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A property of the planar measure of the lemniscates ☆

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Abstract

In this paper, we establish the following conjecture: There exists a constant K such that every lemniscate $E(\alpha, c)$, $\alpha \in \mathbb{C}^n$, $c > 0$, contains a disk $B(\alpha, c)$ with $\mu(E(\alpha, c)) \leqslant K\mu(B(\alpha, c))$, where μ is the planar measure. We prove this conjecture for any family of lemniscates with at the most three foci and for any family of lemniscates where its foci satisfy a suitable condition.

Keywords: Lemniscates; Planar measure; Inequalities

1. Introduction

Let $\mathbb{C}$ be the set of complex numbers and let $\mu(A)$ be the planar measure of the set $A \subset \mathbb{C}$. Fix $n \in \mathbb{N}$, $n \geqslant 2$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $c > 0$, as it is well known the set of points satisfying

$$\left\{ z \in \mathbb{C}\colon \prod_{j=1}^{n} |z - \alpha_j| \leqslant c \right\} \tag{1.1}$$

is called a *lemniscate* in $\mathbb{C}$ and will be designated by $E(\alpha, c)$. The points α_j, $1 \leqslant j \leqslant n$, are called the *foci* of the lemniscate and c its *radius*.

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Several geometric properties over the lemniscates have been extensively studied. A famous lemma of Cartan estimates the size of the lemniscate $E(\alpha, c)$. See [1,7] and [8] for further details and extensions of this lemma. In [4] the authors search on the measure of lemniscatic set; i.e., the intersection of a lemniscate with a disc centered at zero. The problem to estimate the length of the boundary of $E(\alpha, 1)$ is studied in [2,5] and [9]. Other results about the logarithmic capacity and the diameter of a lemniscate can be seem in [4,6] and [9]. We also remark that there are several conjectures in this matter (see [5]).

In this paper we establish the following conjecture about the planar measure of any lemniscate $E(\alpha, c)$ with $\alpha \in \mathbb{C}^n$ and $c > 0$.

Conjecture. *Let $n \in \mathbb{N}$. There exists an absolute constant $\mathcal{K} > 0$ such that for all multi-index $\alpha \in \mathbb{C}^n$ and for all radius c, there exists a circle $B = B(\alpha, c)$ contained in the lemniscate $E(\alpha, c)$ satisfying*

$$\frac{\mu(E(\alpha, c))}{\mu(B(\alpha, c))} \leqslant \mathcal{K}. \tag{1.2}$$

This result for the case of a family of lemniscates with at the most two foci was proved in [3, Lemma 3.3]. Now, we shall prove Conjecture for the case of three foci. Further, if a is a positive number, we shall show the existence of an absolute constant $\mathcal{K} := \mathcal{K}(a) > 0$ verifying (1.2) for all radius c and for all $\alpha \in \mathcal{M}_a$, where

$$\mathcal{M}_a := \left\{\alpha \in \mathbb{C}^n \colon \min_{\alpha_j \neq \alpha_i} |\alpha_j - \alpha_i| \geqslant a \max_{j,i} |\alpha_j - \alpha_i|\right\}.$$

Here, we use the convention $\min_{\alpha_j \neq \alpha_i} |\alpha_j - \alpha_i| = 0$ if α belongs to Δ, the set of multi-index with all its coordinates equals. The last result embraces the case that the foci form a regular polygon.

As we have mentioned in [3, Remark 3.7], if Conjecture is true, we can obtain an extension of the classical Pólya inequality (see [10]) for complex polynomials in L^p spaces, $1 \leqslant p \leqslant \infty$, and an application to multipoint best local approximation.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, we write $R(\alpha) = \{\alpha_j \colon 1 \leqslant j \leqslant n\}$, $|\alpha| = (|\alpha_1|, \ldots, |\alpha_n|)$ and

$$P_\alpha(z) = \prod_{j=1}^{n} (z - \alpha_j).$$

2. Lemniscates with restricted foci

Definition 2.1. For $\alpha \in \mathbb{C}^n$, we define the function $S_\alpha : (-\infty, 0] \to [0, \infty)$ by

$$S_\alpha(r) = \inf\left\{t \geqslant 0 \colon \left|P_\alpha(t)\right| > \left|P_\alpha(r)\right|\right\}.$$

We denote

$$\mathcal{N} := \left\{\alpha \in \mathbb{C}^n \colon \{0, 1\} \subset R(\alpha) \subset [0, 1]\right\}.$$

The following lemma is clear.

Lemma 2.2. *If $\alpha \in \mathcal{N}$, then the function S_α is nonnegative, decreasing, left-continuous on $(-\infty, 0]$ and $|P_\alpha(S_\alpha(r))| = |P_\alpha(r)|$. In addition, the set A_α of discontinuity points of the function S_α is nonempty and has at the most $n - 1$ elements.*

Let $\alpha \in \mathcal{N}$. We suppose that $A_\alpha = \{r_j\colon 1 \leqslant j \leqslant k\}$ where $r_{j-1} > r_j$, $2 \leqslant j \leqslant k$. We denote $r_0 = 0$, $r_{k+1} = -\infty$ and we call

$$s_j = S_\alpha(r_j), \quad t_j = \lim_{r \to r_j^+} S_\alpha(r), \quad 1 \leqslant j \leqslant k,$$

$s_0 = 0$ and $t_{k+1} = \infty$.

We also write $U_j = (r_j, r_{j-1})$ and $I_j = (s_{j-1}, t_j)$, $1 \leqslant j \leqslant k+1$. We will use this notation in the proof of the two following results.

Lemma 2.3. *If $\alpha \in \mathcal{N}$ and S_α is continuous at $r < 0$, then it is differentiable at r and*

$$S'_\alpha(r) = -\frac{|P'_\alpha(r)|}{|P'_\alpha(S_\alpha(r))|}. \tag{2.1}$$

Proof. Let $f_j : I_j \to |P_\alpha|(I_j)$ be the function defined by $f_j(x) = |P_\alpha(x)|$, $1 \leqslant j \leqslant k+1$. Clearly, we have

$$\big(f_j^{-1}\big)'\big(f_j(x)\big) = \frac{1}{|P'_\alpha(x)|}, \quad x \in I_j,\ 1 \leqslant j \leqslant k+1. \tag{2.2}$$

We observe that the function $g(x) = |P_\alpha(x)|$ is differentiable in (r_{k+1}, r_0) and $g'(x) = -|P'_\alpha(x)|$. Since

$$S_\alpha(U_j) = I_j \quad \text{and} \quad f_j\big(S_\alpha(r)\big) = g(r), \quad r \in U_j,\ 1 \leqslant j \leqslant k+1,$$

(2.2) implies

$$\big(f_j^{-1}\big)'\big(g(r)\big) = \frac{1}{|P'_\alpha(S_\alpha(r))|}, \quad r \in I_j,\ 1 \leqslant j \leqslant k+1. \tag{2.3}$$

As $S_\alpha(r) = f_j^{-1}(g(r))$, $r \in U_j$, $1 \leqslant j \leqslant k+1$, from the chain rule and (2.3) we get the lemma. □

Proposition 2.4. *If $\alpha \in \mathcal{N}$, then*

$$\sup_{r<0} \frac{S_\alpha(r)}{|r|} = \max_{r \in A_\alpha} \frac{S_\alpha(r)}{|r|}.$$

Proof. Set the function $f(r) = \frac{S_\alpha(r)}{|r|}$, $r < 0$. By Lemma 2.3, we get

$$f'(r) = \frac{1}{r^2}\left(S_\alpha(r) - \left|\frac{r P'_\alpha(r)}{P'_\alpha(S_\alpha(r))}\right|\right), \quad r \notin A_\alpha.$$

Since $|P_\alpha(S_\alpha(r))| = |P_\alpha(r)|$, the equality

$$P'_\alpha(x) = -P_\alpha(x) \sum_{i=1}^{n} \frac{1}{\alpha_i - x} \tag{2.4}$$

for $x = r$ and $x = S_\alpha(r)$, implies

$$f'(r) = \left|\frac{P_\alpha(S_\alpha(r))}{r^2 P'_\alpha(S_\alpha(r))}\right| \big(\big|L(r)\big| - H(r)\big), \quad r \notin A_\alpha, \tag{2.5}$$

where $L(r)=\sum_{i=1}^{n}\frac{S_\alpha(r)}{\alpha_i-S_\alpha(r)}$ and $H(r)=\sum_{i=1}^{n}\frac{-r}{\alpha_i-r}$. Clearly, for any $r\in U_j$, $1\leqslant j\leqslant k+1$,

$$L(r)=\sum_{\alpha_i>t_j}\frac{S_\alpha(r)}{\alpha_i-S_\alpha(r)}-\sum_{\alpha_i\leqslant s_{j-1}}\frac{S_\alpha(r)}{S_\alpha(r)-\alpha_i}. \tag{2.6}$$

We observe that $L(r)<0$ for $r\notin A_\alpha$. In fact, if $r\in U_{k+1}$ it is obvious. Let j, $1\leqslant j\leqslant k$. A straightforward computation shows that the first term on right member of (2.6) is a decreasing function on U_j, while the second term is an increasing function on U_j. Since $t_j=\lim_{r\to r_j^+}S_\alpha(r)$, from (2.4) we get

$$\lim_{r\to r_j^+}L(r)=0. \tag{2.7}$$

So, $L(r)<0$ for $r\in U_j$.

It is easy to see that H is a decreasing nonnegative function on $(-\infty,0)$ and $|L|$ is an increasing function on U_j, $1\leqslant j\leqslant k+1$. Since, $H(r)$ and $|L(r)|$ tend to n, as r tends to r_{k+1}, then $f'>0$ on U_{k+1}. So,

$$\sup_{r\in U_{k+1}}\frac{S_\alpha(r)}{|r|}=\frac{S_\alpha(r_k)}{|r_k|}. \tag{2.8}$$

We assume that zero is a root of P_α of multiplicity n_0. Clearly, $H(r)$ and $|L(r)|$ tend to n_0, as r tends to r_0. Thus, $f'<0$ on U_1. Consequently,

$$\sup_{r\in U_1}\frac{S_\alpha(r)}{|r|}=\frac{t_1}{|r_1|}. \tag{2.9}$$

For $2\leqslant j\leqslant k$, from (2.7) we have

$$\sup_{r\in U_j}\frac{S_\alpha(r)}{|r|}=\max\left\{\frac{t_j}{|r_j|},\frac{S_\alpha(r_{j-1})}{|r_{j-1}|}\right\}. \tag{2.10}$$

Finally, as $t_j<S_\alpha(r_j)$, $1\leqslant j\leqslant k$, the theorem follows immediately. □

Let $\alpha\in\mathcal{M}_a-\Delta$ and let $C_j(\alpha,c)$, $1\leqslant j\leqslant n$, be the connected component of $E(\alpha,c)$ which contains to α_j. We denote $m_j(\alpha,c)=\max\{|z-\alpha_j|:\ z\in C_j(\alpha,c)\}$ and $m(\alpha,c)=\max\{m_j(\alpha,c):\ 1\leqslant j\leqslant n\}$. Without lost of generality, we assume $m(\alpha,c)=m_1(\alpha,c)$. We consider $\rho_1(\alpha,c)=\min\{|z-\alpha_1|:\ z\in\partial(C_1(\alpha,c))\}$ and $\lambda_1(\alpha)=\max\{|\alpha_j-\alpha_1|:\ 1\leqslant j\leqslant n\}$. Let l_1, $2\leqslant l_1\leqslant n$, be such that $\lambda_1(\alpha)=|\alpha_{l_1}-\alpha_1|$. We call $\beta(\alpha,c)$ to multi-index in $\mathbb{C}^n$ whose jth component is $\frac{\alpha_j-\alpha_1}{\alpha_{l_1}-\alpha_1}$. It is easy to show that

$$z\in E(\alpha,c)\quad\text{if and only if}\quad\frac{z-\alpha_1}{\alpha_{l_1}-\alpha_1}\in E\left(\beta(\alpha,c),\frac{c}{(\lambda_1(\alpha))^n}\right). \tag{2.11}$$

In addition, $|\beta(\alpha,c)|\in\mathcal{N}$, and

$$\left|\beta(\alpha,c)\right|_j>0\quad\text{implies}\quad\left|\beta(\alpha,c)\right|_j\geqslant a. \tag{2.12}$$

From now on, for simplicity, except when it is necessary, we shall omit the dependence on α and c, in each occurrence. We also denote by $D(\alpha_j,\delta)$ the circle in $\mathbb{C}$ of center α_j and radius δ. With this notation we get the following lemma.

Lemma 2.5. *If $r < 0$ and $|P_{|\beta|}(r)| = \frac{c}{\lambda_1^n}$, then*

$$\frac{\mu(E(\alpha, c))}{\mu(D(\alpha_1, \rho_1))} \leqslant n\left(\frac{S_{|\beta|}(r)}{|r|}\right)^2. \tag{2.13}$$

Proof. Let K be the connected component of $E(\beta, \frac{c}{\lambda_1^n})$ which contains to zero. If $\tau = \max_{z\in K}|z|$ and $\gamma = \min_{z\in\partial K}|z|$, (2.11) implies

$$\tau = \frac{m}{\lambda_1} \quad \text{and} \quad \gamma = \frac{\rho_1}{\lambda_1}. \tag{2.14}$$

Let $z_0 \in K$ be such that $|z_0| = \tau$. If $|z_0| > S_{|\beta|}(r)$, from definition of $S_{|\beta|}(r)$ follows that there is t, $S_{|\beta|}(r) < t < |z_0|$, satisfying

$$\left|P_{|\beta|}(t)\right| > \frac{c}{\lambda_1^n}.$$

Since the set $H := \{|z|\colon z \in K\}$ is connect and contains to zero, we have $t \in H$. Let $w \in K$ be such that $t = |w|$. Then

$$\left|P_{|\beta|}(t)\right| \leqslant \left|P_\beta(w)\right| \leqslant \frac{c}{\lambda_1^n},$$

that is a contradiction. So,

$$\tau = |z_0| \leqslant S_{|\beta|}(r). \tag{2.15}$$

Let $z_1 \in \partial K$ be such that $|z_1| = \gamma$. Then

$$\left|P_{|\beta|}\big(-|z_1|\big)\right| \geqslant \left|P_\beta(z_1)\right| = \frac{c}{\lambda_1^n}.$$

Since the function $|P_{|\beta|}(x)|$ is strictly decreasing on $(-\infty, 0]$, we get

$$|r| \leqslant |z_1| = \gamma. \tag{2.16}$$

Finally, as

$$E(\alpha, c) \subset \bigcup_{j=1}^n D(\alpha_j, m), \tag{2.17}$$

from (2.14)–(2.16) follows (2.13). □

Lemma 2.6. *Let $n \in \mathbb{N}$. If b is a positive number, then*

$$I_b := \inf_{\alpha\in\mathcal{N}} \|P_\alpha\|_{[0,b]} \geqslant \left(\frac{b}{2(n+1)}\right)^n,$$

where $\|P\|_A := \sup_{x\in A}|P(x)|$ is the infinite norm of P_α on A.

Proof. Let $\alpha \in \mathcal{N}$. Since the set $R(\alpha) - \{0, 1\}$ has at the most $n-2$ elements, there exists i, $1 \leqslant i \leqslant n-1$ such that if $\alpha_j \notin \{0, 1\}$, then $\alpha_j \notin [\frac{ib}{n+1}, \frac{(i+1)b}{n+1}]$. Consequently,

$$\|P_\alpha\|_{[0,b]} \geqslant \left|P_\alpha\left(\frac{(2i+1)b}{2(n+1)}\right)\right| \geqslant \left(\frac{b}{2(n+1)}\right)^n,$$

and the proof is complete. □

Theorem 2.7. *Let $n \in \mathbb{N}$. There exists a constant $\mathcal{K} = \mathcal{K}(a) > 0$ such that for all multi-index $\alpha \in \mathcal{M}_a$ and for all radius c, there exists a circle $B = B(\alpha, c)$ contained in the lemniscate $E(\alpha, c)$ satisfying*

$$\frac{\mu(E(\alpha, c))}{\mu(B(\alpha, c))} \leqslant \mathcal{K}. \tag{2.18}$$

Proof. For all $\alpha \in \Delta$ and for all $c > 0$, $E(\alpha, c) = B(\alpha, c)$, so (2.18) holds with $\mathcal{K} = 1$. Now, we consider $\alpha \in \mathcal{M}_a - \Delta$ and $c > 0$. Then $|\beta| \in \mathcal{N}$. By Proposition 2.4 and Lemma 2.5, there exists $\epsilon \in A_{|\beta|}$ such that

$$\frac{\mu(E(\alpha, c))}{\mu(B(\alpha, c))} \leqslant n\left(\frac{S_{|\beta|}(\epsilon)}{\epsilon}\right)^2 =: \kappa,$$

where $B(\alpha, c) = D(\alpha_1, \rho_1)$. Our propose is to find a bound of κ, only depending on a. From definition of $S_{|\beta|}(\epsilon)$, we have $a < S_{|\beta|}(\epsilon)$.

Case 1. $S_{|\beta|}(\epsilon) > 1$. We consider

$$I^1 = \max_{\delta \in [0,1]^n} \|P_\delta\|_{[0,1]} \tag{2.19}$$

and $t = \lim_{r \to \epsilon^+} S_{|\beta|}(r)$. Clearly $\|P_{|\beta|}\|_{[0,1]} = |P_{|\beta|}(t)|$. So, from Lemma 2.6 and (2.19), we get

$$0 < I_1 \leqslant \left|P_{|\beta|}\big(S_{|\beta|}(\epsilon)\big)\right| = \left|P_{|\beta|}(t)\right| \leqslant I^1.$$

Let $s > 1$ be such that $s(s-1)^{n-1} = I^1$. Since $|P_{|\beta|}|$ is an increasing function on $[1, \infty)$ and $|P_{|\beta|}(x)| \geqslant x(x-1)^{n-1}$ for $x \geqslant 1$, we get

$$1 < S_{|\beta|}(\epsilon) \leqslant s. \tag{2.20}$$

On the other hand, let $r < 0$ be such that $-r(1-r)^{n-1} = I_1$. Since $|P_{|\beta|}|$ is a decreasing function on $(-\infty, 0]$, and $|P_{|\beta|}(x)| \leqslant -x(1-x)^{n-1}$, $x \leqslant 0$, we have

$$\epsilon \leqslant r < 0. \tag{2.21}$$

Therefore, (2.20) and (2.21) imply that

$$\kappa \leqslant n\left(\frac{s}{r}\right)^2. \tag{2.22}$$

Case 2. $a < S_{|\beta|}(\epsilon) < 1$. We suppose that there is a sequence $(\alpha^{(k)}) \subset \mathcal{M}_a - \Delta$ such that $a < S_{|\beta^{(k)}|}(\epsilon^{(k)}) < 1$ and $\epsilon^{(k)}$ tend to zero, as k tends to infinite. Since $|\beta^{(k)}| \in \mathcal{N}$, we can get a subsequence, which we denote again by $(\alpha^{(k)})$ such that $P_{|\beta^{(k)}|}$ converges uniformly to a polynomial P_γ with $\gamma \in \mathcal{N}$. Thus,

$$\lim_{k \to \infty} \left|P_{|\beta^{(k)}|}\big(S_{|\beta^{(k)}|}(\epsilon^{(k)})\big)\right| = \lim_{k \to \infty} \left|P_{|\beta^{(k)}|}(\epsilon^{(k)})\right| = \left|P_\gamma(0)\right| = 0. \tag{2.23}$$

On the other hand, from definition of $S_{|\beta^{(k)}|}(\epsilon^{(k)})$,

$$\left|P_{|\beta^{(k)}|}(x)\right| \leqslant \left|P_{|\beta^{(k)}|}\big(S_{|\beta^{(k)}|}(\epsilon^{(k)})\big)\right|, \quad x \in [0, a].$$

So, (2.23) implies $P_\gamma = 0$, which is a contradiction. Therefore, there exists a constant $q = q(a) < 0$ such that $\epsilon \leqslant q$. Consequently,

$$\kappa \leqslant \frac{n}{q^2}. \tag{2.24}$$

From (2.22) and (2.24) follows the theorem with $\mathcal{K}(a) = n \max\{\frac{1}{q^2}, (\frac{s}{r})^2\}$. $\square$

3. Lemniscates with three foci

Let $n \geqslant 3$. In this section we assume that the lemniscates have exactly three foci. Let $\mathcal{T}$ denote the family of all multi-index, $\alpha \in \mathbb{C}^n$, with exactly different three coordinates. If $\alpha \in \mathcal{T}$, we put $R(\alpha) = \{\alpha_j\colon 1 \leqslant j \leqslant 3\}$. From now on, for $\alpha \in \mathcal{T} \cap \mathcal{N}$, we assume $0 = \alpha_1 < \alpha_2 < \alpha_3 = 1$,

$$P_\alpha(z) = \prod_{j=1}^{3} (z - \alpha_j)^{n_j},$$

where $n = \sum_{j=1}^{3} n_j$ and we call $t_1 = t_1(\alpha)$ and $t_2 = t_2(\alpha)$ the singular points of P_α in the open intervals (α_1, α_2) and (α_2, α_3), respectively. Since,

$$\sum_{j=1}^{3} n_j \prod_{i \neq j} (t_k - \alpha_i) = 0, \quad 1 \leqslant k \leqslant 2,$$

we have

$$(t_1 - 1)\big(t_1(n_1 + n_2) - n_1\alpha_2\big) = -n_3 t_1 (t_1 - \alpha_2) \tag{3.1}$$

and

$$t_2\big(n_2(t_2 - 1) + n_3(t_2 - \alpha_2)\big) = -n_1(t_2 - \alpha_2)(t_2 - 1). \tag{3.2}$$

An analysis of sign in (3.1) and (3.2) imply that

$$t_1 < \frac{n_1\alpha_2}{n_1 + n_2} < \frac{n_1}{n_1 + n_2} \quad \text{and} \quad t_2 > \frac{n_2 + n_3\alpha_2}{n_2 + n_3} > \frac{n_2}{n_2 + n_3}. \tag{3.3}$$

Lemma 3.1. *Let $\alpha \in \mathcal{T} \cap \mathcal{N}$. Then $\frac{t_1}{\alpha_2}$ and $1 - \frac{t_1}{\alpha_2}$ are bounded away from zero.*

Proof. Suppose that there exists a sequence $(\alpha^{(k)})$ with

$$\lim_{k \to \infty} \frac{t_1^{(k)}}{\alpha_2^{(k)}} = 0 \quad \text{or} \quad \lim_{k \to \infty} \frac{t_1^{(k)}}{\alpha_2^{(k)}} = 1,$$

where $t_1^{(k)} = t_1(\alpha^{(k)})$. We can assume without lost of generality that n_1, n_2 and n_3 are the same for all $k \in \mathbb{N}$. From (3.1), we have

$$\begin{aligned}\big(t_1^{(k)} - 1\big)\left(\frac{t_1^{(k)}}{\alpha_2^{(k)}}(n_1 + n_2) - n_1\right) &= -n_3 t_1^{(k)}\left(\frac{t_1^{(k)}}{\alpha_2^{(k)}} - 1\right) \\ &= -n_3 \frac{t_1^{(k)}}{\alpha_2^{(k)}}\big(t_1^{(k)} - 1\big).\end{aligned} \tag{3.4}$$

Taking limit for k tending to infinity in (3.4) we get in any case that $t_1^{(k)}$ tends to one, as k tends to infinite, which contradicts (3.3). $\square$

Theorem 3.2. *There exists a constant $\mathcal{K} > 0$ such that for all multi-index $\alpha \in \mathcal{T}$ and for all radius c, there exists a circle $B = B(\alpha, c)$ contained in the lemniscate $E(\alpha, c)$ satisfying*

$$\frac{\mu(E(\alpha, c))}{\mu(B(\alpha, c))} \leqslant \mathcal{K}. \tag{3.5}$$

Proof. Using the notation before to Lemma 2.5, for $\alpha \in \mathcal{T}$, $|\beta| \in \mathcal{T} \cap \mathcal{N}$. Here, $0 = |\beta_1| < |\beta_2| < |\beta_3| = 1$. It will be sufficient to prove that

$$\kappa := \max_{r \in \mathcal{A}_{|\beta|}} \frac{S_{|\beta|}(r)}{|r|}$$

is uniformly bounded on α. Let $\epsilon \in \mathcal{A}_{|\beta|}$ be such that $\frac{S_{|\beta|}(\epsilon)}{|\epsilon|} = \kappa$. By simplicity we denote $s = S_{|\beta|}(\epsilon)$.

If $s > \frac{1}{2n}$, in a similar way to the proof of Theorem 2.7, there exists a constant κ_1, only depending on n, such that

$$\kappa \leqslant \kappa_1.$$

Now, we suppose, $s \leqslant \frac{1}{2n}$. Since, $t_1 < |\beta_2| < s$, by definition of the function $S_{|\beta|}$, we know that

$$s^{n_1}\big(s - |\beta_2|\big)^{n_2}(1 - s)^{n_3} = t_1^{n_1}\big(|\beta_2| - t_1\big)^{n_2}(1 - t_1)^{n_3}.$$

Therefore,

$$\left(\frac{s}{|\beta_2|}\right)^{n_1}\left(1 - \frac{s}{|\beta_2|}\right)^{n_2}(1 - s)^{n_3} = \left(\frac{t_1}{|\beta_2|}\right)^{n_1}\left(1 - \frac{t_1}{|\beta_2|}\right)^{n_2}(1 - t_1)^{n_3}. \tag{3.6}$$

We see that $\frac{s}{|\beta_2|}$ is uniformly bounded on α. On the contrary, we can get a sequence $(\alpha^{(k)})$ such that $s^{(k)} \leqslant \frac{1}{2n}$ and

$$\lim_{k \to \infty} \frac{s^{(k)}}{|\beta_2^{(k)}|} = \infty.$$

We can assume without lost of generality that n_1, n_2 and n_3 are the same for all $k \in \mathbb{N}$. Since $1 - s^{(k)} > \frac{2n-1}{2n}$, taking limit for k tending to infinity in (3.6), we obtain

$$\lim_{k \to \infty} \left(\frac{t_1^{(k)}}{|\beta_2^{(k)}|}\right)^{n_1}\left(1 - \frac{t_1^{(k)}}{|\beta_2^{(k)}|}\right)^{n_2}\big(1 - t_1^{(k)}\big)^{n_3} = \infty,$$

a contradiction.

On the other hand, we have that $\frac{|\epsilon|}{|\beta_2|}$ is bounded away from zero. In fact, we know that

$$|\epsilon|^{n_1}\big(|\epsilon| + |\beta_2|\big)^{n_2}\big(|\epsilon| + 1\big)^{n_3} = t_1^{n_1}\big(|\beta_2| - t_1\big)^{n_2}(1 - t_1)^{n_3}.$$

Then, we obtain

$$\left(\frac{|\epsilon|}{|\beta_2|}\right)^{n_1}\left(\frac{|\epsilon|}{|\beta_2|} + 1\right)^{n_2}\big(|\epsilon| + 1\big)^{n_3} = \left(\frac{t_1}{|\beta_2|}\right)^{n_1}\left(1 - \frac{t_1}{|\beta_2|}\right)^{n_2}(1 - t_1)^{n_3}. \tag{3.7}$$

Suppose that for some sequence $(\alpha^{(k)})$,

$$\lim_{k \to \infty} \frac{|\epsilon^{(k)}|}{|\beta_2^{(k)}|} = 0.$$

Since $1 - t_1 > \frac{2n-1}{2n}$, it follows from (3.7) that there exists a subsequence of $\frac{t_1^{(k)}}{|\beta_2^{(k)}|}$, tending to zero or one. It contradicts Lemma 3.1. So, we have proved that there is a constant κ_2, satisfying

$$\kappa = \frac{s}{|\beta_2|} \frac{|\beta_2|}{|\epsilon|} \leqslant \kappa_2.$$

Finally, the theorem follows with $\mathcal{K} = \max\{\kappa_1, \kappa_2\}$. □

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