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De Morgan Heyting algebras satisfying the identity $x^{n(\prime *)} \approx x^{(n+1)(\prime *)}$

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In this paper we investigate the sequence of subvarieties $\mathcal{SDH}_n$ of De Morgan Heyting algebras characterized by the identity $x^{n(\prime *)} \approx x^{(n+1)(\prime *)}$. We obtain necessary and sufficient conditions for a De Morgan Heyting algebra to be in $\mathcal{SDH}_1$ by means of its space of prime filters, and we characterize subdirectly irreducible and simple algebras in $\mathcal{SDH}_1$. We extend these results for finite algebras in the general case $\mathcal{SDH}_n$.

1 Introduction and preliminaries

The De Morgan Heyting algebras were first studied by Monteiro in a paper of paramount importance titled *Sur les Algèbres de Heyting symétriques*, and they are the algebraic counterpart of the symmetric modal propositional calculus of Moisil (see [4, p. 60]). They were later deeply investigated by Sankappanavar in [5].

A *De Morgan Heyting algebra* is an algebra $\langle L, \wedge, \vee, \rightarrow, {}', 0, 1\rangle$ of type $(2, 2, 2, 1, 0, 0)$ such that the structure $\langle L, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a *Heyting algebra* and $\langle L, \wedge, \vee, {}', 0, 1\rangle$ is a *De Morgan algebra* (see [1]), that is, the following identities are satisfied:

(1) $x \wedge (x \rightarrow y) \approx x \wedge y$,
(2) $x \wedge (y \rightarrow z) \approx x \wedge ((x \wedge y) \rightarrow (x \wedge z))$,
(3) $(x \wedge y) \rightarrow x \approx 1$,
(4) $(x \vee y)' \approx x' \wedge y'$,
(5) $(x \wedge y)' \approx x' \vee y'$,
(6) $0' \approx 1$ and $1' \approx 0$,
(7) $x'' \approx x$.

We denote this variety by $\mathcal{DH}$.

In [5], Sankappanavar introduced a sequence $\mathcal{SDH}_n$ of subvarieties of the variety $\mathcal{DH}$, starting with $\mathcal{SDH}_0$ which is the variety of Boolean algebras, and defined within $\mathcal{DH}$ by the identity $x^{n(\prime *)} \approx x^{(n+1)(\prime *)}$. He characterizes the finite simple algebras in the variety $\mathcal{SDH}_1$ and he poses the problem of investigating the lattice of subvarieties of $\mathcal{SDH}_1$.

The present paper is devoted to the investigation of the sequence of subvarieties $\mathcal{SDH}_n$. We obtain necessary and sufficient conditions for an algebra $A \in \mathcal{DH}$ (finite or not) to be in $\mathcal{SDH}_1$ by means of its space of prime filters, and we characterize subdirectly irreducible and simple algebras in $\mathcal{SDH}_1$. Finally we extend these result for finite algebras in the general case $\mathcal{SDH}_n$.

The main tool we use in this paper is a duality between the category of De Morgan Heyting algebras and certain topological spaces, based on the duality developed by Priestley. Next we give a brief summary of the necessary required facts; for further information (see [7] and [8]).

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For a poset (partially ordered set) X and $P \subseteq X$, let $P^{\mathrm{d}} = \{x \in X : x \leq y \text{ for some } y \in P\}$ and $P^{\mathrm{i}} = \{x \in X : x \geq y \text{ for some } y \in P\}$. If $P = \{x\}$ we write x^{i} and x^{d} instead of $\{x\}^{\mathrm{i}}$ and $\{x\}^{\mathrm{d}}$ respectively. P is *decreasing* if $P = P^{\mathrm{d}}$ and P is *increasing* if $P = P^{\mathrm{i}}$.

A triple $(X; \leq, \tau)$ is a *totally order disconnected topological space* if $(X, \leq)$ is a poset, τ is a topology on X, and for $x, y \in X$, if $x \not\leq y$, then there exists a clopen increasing $V \subseteq X$ such that $x \in V$ and $y \notin V$. A *compact* totally order disconnected space is called a *Priestley space*.

In [6] and [7], Priestley showed that the category of bounded distributive lattices and $(0, 1)$–lattice homomorphisms is dually equivalent to the category of Priestley spaces and order preserving continuous functions (see also the survey paper [8]).

If X is a Priestley space, then $\langle \mathbb{D}(X), \cap, \cup, \emptyset, X\rangle$ is the lattice of clopen increasing subsets of X. If the function $f : X \longrightarrow X'$ is a continuous order preserving map, then the map $\mathbb{D}(f) : \mathbb{D}(X') \longrightarrow \mathbb{D}(X)$ defined by $\mathbb{D}(f)(V) = f^{-1}(V)$ is a $(0, 1)$–lattice homomorphism. Conversely, if L is a bounded distributive lattice, then the set of prime filters of L, denoted by $\mathbb{X}(L)$, is a Priestley space, ordered by set inclusion and with the topology having as a sub-basis the sets $\eta_L(a) = \{P \in \mathbb{X}(L) : a \in P\}$ and $\mathbb{X}(L) \setminus \eta_L(a)$ for $a \in L$. If $h : L \longrightarrow L'$ is a $(0, 1)$–lattice homomorphism, then $\mathbb{X}(h) : \mathbb{X}(L') \longrightarrow \mathbb{X}(L)$ defined by $\mathbb{X}(h)(P) = h^{-1}(P)$ is a continuous order preserving map. In addition, $\eta_L : L \longrightarrow \mathbb{D}(\mathbb{X}(L))$ is a lattice isomorphism, and $\varepsilon_X : X \longrightarrow \mathbb{X}(\mathbb{D}(X))$ defined by $\varepsilon_X(x) = \{V \in \mathbb{D}(X) : x \in V\}$ is a homeomorphism and an order isomorphism.

Since Heyting algebras are bounded distributive lattices, the category of Heyting algebras is isomorphic to a subcategory of bounded distributive lattices. A *Heyting space* is a Priestley space $(X, \leq, \tau)$ such that Y^{d} is clopen for every clopen $Y \subseteq X$. If X and X' are Heyting spaces, a (Heyting) morphism is a continuous order-preserving map $\varphi : X \longrightarrow X'$ for which $\varphi(x^{\mathrm{i}}) = \varphi(x)^{\mathrm{i}}$. For a Heyting algebra H and $a \in H$, $\eta(a) = V_a$ denotes the clopen increasing set that represents a. If $a, b \in H$, then, under the duality given above, $a \to b$ corresponds to the clopen increasing set $\big((V_a \cap V_b^{\mathrm{c}})^{\mathrm{d}}\big)^{\mathrm{c}}$, where Y^{c} denotes the complement of Y. As a consequence, Priestley's duality leads us to the following fact: the functors $\mathbb{X}$ and $\mathbb{D}$ establish a dual equivalence between the category of Heyting algebras and the category of Heyting spaces (see [8]).

If X is a Priestley space and $\varphi : X \longrightarrow X$ is an order reversing involutive ($\varphi = \varphi^{-1}$) homeomorphism, then (X, φ) is called a *De Morgan space* [3]. If $\langle M, \wedge, \vee, ', 0, 1\rangle$ is a De Morgan algebra and $\varphi : \mathbb{X}(M) \longrightarrow \mathbb{X}(M)$ is given by $\varphi(P) = P'^{\mathrm{c}}$, where $P' = \{a' \in M : a \in P\}$, then $(\mathbb{X}(M), \varphi)$ is a De Morgan space. φ is called the Birula-Rasiowa transformation. If $\eta(a) = V_a$ denotes the clopen increasing set that represents $a \in M$, then under the duality given above, a' corresponds to the clopen increasing set $\varphi(V_a)^{\mathrm{c}} = \mathbb{X}(M) \setminus \varphi(V_a)$. Conversely, if (X, φ) is a De Morgan space, $\langle \mathbb{D}(X), \cap, \cup, ', \emptyset, X\rangle$ is a De Morgan algebra where for $V \in \mathbb{D}(X)$, $V' = \varphi(V)^{\mathrm{c}}$. The category whose objects are De Morgan spaces with order preserving continuous functions $f : X_1 \longrightarrow X_2$ such that $f \circ \varphi_1 = \varphi_2 \circ f$ as morphisms is dually equivalent to the category of De Morgan algebras and (De Morgan) homomorphisms.

2 Characterization of algebras in $\mathcal{SDH}_1$

In this section we give necessary and sufficient conditions for a De Morgan Heyting algebra L to be in $\mathcal{SDH}_1$ by means of the space of its prime filters.

A *De Morgan Heyting space* is a system $(X; \leq, \tau, \varphi)$ which is both a Heyting space and a De Morgan space. Morphisms in the category of De Morgan Heyting spaces will be functions $f : X \longrightarrow X'$ which are morphisms in the category of Heyting spaces and in the category of De Morgan spaces.

From the observations at the end of Section 1, it is immediate that the category of De Morgan Heyting spaces and the category of De Morgan Heyting algebras are dually equivalent.

Here and subsequently, φ denotes the Birula-Rasiowa transformation.

Lemma 2.1 *Let L be a De Morgan Heyting algebra. For every $M \subseteq \mathbb{X}(L)$ the following properties hold:*

(1) $\varphi(M^{\mathrm{c}}) = \varphi(M)^{\mathrm{c}}$,
(2) $\varphi(M^{\mathrm{d}}) = \varphi(M)^{\mathrm{i}}$,
(3) $\varphi(M^{\mathrm{i}}) = \varphi(M)^{\mathrm{d}}$.

P r o o f. Let us prove (3). If $Q \in \varphi(M^{\mathrm{i}})$, then $Q = \varphi(R)$ with $P \subseteq R$ for some $P \in M$. Therefore $Q = \varphi(R) \subseteq \varphi(P)$, and consequently, $Q \in \varphi(M)^{\mathrm{d}}$. Conversely, if $Q \in \varphi(M)^{\mathrm{d}}$, then $Q \subseteq \varphi(P)$, with $P \in M$. So $P \subseteq \varphi(Q)$, that is, $\varphi(Q) \in M^{\mathrm{i}}$, or equivalently, $Q \in \varphi(M^{\mathrm{i}})$. □

Lemma 2.2 *A De Morgan Heyting algebra L belongs to $\mathcal{SDH}_1$ if and only if for every clopen increasing set $V \in \mathbb{X}(L)$, $V^{\mathrm{d}} = \varphi(V^{\mathrm{d}})^{\mathrm{d}}$.*

P r o o f. Let $L \in \mathcal{SDH}_1$. According to the definitions given above and using that the map $'$ is an involutive anti-isomorphism and that $\mathbb{D}(\mathbb{X}(L)) \cong L$ it follows that

$$V'^{*} = V^{2('*)} \quad \text{for every} \quad V \in \mathbb{D}(\mathbb{X}(L)) \quad \text{is equivalent to} \quad V^{\mathrm{dc}} = \varphi(V^{\mathrm{dc}})^{\mathrm{cdc}} \text{ for every } V \in \mathbb{D}(\mathbb{X}(L)).$$

Since $\varphi(V^{\mathrm{c}}) = \varphi(V)^{\mathrm{c}}$, the expression before can be written as

$$(1) \qquad V^{\mathrm{d}} = \varphi(V^{\mathrm{d}})^{\mathrm{d}}.$$

Taking into account that $\mathbb{X}(\mathbb{D}(X)) \cong X$, it is immediate that a De Morgan Heyting algebra associated with a De Morgan Heyting space (X, g) that satisfies $V^{\mathrm{d}} = \varphi(V^{\mathrm{d}})^{\mathrm{d}}$ with $V \in \mathbb{D}(X)$ belongs to $\mathcal{SDH}_1$, and conversely, the De Morgan Heyting space associated with $L \in \mathcal{SDH}_1$ satisfies the condition (1). □

The following results characterize the space associated with an algebra in $\mathcal{SDH}_1$.

Proposition 2.3 *Each $L \in \mathcal{SDH}_1$ satisfies:*

(1) *If $P \in \mathbb{X}(L)$, then there exists a unique ultrafilter U of L such that $P \subseteq U$.*

(2) *If $P \in \mathbb{X}(L)$ and $P \subseteq U$, where U is an ultrafilter of L, then $\varphi(P) \subseteq U$.*

P r o o f. Let $P \in \mathbb{X}(L)$. By Zorn's Lemma there exists a maximal element $U \in \mathbb{X}(L)$ such that $P \subseteq U$ and a minimal element M in $\mathbb{X}(L)$ such that $M \subseteq P$. Suppose that P is contained in two different ultrafilters, that is, $P \subseteq U_1$ and $P \subseteq U_2$, and $U_1 \neq U_2$. Let M be a minimal prime filter in $\mathbb{X}(L)$ such that $M \subseteq P$. Then $M \subseteq U_1$ and $M \subseteq U_2$. Since M is a minimal filter then $\varphi(M)$ is an ultrafilter. Therefore, $\varphi(M) \neq U_1$ or $\varphi(M) \neq U_2$. Without loss of generality we can assume that $\varphi(M) \neq U_1$. Then $U_1 \not\subseteq \varphi(M)$ and since $\mathbb{X}(L)$ is totally order-disconnected, there exists a clopen increasing set V in $\mathbb{X}(L)$ such that $U_1 \in V$ but $\varphi(M) \notin V$. Clearly $\varphi(M) \notin V^{\mathrm{d}}$, since $\varphi(M) \notin V$ and $\varphi(M)$ is an ultrafilter.

We will now show that $\varphi(M) \in \varphi(V^{\mathrm{d}})^{\mathrm{d}}$, contradicting (1). We have that $U_1 \in V$. So $\varphi\big(U_1^{\mathrm{d}}\big) \subseteq \varphi(V^{\mathrm{d}})$ and thus $\varphi\big(U_1^{\mathrm{d}}\big)^{\mathrm{d}} \subseteq \varphi(V^{\mathrm{d}})^{\mathrm{d}}$. In addition, $M \subseteq U_1$ implies $\varphi(M) \in (\varphi(U_1)^{\mathrm{i}})^{\mathrm{d}} = \varphi\big(U_1^{\mathrm{d}}\big)^{d}$, by Lemma 2.1. Hence $\varphi(M) \in \varphi(V^{\mathrm{d}})^{\mathrm{d}}$.

In order to prove (2) assume that P is a prime filter such that $P \subseteq U$, U an ultrafilter but $\varphi(P) \not\subseteq U$. We will show that there exists a clopen increasing set V in $\mathbb{X}(L)$ that does not verify (1), that is, $L \notin \mathcal{SDH}_1$. Since $\varphi(P) \not\subseteq U$, there exists a clopen increasing set V in $\mathbb{X}(L)$ such that $\varphi(P) \in V$, but $U \notin V$. Obviously, $\varphi(P) \in V^{\mathrm{d}}$. If we suppose that $\varphi(P) \in (\varphi(V^{\mathrm{d}}))^{\mathrm{d}}$, then $\varphi(P) \subseteq \varphi(Q)$ with $Q \subseteq R$ and $R \in V$. Therefore, we have $Q \subseteq P \subseteq U$ and $Q \subseteq R$. Since by (1) Q is contained in a unique ultrafilter U, then necessarily $R \subseteq U$. Consequently $U \in V$, which is impossible. This completes the proof. □

Proposition 2.4 *Let L be a De Morgan Heyting algebra with the following properties:*

1. *If $P \in \mathbb{X}(L)$, then there exists a unique ultrafilter U of L such that $P \subseteq U$.*
2. *If $P \in \mathbb{X}(L)$ and $P \subseteq U$, where U is an ultrafilter of L, then $\varphi(P) \subseteq U$.*

Then $L \in \mathcal{SDH}_1$.

P r o o f. Let L be a De Morgan Heyting algebra under the hypotheses of this proposition and let us prove that $V^{\mathrm{d}} = \varphi(V^{\mathrm{d}})^{\mathrm{d}}$ for every $V \in \mathbb{D}(\mathbb{X}(L))$.

Let $P \in V^{\mathrm{d}}$, for $P \in \mathbb{X}(L)$ and $V \in \mathbb{D}(\mathbb{X}(L))$. Then there exists $Q \in V$ such that $P \subseteq Q$. By condition 2., there exists a unique ultrafilter U satisfying $Q \subseteq U$ and $\varphi(Q) \subseteq U$. Since $Q \in V$ and $\varphi(Q) \subseteq U$, we have that $\varphi(U) \subseteq Q$, that is, $\varphi(U) \in V^{\mathrm{d}}$. In addition, since $P \subseteq Q \subseteq U$, we obtain that $P \subseteq U = \varphi(\varphi(U))$ and then $P \in \varphi(V^{\mathrm{d}})^{\mathrm{d}}$, which establishes $V^{\mathrm{d}} \subseteq \varphi(V^{\mathrm{d}})^{\mathrm{d}}$.

In order to prove the opposite inclusion, suppose that $P \in \varphi(V^{\mathrm{d}})^{\mathrm{d}}$. Then $P \subseteq \varphi(R)$, $R \in V^{\mathrm{d}}$, that is, $R \subseteq Q$ with $Q \in V$. Let U be the unique ultrafilter which contains Q and $\varphi(Q)$. By condition 2., it follows that $R \subseteq Q \subseteq U$ and $\varphi(R) \subseteq U$. Finally, $P \subseteq \varphi(R)$ implies $P \subseteq U$. But $U \in V$ since $Q \in V$ and V is increasing, so we have that $P \in V^{\mathrm{d}}$. □

3 Subdirectly irreducible and simple $\mathcal{SDH}_1$-algebras

In this section we give a characterization of subdirectly irreducible and simple algebras in $\mathcal{SDH}_1$ by means of its space of prime filters. This result was first proved in a different way by Sankappanavar in [5] and just for the finite case.

It is well known that there exists an anti-isomorphism between the lattice of congruences on a Heyting algebra H and the lattice of closed increasing sets in $\mathbb{X}(H)$ (see [8]).

Similarly, there exists an anti-isomorphism between the lattice of congruences on a De Morgan algebra M and the lattice of closed involutive sets in $(\mathbb{X}(M), \varphi)$, where a set Y is involutive if Y satisfies $\varphi(Y) = Y$ (see [3]).

Taking into account these results the following proposition is immediate.

Proposition 3.1 *Let L be a $\mathcal{DH}$-algebra. There exists an anti-isomorphism θ between the lattice of congruences on L and the lattice of closed increasing involutive sets in $\mathbb{X}(L)$ given by $Y \longmapsto \theta(Y)$, where*

$$(a, b) \in \theta(Y) \Longleftrightarrow ((\forall P \in Y)(a \in P \Leftrightarrow b \in P)).$$

Lemma 3.2 *Let L be an $\mathcal{SDH}_1$-algebra and U an ultrafilter in L. Then $Y = \varphi(U)^{\mathrm{i}}$ is an increasing closed and involutive set.*

Proof. Let U be an ultrafilter of L. Clearly $Y = \varphi(U)^{\mathrm{i}}$ is an increasing and closed set.

In order to prove that Y is an involutive set, we note that by Lemma 2.1, we have that

$$\varphi(Y) = Y \Leftrightarrow \varphi(\varphi(U)^{\mathrm{i}}) = \varphi(U)^{\mathrm{i}} \Leftrightarrow \varphi(\varphi(U))^{\mathrm{d}} = \varphi(U)^{\mathrm{i}} \Leftrightarrow U^{\mathrm{d}} = \varphi(U)^{\mathrm{i}}.$$

Suppose that $P \in \varphi(U)^{\mathrm{i}}$, then $\varphi(U) \subseteq P$, that is, $\varphi(P) \subseteq U$. Hence by condition 2. of Proposition 2.4 we have $\varphi(\varphi(P)) = P \subseteq U$. Therefore $P \in U^{\mathrm{d}}$.

If $P \in U^{\mathrm{d}}$, $P \subseteq U$, and again by condition 2. of Proposition 2.4 we obtain $\varphi(U) \subseteq P$, that is $P \in \varphi(U)^{\mathrm{i}}$. □

Theorem 3.3 *Let L be an algebra in $\mathcal{SDH}_1$. If L is subdirectly irreducible, then L has a unique ultrafilter.*

Proof. Let $\{U_i\}_{i \in I}$ be the set of all ultrafilters in L. By Lemma 3.2 we know that $Y_i = \varphi(U_i)^{\mathrm{i}}$ is a closed increasing and involutive set for each $i \in I$. Then, according to Proposition 3.1, for each Y_i there exists a congruence $\theta(Y_i)$ on L.

We consider the set

$$Y = \textstyle\bigcup_{i \in I} Y_i.$$

Observe that $Y = \mathbb{X}(L)$. Then Y is a closed increasing and involutive set and $\theta(Y) = \Delta$, where Δ denotes the identity congruence on L.

Finally, since θ is an anti-isomorphism, we have that

$$\theta(\textstyle\bigcup_{i \in I} Y_i) = \bigcap_{i \in I} \theta(Y_i) = \Delta.$$

This contradicts our assumption that L is subdirectly irreducible. □

Theorem 3.4 *If an algebra $L \in \mathcal{SDH}_1$ has a unique ultrafilter, then L is simple.*

Proof. Let L be an algebra in $\mathcal{SDH}_1$ and suppose that L has only one ultrafilter U. Suppose that θ is a congruence on L and $\theta \neq \nabla$, where ∇ denotes the greatest element in the lattice of congruences on L. Then, by Proposition 3.1, there exists a closed increasing and involutive set $Y \subseteq \mathbb{X}(L)$, such that $\theta(Y) = \theta$. Let us see that $Y = \mathbb{X}(L)$.

Given $Q \in \mathbb{X}(L)$, by Proposition 2.4, $Q, \varphi(Q) \subseteq U$, that is, $\varphi(U) \subseteq Q$. Therefore, taking into account that Y is an involutive increasing set, it follows that $\varphi(U) \in Y$, and hence $Q \in Y$. This shows that $Y = \mathbb{X}(L)$ and consequently, $\theta = \Delta$, which completes the proof. □

Corollary 3.5 *$L \in \mathcal{SDH}_1$ is subdirectly irreducible if and only if L is simple if and only if L has a unique ultrafilter.*

Corollary 3.6 *A finite algebra $L \in \mathcal{SDH}_1$ is subdirectly irreducible if and only if L has a unique atom.*

4 Application: Generation by finite members

In this section, we shall apply the characterization of subdirectly irreducible $\mathcal{SDH}_1$-algebras given in the previous section to show that the variety $\mathcal{SDH}_1$ is generated by its finite members. The proof proceeds through a standard filtration argument.

Lemma 4.1 *If $\langle L, \wedge, \vee, ', \rightarrow, 0, 1\rangle$ is a simple $\mathcal{SDH}_1$-algebra and $N = \{e_1, \dots, e_n\}$ a finite set contained in L, then there exists a finite part L_1 of L and an operation $\rightarrow_1$ with the following properties:*

(a) $\langle L_1, \wedge, \vee, ', \rightarrow_1, 0, 1\rangle$ *is a finite simple $\mathcal{SDH}_1$-algebra,*

(b) $N \subseteq L_1$,

(c) *if $a, b \in L_1$ and $a \rightarrow b \in L_1$, then $a \rightarrow_1 b = a \rightarrow b$.*

Proof. Let L_1 be the De Morgan algebra generated by N. Of course, $N \subseteq L_1$, and since the variety of De Morgan algebras is locally finite, L_1 is finite. Thus, it is possible to define an implication $\rightarrow_1$ on L_1 such that $\langle L_1, \wedge, \vee, ', \rightarrow_1, 0, 1\rangle$ is a De Morgan Heyting algebra. Let us see that $L_1 \in \mathcal{SDH}_1$.

Since L is simple, L has only one ultrafilter U. We will prove that L_1 has only one ultrafilter too, that is, L_1 has a unique atom. Suppose that there exist two distinct atoms a and b in L_1 and let $F(a)$ and $F(b)$ denote the filters generated by a and b in L_1. Consider the filters of L: $U_a = \{x \in L : x \geq a\}$ and $U_b = \{x \in L : x \geq b\}$. U_a and U_b are contained in the unique ultrafilter U in L, so $a, b \in U$. Then we have that $a \wedge b = 0 \in U$. This contradicts the fact that U is an ultrafilter. Therefore, L_1 has a unique atom and, by Proposition 2.4 we can conclude that L_1 is a simple $\mathcal{SDH}_1$-algebra.

In order to prove (c), let $\rightarrow_1$ be the implication defined on L_1 and suppose that $a, b \in L_1$ and $a \rightarrow b \in L_1$.

Recall that the operations $\rightarrow$ and $\rightarrow_1$ have the following properties:

(I) If $a, b \in L$, then for all $x \in L$, the conditions $a \wedge x \leq b$ and $x \leq a \rightarrow b$ are equivalent.

(I_1) If $a, b \in L_1$, then for all $x \in L_1$ the conditions $a \wedge x \leq b$ and $x \leq a \rightarrow_1 b$ are equivalent.

Since $a \rightarrow b \leq a \rightarrow b$, by (I), we have $a \wedge (a \rightarrow b) \leq b$. By (I_1) and $a \rightarrow b \in L_1$ we have $a \rightarrow b \leq a \rightarrow_1 b$.

A similar argument shows that $a \rightarrow_1 b \leq a \rightarrow b$. Thus, $a \rightarrow_1 b = a \rightarrow b$. □

Theorem 4.2 *The variety $\mathcal{SDH}_1$ is generated by its finite members.*

Proof. Suppose we have a term ψ such that $\mathcal{S} \vDash \psi \approx 1$ but $\mathcal{SDH}_1 \nvDash \psi \approx 1$ where $\mathcal{S}$ is the set of all finite and simple algebras in $\mathcal{SDH}_1$. Then there exists a simple algebra $L \in \mathcal{SDH}_1$ with $L \nvDash \psi \approx 1$. Let $\Sigma(\psi)$ be the set of all subterms of ψ, and let $\overline{a} = (a_1, \dots, a_n) \in L^n$ be such that $\psi^L(\overline{a}) \neq 1$. Further, let N stand for $\{\sigma(\overline{a}) : \sigma \in \Sigma(\psi)\}$ and let L_1 be the finite simple algebra in $\mathcal{SDH}_1$ of Lemma 4.1. Then we have that $L_1 \nvDash \psi \approx 1$. This contradicts our assumption. □

5 Subdirectly irreducible algebras in $\mathcal{SDH}_n$

In this section we characterize the spaces of prime filters of finite subdirectly irreducible algebras in $\mathcal{SDH}_n$, for every $n < \omega$.

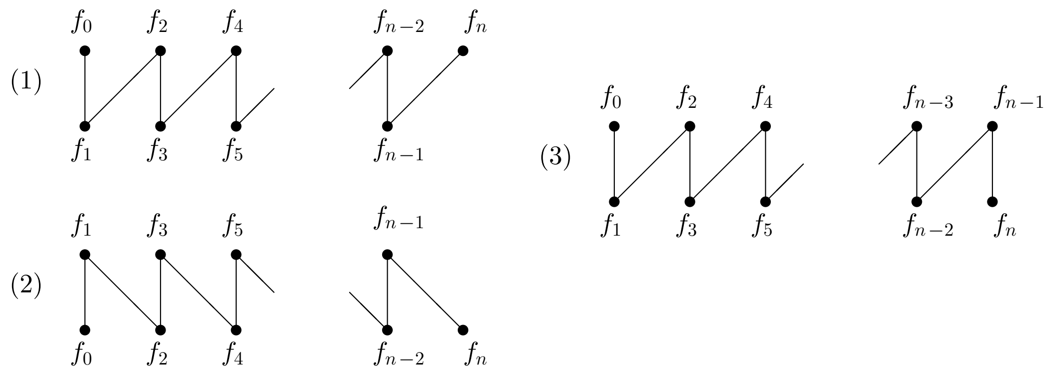
Let R be a poset. Recall that an $(n+1)$-*fence* is a poset $F = \{f_0, \dots, f_n\} \subseteq R$ such that

$$f_0 > f_1, f_1 < f_2, f_2 > f_3, \dots, f_{n-1} < f_n \quad \text{or} \quad f_0 < f_1, f_1 > f_2, f_2 < f_3, \dots, f_{n-1} > f_n$$

if n is even, respectively

$$f_0 < f_1, f_1 > f_2, f_2 < f_3, \dots, f_{n-1} < f_n \quad \text{or} \quad f_0 > f_1, f_1 < f_2, f_2 > f_3, \dots, f_{n-1} > f_n$$

if n is odd, and such that these are all comparabilities between the points. The length of the fence is n. The points f_0 and f_n are called endpoints of the fence (see [2] and [9]).



Pictures of fences

Fences (1) and (2) are the two nonisomorphic fences with an odd number of elements, (3) is the unique (up to isomorphism) fence with an even number of elements.

For a given poset R, let $\mathrm{Max}(R)$ and $\mathrm{Min}(R)$ respectively denote the set of maximal and minimal elements of R. Let $\mathbb{F}^{U,n}$ denotes the set-theoretical union of all fences in $\mathrm{Max}(R) \cup \mathrm{Min}(R)$ of length n with starting point $U \in \mathrm{Max}(R)$, that is, $\mathbb{F}^{U,n}$ consists of those $P \in \mathrm{Max}(R) \cup \mathrm{Min}(R)$ such that there exists a fence of length n with starting point U and containing P.

If L is an algebra in $\mathcal{DH}$ and U is an ultrafilter of L, consider the following fences of length n with starting point U in the poset $\mathrm{Max}(\mathbb{X}(L)) \cup \mathrm{Min}(\mathbb{X}(L))$:

$$\{U_0 = U, M_1, U_2, M_3, U_4, M_5, \ldots, M_{n-1}, U_n\} \quad \text{or} \quad \{U_0 = U, M_1, U_2, M_3, U_4, M_5, \ldots, U_{n-1}, M_n\}$$

where $U_i \in \mathrm{Max}(\mathbb{X}(L))$, $M_i \in \mathrm{Min}(\mathbb{X}(L))$.

In the first of these two fences n is even and its picture is as (1) in the figure and in the second one, n is odd and its picture looks like (3). We will also consider $\{U_0\}$ as a fence of length 0.

Recall that by means of the isomorphism between $L \in \mathcal{DH}$ and $\mathbb{D}(\mathbb{X}(L))$, the operations $'$ and * on L have translations as $V' = \varphi(V)^{\mathrm{c}}$ and $V^* = (V^{\mathrm{d}})^{\mathrm{c}}$, for $V \in \mathbb{D}(\mathbb{X}(L))$.

For every subset V of a poset R, we inductively define $V^{0(\mathrm{di})} = V$ and $V^{(n+1)(\mathrm{di})} = (V^{n(\mathrm{di})})^{\mathrm{di}}$.

Lemma 5.1 *For every $V \in \mathbb{D}(\mathbb{X}(L))$, we have*

$$V^{k(*\prime)} = \begin{cases} \varphi\big(V^{\frac{k-1}{2}(\mathrm{di})\mathrm{d}}\big) & \text{if } k \text{ is odd,} \\ V^{\frac{k}{2}(\mathrm{di})} & \text{if } k \text{ is even.} \end{cases}$$

P r o o f. The proof is by induction on k. If $k = 1$, by Lemma 2.1 we have $V^{*\prime} = \varphi(V^{\mathrm{dc}})^{\mathrm{c}} = \varphi(V^{\mathrm{d}})$. Suppose that the property holds for k. If k is even, then $V^{k(*\prime)} = V^{\frac{k}{2}(\mathrm{di})}$. So

$$V^{(k+1)(*\prime)} = \big(V^{\frac{k}{2}(\mathrm{di})}\big)^{*\prime} = \varphi\big(\big(V^{\frac{k}{2}(\mathrm{di})}\big)^{\mathrm{d}}\big).$$

If k is odd, $V^{k(*\prime)} = \varphi(V^{\frac{k-1}{2}(\mathrm{di})\mathrm{d}})$, thus

$$V^{(k+1)(*\prime)} = \big(\varphi\big(V^{\frac{k-1}{2}(\mathrm{di})\mathrm{d}}\big)\big)^{*\prime} = \varphi\big(\varphi\big(V^{\frac{k-1}{2}(\mathrm{di})\mathrm{d}}\big)^{\mathrm{d}}\big) = \big(\varphi\big(\varphi\big(V^{\frac{k-1}{2}(\mathrm{di})\mathrm{d}}\big)\big)\big)^{\mathrm{i}} = V^{\frac{k-1}{2}(\mathrm{di})\mathrm{di}} = V^{\frac{k+1}{2}(\mathrm{di})}.$$

So the property holds for every k. □

Lemma 5.2 *Let $L \in \mathcal{DH}$. Then $L \in \mathcal{SDH}_n$ if and only if for every $V \in \mathbb{D}(\mathbb{X}(L))$ the following condition holds:*

$$\varphi\big(V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}}\big) = V^{\frac{n+1}{2}(\mathrm{di})} \text{ if } n \text{ is odd,} \quad \text{and} \quad \varphi\big(V^{\frac{n}{2}(\mathrm{di})}\big) = V^{\frac{n}{2}(\mathrm{di})\mathrm{d}} \text{ if } n \text{ is even.}$$

P r o o f. We know that $L \in \mathcal{SDH}_n$ if and only if in $\mathbb{D}(\mathbb{X}(L))$, $V^{n('*)} = V^{(n+1)('*)}$. Taking into account that $'$ is an involution, this equality is equivalent to:

$$V^{(n-1)(*')*} = V^{n(*')*}. \tag{2}$$

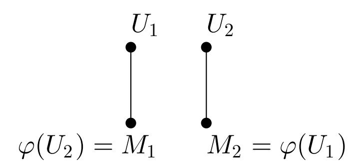
By the previous lemma and the properties of φ we obtain:

If n is even, $V^{(n-1)(*')*} = \varphi\big(V^{\frac{n-2}{2}(\mathrm{di})\mathrm{d}}\big)^{\mathrm{dc}} = \varphi\big(V^{\frac{n-2}{2}(\mathrm{di})\mathrm{di}}\big)^{\mathrm{c}} = \varphi\big(V^{\frac{n}{2}(\mathrm{di})}\big)^{\mathrm{c}}$, and $V^{n(*')*} = \big(V^{\frac{n}{2}(\mathrm{di})}\big)^{\mathrm{dc}}$, so by (2) we get $\varphi\big(V^{\frac{n}{2}(\mathrm{di})}\big) = V^{\frac{n}{2}(\mathrm{di})\mathrm{d}}$.

If n is odd, $V^{(n-1)(*')*} = \big(V^{\frac{n-1}{2}(\mathrm{di})}\big)^{\mathrm{dc}}$, and $V^{n(*')*} = \varphi\big(V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}}\big)^{\mathrm{dc}} = \varphi\big(V^{\frac{n-1}{2}(\mathrm{di})\mathrm{di}}\big)^{\mathrm{c}} = \varphi\big(V^{\frac{n+1}{2}(\mathrm{di})}\big)^{\mathrm{c}}$. Thus by (2), $\varphi\big(V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}}\big) = V^{\frac{n+1}{2}(\mathrm{di})}$.

□

Using the above result it is easy to see that there exist finite $\mathcal{DH}$-algebras that are not $\mathcal{SDH}_n$-algebras for any $n < \omega$. For example, the algebra L whose De Morgan Heyting space is



does not belong to $\mathcal{SDH}_n$ for any $n < \omega$.

In order to give a characterization of finite subdirectly irreducible algebras in $\mathcal{SDH}_n$ we will use the following results whose proofs can be found in [4].

Theorem 5.3 *Let L be a finite De Morgan algebra and let $(\mathbb{X}(L), \varphi)$ be its associated De Morgan space. Let $S_1 + S_2 + \cdots + S_n$ be the decomposition of $\mathbb{X}(L)$ in connected components. Then if $P \in S_i$ and $\varphi(P) \in S_j$ we have $\varphi(S_i) \subseteq S_j$.*

Let $\mathbb{X}(L) = S_1 + S_2 + \cdots + S_n$ be the decomposition of $\mathbb{X}(L)$ in connected components. If $\varphi(S_i) \subseteq S_j$, then it is easily seen that $\varphi(S_i) = S_j$ and $\varphi(S_j) = S_i$, and consequently, $\varphi(S_i + S_j) = S_i + S_j$. In that case we say that $S_i + S_j$ is a *φ-connected component* of $\mathbb{X}(L)$ when $i \neq j$. If $\varphi(S_i) = S_i$, then S_i is a φ-connected component of $\mathbb{X}(L)$. We say that $\mathbb{X}(L)$ is *φ-connected* when $\mathbb{X}(L)$ is connected or $\mathbb{X}(L) = S_i + S_j$, with $\varphi(S_i) \subseteq S_j$, $i \neq j$.

Theorem 5.4 *Let L be a finite De Morgan algebra. L is directly indecomposable if and only if $\mathbb{X}(L)$ is φ-connected.*

For finite algebras in $\mathcal{SDH}_n$ we have the following stronger results.

Lemma 5.5 *Let $L \in \mathcal{SDH}_n$ a finite algebra. If $\mathbb{X}(L)$ is φ-connected, then $\mathbb{X}(L)$ is connected.*

P r o o f. Suppose that $\mathbb{X}(L)$ is φ-connected and that $\mathbb{X}(L) = S_1 + S_2$, where S_1 and S_2 are nonempty connected components such that $\varphi(S_1) = S_2$. Let $U \in S_1$ such that U is an ultrafilter of L and let us consider the clopen increasing set $V = \{U\}$. Then we have $V^{m(\mathrm{di})} \subseteq S_1$ for all m, however $\varphi\big(V^{m(\mathrm{di})}\big) \subseteq S_2$. This shows that $\mathbb{D}(\mathbb{X}(L))$ is not an $\mathcal{SDH}_n-$algebra for any n, a contradiction. Therefore, $\mathbb{X}(L) = S$, where S is connected. □

Proposition 5.6 *Let L be a finite algebra in $\mathcal{SDH}_n$. Then L is directly indecomposable if and only if $\mathbb{X}(L)$ is connected.*

P r o o f. Let L be a directly indecomposable algebra in $\mathcal{SDH}_n$ and suppose that $\mathbb{X}(L) = S_1 + S_2 + \cdots + S_r$, where S_i is a connected component for all i. By Lemma 5.5, $\varphi(S_i) \subseteq S_i$ for all i. Let L_i be the De Morgan Heyting algebra associated with each S_i. We know that $L = L_1 \times \cdots \times L_n$, where L is considered as a $\mathcal{DH}$-algebra. Since L_i is a finite algebra, L_i is a Heyting algebra. In addition, since $L \in \mathcal{SDH}_n$, we obtain that $L_i \in \mathcal{SDH}_n$. Then, L is not a directly indecomposable algebra, a contradiction.

On the other hand, suppose that $L \in \mathcal{SDH}_n$ and $\mathbb{X}(L)$ is a connected space. Therefore $\mathbb{X}(L)$ is φ-connected, and consequently, L is a directly indecomposable De Morgan algebra, so it is a directly indecomposable $\mathcal{SDH}_n$-algebra. □

Proposition 5.7 *Let L be a finite algebra in $\mathcal{DH}$. If $\mathbb{X}(L)$ is connected, then L is simple.*

P r o o f. Suppose that L is not a simple algebra. Then there exists a not trivial congruence θ on L. By Proposition 3.1, there exists an increasing and closed set $Y \subseteq \mathbb{X}(L)$ such that $\theta(Y) = \theta$. Taking into account that $\theta \neq \Delta$ and $\theta \neq \nabla$ we have that $Y \neq \mathbb{X}(L)$ and $Y \neq \emptyset$. So Y is an increasing and involutive clopen set, that is, $\mathbb{X}(L)$ is not a connected component. □

Proposition 5.8 *Let L be a finite algebra in $\mathcal{SDH}_n$. If L is subdirectly irreducible, then $\mathbb{X}(L)$ is connected.*

P r o o f. Let L be a subdirectly irreducible algebra in $\mathcal{SDH}_n$ and suppose that $\mathbb{X}(L) = S_1 + S_2 + \cdots + S_r$, where S_i is a connected component for all i. Clearly, S_i is an increasing and closed subset. Moreover, by Lemma 5.5, S_i is an involutive subset for all i. Then, according to Proposition 3.1, for each S_i, there exists a congruence $\theta(S_i)$ on L.

We have that $\mathbb{X}(L) = \bigcup_{i\in I} S_i$ and $\theta(\mathbb{X}(L)) = \Delta$. Since θ is an anti-isomorphism, it follows that:

$$\theta(\textstyle\bigcup_{i\in I} S_i) = \bigcap_{i\in I} \theta(S_i) = \Delta.$$

This contradicts our assumption that L is subdirectly irreducible. □

Corollary 5.9 *Let L be a finite $\mathcal{SDH}_n$-algebra. L is subdirectly irreducible if and only if $\mathbb{X}(L)$ is connected.*

In what follows, given a poset R, let $\overline{R}$ denote the set $\mathrm{Max}(R) \cup \mathrm{Min}(R)$.

Lemma 5.10 *Let R be a poset, $U \in \mathrm{Max}(R)$ and consider $V = \{U\}$. Then*

$$\overline{V^{n(\mathrm{di})}} = \textstyle\bigcup_{i=0}^{2n} \mathbb{F}^{U,i}.$$

P r o o f. The proof is by induction on n. For $n = 0$, $\overline{V^{0(\mathrm{di})}} = \{U\}$ and $\bigcup_{i=0}^{0} \mathbb{F}^{U,i} = \mathbb{F}^{U,0} = \{U\}$. Suppose that $\overline{V^{n(\mathrm{di})}} = \bigcup_{i=0}^{2n} \mathbb{F}^{U,i}$ and let us prove that $\overline{V^{(n+1)(\mathrm{di})}} = \bigcup_{i=0}^{2(n+1)} \mathbb{F}^{U,i}$. Let $P \in \overline{V^{(n+1)(\mathrm{di})}} = \overline{(V^{n(\mathrm{di})})^{\mathrm{di}}}$. If P is maximal, then there exists $M \in \overline{V^{n(\mathrm{di})\mathrm{d}}}$, M minimal such that $M \subseteq P$. So there exists $S \in \overline{V^{n(\mathrm{di})}}$, S maximal such that $M \subseteq S$. Since by hypothesis $\overline{V^{n(\mathrm{di})}} = \bigcup_{i=0}^{2n} \mathbb{F}^{U,i}$, there exists a fence of length $t \leq 2n$ of the form $UM_1U_2M_3 \ldots M_{t-1}U_t = S$. Consequently the sequence $UM_1U_2M_3 \ldots M_{t-1}SMP$ is a fence of length $t + 2 \leq 2n + 2 = 2(n + 1)$. Thus $P \in \bigcup_{i=0}^{2(n+1)} \mathbb{F}^{U,i}$. If P is minimal, then $P \in \overline{V^{n(\mathrm{di})\mathrm{d}}}$, that is, there exists $S \in \overline{V^{n(\mathrm{di})}}$, S maximal such that $P \subseteq S$. Since $\overline{V^{n(\mathrm{di})}} = \bigcup_{i=0}^{2n} \mathbb{F}^{U,i}$, there exists a fence of length $t \leq 2n$ of the form $UM_1U_2M_3 \ldots M_{t-1}U_t = S$. Then the sequence $UM_1U_2M_3 \ldots M_{t-1}SP$ is a fence of length $t + 1 \leq 2(n + 1)$. Thus $P \in \bigcup_{i=0}^{2(n+1)} \mathbb{F}^{U,i}$, that is, $\overline{V^{(n+1)(\mathrm{di})}} \subseteq \bigcup_{i=0}^{2(n+1)} \mathbb{F}^{U,i}$.

For the opposite inclusion, suppose that $P \in \bigcup_{i=0}^{2(n+1)} \mathbb{F}^{U,i}$. If P is maximal, then there exists a fence of the form $UM_1U_2M_3 \ldots U_{t-2}M_{t-1}U_t = P$ with $t \leq 2n + 2$. Hence $UM_1U_2M_3 \ldots U_{t-2}$ is a sequence of length $t - 2 \leq 2n$. By hypothesis $U_{t-2} \in \overline{V^{n(\mathrm{di})}}$, which implies that $M_{t-1} \in \overline{V^{n(\mathrm{di})\mathrm{d}}}$ and consequently $U_t = P \in \overline{V^{(n+1)(\mathrm{di})}}$. If P is minimal, then there exists a fence of the form $UM_1U_2M_3 \ldots U_{t-1}M_t = P$, with $t \leq 2(n+1)$. Now, t is odd since the sequence ends in a minimal element, so $t < 2(n+1)$ and then, the sequence $UM_1U_2M_3 \ldots U_{t-1}$ is a fence of length $t \leq 2n$. By hypothesis we have that $U_{t-1} \in \overline{V^{n(\mathrm{di})}}$ and since $P \subseteq U_{t-1}$ it follows that $P \in \overline{V^{n(\mathrm{di})\mathrm{d}}} \subseteq \overline{V^{(n+1)(\mathrm{di})}}$. □

The following proposition gives necessary and sufficient conditions on $\mathbb{X}(L)$ for a finite algebra $L \in \mathcal{DH}$ to be in $\mathcal{SDH}_n$.

Theorem 5.11 *Let L be a finite algebra in $\mathcal{DH}$ whose space $\mathbb{X}(L)$ is connected. Then $L \in \mathcal{SDH}_n$ if and only if for every ultrafilter U of L*

$$\mathrm{Max}(\mathbb{X}(L)) \subseteq \textstyle\bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}, \text{ if } n \text{ is odd,} \quad \text{and} \quad \mathrm{Min}(\mathbb{X}(L)) \subseteq \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}, \text{ if } n \text{ is even.}$$

P r o o f. Let L be a finite algebra in $\mathcal{DH}$ whose space $\mathbb{X}(L)$ is connected. Suppose that $L \in \mathcal{SDH}_n$. Observe that since $\mathbb{X}(L)$ is finite and connected, if W is a decreasing subset of $\mathbb{X}(L)$ and $W \neq \mathbb{X}(L)$, then $|W| < |W^{\mathrm{i}}|$, and consequently, $|\varphi(W)| < |W^{\mathrm{i}}|$.

So if n is odd, U an ultrafilter and we consider the clopen increasing subset $V = \{U\}$, then $V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}} = \mathbb{X}(L)$ since otherwise we would have $\varphi\big(V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}}\big) \neq V^{\frac{n+1}{2}(\mathrm{di})}$, a contradiction as $L \in \mathcal{SDH}_n$ (Lemma 5.2). If we suppose that there exists a maximal U_1 such that $U_1 \notin \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}$, then by Lemma 5.10, $U_1 \notin \overline{V^{\frac{n-1}{2}(\mathrm{di})}}$, that is, $U_1 \notin \overline{V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}}}$ which is a contradiction. Hence, $\mathrm{Max}(\mathbb{X}(L)) \subseteq \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}$

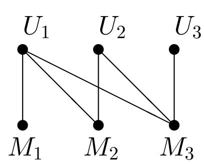
Similarly, now suppose n is even, U an ultrafilter and we consider the clopen increasing subset $V = \{U\}$, then $V^{\frac{n}{2}(\mathrm{di})} = \mathbb{X}(L)$ since otherwise we would have $\varphi\big(V^{\frac{n}{2}(\mathrm{di})}\big) \neq V^{\frac{n}{2}(\mathrm{di})\mathrm{d}}$, a contradiction as $L \in \mathcal{SDH}_n$ (Lemma 5.2). If we suppose that there exists $\varphi(U_0) \in \mathrm{Min}(\mathbb{X}(L))$ such that $\varphi(U_0) \notin \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}$, then $\varphi(U_0) \notin \bigcup_{i=0}^{n} \mathbb{F}^{U,i}$ since n is even and by Lemma 5.10 it follows that $\varphi(U_0) \notin \overline{V^{\frac{n}{2}(\mathrm{di})}}$, a contradiction. Hence, $\mathrm{Min}(\mathbb{X}(L)) \subseteq \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}$.

For the converse, suppose that $\mathrm{Max}(\mathbb{X}(L)) \subseteq \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}$ if n is odd and $\mathrm{Min}(\mathbb{X}(L)) \subseteq \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}$ if n is even. Let V be a clopen increasing nonempty subset of $\mathbb{X}(L)$ and let $U \in V$, U an ultrafilter. Consider the clopen increasing subset $V' = \{U\}$.

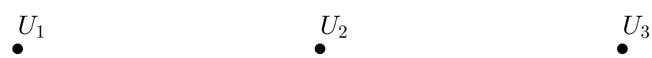
If n is odd, by Lemma 5.10 we have that $\overline{V'^{\frac{n-1}{2}(\mathrm{di})}} = \bigcup_{i=1}^{n-1} \mathbb{F}^{U,i}$, and $\mathrm{Max}(\mathbb{X}(L)) \subseteq \overline{V'^{\frac{n-1}{2}(\mathrm{di})}}$, by hypothesis. Since $V' \subseteq V$ we have $V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}} = \mathbb{X}(L)$. So $\varphi(V^{\frac{n-1}{2}(\mathrm{di})\mathrm{d}}) = V^{\frac{n+1}{2}(\mathrm{di})}$, consequently, $L \in \mathcal{SDH}_n$.

If n is even, then $\overline{V'^{\frac{n}{2}(\mathrm{di})}} = \bigcup_{j=0}^{n} \mathbb{F}^{U,j}$, and by hypothesis we have $\mathrm{Min}(\mathbb{X}(L)) \subseteq \bigcup_{i=0}^{n-1} \mathbb{F}^{U,i}$. Therefore $\mathrm{Max}(\mathbb{X}(L)) \cup \mathrm{Min}(\mathbb{X}(L)) \subseteq \bigcup_{j=0}^{n} \mathbb{F}^{U,j}$, that is, $V^{\frac{n}{2}(\mathrm{di})} = \mathbb{X}(L)$. So $\varphi\big(V^{\frac{n}{2}(\mathrm{di})}\big) = V^{\frac{n}{2}(\mathrm{di})\mathrm{d}}$, and then L is an element of $\mathcal{SDH}_n$. □

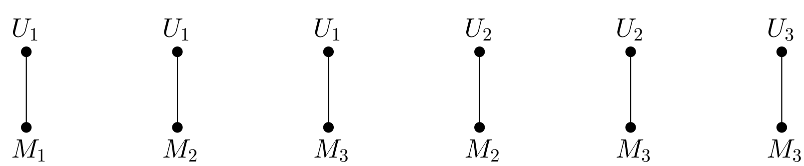
For example, let L be a De Morgan Heyting algebra with space $\mathbb{X}(L)$ such that $\mathrm{Max}(\mathbb{X}(L)) \cup \mathrm{Min}(\mathbb{X}(L))$ is given by the following figure:



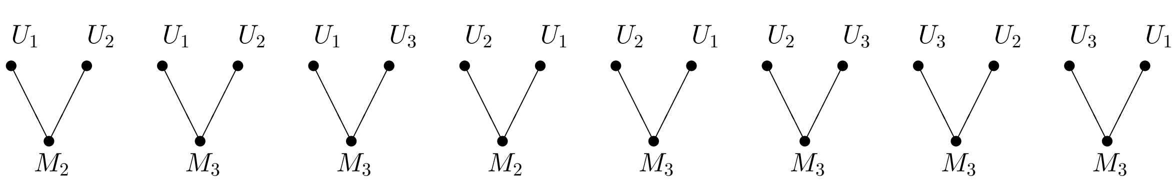
Fences of length 0:



Fences of length 1:



Fences of length 2:



$$\mathbb{F}^{U_1,0} = \{U_1\}, \quad \mathbb{F}^{U_1,1} = \{U_1, M_1, M_2, M_3\}, \quad \mathbb{F}^{U_1,2} = \{U_1, U_2, U_3, M_2, M_3\}$$
$$\mathbb{F}^{U_2,0} = \{U_2\}, \quad \mathbb{F}^{U_2,1} = \{U_2, M_2, M_3\}, \quad \mathbb{F}^{U_2,2} = \{U_1, U_2, U_3, M_2, M_3\}$$
$$\mathbb{F}^{U_3,0} = \{U_3\}, \quad \mathbb{F}^{U_3,1} = \{U_3, M_3\}, \quad \mathbb{F}^{U_3,2} = \{U_1, U_2, U_3, M_3\}$$

Then

$$\bigcup_{i=0}^{0} \mathbb{F}^{U_1,i} = \{U_1\}, \quad \bigcup_{i=0}^{1} \mathbb{F}^{U_3,i} = \{U_3, M_3\}, \quad \bigcup_{i=0}^{2} \mathbb{F}^{U_1,i} = \overline{\mathbb{X}(L)},$$

$$\bigcup_{i=0}^{2} \mathbb{F}^{U_2,i} = \{U_1, U_2, U_3, M_2, M_3\}, \qquad \bigcup_{i=0}^{2} \mathbb{F}^{U_3,i} = \{U_1, U_2, U_3, M_3\}.$$

Therefore $L \notin \mathcal{SDH}_1$, $L \notin \mathcal{SDH}_2$ but $L \in \mathcal{SDH}_3$, and consequently, $L \in \mathcal{SDH}_n$ for $n \geq 3$.

Corollaries 5.12 and 5.13 provide a quite simple characterization for finite subdirectly irreducible algebras in $\mathcal{SDH}_2$ and $\mathcal{SDH}_3$.

Corollary 5.12 *Let L be a finite algebra in $\mathcal{DH}$ such that $\mathbb{X}(L)$ is connected. Then $L \in \mathcal{SDH}_2$ if and only if $\mathrm{Max}(\mathbb{X}(L)) \cup \mathrm{Min}(\mathbb{X}(L))$ is a complete bipartite graph, that is, $\varphi(U_1) \subseteq U_2$ for all ultrafilters U_1, U_2 of L.*

Proof. Suppose that $L \in \mathcal{SDH}_2$ and let U_1, U_2 ultrafilters of L. We know that $\mathrm{Min}(\mathbb{X}(L)) \subseteq \mathbb{F}^{U_2,0} \cup \mathbb{F}^{U_2,1}$ by Theorem 5.11. Since $\varphi(U_1) \in \mathrm{Min}(\mathbb{X}(L))$, we have $\varphi(U_1) \in \mathbb{F}^{U_2,0}$ or $\varphi(U_1) \in \mathbb{F}^{U_2,1}$, that is, $\varphi(U_1) \in \mathbb{F}^{U_2,1}$. Therefore, $\varphi(U_1) \subseteq U_2$.

Conversely, let U_2 be an ultrafilter of L and $M \in \mathrm{Min}(\mathbb{X}(L))$. Then $M = \varphi(U_1)$ where U_1 is an ultrafilter. By hypothesis, $\varphi(U_1) \subseteq U_2$. This shows that $\varphi(U_1) \in \mathbb{F}^{U_2,1}$, so $\mathrm{Min}(\mathbb{X}(L)) \subseteq \mathbb{F}^{U_2,0} \cup \mathbb{F}^{U_2,1}$. □

Corollary 5.13 *Let L be a finite algebra in $\mathcal{DH}$ whose dual space is connected. Then $L \in \mathcal{SDH}_3$ if and only if for every pair $U_1, U_2 \in \mathrm{Max}(\mathbb{X}(L))$ there exists $M \in \mathrm{Min}(\mathbb{X}(L))$ such that $M \subseteq U_1$ and $M \subseteq U_2$.*

Proof. Suppose that $L \in \mathcal{SDH}_3$ and let U_1, U_2 distinct ultrafilters of L. By Theorem 5.11, $\mathrm{Max}(\mathbb{X}(L))$ is a subset of $\mathbb{F}^{U_2,0} \cup \mathbb{F}^{U_2,1} \cup \mathbb{F}^{U_2,2}$. Then $U_1 \in \mathbb{F}^{U_2,0} \cup \mathbb{F}^{U_2,1} \cup \mathbb{F}^{U_2,2}$. Since U_1 is an ultrafilter, $U_1 \in \mathbb{F}^{U_2,2}$. That is, there exists a fence $\{U_2, M, U_1\}$ where $M \subseteq U_2$ and $M \subseteq U_1$.

Conversely, let U be an ultrafilter of L. If $\mathrm{Max}(\mathbb{X}(L)) = \{U\}$, then $L \in \mathcal{SDH}_1$ and, in particular, $L \in \mathcal{SDH}_3$. Suppose that there exists $U_1 \in \mathrm{Max}(\mathbb{X}(L))$, $U_1 \neq U$. By hypothesis there exists $M \in \mathrm{Min}(\mathbb{X}(L))$ such that $M \subseteq U$ and $M \subseteq U_1$. Thus, $U_1 \in \mathbb{F}^{U,2}$, so $\mathrm{Max}(\mathbb{X}(L)) \subseteq \mathbb{F}^{U,0} \cup \mathbb{F}^{U,1} \cup \mathbb{F}^{U,2}$. This finishes the proof. □

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References

[1] R. Balbes and P. Dwinger, Distributive Lattices (University of Missouri Press, 1974).
[2] T. S. Blyth, Lattices and Ordered Algebraic Structures. Universitext (Springer-Verlag, 2005).
[3] T. S. Blyth and J. C. Varlet, Ockham Algebras (Oxford University Press, 1994).
[4] A. Monteiro, Sur Les Algèbres de Heyting symétriques, Portugaliae Math. **39**, 1–237 (1980).
[5] H. P. Sankappanavar, Heyting algebras with a dual lattice endomorphism, Z. Math. Log. Grundl. Math. **33**, 565–573 (1987).
[6] H. A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, Bull. Lond. Math. Soc. **2**, 186–190 (1970).
[7] H. A. Priestley, Ordered topological spaces and the representation of distributive lattices, Proc. Lond. Math. Soc. **24**, 507–530 (1972).
[8] H. A. Priestley, Ordered sets and duality for distributive dattices, Ann. Discrete Math. **23**, 39–60 (1984).
[9] B. S. W. Schröder, Ordered sets. An introduction (Birkhäuser, 2003).