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A study of the orthogonal polynomials associated with the quantum harmonic oscillator on constant curvature spaces

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Recently, Cariñena, *et al.* [Ann. Phys. **322**, 434 (2007)] introduced a new family of orthogonal polynomials that appear in the wave functions of the quantum harmonic oscillator in two-dimensional constant curvature spaces. They are a generalization of the Hermite polynomials and will be called *curved Hermite polynomials* in the following. We show that these polynomials are naturally related to the relativistic Hermite polynomials introduced by Aldaya *et al.* [Phys. Lett. A **156**, 381 (1991)], and thus are Jacobi polynomials. Moreover, we exhibit a natural bijection between the solutions of the quantum harmonic oscillator on negative curvature spaces and on positive curvature spaces. At last, we show a maximum entropy property for the ground states of these oscillators. © *2009 American Institute of Physics.*
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I. INTRODUCTION

The relativistic Hermite polynomials (RHPs) were introduced in 1991 by Aldaya *et al.*[2] in a generalization of the theory of the quantum harmonic oscillator to the relativistic context: the resulting wave functions verify the Klein–Gordon equation associated with the anti-de Sitter metric

$$ds^2 = c^2\alpha^2 dt^2 - \alpha^{-2}dx^2,$$

where $\alpha = \sqrt{1+(\omega^2/c^2)x^2}$. In a study by Nagel,[3] these polynomials were later related to the Gegenbauer polynomials. For this reason, as underlined by Ismail in Ref. 4, they do not deserve any special study since their properties can be deduced from those of the well-known Jacobi polynomials—a class of polynomials that includes the Gegenbauer polynomials. In a later paper,[5] Aldaya *et al.* gave a group-theoretical construction of the RHP and derived the creation and annihilation operators and a Bargmann–Fock representation. Further studies include the determination of the distribution of zeros of the wave function of the relativistic harmonic oscillator[6] by Zarzo *et al.*

Recently, Cariñena *et al.*[1] studied the behavior of the quantum harmonic oscillator on the sphere $\mathcal{S}^2$ and on the hyperbolic plane and showed that the Schrödinger equation can be analytically solved; the solutions are an extension of the wave functions of the quantum harmonic oscillator where the usual Hermite polynomials are replaced by some new polynomials—that we will call here *curved Hermite polynomials* (CHPs). In Sec. II, we show that these CHPs are in fact Jacobi polynomials. Then we provide a geometric link between the wave functions on constant negative and positive curvature spaces. Section V exhibits an entropic characterization of these oscillators.

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TABLE I. Examples of RH, Gegenbauer, and CH polynomials.

n	RHP $H_n^\mu(X)$	Gegenbauer $C_n^\nu(X)$	CHP $\mathcal{H}_n^{\mathcal{N}}(X)$
0	1	1	1
1	$2X$	$2\nu X$	$\left(2-\frac{1}{\mathcal{N}}\right)X$
2	$2\left(-1+X^2\left(2+\frac{1}{\mu}\right)\right)$	$2\nu(\nu+1)X^2-\nu$	$\left(2-\frac{3}{\mathcal{N}}\right)\left(-1+2\left(1-\frac{1}{\mathcal{N}}\right)X^2\right)$
3	$4\left(1+\frac{1}{\mu}\right)\left(-3X+X^3\left(2+\frac{1}{\mu}\right)\right)$	$2\nu(\nu+1)\left(-X+\frac{2(\nu+2)}{3}X^3\right)$	$\left(2-\frac{3}{\mathcal{N}}\right)\left(2-\frac{5}{\mathcal{N}}\right)\left(-3X+2\left(1-\frac{2}{\mathcal{N}}\right)X^3\right)$

II. RELATIVISTIC HERMITE AND CURVED HERMITE POLYNOMIALS: DEFINITIONS AND NOTATIONS

A. Relativistic Hermite polynomials

The RHP H_n^μ of degree n and parameter $\mu \neq 0$ is defined[2] by the Rodrigues formula

$$H_n^\mu(X) = (-1)^n\left(1+\frac{X^2}{\mu}\right)^{\mu+n}\frac{d^n}{dX^n}\left(1+\frac{X^2}{\mu}\right)^{-\mu}. \tag{1}$$

Examples of these polynomials are given in Table I. They are extensions of the classical Hermite polynomials

$$H_n(X) = (-1)^n \exp(X^2)\frac{d^n}{dX^n}\exp(-X^2) \tag{2}$$

that can be obtained as the limit case

$$\lim_{\mu\to+\infty} H_n^\mu(X) = H_n(X).$$

The RHPs are orthogonal on the real line in the following sense:

$$\int_{-\infty}^{+\infty} H_n^\mu(X)H_m^\mu(X)\left(1+\frac{X^2}{\mu}\right)^{-\mu-1-(m+n)/2}dX = c_n\delta_{m,n}, \quad \forall\, \mu > -\frac{1}{2}, \quad \forall\, m,n \in \mathbb{N}. \tag{3}$$

with

$$c_n = \pi\frac{n!\,2^{1-2\mu}\Gamma(2\mu+n)}{\mu^{n-1/2}(n+\mu)\Gamma^2(\mu)}.$$

We remark that (3) expresses an unconventional orthogonality since it holds with respect to varying weights, that is, to a measure which depends on the degrees of the polynomials.

B. Gegenbauer polynomials

The Gegenbauer polynomial C_n^ν of degree n and parameter $\nu \neq 0$ is defined by the Rodrigues formula

$$C_n^\nu(X) = \alpha_{n,\nu}(-1)^n(1-X^2)^{1/2-\nu}\frac{d^n}{dX^n}(1-X^2)^{n+\nu-1/2},$$

with

$$\alpha_{n,\nu} = \frac{(2\nu)_n}{2^n n!\left(\nu + \frac{1}{2}\right)_n}$$

[we use Pochhammer's notation $(a)_n = \Gamma(a+n)/\Gamma(a)$]. These polynomials are orthogonal with respect to the measure $(1-X^2)^{\nu-1/2}dX$ on $[-1,+1]$: for $\nu \neq 0$,

$$\int_{-1}^{+1} C_n^\nu(X) C_m^\nu(X)(1-X^2)^{\nu-1/2} dX = \pi \frac{2^{1-2\nu}\Gamma(n+2\nu)}{n!\,(n+\nu)\Gamma(\nu)} \delta_{m,n}.$$

C. Curved Hermite polynomials

The CHP of degree n and parameter $\mathcal{N} \neq 0$ is defined by the Rodrigues formula

$$\mathcal{H}_n^{\mathcal{N}}(X) = (-1)^n \left(1 + \frac{X^2}{\mathcal{N}}\right)^{\mathcal{N}+1/2} \frac{d^n}{dX^n} \left(1 + \frac{X^2}{\mathcal{N}}\right)^{n-\mathcal{N}-1/2},$$

where $X \in \mathbb{R}$ for $\mathcal{N} > 0$ and $X \in [-\sqrt{-\mathcal{N}}, +\sqrt{-\mathcal{N}}]$ when $\mathcal{N} < 0$. Since the cases $\mathcal{N} > 0$ and $\mathcal{N} < 0$ differ greatly, we will denote the CHPs with negative parameter $\mathcal{N}$ as

$$\mathcal{H}_n^{\mathcal{N}}(X) = \mathfrak{C}_n^\nu(X), \quad \nu = -\mathcal{N} > 0.$$

The CHPs are orthogonal with respect to the measure $(1+X^2/\mathcal{N})^{-\mathcal{N}-1/2}dX$ on the real line for $\mathcal{N} > 0$,

$$\int_{\mathbb{R}} \mathcal{H}_n^{\mathcal{N}}(X)\mathcal{H}_m^{\mathcal{N}}(X)\left(1 + \frac{X^2}{\mathcal{N}}\right)^{-\mathcal{N}-1/2} dX = a_n \delta_{m,n}, \quad m+n < 2\mathcal{N}, \tag{4}$$

and on the interval $[-\sqrt{\nu}, +\sqrt{\nu}]$ with respect to the measure $(1-X^2/\nu)^{\nu-1/2}dX$ when $\mathcal{N} < 0$:

$$\int_{-\sqrt{\nu}}^{+\sqrt{\nu}} \mathfrak{C}_n^\nu(X)\mathfrak{C}_m^\nu(X)\left(1 - \frac{X^2}{\nu}\right)^{\nu-1/2} dX = b_n \delta_{m,n}, \quad \forall\, m,n \in \mathbb{N},$$

for some positive constants a_n and b_n.

III. LINKS BETWEEN RELATIVISTIC HERMITE AND CURVED HERMITE POLYNOMIALS

The following theorems show that the family of CHPs is related to the set of RHPs and Gegenbauer polynomials in a simple way.

A. Link between CHP with positive parameter and RHP

Theorem: *The CHP $\mathcal{H}_n^{\mathcal{N}}(X)$ of degree n and parameter $\mathcal{N} > 0$ is related to the RHP $H_n^\mu(X)$ of the same degree n and parameter μ* as

$$\mathcal{H}_n^{\mathcal{N}}(X) = \left(\frac{\mu}{\mathcal{N}}\right)^{n/2} H_n^\mu\left(X\sqrt{\frac{\mu}{\mathcal{N}}}\right), \tag{5}$$

with

$$\mu = \mathcal{N} + 1/2 - n.$$

Proof: Denote $\mu = \mathcal{N} + \frac{1}{2} - n$; then

$$\mathcal{H}_n^{\mathcal{N}}(X) = (-1)^n\left(1+\frac{X^2}{\mathcal{N}}\right)^{\mu+n}\frac{d^n}{dX^n}\left(1+\frac{X^2}{\mathcal{N}}\right)^{-\mu}.$$

But by the Rodrigues formula (1)

$$(-1)^n\left(1+\frac{X^2}{\mathcal{N}}\right)^{\mu+n}\frac{d^n}{dX^n}\left(1+\frac{X^2}{\mathcal{N}}\right)^{-\mu} = \left(\frac{\mu}{\mathcal{N}}\right)^{n/2}H_n^{\mu}\left(X\sqrt{\frac{\mu}{\mathcal{N}}}\right),$$

so that the result holds. □

B. Link between CHP with negative parameter and Gegenbauer polynomials

The same kind of result is now obtained for the CHPs with negative parameter, where the Gegenbauer polynomials now play the role of the RHPs.

Theorem: *The CHP $\mathfrak{C}_n^{\nu}(X)$ of degree n and parameter $\mathcal{N}=-\nu<0$ is related to the Gegenbauer polynomial C_n^{ν} of the same degree n and parameter ν as*

$$\mathfrak{C}_n^{\nu}(X) = \frac{1}{\alpha_{n,\nu}}\nu^{-n/2}C_n^{\nu}\left(\frac{X}{\sqrt{\nu}}\right). \tag{6}$$

Proof: With $\nu=-\mathcal{N}$, we deduce

$$\mathfrak{C}_n^{\nu}(X) = (-1)^n\left(1-\frac{X^2}{\nu}\right)^{1/2-\nu}\frac{d^n}{dX^n}\left(1-\frac{X^2}{\nu}\right)^{n+\nu-1/2}.$$

It can be easily checked that

$$\left(1-\frac{X^2}{\nu}\right)^{1/2-\nu}\frac{d^n}{dX^n}\left(1-\frac{X^2}{\nu}\right)^{n+\nu-1/2} = \frac{1}{\alpha_{n,\nu}}\left(\frac{1}{\nu}\right)^{n/2}C_n^{\nu}\left(\frac{X}{\sqrt{\nu}}\right),$$

so that the result holds. □

C. Link between CHP with positive parameter and CHP with negative parameter

Nagel's identity[3]

$$H_n^{\mu}(X) = \frac{n!}{\mu^{n/2}}\left(1+\frac{X^2}{\mu}\right)^{n/2}C_n^{\mu}\left(\frac{X/\sqrt{\mu}}{\sqrt{1+\frac{X^2}{\mu}}}\right) \tag{7}$$

shows that the RHPs $H_n^{\mu}(X)$ are Gegenbauer polynomials $C_n^{\mu}(X)$ in a different variable;[4] the following theorem shows that the same kind of connection can be derived between CHPs with positive parameter $\mathcal{H}_n^{\mathcal{N}}(X)$ and CHPs with negative parameter $\mathfrak{C}_n^{\nu}(X)$.

Theorem: *The CHP $\mathcal{H}_n^{\mathcal{N}}(X)$ of degree n and parameter $\mathcal{N}>0$ is related to the CHP $\mathfrak{C}_n^{\nu}(X)$ of the same degree n and parameter ν by the following formula:*

$$\mathcal{H}_n^{\mathcal{N}}(X\sqrt{\mathcal{N}}) = \alpha_{n,\nu}n!\left(\frac{\nu}{\mathcal{N}}\right)^{n/2}(1+X^2)^{n/2}\mathfrak{C}_n^{\nu}\left(\frac{X\sqrt{\nu}}{\sqrt{1+X^2}}\right), \tag{8}$$

where

$$\nu = \mathcal{N}+1/2-n.$$

Proof: This is a direct consequence of Nagel's identity (7) and equalities (5) and (6). □

These results are summarized in Table II.

TABLE II. Summary of the links between RHP and CHP.

RH $H_n^{\mu}(X)$	$\overset{(5)}{\rightarrow}$	CH $\mathcal{H}_n^{\mathcal{N}}(X)$ with $\mathcal{N}>0$
(7) $\downarrow$		$\downarrow$(8)
Gegenbauer $C_n^{\nu}(X)$	$\overset{(6)}{\rightarrow}$	CH $\mathfrak{C}_n^{\nu}(X)$ with $\mathcal{N}=-\nu<0$

IV. A GEOMETRIC CORRESPONDENCE

To each of the families of orthogonal polynomials studied above is associated a set of orthogonal wave functions that we denote

$$\mathfrak{c}_n^{\nu}(X) = (1-X^2)^{\nu/2-1/4}C_n^{\nu}(X), \quad \mathfrak{h}_n^{\mathcal{N}}(X) = \left(1+\frac{X^2}{\mathcal{N}}\right)^{-\mathcal{N}/2-1/4}\mathcal{H}_n^{\mathcal{N}}(X)$$

and

$$h_n^{\mu}(X) = \left(1+\frac{X^2}{\mu}\right)^{-(\mu+1+n)/2}H_n^{\mu}(X)$$

in the cases of the Gegenbauer polynomial, CHP, and RHP, respectively. We study first a geometric interpretation of Nagel's identity in the case of the relativistic harmonic oscillator.

A. The relativistic harmonic oscillator

Let us denote

$$f_{n,\mu}(X) = |h_n^{\mu}(X)|^2 \quad \text{and} \quad g_{n,\nu}(Y) = |\mathfrak{c}_n^{\nu}(Y)|^2$$

the probability densities associated with the orthogonal functions defined above. A geometric interpretation of Nagel's identity is as follows.

Theorem: *If a random variable X is distributed according to $f_{n,\mu}$ then the random variable*

$$Y = \frac{X/\sqrt{\mu}}{\sqrt{1+\frac{X^2}{\mu}}} \tag{9}$$

is distributed according to $g_{n,\nu}$ with $\nu=\mu$.

Proof: The distribution of Y defined by (9) is

$$f_Y(Y) = (1-Y^2)^{-3/2}f_{n,\mu}\left(\frac{\sqrt{\mu}Y}{\sqrt{1-Y^2}}\right)$$

so that by Nagel's identity

$$f_Y(Y) = (1-Y^2)^{n+\mu-1/2}\left(1+\frac{Y^2}{1-Y^2}\right)^n|C_n^{\mu}(Y)|^2 = (1-Y^2)^{\mu-1/2}|C_n^{\mu}(Y)|^2 = g_{n,\nu}(Y).$$

□

The application defined by (9) maps $\mathbb{R}$ to the interval $[-\sqrt{\mu};+\sqrt{\mu}]$; the inverse application $Y\mapsto X=Y\sqrt{\mu}/\sqrt{1-Y^2}$ is nothing but the two-dimensional (2D) version of the gnomonic projection. It is illustrated in Fig. 1.

B. The harmonic oscillator on the sphere and on the hyperbolic plane

We now extend the preceding result to the case of the harmonic oscillator on spaces of constant curvature. All scaling constants are set to unity for simplicity.

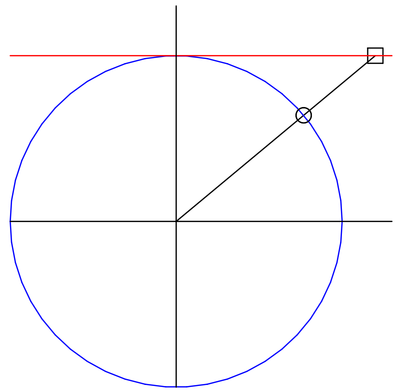

FIG. 1. (Color online) The 2D gnomonic projection: the point □ on the horizontal line is the gnomonic projection of the point ○ on the circle of radius $\sqrt{\mu}$; the abscissas of these points are, respectively, X and Y as given by (9).

Theorem: *Consider the harmonic oscillator on the hyperbolic plane*[1] *described by its coordinates* (x,y) *and with probability density*

$$f_{m,n,\mu}(z,y) = |\mathfrak{h}_n^{\mu-m-1/2}(y)|^2 |\mathfrak{h}_m^{\mu}(z)|^2$$

(with $z=x/\sqrt{1+y^2}$*). If this system is transformed as*

$$X = \frac{x}{\sqrt{1+x^2+y^2}}, \quad Y = \frac{y}{\sqrt{1+x^2+y^2}} \Leftrightarrow x = \frac{X}{\sqrt{1-X^2-Y^2}}, \quad y = \frac{Y}{\sqrt{1-X^2-Y^2}} \tag{10}$$

then the new system described by coordinates (X,Y) *has probability density*

$$g_{m,n,\nu}(Z,Y) = |\mathfrak{c}_m^{\nu+n+1/2}(Y)|^2 |\mathfrak{c}_n^{\nu}(Z)|^2,$$

where

$$Z = \frac{X}{\sqrt{1-Y^2}} \quad \text{and} \quad \nu = \mu - m - n.$$

Proof: As a function of variables (x,y), the density of the harmonic oscillator writes, in terms of Gegenbauer polynomials, as

$$f_{m,n,\mu}(x,y) = (1+y^2)^n (1+x^2+y^2)^{-\mu+m-1/2} \left| C_n^{\mu-m-n}\left(\frac{y}{\sqrt{1+y^2}}\right)\right|^2 \left| C_m^{\mu-m+1/2}\left(\frac{x}{\sqrt{1+x^2+y^2}}\right)\right|^2.$$

We now perform the change of variable (10); the distribution of the new system is obtained as

$$\tilde{f}_{m,n,\mu}(X,Y)d\mu(X,Y) = f_{m,n,\mu}(x,y)d\mu(x,y)$$

with the measure[1]

$$d\mu(x,y) = \frac{dxdy}{\sqrt{1+x^2+y^2}}, \quad d\mu(X,Y) = \frac{dXdY}{\sqrt{1-X^2-Y^2}}.$$

Since the Jacobian of the transformation $(x,y) \mapsto (X,Y)$ is

$$J = (1+x^2+y^2)^{-2} = (1-X^2-Y^2)^2,$$

we deduce

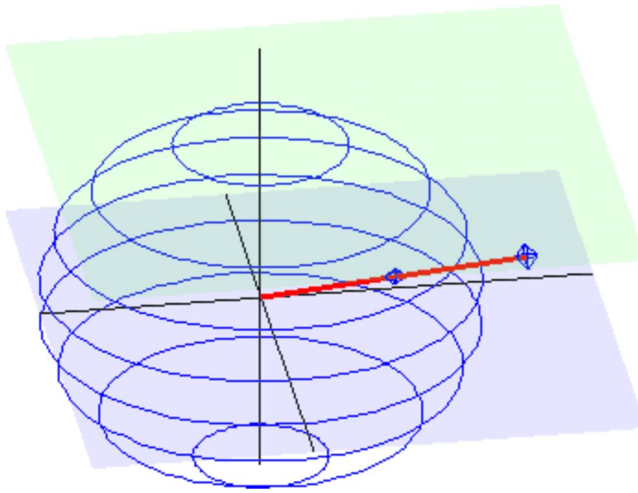

FIG. 2. (Color online) The three-dimensional gnomonic projection: the ◇ point $(x,y,1)$ on the $z=1$ horizontal plane is the gnomonic projection of the ○ point $(X,Y,\sqrt{1-X^2-Y^2})$ on the sphere; coordinates (X,Y) are related to coordinates (x,y) via Eq. (10).

$$\begin{aligned}\widetilde{f}_{m,n,\mu}(X,Y) &= (1-X^2-Y^2)^{-1}(1-X^2-Y^2)^{\mu-m+1/2}\left(1+\frac{Y^2}{1-X^2-Y^2}\right)^n \\ &\quad\times\left|C_n^{\mu-m-n}\left(\frac{Y}{\sqrt{1-X^2}}\right)\right|^2|C_m^{\mu-m+1/2}(X)|^2 \\ &= (1-X^2-Y^2)^{\mu-n-m-1/2}(1-X^2)^n\left|C_n^{\mu-m-n}\left(\frac{Y}{\sqrt{1-X^2}}\right)\right|^2|C_m^{\mu-m+1/2}(X)|^2.\end{aligned}$$

But since the distribution of the harmonic oscillator on the sphere reads, in terms of Gegenbauer polynomials, as

$$g_{m,n,\nu}(X,Y) = (1-Y^2)^{m-1/2}(1-X^2-Y^2)^{\nu}|C_n^{\nu+m+1/2}(Y)|^2\left|C_m^{\nu}\left(\frac{X}{\sqrt{1-Y^2}}\right)\right|^2, \tag{11}$$

we deduce that

$$\widetilde{f}_{m,n,\mu}(X,Y) = g_{n,m,\nu}(Y,X),$$

with

$$\nu = \mu - m - n.$$

□

We note that transformation (10) expresses the fact that the point $[x,y,z]$ is the gnomonic projection of the point $[X,Y,Z]$. This is illustrated in Fig. 2. We note also that the gnomonic projection is the tool used by Higgs[8] to derive the potentials that ensure the closedness of the orbits of a particle moving on the sphere under the action of a conservative central force.

V. AN ENTROPIC APPROACH

In the nonextensive statistical theory,[7] the classical Shannon entropy of a probability density f_X,

$$H = -\int f_X \log f_X,$$

is replaced by the so-called Tsallis entropy

$$H_q = \frac{1}{1-q}\int (f_X - f_X^q),$$

where q is a positive real parameter called the nonextensivity parameter. It can be checked by the L'Hospital rule that the Shannon entropy is the limit case

$$H = \lim_{q\to 1} H_q.$$

The "nonextensivity" term comes from the fact that if A and B are two independent systems with respective entropies $H_q(A)$ and $H_q(B)$ then the entropy of the whole system (A,B) is not the sum of both entropies but

$$H_q(A,B) = H_q(A) + H_q(B) - (1-q)H_q(A)H_q(B).$$

In the standard $q=1$ case, the canonical distribution—that is, the distribution with maximum entropy and given variance σ^2—is the Gaussian distribution

$$f_X(X) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{X^2}{2\sigma^2}\right), \tag{12}$$

and the polynomials orthogonal with respect to the Gaussian measure on the real line are the Hermite polynomials (2); the Hermite *functions* are defined as

$$h_n(X) = \exp\left(-\frac{X^2}{2}\right)H_n(X), \quad n \geq 0,$$

and verify the orthogonality property

$$\int_{\mathbb{R}} h_n(X)h_m(X)dX = \sqrt{\pi}2^n n!\,\delta_{m,n}, \quad \forall\, m,n \in \mathbb{N}.$$

In the nonextensive case, the canonical distributions with variance σ^2 are called q-Gaussian distributions and read, for $q<1$,

$$f_X(X;q) = \frac{\Gamma\left(\frac{2-q}{1-q}+\frac{1}{2}\right)}{\Gamma\left(\frac{2-q}{1-q}\right)\sigma\sqrt{\pi d}}\left(1-\frac{X^2}{d\sigma^2}\right)_+^{1/(1-q)}, \quad d = 2\frac{2-q}{1-q}+1,$$

with notation $(x)_+=\max(0,x)$, and for $1<q<\frac{5}{3}$,

$$f_X(X;q) = \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)\sigma\sqrt{\pi(m-2)}}\left(1+\frac{X^2}{(m-2)\sigma^2}\right)^{1/(1-q)}, \quad m = \frac{2}{q-1}-1.$$

It can be easily checked that the limit case $\lim_{q\to 1} f_X(X;q)$ coincides with the Gaussian distribution (12).

To our best knowledge, the polynomials orthogonal with respect to the q-Gaussian distributions have not been studied in the nonextensive theory. They can be deduced from the results of

TABLE III. polynomials orthogonal with respect to the q-Gaussian measure and the corresponding orthogonal functions.

	q	Orthogonal function	Domain
Gegenbauer polynomials	$\frac{2\nu-3}{2\nu-1}<1$	$\mathfrak{c}_n^{\nu}(X)=(1-X^2)^{\nu/2-1/4}C_n^{\nu}(X)$	$[-1;1]$
CHPs	$\frac{2\mathcal{N}+3}{2\mathcal{N}+1}>1$	$\mathfrak{h}_n^{\mathcal{N}}(X)=\left(1+\frac{X^2}{\mathcal{N}}\right)^{-\mathcal{N}/2-1/4}\mathcal{H}_n^{\mathcal{N}}(X)$	$\mathbb{R}$
RHPs	$\frac{2+\mu+\frac{m+n}{2}}{1+\mu+\frac{m+n}{2}}>1$	$h_n^{\mu}(X)=\left(1+\frac{X^2}{\mu}\right)^{-(\mu+1+n)/2}H_n^{\mu}(X)$	$\mathbb{R}$

Sec. II and are indicated in Table III: note that we consider here the q-Gaussian distributions with scaling constants normalized to unity. In the case $q<1$, these polynomials coincide with the Gegenbauer polynomials. In the case $q>1$, both the CHPs and the RHPs are orthogonal with respect to the q-Gaussian measure. In all cases, the corresponding orthogonal function and the domain of definition I are given; each of the corresponding orthogonal function—let us call it generically $w_n(X)$—verifies the orthogonality property

$$\int_I w_n(X)w_m(X)dX=K_n\delta_{m,n}$$

for some positive constant K_n.

It turns out that, as mentioned in Sec. I, the orthogonal functions above cited describe the behaviors of physically significant systems:

- As shown in Ref. 1 the probability density that describes the harmonic oscillator on a 2D surface of constant negative curvature κ (typically the hyperbolic plane) is

$$f_{m,n,\mathcal{N}}(Y,Z)=|\mathfrak{h}_n^{\mathcal{N}-m-1/2}(Y)|^2|\mathfrak{h}_m^{\mathcal{N}}(Z)|^2, \quad (13)$$

  with $Z=X/\sqrt{1+Y^2}$ (note that y and z are not independent variables). The parameter $\mathcal{N}$ here is defined as

$$\mathcal{N}=-\frac{m\alpha}{\hbar\kappa}>0,$$

  where m is the mass of the oscillator.
- As shown in the same reference, the probability density that describes the harmonic oscillator on a 2D surface of constant positive curvature κ (typically the sphere) is

$$g_{m,n,\nu}(Y,Z)=|\mathfrak{c}_n^{\nu+m+1/2}(Y)|^2|\mathfrak{c}_m^{\nu}(Z)|^2, \quad (14)$$

  with $Z=X/\sqrt{1-Y^2}$ and

$$\nu=\frac{m\alpha}{\hbar\kappa}>0.$$

- The harmonic oscillator in the relativistic context as described in Ref. 2 has probability density

$$f_{n,\mu}(X)=|h_n^{\mu}(X)|^2,$$

  where the parameter $\mu>0$ is defined as

TABLE IV. Values of the nonextensivity parameter q associated with the three harmonic oscillators.

Positive curvature κ	$\frac{2\nu-3}{2\nu-1}$	$\frac{\nu+m-1}{\nu+m}$
Negative curvature κ	$\frac{2\mathcal{N}+3}{2\mathcal{N}+1}$	$\frac{\mathcal{N}-m+1}{\mathcal{N}-m}$
RHP	$1+\frac{1}{1+\frac{mc^2}{\hbar\mu}+\frac{m+n}{2}}$	

$$\mu = \frac{mc^2}{\hbar\omega}$$

so that the nonrelativistic limit $c \to +\infty$ corresponds to the classical $\mu \to +\infty$ Hermite polynomials.

Thus the behavior of the harmonic oscillator—in either the relativistic case or the case of constant curvature geometries—can be related to the nonextensive framework, giving in each case an explicit physical interpretation of the nonextensivity parameter q—a crucial problem in the nonextensivity context—in terms of the physical constants of the harmonic oscillator, as shown in Table IV. We note that in the case of the harmonic oscillator on the sphere or of the hyperbolic plane, the densities (13) and (14) are separable functions in the variables $Z=X/\sqrt{1+Y^2}$ and $Z=\frac{X}{\sqrt{1-Y^2}}$, respectively, and Y, each term inducing a different value of q.

VI. CONCLUSION

We have shown that the CHPs are Jacobi polynomials and we have exhibited their links with RHPs; moreover, we have shown that there exists a natural bijection between the negative and the positive curvature cases and a geometric interpretation of this bijection. These results hold in the 2D case only. An important consequence of our results is that they provide a geometrical and physical interpretation of the nonextensivity parameter q.

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