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Quasi-stationary states of the NRT nonlinear Schrödinger equation

I.V. Toranzo [a], A.R. Plastino [a,b], J.S. Dehesa [a], A. Plastino [a,c,*]

[a] Instituto Carlos I de Física Teórica y Computacional and Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada, 18071-Granada, Spain

[b] CeBio y Secretaria de Investigaciones, Universidad Nacional Noroeste-Buenos Aires-UNNOBA y CONICET, R. Saenz Peña 456, Junin, Argentina

[c] National University La Plata, IFLP-CCT-Conicet, C.C. 727, 1900, La Plata, Argentina

HIGHLIGHTS

- We explore quasi-stationary solutions of a nonlinear Schrödinger equation based upon the non-extensive thermostatistics.
- We show that the NRT equation with quadratic potentials admits q-Gaussian quasi-stationary solutions.
- We obtain an exact quasi-stationary solution for the Moshinsky model.

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ABSTRACT

With regards to the nonlinear Schrödinger equation recently advanced by Nobre, Rego-Monteiro, and Tsallis (NRT), based on Tsallis q-thermo-statistical formalism, we investigate the existence and properties of its quasi-stationary solutions, which have the time and space dependences "separated" in a q-deformed fashion. One recovers the normal factorization into purely spatial and purely temporal factors, corresponding to the standard, linear Schrödinger equation, when the deformation vanishes ($q = 1$). We discuss various specific examples of exact, quasi-stationary solutions of the NRT equation. In particular, we obtain a quasi-stationary solution for the Moshinsky model, providing the first example of an exact solution of the NRT equation for a system of interacting particles.

1. Introduction

1.1. Preliminaries

Recently, Nobre, Rego-Monteiro and Tsallis [1,2] have advanced an interesting new version of the nonlinear Schrödinger equation (NLSE), an intriguing proposal that can be regarded as part of a plan to explore nonlinear versions of some of the fundamental equations of physics, a research road that has been actively traversed recently [3,4]. Other nonlinear versions of the Schrödinger equation, previously discussed in the literature, have found application in a variety of areas (fiber optics and water waves, for instance) [4]. One of these equations, a most studied one, involves a cubic nonlinearity in the wave function. In quantal scenarios the NLSE usually governs the behavior of a single-particle's wave function, which, in turn, provides an effective, mean-field description of a quantum many-body system. One can mention the Gross–Pitaevskii equation, used in

* Corresponding author at: National University La Plata, IFLP-CCT-Conicet, C.C. 727, 1900, La Plata, Argentina. Tel.: +54 22145239995; fax: +54 2214523995.
E-mail addresses: plastino@fisica.unlp.edu.ar, angeloplastino@gmail.com (A. Plastino).

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the study of Bose–Einstein condensates [5]. The cubic nonlinear term appearing in the Gross–Pitaevskii equation describes the short-range interactions between the constituents of the condensate. The NLSE for the system's (effective) single-particle wave function is found by assuming a Hartree–Fock-like form for the global many-body wave function, with a Dirac delta form for the inter-particle potential.

The NRT equation derives from the thermo-statistical formalism based upon the Tsallis S_q non-additive, power-law information measure. Applications of the functional S_q involve diverse physical systems and processes, having attracted much attention in the past 20 years (see, for example, [6–14], and references therein). In particular, the S_q entropy has proved to be useful for the analysis of diverse problems in quantum physics [15–23].

The NRT-NLSE governing the field $\Phi(x,t)$ ("wave function") for a particle of mass m reads [1,2],

$$i\hbar \frac{\partial}{\partial t}\left[\frac{\Phi(x,t)}{\Phi_0}\right] = -\frac{1}{2-q}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left[\frac{\Phi(x,t)}{\Phi_0}\right]^{2-q}, \tag{1}$$

where the scaling constant Φ_0 guarantees an appropriate physical normalization for the different terms appearing in the equation, while i is the imaginary unit, $\hbar$ is the Planck constant, and q is a real parameter typical of non-extensive thermo-statistics [7]. From here on we use $\Psi(x,t) = \frac{\Phi(x,t)}{\Phi_0}$.

From Ref. [1] one gathers that the wave equation (1) admits time-dependent solutions having the "q-plane wave" form,

$$\Phi(x,t) = \Phi_0 [1 + (1-q)i(kx - wt)]^{\frac{1}{1-q}}, \tag{2}$$

with k and w real parameters having, respectively, dimensions of inverse length and inverse time (that is, k can be regarded as a wave number and w as a frequency). In the limit $q \to 1$, the q-plane waves (2) become the plane wave solutions $\Phi_0 \exp(-i(kx - wt))$ of the standard, linear Schrödinger equation describing a free particle of mass m.

1.2. Motivation and goals

In this paper *we introduce the notion of quasi-stationarity within the context of the NRT equation*. In order to explain it we recall that the time-dependent Schrödinger equation for a potential function $V(x)$ arises naturally from the full time-dependent Schrödinger equation if one proposes solutions in which the spatial and temporal variables are "separated" in the sense of factorability,

$$\Psi(x,t) = \phi(x)\exp[-g(t)]; \quad g(t) = iEt/\hbar, \tag{3}$$

which, when inserted into Schrödinger's time-dependent equation leads to the well known eigenvalue equation,

$$-\frac{1}{2}\frac{\hbar^2}{2m}\frac{d\phi(x)}{dx} + V(x)\phi(x) = E\phi(x), \tag{4}$$

where E stands for the energy eigenvalue. Further, since the ground state $\phi_o(x)$ has no nodes, it can always be cast in the form

$$\phi_o(x) = \exp[-f(x)], \tag{5}$$

so that the typical stationary form one is familiar with can be cast in the form

$$\Psi(x,t) = \exp[-(f(x) + g(t))]. \tag{6}$$

How close we can come to the (3)–(6) environment when dealing with Eq. (1)? The natural answer is that we do so if we have solutions to (1) of the form

$$\Phi(x,t) = e_q[-(f(x) + g(t))], \tag{7}$$

which we call quasi-stationary, using the so-called q-exponentials [15–23], defined via

$$e_q(x) = [1 + (1-q)x]^{1/(1-q)}. \tag{8}$$

In the limit $q \to 1$ one finds that $e_q(x) \to \exp(x)$. An important difference between these two kinds of exponential is that the usual exponential factorability does not hold for $q \neq 1$, i.e.,

$$e_q(x+y) \neq e_q(x)e_q(y) \quad \text{for } q \neq 1. \tag{9}$$

Why should this be of interest? Stationary solutions of relevant evolution equations usually provide valuable descriptions regarding the structural features of physical systems. For instance, stationary solutions of Schrödinger's equation account for the main characteristics of atomic, molecular and nuclear systems. It is thus of great interest to understand the properties of quasi-stationary solutions of the recently proposed NRT equation. In particular, one wishes to identify those scenarios leading to exact analytical solutions. Beyond the conceptual interest that exact analytical solutions accrue, such solutions usually constitute useful tools for testing either numerical procedures or analytical approximate schemes for tackling more general solutions to the NRT equation.

More generally, we know that the q-extension of other fundamental equations of physics has constituted a fruitful task that yielded considerable insights [15–23]. Among a host of possibilities, we cite here, as just one example, the nonlinear diffusion and Fokker–Planck equations, which have encountered countless applications in various areas of scientific endeavor [24]. Accordingly, *it is our goal here to study the main mathematical properties of our concept of quasi-stationary solutions to the NRT equation*. Before doing that we will first summarily review, below, the main properties of the NRT q-extension of Schrödinger's equation.

2. Basic properties of the NRT equation

Our q-plane wave solutions (2) propagate at a constant velocity $c = w/k$ without shape-modification, thus displaying soliton-like behavior. In contrast to the $q \to 1$ case yielding standard plane waves, the solutions for $q \neq 1$ lack a spatially constant modulus. Indeed (defining $\psi = \Phi/\Phi_0$),

$$|\psi(x,t)|^2 = \left[1 + (1-q)^2(kx - wt)^2\right]^{\frac{1}{1-q}}, \tag{10}$$

which corresponds, for $1 < q < 3$, to a normalizable q-Gaussian centered at $x = wt/k$. Accordingly, the q-plane wave solution describes a phenomenon characterized by a certain degree of spatial localization. In Ref. [2] one finds a field-theoretical approach to the NRT equation and it is shown that the equation derives from a variational principle. Also, the NRT equation is formally related to the nonlinear Fokker–Planck equation, with a diffusion term depending on a power of the density. Evolution equations like the nonlinear Fokker–Planck equation, and their relations with the q-thermostatistical formalism, have received intense attention recently [24–32]. The formal resemblance between the NRT Schrödinger equation and the nonlinear Fokker–Planck equation notwithstanding, major differences between these two types of equations exist. For example, the nonlinear Fokker–Planck equation does not admit q-plane wave solutions of the form (2), that propagate without shape-change.

An interesting property of the solutions (2) deserves mentioning [1]: they are consistent with the celebrated de Broglie relations [33],

$$E = \hbar w, \qquad p = \hbar k, \tag{11}$$

connecting, respectively, energy with frequency and momentum with wave number. The q-plane wave (2) satisfies the Eq. (1) if and only if the parameters w and k comply with the relation

$$w = \frac{\hbar k^2}{2m}, \tag{12}$$

which, in conjunction with (11), leads to the standard relation between linear momentum and kinetic energy,

$$E = \frac{p^2}{2m}. \tag{13}$$

This in turn suggests that the q-plane waves (2) represent particles of mass m with kinetic energy $\hbar w$ and momentum $\hbar k$ [1]. We are now prepared for a discussion of the present contributions to the subject.

3. A general ansatz for stationary and quasi-stationary solutions of the NRT equation

A rather general scenario surrounding the equation

$$i\hbar\frac{\partial\Psi(\vec{x},t)}{\partial t} = -\frac{1}{2-q}\nabla_n^2\Psi(\vec{x},t)^{2-q} + V(\vec{x})\Psi(\vec{x},t)^q; \quad q > 1, \tag{14}$$

in which $\Psi(\vec{x},t)$ is our quasi-stationary ansatz (7)

$$\Psi(\vec{x},t) = [1 - (1-q)(f(\vec{x}) + g(t))]^{\frac{1}{1-q}}; \quad q > 1, \tag{15}$$

with

$$\Psi(\vec{x},t) = \Psi(x_1, x_2, \ldots, x_N, t) \quad \text{accompanied by } f(\vec{x}) = f(x_1, x_2, \ldots, x_n)$$

with Cartesian coordinates $(x_1, x_2, \ldots, x_N)$ is now to be developed. The nonlinear partial differential equation that we obtain when we place our ansatz into the NRT equation becomes

$$-\frac{\hbar^2}{2m}[1 - (1-q)(f(\vec{x}) + g(t))]\nabla^2 f(\vec{x}) - \frac{\hbar^2}{2m}[\nabla f(\vec{x})]^2 + i\hbar\dot{g}(t) + V(\vec{x}) = 0, \tag{16}$$

with

$$\nabla^2 f(\vec{x}) = \sum_{i=1}^{N} \frac{\partial^2 f(\vec{x})}{\partial x_i^2},$$

and

$$[\nabla f(\vec{x})]^2 = (\nabla f(\vec{x})) \cdot (\nabla f(\vec{x})) = \sum_{i=1}^{N} \left(\frac{\partial f(\vec{x})}{\partial x_i} \right)^2.$$

Recast (16) as

$$i\hbar \dot{g}(t) - (1-q)\frac{-\hbar^2}{2m} g(t) \nabla^2 f(\vec{x}) = \frac{-\hbar^2}{2m}[1-(1-q)f(\vec{x})]\nabla^2 f(\vec{x}) + \frac{\hbar^2}{2m}[\nabla f(\vec{x})]^2 - V(\vec{x}), \tag{17}$$

where we appreciate that the right hand side of the above equation depends only on $\vec{x}$, which entails that the left hand side *cannot depend upon the time*. Three possibilities can be envisaged to fulfill this time-independence requirement.

The most straightforward way of obtaining a time-independent left hand side in Eq. (17) is to consider solutions described by our quasi-stationary ansatz with $g(t) = 0$. In this instance Eq. (17) reduces to

$$g(t) = 0 \Rightarrow \frac{-\hbar^2}{2m}[1-(1-q)(f(\vec{x}))]\nabla^2 f(\vec{x}) + \frac{\hbar^2}{2m}[\nabla f(\vec{x})]^2 - V(\vec{x}) = 0. \tag{18}$$

An $f(\vec{x})$ complying with this equation leads to an exact solution to the NRT equation with potential $V(\vec{x})$ having the quasi-stationary form. These solutions become strictly stationary (that is, independent of time).

Next, we consider solutions with $\nabla^2 f(\vec{x}) = 0$ and $g(t) = iwt$, with $w =$ a real constant. Accordingly, Eq. (17) yields

$$\frac{\hbar^2}{2m}[\nabla f(\vec{x})]^2 - V(\vec{x}) + \hbar\, w = 0. \tag{19}$$

Finally, we consider that one has $\nabla^2 f(\vec{x}) = D$, with D a real constant and

$$\dot{g}(t) = iw + \frac{i\hbar}{2m}(q-1)D, \tag{20}$$

entailing

$$-\frac{\hbar^2}{2m}[1-(1-q)f(\vec{x})]D + \frac{\hbar^2}{2m}[\nabla f(\vec{x})]^2 - V(\vec{x}) + \hbar\, w = 0. \tag{21}$$

Recall that in the ordinary Schrödinger's instance, given a wave function ψ with no nodes, one can immediately show that it is the ground state eigenfunction of the potential function

$$V(\vec{x}) = \frac{\nabla^2 \psi}{\psi}, \tag{22}$$

which is conveniently shifted so that its ground-state energy is zero. The particular instantiation of our quasi-stationary ansatz represented by $g(t) = 0$ corresponds to a q-extension of (22). The associated equation connecting $V(\vec{x})$ with $\Psi(\vec{x})$ (which is equivalent to (18)) becomes

$$V(\vec{x}) = \frac{1}{2-q}\frac{1}{\Psi^q}\nabla^2\left[\Psi^{2-q}\right]. \tag{23}$$

Some comments are in order. First, the set of potential solutions $V(\vec{x})$ admitting a quasi-stationary solution with $g(t) = 0$ (that is, admitting a strictly stationary solution) comprises potentials with a wide variety of possible shapes. However, not all potentials $V(\vec{x})$ admit a solution of this kind. Second, when one passes to the case of time-dependent quasi-stationary solutions, the sets of possible shapes for $V(\vec{x})$ is severely restricted. As an example, we mention that solutions with $g(t) = iwt$, with w real, exist only for potentials of the form

$$V(\vec{x}) = \frac{\hbar}{2m}[\nabla f(\vec{x})]^2 + \hbar w, \tag{24}$$

with $f(\vec{x})$ a harmonic function.

We are now going to consider some particular examples of (quasi-)stationary solutions.

3.1. Quadratic potential

We are going to consider as a first example for the current scenario an N-dimensional system with a quadratic potential. Let us consider a wave function of the type

$$\Psi = A\left[1-(1-q)\sum_i^N \lambda_i x_i^2\right]^{1/(1-q)}, \tag{25}$$

with A a suitable normalization constant. It is clear that this q-Gaussian wave function corresponds to our quasi-stationary ansatz with $g(t)=0$ and $f(\vec{x})=\sum_i \lambda_i x_i^2$ being an homogeneous quadratic function of the spatial coordinates. If now we set

$$V(\vec{x}) = \sum_{i=1}^N \frac{1}{2} m w_i^2 x_i^2 - V_o, \quad \text{with} \tag{26}$$

$$w_i^2 = \frac{\hbar^2 A^{2(1-q)}}{m^2}\left[4\lambda_i^2 - 2(1-q)\lambda_i \sum_{j=1}^N \lambda_j\right], \quad \text{and} \tag{27}$$

$$V_o = m^{-1}\hbar^2 A^{2(1-q)} \sum_{j=1}^N \lambda_j, \tag{28}$$

it can be verified, after some algebra, that the NRT nonlinear equation corresponding to the N-dimensional harmonic oscillator potential (26) admits the stationary solution (25) exhibiting the q-Gaussian form. We pass now to a second example.

3.2. Shifted-attractive delta potential

Consider now the stationary wave function

$$\psi(x) = e_q(-\beta x); \quad x \geq 0$$
$$\psi(x) = e_q(\beta x); \quad x \leq 0, \tag{29}$$

with $A,\ \beta$ two real, positive constants. It is possible to verify that (29) is a stationary solution to the NLSE that is occupying our attention here, associated with a potential V

$$V(x) = V_o - D\delta(x), \tag{30}$$

with $V_o,\ D$ two real, positive constants. Our four parameters $A,\ \beta,\ V_o,\ D$ are related through

$$D = \hbar^2 \beta/m, \tag{31}$$

$$V_0 = \hbar^2 \beta^2 A^{2(1-q)}/2m. \tag{32}$$

Interestingly enough, Eq. (31), relating the "width" β of the stationary solution to the strength D of the attractive delta potential is independent of the Tsallis parameter q. In the $q \to 1$ limit the stationary solution (29) of the NRT equation yields the well-known bound state of the one-dimensional delta potential.

3.3. Free-particle travelling q-plane solutions

Here we have $\nabla^2 f(\vec{x}) = 0$ plus $g(t) = iwt$, with w a real constant, yielding $\frac{\hbar^2}{2m}|\nabla f(\vec{x})|^2 - V + \hbar w = 0$. The simplest, and arguably most important instantiation is given by the NRT q-plane wave associated with a spatial dependence of the type

$$f(\vec{x}) = i\vec{k}.\vec{x}, \tag{33}$$

leading to

$$\frac{\hbar^2 \vec{k}^2}{2m} + V = \hbar w, \tag{34}$$

that can only be realized for a constant potential V. Eq. (34) admits a clear interpretation in terms of the de Broglie relations. Of course, many other possibilities are feasible in this scenario, given by $f(\vec{x}) = ih(x)$, with h a real harmonic function, i.e., $\nabla^2 h = 0$. As an example, consider

$$f(\vec{x}) = i\sum_{j,k=1}^N a_{j,k} x_j x_k; \qquad \sum_{i=1}^N a_{i,i} = 0; \quad a_{j,l} \text{ real}, \tag{35}$$

which leads to an exact quasi-stationary solution of (19). Although we will not explore this more general case further here, we mention that in this case the NRT equation reduces to

$$-\frac{1}{2-q}\frac{\hbar^2}{2m}\nabla^2\Psi^{2-q} + V\Psi^q = \hbar w \Psi^q, \tag{36}$$

which resembles the stationary (time-independent) Schrödinger equation, with Ψ^q playing the role of an energy eigenvalue.

4. The NRT equation and the Moshinsky model

The Moshinsky model [34] (sometimes referred to as the "harmonium") constitutes an exactly solvable quantum system that has been intensively studied in recent years as a testing ground for investigating the validity of several approximations in atomic and molecular physics and also to explore, within a tractable scenario, fundamental issues such as the entanglement-features of atomic systems (see Refs. [35,36] and references therein). The potential function characterizing the Moshinsky model in two dimensions has the form

$$V(x, y) = \frac{1}{2}\omega^2\left(x^2 + y^2 + \kappa(x-y)^2\right) - V_o, \tag{37}$$

so that we are led to the NLSE

$$i\frac{\partial}{\partial t}\Psi(x, y, t) = -\frac{1}{2-q}\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[\Psi(x, y, t)]^{2-q} + V(x, y)[\Psi(x, y, t)]^q, \tag{38}$$

where ω is the natural frequency associated with the (confining) harmonic oscillator potential, κ determines the strength of the inter-particle interaction, and V_o is a constant. Here we use atomic units ($\hbar = m = 1$). We now perform a change of variables, going to center-of-mass and relative coordinates.

$$R = \frac{1}{\sqrt{2}}(x + y), \tag{39}$$

$$r = \frac{1}{\sqrt{2}}(x - y). \tag{40}$$

In terms of these new coordinates the NRT-Moshinsky equation reads

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2(2-q)}\left(\frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial r^2}\right)\Psi^{2-q} + (1/2)[\omega^2 R^2 + \Lambda^2 r^2 - 2V_o]\Psi^q, \tag{41}$$

where $\Lambda^2 = (2k+1)\omega^2$. This equation admits the quasi-stationary solution

$$\Psi = A[1-(1-q)(\lambda_1 R^2 + \lambda_2 r^2)]^{1/(1-q)} \equiv Ae_q[\lambda_1 R^2 + \lambda_2 r^2], \tag{42}$$

where the parameters λ_i verify

$$\omega^2 = A^{2(1-q)}[4\lambda_1^2 - 2(1-q)\lambda_1(\lambda_1 + \lambda_2)], \tag{43}$$

$$\Lambda^2 = A^{2(1-q)}[4\lambda_2^2 - 2(1-q)\lambda_2(\lambda_1 + \lambda_2)]. \tag{44}$$

The wave function (42) constitutes the first case reported in the literature of a NRT solution for a system of interacting particles. In the limit $q \to 1$ it reduces to the ground state function associated with the Moshinsky model's potential. It is instructive to compare the $q = 1$ and $q \neq 1$ cases. For the latter one, due to nonlinear nature of the NRT equation, the wave function is no longer factorized into a part depending on the center-of-mass position and a part dependent on the relative position, which illustrates the fact that in a NRT scenario the center-of-mass motion is not, in general, separated from the internal motion of a composite system.

5. Conclusions

We have investigated a particular family of exact analytical solutions of the NRT nonlinear Schrödinger equation. These solutions are quasi-stationary in the sense of having a particularly simple time dependence that resembles that of the phase factor $\exp(-iEt/\hbar)$, associated with stationary solutions of the ordinary Schrödinger equation.

As a matter of fact, our quasi-stationary solutions display time and spatial dependences that are "separated" in such a way that they become factorized in the limit $q \to 1$. A particular instantiation of this quasi-separation is given by the q-plane wave solutions for the free-particle case of the NRT equation discovered by Nobre et al. [1].

We also investigated other quasi-stationary solutions that exhibit strict time-independence. They are the counterparts, in the q-nonlinear case, of the ground state eigenfunction of the linear Schrödinger equation for potential functions $V(x)$ that are appropriately shifted so that the ground state energy is zero. In the particular case of a (not necessarily isotropic) N-dimensional harmonic potential, the associated quasi-stationary solutions were found to have an N-dimensional q-Gaussian form.

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