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Bounded rank-one perturbations in sampling theory ☆

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ABSTRACT

Sampling theory concerns the problem of reconstruction of functions from the knowledge of their values at some discrete set of points. In this paper we derive an orthogonal sampling theory and associated Lagrange interpolation formulae from a family of bounded rank-one perturbations of a self-adjoint operator that has only discrete spectrum of multiplicity one.

1. Introduction

Sampling theory is concerned with the problem of reconstruction of functions, in a pointwise manner, from the knowledge of their values at a prescribed discrete set of points. Resolution of concrete situations in this theory generally involves the characterization of a class (usually a linear set) of functions to be interpolated, the specification of a set of sampling points to be used for all the functions in the given class, and the derivation of an interpolation formula.

The cornerstone for many works on sampling theory is the Kramer sampling theorem [17] and its analytic extension [11]. Orthogonal sampling formulae often arise as realizations of this theorem. The celebrated Whittaker–Shannon–Kotel'nikov sampling theorem [16,24,31] is also a particular case of the Kramer's theorem, although historically the former came first and motivated the latter (for a recent survey on these issues see [9]).

Orthogonal sampling formulae have been obtained in connection with differential and difference self-adjoint boundary value problems (see for instance [4,13,32] and, of course, the paper due to Kramer himself [17]), and also by resorting to Green's functions methods [5,6], among other ODE's techniques. These results suggest that the spectral theory of operators should provide a unifying approach to sampling theory. Following this idea, a general method for obtaining analytic, orthogonal sampling formulae has been derived in [25] on the basis of the theory of representation of simple symmetric operators due to M.G. Krein [18–21]. Roughly speaking, the technique given in [25] is based on the following: By [18–20], every closed simple symmetric operator A in a Hilbert space $\mathcal{H}$ generates a bijective isomorphism between $\mathcal{H}$ and a space of functions $\widehat{\mathcal{H}}$ with certain analytic properties. If the operator A satisfies some additional conditions, all of its self-adjoint extensions have discrete spectrum and every function f in $\widehat{\mathcal{H}}$ can be uniquely reconstructed, as long as one knows the value of f at the spectrum of any self-adjoint extension of A.

The self-adjoint extensions of a symmetric operator with deficiency indices $(1,1)$ constitute a family of singular rank-one perturbations of one of these self-adjoint extensions [3, Sections 1.1–1.3]. Therefore, the methods developed in [25] also hold

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for singular rank-one perturbations, provided that these operators correspond to a family of self-adjoint extensions of some simple symmetric (hence densely defined) operator. We cannot use, however, a family of bounded rank-one perturbations in applications to sampling theory without making substantial changes to the results of [25]. Krein's approach to symmetric operators with equal deficiency indices does not work for bounded rank-one perturbations since operators of this kind may only be seen as self-adjoint extensions of a certain not densely defined, Hermitian operator [3, Section 1.1].

The main motivation of the present work is to develop a method in sampling theory analogous to [25] for the case of bounded rank-one perturbations. Accordingly, this work illustrates the use of spectral and operator methods in dealing with various aspects of sampling theory. It is worth pointing out that, in a sense, this article (and also [25]) may be seen as a particular realization of the abstract sampling theory on reproducing kernel Hilbert spaces developed in [15,23] (see also [12]). Indeed, as we shall see, we obtain in this work, from the spectral properties of a family of rank-one perturbations, the objects assumed as a starting point of the investigations carried out in [12,15,23].

With the motivation stated above in mind, we begin by constructing a representation space for bounded rank-one perturbations in some sense similar to that of Krein for simple symmetric operators. Elements of this representation space are the functions to be interpolated. Then we obtain a general Kramer-type analytic sampling formula which turns out to be a Lagrange interpolation formula. We also characterize the space of interpolated functions as a space of meromorphic functions that is related to a de Branges space. Examples are discussed in the last part of this work.

2. Preliminaries

The following review is based on the classical theory of rank-one perturbations of self-adjoint operators, as discussed in detail by Donoghue [10], Simon [27], Gesztesy and Simon [14], and Albeverio and Kurasov [3].

In a Hilbert space $\mathcal{H}$, let us consider a self-adjoint operator A with discrete and simple spectrum. Discreteness means that the spectrum consists only of isolated eigenvalues of finite multiplicity. Simplicity means that there is a vector $\mu \in \mathcal{H}$, called cyclic, such that $\{(A - zI)^{-1}\mu\colon\ z \in \mathbb{C}\}$ is a total set in $\mathcal{H}$. Because of the simplicity, the eigenspaces $\operatorname{Ker}(A - xI)$, where x is in the spectrum, are one-dimensional. The conditions imposed on A imply that the space $\mathcal{H}$ is separable.

Let us fix a cyclic vector μ and assume $\|\mu\| = 1$. We define the family of bounded rank-one perturbations of A as follows.

$$A_h := A + h\langle\mu, \cdot\rangle\mu, \quad h \in \mathbb{R}, \tag{2.1}$$

where the inner product is taken, from now on, anti-linear in its first argument. Naturally, $\operatorname{Dom}(A_h) = \operatorname{Dom}(A)$ for any $h \in \mathbb{R}$. Elementary perturbation theory implies that all the elements of (2.1) have discrete spectrum. Moreover, since μ is cyclic for every A_h with $h \in \mathbb{R}$, it follows that the spectrum of A_h, henceforth denoted by $\operatorname{Sp}(A_h)$, is simple for $h \in \mathbb{R}$ [10, Section 1]. As pointed out in [3, Section 1.1], the operators A_h may be seen as self-adjoint extensions of some Hermitian operator, in the sense of [30], with non-dense domain.

Alongside (2.1), we consider the family of functions

$$F_h(z) := \big\langle\mu, (A_h - zI)^{-1}\mu\big\rangle, \quad z \notin \operatorname{Sp}(A_h),\ h \in \mathbb{R}. \tag{2.2}$$

In the sequel F_0 will be denoted by F. By the spectral theorem, $F_h(z)$ is the Borel transform of the spectral function $m_h(t) = \langle\mu, E_h(t)\mu\rangle$, where $E_h(t)$ is the spectral resolution of the identity corresponding to A_h. Hence, $F_h(z)$ is a Herglotz meromorphic function having simple poles at the eigenvalues of A_h.

From the second resolvent identity [30, Theorem 5.13] one obtains

$$(A_h - zI)^{-1} = (A - zI)^{-1} - h\big\langle(A_h - \bar{z}I)^{-1}\mu, \cdot\big\rangle(A - zI)^{-1}\mu, \quad h \in \mathbb{R}. \tag{2.3}$$

This equation yields the well-known Aronzajn–Krein formula [27, Eq. (1.13)]

$$F_h(z) = \frac{F(z)}{1 + hF(z)}, \quad h \in \mathbb{R}. \tag{2.4}$$

The spectral properties of the whole family (2.1) are contained in (2.4). Indeed, one can easily show that the function $F_h/F_{h'}$, $h \neq h'$, is also Herglotz and its zeros and poles are given by the poles of $F_{h'}$ and F_h respectively [26]. Thus, the spectra of any two different elements of the family (2.1) interlace, i.e., between two neighboring eigenvalues of one operator there is one and only one eigenvalue of any other. Also, (2.4) implies that $x_0 \in \mathbb{R}$ is a pole of F_h if and only if

$$\frac{1}{F(x_0)} + h = 0. \tag{2.5}$$

Therefore, for any $x \in \mathbb{R}$ which is not a zero of F, there exists a unique $h \in \mathbb{R}$ such that x is an eigenvalue of A_h. One may extend this result to every $x \in \mathbb{R}$ by considering an infinite coupling constant $h = \infty$ in (2.1) (see [27, Section 1.5], [3, Section 1.1.2]). From the properties of $F(z)$, it is shown that A_∞ also has simple discrete spectrum (see footnote in [1, p. 55]) and $\operatorname{Sp}(A_\infty) = \{x \in \mathbb{R}\colon\ F(x) = 0\}$. Thus, for any $x \in \mathbb{R}$, there exists a unique $h \in \mathbb{R} \cup \{\infty\}$ such that x is an eigenvalue of A_h.

The main peculiarity of A_∞, that separates it from the family A_h with finite h, is its domain. Indeed, for the kind of perturbations considered here, the domain of A_∞ is the set $\{\varphi \in \operatorname{Dom}(A)\colon\ \langle\varphi, \mu\rangle = 0\}$ ([3, Section 1.1.1], [27, Theorem 1.15]).

3. Sampling theory

Based on the theory of rank-one perturbations, we construct in this section a linear space of meromorphic functions $\widehat{\mathcal{H}}_\mu$ and derive an interpolation formula valid for all the elements in $\widehat{\mathcal{H}}_\mu$.

In our considerations below, the vector-valued function

$$\xi(z) := \frac{(A-\bar{z}I)^{-1}\mu}{F(\bar{z})}, \quad z \in \mathbb{C} \setminus \operatorname{Sp}(A_\infty),$$

plays an important role. Note that $\xi(z)$ is well defined for $z \in \operatorname{Sp}(A)$ and is analytic in its domain of definition.

Lemma 1. *For any $x \in \mathbb{R} \setminus \operatorname{Sp}(A_\infty)$, there exists (a unique) $h \in \mathbb{R}$ such that*

$$\xi(x) \in \operatorname{Ker}(A_h - xI). \tag{3.1}$$

Similarly,

$$(A - xI)^{-1}\mu \in \operatorname{Ker}(A_\infty - xI) \tag{3.2}$$

for every $x \in \operatorname{Sp}(A_\infty)$.

Proof. We first consider $x \notin \operatorname{Sp}(A) \cup \operatorname{Sp}(A_\infty)$. Let $h \neq 0$ be such that $x \in \operatorname{Sp}(A_h)$ (we already know that there is always such h). We have

$$A_h\xi(x) = \frac{1}{F(x)} A_h(A-xI)^{-1}\mu = \left(\frac{1}{F(x)} + h\right)\mu + \frac{x}{F(x)}(A-xI)^{-1}\mu.$$

The first assertion of the lemma follows from the last expression and (2.5). When $x \in \operatorname{Sp}(A)$, the statement follows by a limiting argument based on the fact that A is closed.

We now prove the last assertion of the lemma. Define $P := \langle \mu, \cdot \rangle \mu$. By virtue of [27, Theorem 1.18, Remark 2], there is a cyclic vector η that obeys

$$(A_\infty - zI)^{-1}\eta = \frac{1}{F(z)}(I-P)(A-zI)^{-1}\mu, \quad z \notin \operatorname{Sp}(A) \cup \operatorname{Sp}(A_\infty). \tag{3.3}$$

Clearly, $\eta \in \operatorname{Dom}(A_\infty)$. We compute the projection of η along the eigenspace associated to $x \in \operatorname{Sp}(A_\infty)$. Using (3.3) and some $\epsilon > 0$ sufficiently small, we obtain

$$\begin{aligned}\left[E_\infty(x+0) - E_\infty(x-0)\right]\eta &= \frac{1}{2\pi \mathrm{i}} \int\limits_{|x-z|=\epsilon} (A_\infty - zI)^{-1}\eta \, dz = \frac{1}{2\pi \mathrm{i}} \int\limits_{|x-z|=\epsilon} \left[\frac{1}{F(z)}(A-zI)^{-1} - I\right]\mu \, dz \\ &= \left(\operatorname*{Res}_{z=x} \frac{1}{F(z)}\right)(A-xI)^{-1}\mu.\end{aligned}$$

Since the last expression is different from zero, (3.2) is proven. □

Definition 1. For any $\varphi \in \mathcal{H}$, let Φ_μ be the mapping given by

$$(\Phi_\mu\varphi)(z) := \langle \xi(z), \varphi \rangle, \quad z \in \mathbb{C} \setminus \operatorname{Sp}(A_\infty).$$

We sometimes shall denote $\Phi_\mu\varphi$ by $\widehat{\varphi}$.

The mapping Φ_μ is a linear injective operator from $\mathcal{H}$ onto a certain space of meromorphic functions $\widehat{\mathcal{H}}_\mu := \Phi_\mu\mathcal{H}$. The injectivity may be verified with the aid of (3.1). Some properties that characterize the set $\widehat{\mathcal{H}}_\mu$ will be accounted for in the next section.

Proposition 1. *Given some fixed $h \in \mathbb{R}$, let $\{x_j\}_j = \operatorname{Sp}(A_h)$. Define $G_h(z) := 1/F_h(z) = h + 1/F(z)$. Then, for every $f(z) \in \widehat{\mathcal{H}}_\mu$, we have*

$$f(z) = \sum_{x_j \in \operatorname{Sp}(A_h)} \frac{G_h(z)}{(z-x_j)G_h'(x_j)} f(x_j), \quad z \in \mathbb{C} \setminus \operatorname{Sp}(A_\infty). \tag{3.4}$$

The series converges absolutely and uniformly on compact subsets of the domain.

Proof. Because of the assertion (3.1) of Lemma 1, $\{\xi(x_j)\}_j$ is a complete orthogonal set in $\mathcal{H}$. Hence

$$\widehat{\varphi}(z)=\langle\xi(z),\varphi\rangle=\sum_{x_j\in\mathrm{Sp}(A_h)}\frac{\langle\xi(z),\xi(x_j)\rangle}{\|\xi(x_j)\|^2}\widehat{\varphi}(x_j),\tag{3.5}$$

where the series converges absolutely by virtue of the Cauchy–Schwarz inequality and uniformly on compacts because $\|\xi(z)\|$ is continuous in $\mathbb{C}\setminus\mathrm{Sp}(A_\infty)$; see the proof of [25, Proposition 1] for a more detailed argument of this sort.

Now, the first resolvent identity implies

$$\langle\xi(z),\xi(w)\rangle=(z-\overline{w})^{-1}\left[\frac{1}{F(\overline{w})}-\frac{1}{F(z)}\right].$$

In conjunction with (2.5) and the convention $1/\infty=0$ when $h=0$, the last equation gives rise to the identity

$$\langle\xi(z),\xi(x_j)\rangle=-(z-x_j)^{-1}\left[h+\frac{1}{F(z)}\right].$$

Finally, note that

$$G_h'(w)=-\frac{1}{F^2(w)}F'(w)=-\frac{1}{F^2(w)}\langle\mu,(A-wI)^{-2}\mu\rangle=-\langle\xi(w),\xi(\overline{w})\rangle.$$

Evaluation of the last expression at $w=x_j$ yields the desired result. □

Remark 1. Eq. (3.5) is an orthogonal sampling formula of Kramer-type [17]. Since the function $G_h(z)$ has simple zeros at the points of $\mathrm{Sp}(A_h)$, expression (3.4) is indeed a Lagrange interpolation formula.

4. Spaces of interpolated functions

The present section is devoted to the characterization of the space of functions $\widehat{\mathcal{H}}_\mu$ introduced by means of the mapping Φ_μ of Definition 1. Note that $\widehat{\mathcal{H}}_\mu$ is determined by the way we have defined $\xi(z)$, which in turn only depends on the choice of the operator A and the cyclic vector μ. More precisely, if we define

$$\xi_h(z):=\frac{(A_h-\bar{z}I)^{-1}\mu}{F_h(\bar{z})},\quad z\in\mathbb{C}\setminus\mathrm{Sp}(A_\infty),\ h\in\mathbb{R},$$

then (2.3) and (2.4) imply that $\xi_h(z)=\xi_0(z)=\xi(z)$ so $\widehat{\mathcal{H}}_\mu$ has no dependency on the perturbation parameter h. This parameter however does enter in the selection of the sampling points and, as we will see below, it also determines different choices for the inner product associated to $\widehat{\mathcal{H}}_\mu$.

The following statement gives an explicit description of $\widehat{\mathcal{H}}_\mu$.

Proposition 2.

$$\widehat{\mathcal{H}}_\mu=\left\{f(z)=c+\sum_{x_n\in\mathrm{Sp}(A_\infty)}\frac{c_n}{z-x_n}:\ c,c_n\in\mathbb{C},\ \sum_{x_n\in\mathrm{Sp}(A_\infty)}|c_n|^2F'(x_n)<\infty\right\},$$

where the series above converges absolutely and uniformly on compact subsets of $\mathbb{C}\setminus\mathrm{Sp}(A_\infty)$.

Proof. Let G denote the set defined by the right-hand side of the statement.

Given $x_n\in\mathrm{Sp}(A_\infty)$, it follows from (3.2) of Lemma 1 that $\omega(x_n):=(A-x_nI)^{-1}\mu$ is the associated eigenvector (up to normalization). Taking into account the first resolvent identity, we get

$$\widehat{\omega(x_n)}(z)=\frac{1}{F(z)}\langle(A-\bar{z}I)^{-1}\mu,(A-x_nI)^{-1}\mu\rangle=\frac{1}{F(z)}\left[\frac{F(z)}{z-x_n}+F(x_n)\right]=\frac{1}{z-x_n}.$$

The expression after the first equality above also implies that $\|\omega(x_n)\|^2=F'(x_n)$. Recalling the definition of $\mathrm{Dom}(A_\infty)$, it follows that the set

$$B:=\{\mu\}\cup\left\{\|\omega(x_n)\|^{-1}\omega(x_n)\right\}_{x_n\in\mathrm{Sp}(A_\infty)}$$

is an orthonormal basis in $\mathcal{H}$. Now, take an arbitrary element φ of $\mathcal{H}$ and expand it on the basis B. By applying Φ_μ to φ and using the Cauchy–Schwarz inequality, we conclude that $\widehat{H}_\mu\subset G$.

The inclusion $G\subset\widehat{H}_\mu$ follows from noticing that any $f(z)\in G$ is the image under Φ_μ of an element in $\mathcal{H}$ of the form $c\mu+\sum_{x_n\in\mathrm{Sp}(A_\infty)}c_n\omega(x_n)$. □

Note that, by virtue of Proposition 2, the only entire functions in $\widehat{\mathcal{H}}_\mu$ are the constant functions. Also, one easily verifies that any constant function in $\widehat{\mathcal{H}}_\mu$ is the image under Φ_μ of a vector in $\mathrm{Span}\{\mu\}$.

Lemma 2. *The space $\widehat{\mathcal{H}}_\mu$ has the following properties*:

(i) *If $f(z) \in \widehat{\mathcal{H}}_\mu$ has a non-real zero w, then $g(z) := \frac{z-\overline{w}}{z-w} f(z)$ also belongs to $\widehat{\mathcal{H}}_\mu$.*
(ii) *The evaluation functional $f(\cdot) \mapsto f(z)$ is continuous for every $z \in \mathbb{C} \setminus \operatorname{Sp}(A_\infty)$.*
(iii) *For every $f(z) \in \widehat{\mathcal{H}}_\mu$, $f^*(z) := \overline{f(\overline{z})}$ belongs to $\widehat{\mathcal{H}}_\mu$.*

Proof. We have $f(z) = \langle \xi(z), \varphi \rangle$ for some $\varphi \in \mathcal{H}$. Given w such that $f(w) = 0$, consider $\eta = (A - \overline{w}I)(A - wI)^{-1}\varphi$. A short computation yields $g(z) = \langle \xi(z), \eta \rangle$, thus showing (i). Assertion (ii) is rather obvious so the proof is omitted. On the basis of Proposition 2 one verifies (iii). □

In what follows we show that $\widehat{\mathcal{H}}_\mu$ can be endowed with several Hilbert space structures, each one determined by the spectral functions $m_h(x)$, $h \in \mathbb{R}$.

Lemma 3. *Let $h \in \mathbb{R}$ and $\{x_j\}_j = \operatorname{Sp}(A_h)$. Then the spectral function $m_h(x)$ is given by*

$$m_h(x) = \sum_{x_j \leqslant x} \left\| \xi(x_j) \right\|^{-2}.$$

Proof. Let us recall first the following well-known results [27, Theorem 1.6]

$$\lim_{\epsilon \to 0} \epsilon \operatorname{Re} F_h(x + \mathrm{i}\epsilon) = 0,$$
$$\lim_{\epsilon \to 0} \epsilon \operatorname{Im} F_h(x + \mathrm{i}\epsilon) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{\epsilon^2 \, dm_h(y)}{(y-x)^2 + \epsilon^2} = m_h\big(\{x\}\big),$$

and also the identity

$$(A_h - zI)^{-1}\mu = \frac{1}{1 + hF(z)}(A - zI)^{-1}\mu.$$

Consider $x_j \in \operatorname{Sp}(A_h)$. It suffices to verify that $m_h(\{x_j\}) = \|\xi(x_j)\|^{-2}$. By resorting to the equalities mentioned above, a straightforward computation shows that

$$\begin{aligned}
\big\langle \xi(x_j - i\epsilon), \xi(x_j - i\epsilon) \big\rangle &= \frac{1}{|F(x_j + i\epsilon)|^2} \big\langle \big(A - (x_j + i\epsilon)I\big)^{-1}\mu, \big(A - (x_j + i\epsilon)I\big)^{-1}\mu \big\rangle \\
&= \frac{1}{|F_h(x_j + i\epsilon)|^2} \big\langle \big(A_h - (x_j + i\epsilon)I\big)^{-1}\mu, \big(A_h - (x_j + i\epsilon)I\big)^{-1}\mu \big\rangle \\
&= \frac{1}{[\epsilon \operatorname{Re} F_h(x_j + i\epsilon)]^2 + [\epsilon \operatorname{Im} F_h(x_j + i\epsilon)]^2} \int_{\mathbb{R}} \frac{\epsilon^2 \, dm_h(y)}{(y - x_j)^2 + \epsilon^2} \\
&\to \frac{1}{m_h(\{x_j\})}, \quad \epsilon \to 0.
\end{aligned}$$

The proof is now complete. □

Remark 2. For $h \neq 0$, this result is in fact statement (ii) of [27, Theorem 2.2].

Proposition 3. *For arbitrary $h \in \mathbb{R}$, the map Φ_μ is a unitary transformation from $\mathcal{H}$ onto $L^2(\mathbb{R}, dm_h)$.*

Proof. Φ_μ is a linear isometry from $\mathcal{H}$ into $L^2(\mathbb{R}, dm_h)$. Indeed,

$$\big\langle \widehat{\varphi}(\cdot), \widehat{\psi}(\cdot) \big\rangle_h := \int_{\mathbb{R}} \big\langle \varphi, \xi(x) \big\rangle \big\langle \xi(x), \psi \big\rangle \, dm_h(x) = \sum_{x_j \in \operatorname{Sp}(A_h)} \frac{\langle \varphi, \xi(x_j) \rangle \langle \xi(x_j), \psi \rangle}{\|\xi(x_j)\|^2} = \langle \varphi, \psi \rangle.$$

Now consider $f(x) \in L^2(\mathbb{R}, dm_h)$. This means that

$$\big\| f(\cdot) \big\|_h^2 = \sum_{x_j \in \operatorname{Sp}(A_h)} \frac{|f(x_j)|^2}{\|\xi(x_j)\|^2} < \infty.$$

Define

$$\eta = \sum_{x_j \in \mathrm{Sp}(A_h)} \frac{f(x_j)}{\|\xi(x_j)\|^2} \xi(x_j),$$

which is clearly an element in $\mathcal{H}$. It is not difficult to verify that $\|f(\cdot) - \widehat{\eta}(\cdot)\|_h = 0$. □

Remark 3.

1. Proposition 3 also implies that $\widehat{\mathcal{H}}_\mu$ becomes a Hilbert space when it is equipped with the inner product $\langle\cdot,\cdot\rangle_h$. By statements (i) and (iii) of Lemma 2, $\widehat{\mathcal{H}}_\mu$ satisfies
$$\left\| \frac{(\cdot) - \overline{w}}{(\cdot) - w} f(\cdot) \right\|_h = \|f(\cdot)\|_h \quad \text{and} \quad \|f^*(\cdot)\|_h = \|f(\cdot)\|_h,$$
where w is a non-real zero of $f(z)$.
2. Statement (ii) of Lemma 2 implies that $\widehat{\mathcal{H}}_\mu$ is a reproducing kernel Hilbert space [23]. A short computation using the unitarity of Φ_μ shows that the reproducing kernel is $k(z,w) := \langle \xi(z), \xi(w)\rangle$, in others words,
$$f(w) = \langle k(\cdot, w), f(\cdot)\rangle_h, \quad w \in \mathbb{C} \setminus \mathrm{Sp}(A_\infty),$$
for every $f(z) \in \widehat{\mathcal{H}}_\mu$. A similar expression for $k(z,w)$ has been obtained in [25] and also discussed in a more abstract setting in [12,15,23].
3. Let us define $C := \Phi_\mu^{-1} \widehat{C} \Phi_\mu$, where $(\widehat{C} f)(z) = \overline{f(\bar{z})}$ for $f \in \widehat{\mathcal{H}}_\mu$. Because of statement (iii) of Lemma 2 and Proposition 3, it follows that C is a complex conjugation with respect to which both A and μ are real.

The previous results show that $\widehat{\mathcal{H}}_\mu$ looks like an axiomatic de Branges space [8, Theorem 23], the difference being that $\widehat{\mathcal{H}}_\mu$ is a space of meromorphic functions.

Since $\mathrm{Sp}(A_\infty)$ is void of finite accumulation points, there exists entire functions having $\mathrm{Sp}(A_\infty)$ as its zero set [22, Vol. 2, Section 4.6, Theorem 10.1]. Let $p(z)$ be one of these entire functions chosen so that $\overline{p(z)} = p(\bar{z})$. This last property is found among the entire functions with all zeros in $\mathrm{Sp}(A_\infty)$ since $\mathrm{Sp}(A_\infty) \subset \mathbb{R}$.

Now define $\widehat{\mathcal{B}}_\mu := p(z)\widehat{\mathcal{H}}_\mu$. Since the functions in $\widehat{\mathcal{H}}_\mu$ may have simple poles only at $\mathrm{Sp}(A_\infty)$, $\widehat{\mathcal{B}}_\mu$ is a linear space of entire functions. We then have the following corollary to Lemma 2 and Proposition 3, whose proof is omitted:

Corollary 1. *The space $\widehat{\mathcal{B}}_\mu$, equipped with the inner product*

$$\langle F(\cdot), G(\cdot)\rangle_h := \int_{\mathbb{R}} \frac{\overline{F(x)}G(x)}{|p(x)|^2} \, dm_h(x),$$

is an axiomatic de Branges space. Moreover, the mapping $f(z) \mapsto F(z) = p(z)f(z)$ is a linear isometry from $\widehat{\mathcal{H}}_\mu$ onto $\widehat{\mathcal{B}}_\mu$.

In view of Corollary 1 and the results of Section 3, there are sampling formulae analogous to (3.4) and (3.5) for functions in the de Branges space $\widehat{\mathcal{B}}_\mu$. We consider that sampling in both spaces, $\widehat{\mathcal{H}}_\mu$ and $\widehat{\mathcal{B}}_\mu$, may be of interest in applications.

We conclude this section with a comment about the representation of the operators A_h as operators on $\widehat{\mathcal{H}}_\mu$. A simple computation shows that, for every $h \in \mathbb{R}$, A_h is transformed by Φ_μ into a quasi-multiplication operator, in the sense that

$$\widehat{A_h \varphi}(z) = \frac{1}{F_h(z)} \langle \mu, \varphi\rangle + z\widehat{\varphi}(z) \tag{4.1}$$

for every $\varphi \in \mathrm{Dom}(A)$. This is obviously the multiplication operator in $\Phi_\mu \, \mathrm{Dom}(A_\infty)$. Moreover, (4.1) reduces to the multiplication operator in a weak sense; indeed,

$$\langle \widehat{\varphi}(\cdot), \widehat{A_h \psi}(\cdot)\rangle_h = \langle \widehat{\varphi}(\cdot), (\cdot)\widehat{\psi}(\cdot)\rangle_h$$

for every $\varphi, \psi \in \mathrm{Dom}(A)$.

5. Examples

Rank-one perturbations of a Jacobi matrix. Consider the following semi-infinite Jacobi matrix

$$\begin{pmatrix} q_1 & b_1 & 0 & 0 & \cdots \\ b_1 & q_2 & b_2 & 0 & \cdots \\ 0 & b_2 & q_3 & b_3 & \\ 0 & 0 & b_3 & q_4 & \ddots \\ \vdots & \vdots & & \ddots & \ddots \end{pmatrix} \tag{5.1}$$

with $q_n \in \mathbb{R}$ and $b_n > 0$ for $n \in \mathbb{N}$, and define in the Hilbert space $l^2(\mathbb{N})$ the operator J in such a way that its matrix representation with respect to the canonical basis $\{\delta_n\}_{n=1}^{\infty}$ in $l^2(\mathbb{N})$ is (5.1). By this definition, J (cf. [2, Section 47]) is the minimal closed symmetric operator satisfying

$$\langle \delta_n, J\delta_n \rangle = q_n, \qquad \langle \delta_{n+1}, J\delta_n \rangle = \langle \delta_n, J\delta_{n+1} \rangle = b_n,$$
$$\langle \delta_n, J\delta_{n+k} \rangle = \langle \delta_{n+k}, J\delta_n \rangle = 0, \quad n \in \mathbb{N},\ k \in \mathbb{N} \setminus \{1\}.$$

The Jacobi operator J may have deficiency indices $(1,1)$ or $(0,0)$ [1, Chapter 4, Section 1.2], [28, Corollary 2.9]. For this example we consider J to be self-adjoint, i.e., the case of deficiency indices $(0,0)$. We also assume that J has only discrete spectrum. Our family of self-adjoint operators is given by

$$J_h := J + h\langle \delta_1, \cdot \rangle \delta_1, \quad h \in \mathbb{R}. \tag{5.2}$$

It is relevant to note that δ_1 is a cyclic vector for J since the matrix elements b_n are always assumed to be different from zero.

One can study J through the following second order difference system

$$b_{n-1} f_{n-1} + q_n f_n + b_n f_{n+1} = z f_n, \quad n > 1,\ z \in \mathbb{C}, \tag{5.3}$$

with boundary condition

$$q_1 f_1 + b_1 f_2 = z f_1. \tag{5.4}$$

If one sets $f_1 = 1$, then f_2 is completely determined by (5.4). Having f_1 and f_2, Eq. (5.3) gives all the other elements of a sequence $\{f_n\}_{n=1}^{\infty}$ that formally satisfies (5.3) and (5.4). f_n is a polynomial of z of degree $n-1$, so we denote $f_n =: P_{n-1}(z)$. The polynomials $P_n(z)$, $n = 0, 1, 2, \ldots$, are referred to as the polynomials of the first kind associated with the matrix (5.1). The polynomials of the second kind $Q_n(z)$, $n = 0, 1, 2, \ldots$, associated with (5.1) are defined as the solutions of

$$b_{n-1} f_{n-1} + q_n f_n + b_n f_{n+1} = z f_n, \quad n \in \mathbb{N} \setminus \{1\},$$

under the assumption that $f_1 = 0$ and $f_2 = b_1^{-1}$. Then

$$Q_{n-1}(z) := f_n, \quad \forall n \in \mathbb{N}.$$

$Q_n(z)$ is a polynomial of degree $n-1$.

Let $P(z) = \{P_n(z)\}_{n=0}^{\infty}$ and $Q(z) = \{Q_n(z)\}_{n=0}^{\infty}$. Then, classical results in the theory of Jacobi matrices [1] give us the following expression for $\xi(z)$ defined in Section 3,

$$\xi(z) = P(z) + \frac{1}{F(z)} Q(z),$$

where $F(z)$ is the function given by (2.2) with $h = 0$. In this context $F(z)$ is referred to as the Weyl function of J and may be determined by

$$F(z) = -\lim_{n\to\infty} \frac{1}{w_n(z)}, \qquad w_n(z) := \frac{P_n(z)}{Q_n(z)}, \tag{5.5}$$

where the convergence is uniform on any compact subset of $\mathbb{C} \setminus \mathrm{Sp}(J)$ [1, Sections 2.4, 4.2].

The operator J_∞ corresponds in this case to the operator in $l^2(2,\infty)$ whose matrix representation is (5.1) with the first column and row removed.

For any $f \in \widehat{\mathcal{H}}_{\delta_1}$ there is a sequence $\{\varphi_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$ such that

$$f(z) = \sum_{k=1}^{\infty} \left(P_{k-1}(z)\varphi_k + \frac{\varphi_k}{F(z)} Q_{k-1}(z) \right).$$

Note that the poles of f are the eigenvalues of J_∞.

By Proposition 1 we have the following interpolation formula

$$f(z) = \lim_{n\to\infty} \sum_{x_j \in \mathrm{Sp}(J_h)} \frac{h - w_n(z)}{(x_j - z) w_n'(x_j)} f(x_j), \quad h \in \mathbb{R}. \tag{5.6}$$

Indeed, one can write (5.6) on the basis of (3.4) using the uniform convergence of the limit and the series in (5.5) and (3.4), respectively, and the fact that $w_n'(z)$ is also uniform convergent.

One-dimensional harmonic oscillator. On $\mathcal{H} = L^2(\mathbb{R}, dx)$, we consider the differential operator

$$A := -\frac{d^2}{dx^2} + x^2,$$

which is essentially self-adjoint on $C_0^\infty(\mathbb{R})$. The eigenvalues are $2n+1$ for $n \in \mathbb{N} \cup \{0\}$; the corresponding eigenfunctions are

$$\phi_n(x) = \pi^{-1/4}\left(2^n n!\right)^{-1/2} e^{-x^2/2} H_n(x),$$

where $H_n(x)$ are the Hermite polynomials.

A cyclic vector for the operator A is

$$\mu(x) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{1/2}} \phi_n(x) = \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}(x^2 - 2\sqrt{2}x + 1)}, \tag{5.7}$$

where the last equality follows from a quick look to the generating function of the Hermite polynomials.

Now, for every $z \notin \operatorname{Sp}(A)$,

$$F(z) = \langle \mu, (A_h - zI)^{-1}\mu \rangle = \sum_{n=0}^{\infty} \frac{1}{n!(2n+1-z)}.$$

An elementary argument involving series of partial fractions then shows that

$$F(z) = \frac{1}{2\cos\frac{\pi z}{2}} \int_{-\pi}^{\pi} e^{-\cos\theta + i\sin\theta} e^{i\frac{1-z}{2}\theta}\, d\theta. \tag{5.8}$$

(See, for instance, [7].)

The mapping Φ_μ is defined by the vector-valued function

$$\xi(z; x) = \frac{[(A - \bar{z}I)^{-1}\mu](x)}{F(\bar{z})} = \frac{1}{F(\bar{z})} \int_{-\infty}^{\infty} K(z; x, y)\mu(y)\, dy,$$

where the Green's function $K(z; x, y)$ is given by (see [29])

$$K(z; x, y) = -\frac{\pi^{1/2}}{2\Gamma(\frac{1+z}{2})\cos\frac{\pi z}{2}} \times \begin{cases} D_{\frac{z-1}{2}}(2^{1/2}x) D_{\frac{z-1}{2}}(-2^{1/2}y), & y \leqslant x, \\ D_{\frac{z-1}{2}}(-2^{1/2}x) D_{\frac{z-1}{2}}(2^{1/2}y), & y > x. \end{cases} \tag{5.9}$$

In the last expression, $D_p(x)$ denotes the parabolic cylinder function of order p. Therefore, the associated linear space $\widehat{\mathcal{H}}_\mu$ is the set of functions of the form

$$f(z) = \frac{1}{F(z)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(z; x, y)\mu(y)\varphi(x)\, dy\, dx, \quad \varphi(x) \in L^2(\mathbb{R}, dx). \tag{5.10}$$

These functions may have simple poles at the zeros of the function $F(z)$.

As we have said before, different sets of sampling points are given by $\operatorname{Sp}(A + h\langle \mu, \cdot \rangle \mu)$, $h \in \mathbb{R}$. For $h = 0$, we have $\operatorname{Sp}(A) = \{2n+1\colon\ n \in \mathbb{N} \cup \{0\}\}$ so in that case

$$f(z) = \sum_{n=0}^{\infty} \frac{G(z)}{(z - 2n - 1)G'(2n+1)} f(2n+1), \quad G(z) := 1/F(z),$$

for every $f(z)$ of the form (5.10).

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