2005) are known to be NP-complete over bipartite graphs. This implies that (γ, μ)-coloring and list-coloring over this class are also NP-complete, and that the four problems are NP-complete on comparability graphs, a widely studied subclass of perfect graphs which includes bipartite graphs. Moreover, list-coloring is NP-complete even over complete bipartite graphs (Jansen and Scheffler 1997).

For complements of bipartite graphs, precoloring extension can be solved in polynomial time (Hujter and Tuza 1996), but list-coloring is NP-complete (Jansen 1997). The same holds for *cographs*, i.e., graphs with no induced P_4 (or P_4-free) (Hujter and Tuza 1996; Jansen and Scheffler 1997). For this class of graphs, μ-coloring is polynomial (Bonomo and Cecowski 2005). Cographs are a subclass of *distance-hereditary* graphs, another known subclass of perfect graphs. A graph is distance-hereditary if any two vertices are equidistant in every connected induced subgraph containing them.

Two known subclasses of cographs are *trivially perfect* and *threshold* graphs. A graph is trivially perfect if it is $\{C_4, P_4\}$-free. A graph G is threshold if G and $\overline{G}$ are trivially perfect. This last class includes complete split graphs.

The *line graph* of a graph is the intersection graph of its edges. The edge coloring problem (equivalent to coloring the line graph) is NP-complete in general (Holyer 1981) but can be solved in polynomial-time for complete graphs and bipartite graphs (König 1916). It is known that precoloring extension is NP-complete on line graphs of complete bipartite graphs $K_{n,n}$ (Colbourn 1984), and list-coloring is NP-complete on line graphs of complete graphs (Kubale 1992).

A good survey on variations of the coloring problem appears in Tuza (1997). Graph classes and graph theory properties not defined here can be found in Brandstädt et al. (1999), Golumbic (2004).

3 New results

In this section we introduce new results on the computational complexity of the previously mentioned coloring problems over the graph classes described in Sect. 2 and related classes. We first analyze different subclasses of perfect graphs and in Sect. 3.2 we study a non-perfect class: line graphs of complete graphs.

3.1 Subclasses of perfect graphs

3.1.1 Interval graphs

In order to prove that the μ-coloring problem over interval graphs is NP-complete we will show a reduction from the coloring problem over circular-arc graphs, which is NP-complete (Garey et al. 1980). The proof is similar to the one given in Biro et al. (1992) for precoloring extension over interval graphs.

Theorem 1 *The μ-coloring problem over interval graphs is NP-complete.*

Proof An instance of the coloring problem over circular-arc graphs is given by a circular-arc graph G and an integer $k \geq 1$, and consists in deciding whether G can be k-colored or not. Let G be a circular-arc graph and k be an integer greater than zero. Let $A = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ be a circular-arc representation of G (i.e., a collection of arcs over the unit circle $[0, 2\pi)$ such that G is the intersection graph of A). For $i = 1, \ldots, n$, we call v_i the vertex of G corresponding to the arc (a_i, b_i).

Let A_0 be the set of arcs from A containing the point 0. We can suppose w.l.o.g. $A_0 = \{(a_1, b_1), \ldots, (a_t, b_t)\}$. We can also suppose $t \leq k$, otherwise G is clearly not k-colorable. Define

$$I = (A \backslash A_0) \cup \{(a_i, 2\pi) : i = 1, \ldots, t\} \cup \{(0, b_i) : i = 1, \ldots, t\}$$

to be a family of arcs over the unit circle. Since $a < b$ for every arc $(a, b) \in I$, we can see I as a family of intervals on the real line. Let H be the interval graph induced by I. For $i = 1, \ldots, t$, we call w_i and w_i' the vertices of H corresponding to the intervals $(a_i, 2\pi)$ and $(0, b_i)$, respectively. For $i = t + 1, \ldots, n$, we call w_i the vertex corresponding to the interval (a_i, b_i). Moreover, let $\mu : V(H) \to \mathbb{N}$ be defined by

$$\mu(w_i) = \begin{cases} i & \text{if } i = 1, \ldots, t \\ k & \text{otherwise} \end{cases} \quad \text{for } i = 1, \ldots, n$$

$$\mu(w_i') = i \quad \text{for } i = 1, \ldots, t$$

This construction is clearly polynomial. We claim that G is k-colorable if and only if H is μ-colorable.

Assume first that G is k-colorable and let $c : V(G) \to \mathbb{N}$ be a coloring of G using at most k colors. The vertices $v_1, \ldots, v_t$ corresponding to arcs of A_0 form a complete graph, hence we can reorder the colors of c in such a way that $c(v_i) = i$, for $i = 1, \ldots, t$. Now, the function $d : V(H) \to \mathbb{N}$ defined by

$$d(w_i) = c(v_i) \quad \text{for } i = 1, \ldots, n$$

$$d(w_i') = c(v_i) \quad \text{for } i = 1, \ldots, t$$

is a μ-coloring of H and, therefore, H is μ-colorable.

On the other hand, assume that H is μ-colorable and let $d : V(H) \to \mathbb{N}$ be a μ-coloring of H. Since the vertices $w_1, \ldots, w_t$ form a complete subgraph and $\mu(w_i) = i$ for $i = 1, \ldots, t$, then we have $d(w_i) = i$ for $i = 1, \ldots, t$. A similar analysis shows $d(w_i') = i$ for $i = 1, \ldots, t$.

Consider now the function $c : V(G) \to \mathbb{N}$ defined by $c(v_i) = d(w_i)$ for $i = 1, \ldots, n$. Since $t \leq k$ and $d(w_i) \leq \mu(w_i)$ for $i = 1, \ldots, n$, it holds that $c(v_i) \leq k$ for $i = 1, \ldots, n$. We claim that c is a valid k-coloring of G. To this end, let $v_i v_j \in E(G)$ be an edge of G. The following case analysis shows that $c(v_i) \neq c(v_j)$:

- If $i, j > t$ or $i, j \leq t$, then $c(v_i) = d(w_i) \neq d(w_j) = c(v_j)$.
- If $i \leq t$ and $j > t$, then either the interval (a_j, b_j) intersects the interval $(a_i, 2\pi)$ (in which case $c(v_i) = d(w_i) \neq d(w_j) = c(v_j)$), or the interval (a_j, b_j) intersects the interval $(0, b_i)$ (in which case $c(v_i) = d(w_i) = i = d(w_i') \neq d(w_j) = c(v_j)$). In both cases we get $c(v_i) \neq c(v_j)$.
- If $i > t$ and $j \leq t$, a similar argument shows $c(v_i) \neq c(v_j)$.

Hence, the graph G is k-colorable. □

With this result and the NP-completeness of precoloring extension on interval graphs, it follows that the four problems considered are NP-complete also for chordal graphs, one of the most studied subclasses of perfect graphs, which is a superclass of interval graphs.

3.1.2 Complete bipartite graphs

The next theorem uses combinatorial arguments to prove that (γ,μ)-coloring problem is polynomial in complete bipartite graphs. If $G=(V,E)$ is a graph and $\gamma,\mu: V\to\mathbb{N}$, we define $\gamma_{\min}=\min\{\gamma(v): v\in V\}$ and $\mu_{\max}=\max\{\mu(v): v\in V\}$.

Theorem 2 *The (γ,μ)-coloring problem in complete bipartite graphs can be solved in polynomial time.*

Proof Let $G=(V,E)$ be a complete bipartite graph, with bipartition $V_1\cup V_2$, and let $\gamma,\mu: V\to\mathbb{N}$ such that $\gamma(v)\le\mu(v)$ for every $v\in V$. Let $K_0=\{\gamma_{\min},\dots,\mu_{\max}\}$, and consider the following procedure:

set $K := K_0$; {available colors}
set $F := \emptyset$; {uniquely colorable vertices}
while there exists some non-colored vertex $v\in V$ such that
$K\cap\{\gamma(v),\dots,\mu(v)\}$ is a singleton, say $\{i\}$:
  Let $j\in\{1,2\}$ such that $v\in V_j$;
  Assign color i to all the vertices w in V_j such that
  $\gamma(w)\le i\le\mu(w)$ (note that this includes the vertex v);
  set $K := K\backslash\{i\}$;
  set $F := F\cup\{v\}$;
end;

Upon termination of this procedure, we are left with a set $C\subseteq V$ of colored vertices. Moreover, the set $F\subseteq C$ contains uniquely colorable vertices and so, each vertex of this set is assigned the only possible color in any valid (γ,μ)-coloring of G. We now show that G is (γ,μ)-colorable if and only if $K\cap\{\gamma(v),\dots,\mu(v)\}\ne\emptyset$ for every $v\in V\backslash C$. Assume there exists some $v\in V\backslash C$ such that $K\cap\{\gamma(v),\dots,\mu(v)\}=\emptyset$, and suppose w.l.o.g. $v\in V_1$. For every $j=\gamma(v),\dots,\mu(v)$, there exists some $w\in V_2\cap F$ such that the procedure has assigned the color j to w, and this is the only possible color for w in any (γ,μ)-coloring. Hence v cannot be assigned any color in $\{\gamma(v),\dots,\mu(v)\}$ and, therefore, G is not (γ,μ)-colorable.

On the other hand, suppose $K\cap\{\gamma(v),\dots,\mu(v)\}$ contains at least two colors for every $v\in V\backslash C$. Let $K=\{i_1,\dots,i_k\}$ with $i_t<i_{t+1}$ for $t=1,\dots,k-1$. Since each vertex in $V_1\backslash C$ (resp. $V_2\backslash C$) admits at least two consecutive colors in K (note that they are not necessarily consecutive in K_0), then we can color $V_1\backslash C$ with colors in $\{i_j \text{ in } K : j \text{ is odd}\}$ and we can color $V_2\backslash C$ with colors in $\{i_j \text{ in } K : j \text{ is even}\}$, thus obtaining a valid (γ,μ)-coloring of G. This procedure is clearly polynomial in the number of vertices of G. □

This result implies that μ-coloring over complete bipartite graphs can be solved in polynomial time.

Remark 1 The final observation in the proof of the previous theorem can be generalized as follows. Let $G=(V,E)$ be an arbitrary k-colorable graph, and let $\gamma,\mu: V\to\mathbb{N}$ such that, for each vertex v in V, $\mu(v)-\gamma(v)+1\ge k$. Then G is (γ,μ)-colorable: just take a coloring c of G with k colors, and then for each vertex v of G, assign to it a color $c'(v)$ such that $\gamma(v)\le c'(v)\le\mu(v)$ and $c'(v)\equiv c(v) \bmod k$.

3.1.3 Split graphs

We first prove that for general split graphs the μ-coloring problem is NP-complete. We use a reduction from the dominating set problem on split graphs, which is NP-complete (Bertossi 1984; Corneil and Perl 1984).

Theorem 3 *The μ-coloring problem over split graphs is NP-complete.*

Proof An instance of the dominating set problem on split graphs is given by a split graph G and an integer $k \geq 1$, and consists in deciding if there exists a subset D of $V(G)$, with $|D| \leq k$, and such that every vertex of $V(G)$ either belongs to D or has a neighbor in D. Such a set is called a *dominating set*.

Let G be a split graph and k be an integer greater than zero. We will construct a split graph G' and a function $\mu : V(G') \to \mathbb{N}$ such that G' is μ-colorable if and only if G admits a dominating set of cardinality at most k. Let K and I such that $V(G) = K \cup I$, K is a complete and I is an independent set in G. We may assume w.l.o.g. that G does not have isolated vertices and $k \leq |K|$. The graph G' is defined as follows: $V(G') = K \cup I$; K is a complete and I is an independent set in G'; for every pair of vertices $v \in K$ and $w \in I$, $vw \in E(G')$ if and only if $vw \notin E(G)$. Define $\mu(v) = |K|$ for every $v \in K$, and $\mu(w) = k$ for every $w \in I$.

Suppose first that G admits a dominating set D with $|D| \leq k$. Since G has no isolated vertices, G admits such a set $D \subseteq K$. Let us define a μ-coloring of G' as follows: color the vertices of D using different colors from 1 to $|D|$; color the remaining vertices of K using different colors from $|D| + 1$ to $|K|$; for each vertex w in I, choose w' in D such that $ww' \in E(G)$ and color w with the color used by w'.

Suppose now that G' is μ-colorable, and let $c : V(G') \to \mathbb{N}$ be a μ-coloring of G'. Since $\mu(v) = |K|$ for every $v \in K$ and K is complete in G', it follows that $c(K) = \{1, \ldots, |K|\}$. Since $k \leq |K|$, for each vertex $w \in I$ there is a vertex $w' \in K$ such that $c(w) = c(w') \leq k$. Then $ww' \notin E(G')$, so $ww' \in E(G)$. Thus the set $\{v \in K : c(v) \leq k\}$ is a dominating set of G of size k. □

This result implies that (γ, μ)-coloring over split graphs is NP-complete too. At this moment, split graphs is the only class where we know that the computational complexity of μ-coloring and precoloring extension is different, unless P = NP.

Now, integer programming techniques are employed to prove the polynomiality of the (γ, μ)-coloring problem for complete split graphs.

Theorem 4 *The (γ, μ)-coloring problem in complete split graphs can be solved in polynomial time.*

Proof Let $G = (V, E)$ be a complete split graph with partition $V = K \cup I$, where K is a complete graph and I is an independent set. For $0 < j \leq i \leq \mu_{\max}$, let $L_{i,j} = |\{v \in K : j \leq \gamma(v) \text{ and } \mu(v) \leq i\}|$. We reduce the problem of finding a (γ, μ)-coloring of G to a linear programming feasibility problem. For $j = 1, \ldots, \mu_{\max}$, we define the integer variable x_j to be the number of colors from the set $\{1, \ldots, j\}$ assigned to vertices of K and, based on this definition, we consider the following linear program:

$$x_0 = 0 \tag{1}$$

$$x_{j+1} - x_j \geq 0 \quad \forall j \in \{0, \ldots, \mu_{\max} - 1\} \tag{2}$$

$$x_{j+1} - x_j \leq 1 \quad \forall j \in \{0, \ldots, \mu_{\max} - 1\} \tag{3}$$

$$x_i - x_{j-1} \geq L_{i,j} \quad \forall i, j : 0 < j \leq i \leq \mu_{\max} \tag{4}$$

$$x_{\mu(v)} - x_{\gamma(v)-1} \leq \mu(v) - \gamma(v) \quad \forall v \in I \tag{5}$$

We may assume that every color between 1 and $\mu_{\max}$ belongs to the interval $[\gamma(v), \mu(v)]$, for some $v \in V$. Furthermore, we may assume $\mu(v) - \gamma(v) \leq d(v)$ for every $v \in K \cup I$, implying that the number of variables and constraints is polynomial in the size of G. All the constraints take the form $x_j - x_k \geq \alpha_{jk}$ or $x_j = \alpha_j$, hence the constraint matrix is totally unimodular, implying that the associated polytope is integral (see for example Nemhauser and Wolsey 1988). To complete the proof, we verify that G is (γ, μ)-colorable if and only if the linear program (1)–(5) is feasible.

Assume first G is (γ, μ)-colorable. Let $x_0 = 0$ and, for $j = 1, \ldots, \mu_{\max}$, let x_j be the number of colors from $\{1, \ldots, j\}$ assigned to vertices of K. Constraints (1) to (3) are clearly verified. Since K is a complete subgraph, then $|K|$ different colors are assigned to the vertices of K, hence constraints (4) hold. Finally, since every vertex $v \in I$ is assigned a color between $\gamma(v)$ and $\mu(v)$, and v is adjacent to every vertex in K, then K cannot use all the colors in $\{\gamma(v), \ldots, \mu(v)\}$ and, therefore, constraints (5) are verified. Thus, the linear program (1)–(5) admits a feasible solution.

Conversely, assume the linear program (1)–(5) is feasible and let x be an integer solution, which exists since the associated polytope is integral. We shall verify that G admits a (γ, μ)-coloring. Let $M = \{j : 1 \leq j \leq \mu_{\max} \text{ and } x_j - x_{j-1} = 1\}$. We construct a bipartite graph B with vertex set $K \cup M$, and such that $v \in K$ is adjacent to $j \in M$ if and only if $\gamma(v) \leq j \leq \mu(v)$. Any (γ, μ)-coloring of K using a subset of M as color set corresponds to a matching of B of size $|K|$. Moreover, by Hall's Theorem, such a matching exists if and only if for every subset R of K, the neighborhood of R in M has at least $|R|$ vertices (Hall 1935).

Let R be a subset of K, and let $M_R \subseteq M$ be the neighborhood of R in B. Let $i_1, \ldots, i_t$ be the elements of M in (strictly) increasing order, and partition $M_R = M_R^1 \cup \cdots \cup M_R^k$ such that M_R^j is a maximal interval in M_R (i.e., $M_R^j = \{i_{p_j}, i_{p_j+1}, \ldots, i_{q_j}\}$ for some p_j and q_j, and $i_{p_j-1}, i_{q_j+1} \notin M_R$). Since the neighborhood of every vertex of K is an interval in M, then we can partition R in k disjoint sets $R_1, \ldots, R_k$ such that the neighborhood of R_i in M is exactly M_R^i, for $i = 1, \ldots, k$. Therefore, $|M_R| = \sum_{i=1}^k |M_R^i|$ and $|R| = \sum_{i=1}^k |R_i|$. In order to complete the proof, we verify $|M_R^i| \geq |R_i|$ for $i = 1, \ldots, k$.

Let $M' = M \cup \{0, \mu_{\max} + 1\}$. For $i = 1, \ldots, k$, define a_i to be the maximum value in M' such that every element from M_R^i is strictly greater than a_i, and define b_i to be the minimum value in M' such that every element from M_R^i is strictly less than b_i. We have $|R_i| \leq L_{b_i-1, a_i+1}$ and, since x verifies (2)–(4), then $|M_R^i| \geq L_{b_i-1, a_i+1}$. We conclude that B admits a matching of size $|K|$ and, therefore, K is (γ, μ)-colorable. Since x verifies (5) and I is an independent set, then this (γ, μ)-coloring of K can be extended to a (γ, μ)-coloring of G. □

This theorem implies that μ-coloring over complete split graphs can be solved in polynomial time.

3.1.4 Line graphs of complete bipartite graphs

Considering these coloring variations applied to edge coloring, we have the following result.

Theorem 5 *The μ-coloring problem over line graphs of complete bipartite graphs is NP-complete.*

Proof We will show a reduction from precoloring extension of line graphs of bipartite graphs, which is NP-complete (Colbourn 1984), to μ-coloring of line graphs of complete bipartite graphs. The former takes as input a bipartite graph $B = (V_1 \cup V_2, E)$, an integer $k \geq 1$, and a partial edge-precoloring $f : E_1 \subseteq E \to \{1, \ldots, k\}$, and consists in deciding whether f can be extended to a valid k-edge-coloring of B or not. The second takes as input a complete bipartite graph $K_{n,n}$, a function μ, and consists in deciding whether B' can be μ-edge-colored or not.

Let $B = (V_1 \cup V_2, E)$, $k \geq 1$, $f : E_1 \subseteq E \to \{1, \ldots, k\}$ be an instance of precoloring extension of line graphs of bipartite graphs.

Construct a new graph $B' = (V_1' \cup V_2', E')$ with

$$\begin{aligned}
V_1' &= V_1 \cup \{w_{v'v} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1\} \\
&\qquad \cup \{z_{vv'j} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1,\ 1 \leq j < f(vv')\} \\
V_2' &= V_2 \cup \{w_{vv'} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1\} \\
&\qquad \cup \{z_{v'vj} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1,\ 1 \leq j < f(vv')\} \\
E' &= (E \setminus E_1) \cup \{v\, w_{vv'} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1\} \\
&\qquad \cup \{v'\, w_{v'v} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1\} \\
&\qquad \cup \{w_{vv'}\, z_{vv'j} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1,\ 1 \leq j < f(vv')\} \\
&\qquad \cup \{w_{v'v}\, z_{v'vj} : v \in V_1, v' \in V_2 \text{ and } vv' \in E_1,\ 1 \leq j < f(vv')\}
\end{aligned}$$

Define $\mu : E' \to \mathbb{N}$ as follows: $\mu(e) = k$ for $e \in E \setminus E_1$; $\mu(v\, w_{vv'}) = \mu(v'\, w_{v'v}) = f(vv')$ for $vv' \in E_1$; $\mu(w_{vv'}\, z_{vv'j}) = \mu(w_{v'v}\, z_{v'vj}) = j$ for $vv' \in E_1$, $1 \leq j < f(vv')$.

Finally, let $n = \max\{|V_1'|, |V_2'|\}$. Add the required vertices and edges to B' in order to obtain $K_{n,n}$, and extend μ by defining $\mu(e) = 2n - 1$ for each new edge e (this upper bound allows to color correctly the new edges because they have $2n - 2$ incident edges). It is not difficult to see that the transformation is polynomial, and that f can be extended to a valid k-edge-coloring of B if and only if $K_{n,n}$ can be μ-edge-colored. □

3.2 A non-perfect class: line graphs of complete graphs

Finally, we analyze the class of line graphs of complete graphs. Again, we have to consider the edge coloring of complete graphs.

Theorem 6 *The μ-coloring problem over line graphs of complete graphs is NP-complete.*

Proof We show a reduction from the edge coloring problem, which is NP-complete (Holyer 1981), to the edge μ-coloring problem of complete graphs, which is equivalent to the μ-coloring problem over line graphs of complete graphs. The edge coloring problem takes as input a graph G with n vertices, and consists in deciding whether the edges of G can be colored with $\Delta(G)$ colors or not, where $\Delta(G)$ is the maximum degree of the vertices of G. The reduction consists in extending G to the complete graph K_n, and then defining $\mu : E(K_n) \to \mathbb{N}$ such that $\mu(e) = \Delta(G)$ if $e \in E(G)$ and $\mu(e) = 2n - 3$, otherwise (this upper bound allows to color correctly the new edges because they have $2n - 4$ incident edges). It is easy to see that G can be $\Delta(G)$-edge-colored if and only if K_n can be μ-edge-colored. □

This result implies that (γ, μ)-coloring over line graphs of complete graphs is NP-complete too.

Theorem 7 *The precoloring extension problem over line graphs of complete graphs is NP-complete.*

Proof We provide a reduction from the precoloring extension problem over line graphs of complete bipartite graphs, which is NP-complete (Colbourn 1984), to the edge precoloring extension problem of complete graphs, which is equivalent to the precoloring extension problem over line graphs of complete graphs. The former takes as input the complete bipartite graph $K_{n,n} = (V_1 \cup V_2, E)$ on $2n$ vertices, an integer k, and a partial edge-precoloring $f : E' \subseteq E \to \{1, \dots, k\}$, and consists in deciding whether f can be extended to a valid k-edge-coloring of $K_{n,n}$ or not.

Consider the case n even first. We extend the graph $K_{n,n}$ to the complete graph K_{2n} by adding an edge between every pair of vertices in V_1 and an edge between every pair of vertices in V_2. Denote by E_1 (resp. E_2) the set of edges joining pairs of vertices in V_1 (resp. V_2). Since V_1 (resp. V_2) induces a complete graph on (even) n vertices, then E_1 (resp. E_2) can be optimally edge-colored with $n - 1$ colors. We precolor the edges in E_1 (resp. E_2) with such an optimal edge-coloring using colors $k + 1, \dots, k + n - 1$, and we maintain the original precoloring f for the precolored edges in E. Since every vertex in V_1 (resp. V_2) is incident to an edge precolored with color c, for each $c \in \{k+1, \dots, k+n-1\}$, then this new precoloring can be extended to a $(k+n-1)$-edge-coloring of K_{2n} if and only if f can be extended to a k-edge-coloring of $K_{n,n}$.

Consider now the case n odd. We cannot directly apply the previous procedure in this case, since for odd n the chromatic index of K_n is n, hence some edge in E could be assigned a color in $\{k + 1, \dots, k + n\}$. In order to handle this situation, we first construct a graph $K_{2n,2n}$ with bipartition $V_{11} \cup V_{12}$ and $V_{21} \cup V_{22}$ (each set V_{ij} has n vertices). Define the partial precoloring f' in the following way: color the edges joining vertices of V_{11} with vertices of V_{22} (resp. V_{12} and V_{21}) with an optimal n-color edge-coloring using colors $k + 1, \dots, k + n$, and the edges joining vertices of V_{11} with vertices of V_{21} (resp. V_{12} and V_{22}) with the precoloring f. This new graph admits a precoloring extension with $k + n$ colors if and only if the original graph admits a precoloring extension with k colors. To complete the proof, we now apply the procedure for the even case to the newly constructed graph, thus obtaining a complete graph on $4n$ vertices which admits a precoloring extension with $(k + 3n - 1)$ colors if and only if f' can be extended to a $k + n$-edge-coloring of $K_{2n,2n}$. □

4 General results

Since all these problems are NP-complete in the general case, there are polynomial-time reductions from each one to any other one. The reductions we suggest in the following theorems involve changes in the graph, but are closed within some graph classes. Therefore, they can be applied to prove that the problems are polynomially equivalent in those classes.

Theorem 8 *Let $\mathcal{F}$ be a family of graphs such that every graph in $\mathcal{F}$ has minimum degree at least two. Then list-coloring, (γ, μ)-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.*

Proof Let (G, L) be an instance of list-coloring over $\mathcal{F}$-free graphs, consisting of an $\mathcal{F}$-free graph $G = (V, E)$ and a list $L(v) \subseteq \{1, \dots, k\}$ of colors for every $v \in V$. We may assume $\bigcup_{v \in V} L(v) = \{1, \dots, k\}$. For $v \in V$, define $\bar{L}(v) = \{1, \dots, k\} \setminus L(v)$ to be the set of forbidden colors for the vertex v. We shall reduce this instance to an instance of precoloring extension over $\mathcal{F}$-free graphs. To this end, we construct a new graph $H = (V', E')$ with

$$V' = V \cup \{w_{vj} : v \in V \text{ and } j \in \bar{L}(v)\}$$
$$E' = E \cup \{v\, w_{vj} : v \in V \text{ and } j \in \bar{L}(v)\}$$

In other words, for every vertex $v \in V$ and every color $j \in \bar{L}(v)$, we add a new vertex w_{vj} adjacent to v. Furthermore, for every $v \in V$ and every $j \in \bar{L}(v)$, we precolor the vertex w_{vj} with color j. Since G is an $\mathcal{F}$-free graph and all the vertices added to G by the construction have degree one, then H does not contain any induced subgraph from $\mathcal{F}$. Moreover, G is list-colorable if and only if the precoloring of H can be extended to a k-coloring. We can, therefore, reduce list-coloring over $\mathcal{F}$-free graphs to precoloring extension over $\mathcal{F}$-free graphs and conversely, hence precoloring extension, (γ, μ)-coloring, and list-coloring are polynomially equivalent over this class. □

Theorem 9 *Let $\mathcal{F}$ be a family of graphs satisfying the following property: for every graph G in $\mathcal{F}$, no connected component of G is complete, and for every cutpoint v of G, no connected component of $G \setminus v$ is complete. Then list-coloring, (γ, μ)-coloring, μ-coloring and precoloring extension are polynomially equivalent in the class of $\mathcal{F}$-free graphs.*

Proof Since $\mathcal{F}$ satisfies the conditions of Theorem 8, it follows that list-coloring, (γ, μ)-coloring, and precoloring extension are polynomially equivalent over the class of $\mathcal{F}$-free graphs. It suffices now to show a reduction from (γ, μ)-coloring on $\mathcal{F}$-free graphs to μ-coloring on $\mathcal{F}$-free graphs.

Let (G, γ, μ) be an instance of (γ, μ)-coloring over $\mathcal{F}$-free graphs, consisting of an $\mathcal{F}$-free graph $G = (V, E)$ and two functions $\gamma, \mu : V \to \mathbb{N}$ such that $\gamma(v) \leq \mu(v)$ for every $v \in V$. We may assume $\mu(v) - \gamma(v) \leq d(v)$ for every $v \in V$, and that all the intervals cover the set $\{1, \dots, \mu_{\max}\}$, implying that $\mu_{\max}$ is polynomial in the size of G. We shall reduce this instance to an instance of μ-coloring over $\mathcal{F}$-free graphs. To this end, we construct a new graph $H = (V', E')$ with

$$V' = V \cup \{w_{vj} : v \in V \text{ and } 1 \leq j < \gamma(v)\}$$
$$E' = E \cup \{v\, w_{vj} : v \in V \text{ and } 1 \leq j < \gamma(v)\}$$
$$\cup \{w_{vj}\, w_{vt} : v \in V \text{ and } 1 \leq j < t < \gamma(v)\}$$

In other words, for every vertex $v \in V$ we add a complete subgraph on $\gamma(v) - 1$ vertices, all of them joined to v. Furthermore, we keep $\mu(v)$ for every $v \in V$ and set $\mu(w_{vj}) = j$ for every $v \in V$ and every $j = 1, \dots, \gamma(v) - 1$. Note that any μ-coloring of H assigns color j to w_{vj}, for $v \in V$ and $j = 1, \dots, \gamma(v) - 1$, hence precluding the colors in $\{1, \dots, \gamma(v) - 1\}$ for the vertex v. Therefore, G is (γ, μ)-colorable if and only if H is μ-colorable.

Finally, we verify that the construction of H ensures that H does not contain any induced subgraph from $\mathcal{F}$. Suppose the contrary, i.e., assume H contains some induced subgraph $S \in \mathcal{F}$. Denote by $V^{\text{new}} = V' \setminus V$ the vertices of H added to G by the previous construction. Since G is an $\mathcal{F}$-free graph, then S must contain at least one vertex from V^{new}. Moreover, since no connected component of S is complete and every connected component of H induced

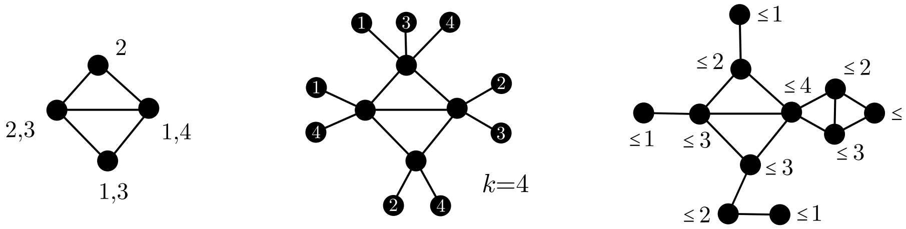
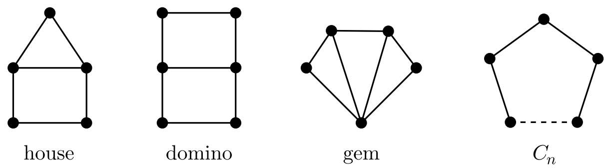


Fig. 1 Example of reductions of Theorems 8 and 9. From *left* to *right*, a list-coloring instance and its corresponding precoloring extension and μ-coloring instances, respectively

Fig. 2 Forbidden induced subgraphs for distance-hereditary graphs



by V^{new} is complete, then every connected component of S must contain at least one vertex from V. Let C be a connected component of S containing vertices of V^{new}, and let $v \in C \cap V$ such that v has some neighbor in $C \cap V^{\text{new}}$. By construction, and since C is not complete, v is a cutpoint of C, and the neighbors of v in $C \cap V^{\text{new}}$ form a complete connected component M of $C \setminus v$ (in order to see that v is a cutpoint of C, recall that every vertex in $C \cap V$, different from v, does not have adjacencies in M). Therefore, S admits a cutpoint v such that some connected component of $S \setminus v$ is complete, contradicting the fact that $S \in \mathcal{F}$. □

An example of these reductions is shown in Fig. 1, where we can see a list-coloring instance and its corresponding precoloring extension and μ-coloring instances.

Please note that, since odd holes and antiholes satisfy the conditions of the theorems above, then these results are applicable for many subclasses of perfect graphs. For example, since distance-hereditary graphs are equivalent to {house, domino, gem, $\{C_n\}_{n\geq 5}$}-free graphs (Bandelt and Mulder 1986) (see Fig. 2 for the definition of each one of these graphs), we obtain the following result as a corollary of Theorem 9 and the fact that list-coloring is NP-complete for distance-hereditary graphs (Jansen and Scheffler 1997).

Corollary 1 *The (γ, μ)-coloring, μ-coloring and precoloring extension problems are NP-complete for distance-hereditary graphs.*

5 Summary of complexity results

We summarize all the results about these coloring problems in Table 1. As this table shows, unless P = NP, μ-coloring and precoloring extension are strictly more difficult than vertex coloring (due for example to interval and bipartite graphs). On the other hand, list-coloring is strictly more difficult than (γ, μ)-coloring, due to complete split and complete bipartite graphs, and (γ, μ)-coloring is strictly more difficult than precoloring extension, due to split graphs.

Table 1 Complexity table for coloring problems. Boldfaced results have been obtained here. "NP-c" indicates an NP-complete problem, "P" a polynomial problem, and "?" an open problem. Results with no reference are trivial or can be directly deduced from the other ones

Class	Coloring	PrExt	μ-col.	(γ, μ)-col.	List-col.
Complete bipartite	P	P	**P**	**P**	NP-c (Jansen and Scheffler 1997)
Bipartite	P	NP-c (Hujter and Tuza 1993)	NP-c (Bonomo and Cecowski 2005)	NP-c	NP-c (Kubale 1992)
Cographs	P (Grötschel et al. 1981)	P (Hujter and Tuza 1996)	P (Bonomo and Cecowski 2005)	?	NP-c (Jansen and Scheffler 1997)
Distance-hereditary	P (Grötschel et al. 1981)	**NP-c**	**NP-c**	**NP-c**	NP-c (Jansen and Scheffler 1997)
Interval	P (Grötschel et al. 1981)	NP-c (Biro et al. 1992)	**NP-c**	NP-c	NP-c
Unit interval	P	NP-c (Marx 2006)	?	NP-c	NP-c
Split	P	P (Hujter and Tuza 1996)	**NP-c**	**NP-c**	NP-c
Complete split	P	P	**P**	**P**	NP-c (Jansen and Scheffler 1997)
Trivially perfect	P	P	P	?	NP-c
Threshold	P	P	P	?	NP-c
Line of $K_{n,n}$	P (König 1916)	NP-c (Colbourn 1984)	**NP-c**	NP-c	NP-c
Complement of bipartite	P (Grötschel et al. 1981)	P (Hujter and Tuza 1996)	?	?	NP-c (Jansen 1997)
Line of K_n	P (König 1916)	**NP-c**	**NP-c**	**NP-c**	NP-c (Kubale 1992)

It remains as an open problem to know if there exists some class of graphs where (γ, μ)-coloring is NP-complete and μ-coloring can be solved in polynomial time. Among the classes considered in this work, the candidate classes are COGRAPHS, UNIT INTERVAL, TRIVIALLY PERFECT, THRESHOLD and COMPLEMENT OF BIPARTITE.

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