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Cohomology of Associative H-Pseudoalgebras

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Abstract. We define the cohomology of associative H-pseudoalgebras, and we show that it describes module extensions, abelian pseudoalgebra extensions, and pseudoalgebra first-order deformations. The same results for the special case of associative conformal algebras are also described in details.

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1 Introduction

Since the pioneering papers [5] and [6], there has been a great deal of work towards the understanding of the algebraic structure underlying the notion of the operator product expansion (OPE) of chiral fields of a conformal field theory. The singular part of the OPE encodes the commutation relations of fields, which leads to the notion of a Lie conformal algebra introduced by V. Kac [14]. In the past few years a structure theory [10], a representation theory [8, 9] and a cohomology theory [4] of finite Lie conformal algebras have been developed.

In [1], Bakalov, D'Andrea and Kac developed a theory of "multi-dimensional" Lie conformal algebras, called Lie H-pseudoalgebras, where H is a Hopf algebra. They also solved classification problems and developed the cohomology theory. In [2, 3], they continued with the representation theory, classifying the irreducible modules over finite simple Lie H-pseudoalgebras.

In the present work, we study *associative* H-pseudoalgebras and the particular case of associative conformal algebras, that is, when $H = \mathbb{C}[\partial]$. The associative H-pseudoalgebra has not been studied to the extent it needs. Important results for associative conformal algebras have been obtained by Kolesnikov (see [16]), where an analog of the Wedderburn theorem for associative conformal algebras was proved. In [12], Dolguntseva defined the cohomology groups of associative H-pseudoalgebras, and proved an analog of Hochschild's theorem for such algebras, establishing a relationship between extensions of the algebras and the second cohomology group. The explicit computations of the second cohomology group for the main examples of associative conformal algebras, Cend_n and Cur_n, were presented in [13]. In [15], the classification of irreducible subalgebras of the associative conformal algebra Cend_n was presented. In [17, 18], all semisimple algebras of conformal endomorphisms which have the trivial second Hochschild cohomology group with coefficients in every conformal bimodule were described. As a consequence, a complete solution of the radical splitting problem was stated in the class of associative conformal algebras with a finite faithful representation. In [7], we described the finite irreducible modules over $\mathrm{Cend}_{n,p}$ (a family of infinite subalgebras of Cend_n); we also classified certain extensions of irreducible modules over $\mathrm{Cend}_{n,p}$, and obtained all the automorphisms of $\mathrm{Cend}_{n,p}$.

As we pointed out, the cohomology of associative H-pseudoalgebras was defined in [12], but it was only used there to describe the extensions of algebras via the second cohomology group. In the present work, we develop in full detail the zero, first and second cohomologies of associative H-pseudoalgebras.

The zero cohomology deserves special attention. The zero differential map d_0 is not explicitly written in any paper, and the general formula for the differential maps given in [12] does not apply. So this is the first time where the zero cohomology group is described. The image of d_0 is what we call the set of inner derivations, and we prove that they are derivations; that is, we present a proof for the assertion that the composition of differentials $d_1 \circ d_0$ is zero. This is one of the new results of this work.

For an associative H-pseudoalgebra A, and for any pair of left A-modules M and N, we provide a new structure of A-bimodule on $\mathrm{Chom}(M,N)$, where $\mathrm{Chom}(M,N)$ is the conformal analog of the Hom functor for associative algebras (see [1]). Then one of our main results is Theorem 4.4, where we establish that the extensions of modules, of M by N, are in one-to-one correspondence with the elements of the first cohomology group of A with coefficient in $\mathrm{Chom}(M,N)$. Finally, we present another main result, given by the classification of first-order deformations of an associative H-pseudoalgebra in terms of the second cohomology group (see Theorem 5.4).

At the end of this work we apply these results to the particular example of associative conformal algebras. In this case, the n-cochains are defined by using only $n-1$ variables, instead of the n variables used in the Lie conformal algebra case in [4]. Our situation is similar to the corrected version presented in [11].

In Section 2, we present the basic definitions and notations. In Section 3, we define the Hochschild cohomology for an associative H-pseudoalgebra A over an A-bimodule. Then we study in more details the zero, first and second cohomologies. In Section 4, we describe the extensions of modules over an associative H-pseudoalgebra. In Section 5, we describe the abelian extensions and the first-order deformations in terms of the corresponding second cohomology group. In Section 6, we apply these results to the particular example of associative conformal algebras.

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over a field $\mathbb{F}$ of characteristic 0.

2 Definitions and Notation

Let H be a Hopf algebra with comultiplication Δ and counit ε. A more conceptual approach to the theory of associative conformal algebras, their identities, modules, cohomology, etc., is provided by the notion of an H-pseudoalgebra introduced in [4]. Indeed, in ordinary algebras, all basic definitions may be stated in terms of linear spaces, polylinear maps, and their compositions. For H-pseudoalgebras, the base field is replaced with the Hopf algebra H, the class of linear spaces is replaced with the class $\mathcal{M}(H)$ of left H-modules and the role of n-linear maps is played by $H^{\otimes n}$-linear maps of the form

$$\varphi : \ V_1 \otimes \cdots \otimes V_n \longrightarrow H^{\otimes n} \otimes_H V, \quad V_i, V \in \mathcal{M}(H),$$

where $H^{\otimes n} = H \otimes \cdots \otimes H$ and we define the right action of H on $H^{\otimes n}$ by setting $(h_1 \otimes \cdots \otimes h_n) \cdot h = (h_1 \otimes \cdots \otimes h_n)\, \Delta^{(n-1)}(h)$, where

$$\Delta^{(n-1)} := (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \cdots (\Delta \otimes \mathrm{id})\Delta : \ H \longrightarrow H^{\otimes n}$$

is the iterated comultiplication for $n > 1$, and $\Delta^{(0)} := \mathrm{id}$. The map φ is called $H^{\otimes n}$-linear if

$$\varphi(h_1 a_1 \otimes \cdots \otimes h_n\, a_n) = \big((h_1 \otimes \cdots \otimes h_n) \otimes_H 1\big)\, \varphi(a_1 \otimes \cdots \otimes a_n)$$

for $h_i \in H$ and $a_i \in V_i$.

Assume that V_1, V_2 and V_3 are left H-modules on which some $H^{\otimes 2}$-linear operation $*\colon V_1 \otimes V_2 \to H^{\otimes 2} \otimes_H V_3$ is defined. Note that $*$ naturally extends to

$$* : \ (H^{\otimes n} \otimes_H V_1) \otimes (H^{\otimes m} \otimes_H V_2) \longrightarrow H^{\otimes (n+m)} \otimes_H V_3$$

by taking

$$\begin{aligned} &\big((h_1 \otimes \cdots \otimes h_n) \otimes_H v_1\big) * \big((g_1 \otimes \cdots \otimes g_m) \otimes_H v_2\big) \\ = &\big((h_1 \otimes \cdots \otimes h_n \otimes g_1 \otimes \cdots \otimes g_m) \otimes_H 1\big)\big((\Delta^{(n-1)} \otimes \Delta^{(m-1)}) \otimes_H \mathrm{id}\big)(v_1 * v_2). \end{aligned} \tag{2.1}$$

This formula reflects the composition rule of polylinear maps in $\mathcal{M}(H)$ (see [1] for details).

An *H-pseudoalgebra* is a left H-module A together with an $H^{\otimes 2}$-linear map

$$* : \ A \otimes A \longrightarrow H^{\otimes 2} \otimes_H A \quad \text{given by} \quad a \otimes b \longmapsto a * b,$$

called the *pseudoproduct* (similar to the definition of an ordinary algebra as a linear space equipped with a bilinear product map). For the purpose of defining the associativity of a pseudoproduct, we extend it from $A \otimes A \to H^{\otimes 2} \otimes_H A$ to $(H^{\otimes 2} \otimes_H A) \otimes A \to H^{\otimes 3} \otimes_H A$, and to $A \otimes (H^{\otimes 2} \otimes_H A) \to H^{\otimes 3} \otimes_H A$, by using the composition rules in (2.1) with $A = V_1 = V_2 = V_3$:

$$(f \otimes_H a) * b = \sum_i (f \otimes 1)(\Delta \otimes \mathrm{id})(g_i) \otimes_H c_i,$$

$$a * (f \otimes_H b) = \sum_i (1 \otimes f)(\mathrm{id} \otimes \Delta)(g_i) \otimes_H c_i,$$

where $a * b = \sum_i g_i \otimes_H c_i$.

An H-pseudoalgebra is called *associative* if it satisfies the usual equality (in $H^{\otimes 3} \otimes_H A$):

$$(a * b) * c = a * (b * c). \tag{2.2}$$

In more detail, each term of (2.2) is explicitly given by the following formulas: if $a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i$ and $e_i * c = \sum_j (f_{ij} \otimes g_{ij}) \otimes_H e_{ij}$, then

$$(a * b) * c = \sum_{i,j} (f_i f_{ij(1)} \otimes g_i f_{ij(2)} \otimes g_{ij}) \otimes_H e_{ij} \in H^{\otimes 3} \otimes_H A.$$

Similarly, if we write $b * c = \sum_i (h_i \otimes l_i) \otimes_H d_i$ and $a * d_i = \sum_j (h_{ij} \otimes l_{ij}) \otimes_H d_{ij}$, then

$$a * (b * c) = \sum_{i,j} (h_{ij} \otimes h_i l_{ij(1)} \otimes l_i l_{ij(2)}) \otimes_H d_{ij} \in H^{\otimes 3} \otimes_H A.$$

Definition 2.1. Let A be an associative H-pseudoalgebra.

(a) A *left A-module* is a left H-module M together with an $H^{\otimes 2}$-linear map $\overset{M}{*} \colon A \otimes M \to H^{\otimes 2} \otimes_H M$ such that $(a * b) \overset{M}{*} u = a \overset{M}{*} (b \overset{M}{*} u)$ for all $a, b \in A$ and $u \in M$.

(b) A *right A-module* is a left H-module M together with an $H^{\otimes 2}$-linear map $\overset{M}{*} \colon M \otimes A \to H^{\otimes 2} \otimes_H M$ such that $(u \overset{M}{*} a) \overset{M}{*} b = u \overset{M}{*} (a * b)$ for all $a, b \in A$ and $u \in M$. In general, we will simply write $*$ instead of $\overset{M}{*}$.

(c) A *bimodule* over A is a left and right A-module M satisfying

$$(a * u) * b = a * (u * b).$$

If $H = \mathbb{C}$, then all these definitions correspond to the usual associative algebras and their modules.

3 Hochschild Cohomology for Associative *H*-Pseudoalgebras

Let us describe the Hochschild cohomology for an associative H-pseudoalgebra A and a bimodule M over A (see [12]). The space of n-cochains $C^n(A, M)$ consists of all $H^{\otimes n}$-linear maps

$$\varphi : \; A^{\otimes n} \longrightarrow H^{\otimes n} \otimes_H M. \tag{3.1}$$

The differential $d_n \colon C^n(A, M) \to C^{n+1}(A, M)$ is defined similarly to the ordinary one, and we assume the compositions of polylinear maps in $\mathcal{M}(H)$:

$$\begin{aligned}(d_n \varphi)(a_1, \dots, a_{n+1}) = {} & a_1 * \varphi(a_2, \dots, a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \varphi(a_1, \dots, a_i * a_{i+1}, \dots, a_{n+1}) \\ & + (-1)^{n+1} \varphi(a_1, \dots, a_n) * a_{n+1}.\end{aligned} \tag{3.2}$$

In the first and the last summands in (3.2), we use the following conventions that correspond to the composition defined in (2.1). If $a * u = \sum_i f_i \otimes_H u_i \in H^{\otimes 2} \otimes_H M$ for $a \in A$ and $u \in M$, then for any $f \in H^{\otimes n}$ we set

$$a * (f \otimes_H u) = \sum_i (1 \otimes f)(\mathrm{id} \otimes \Delta^{(n-1)})(f_i) \otimes_H u_i \in H^{\otimes (n+1)} \otimes_H M. \tag{3.3}$$

Similarly, if $u * a = \sum_i g_i \otimes_H u_i \in H^{\otimes 2} \otimes_H M$ for $a \in A$ and $u \in M$, then for any $g \in H^{\otimes n}$ we set

$$(g \otimes_H u) * a = \sum_i (g \otimes 1)(\Delta^{(n-1)} \otimes \mathrm{id})(g_i) \otimes_H u_i \in H^{\otimes(n+1)} \otimes_H M. \tag{3.4}$$

Finally, it remains to describe the composition used in the second summand in (3.2). For $g \in H^{\otimes 2}$ and $\varphi \in C^n(A, M)$, we set

$$\begin{aligned} &\varphi(b_1, \dots, b_{i-1}, g \otimes_H b_i, b_{i+1}, \dots, b_n) \\ &= \big[(1^{\otimes(i-1)} \otimes g \otimes 1^{\otimes(n-i)})(\mathrm{id}^{\otimes(i-1)} \otimes \Delta \otimes \mathrm{id}^{\otimes(n-i)}) \otimes_H \mathrm{id}_M\big]\varphi(b_1, \dots, b_n) \\ &\in H^{\otimes(n+1)} \otimes_H M. \end{aligned} \tag{3.5}$$

Direct computations show $d_{n+1} \circ d_n = 0$. If $d_n \varphi = 0$, then φ is called an *n-cocycle*. In addition, a cochain $\varphi \in C^n(A, M)$ is called an *n-coboundary* if there exists an $(n-1)$-cochain ψ such that $d_n \psi = \varphi$. Denote by $Z^n(A, M)$ and $B^n(A, M)$ the subspaces of n-cocycles and n-coboundaries, respectively. The quotient space $H^n(A, M) = Z^n(A, M)/B^n(A, M)$ is called the n-th Hochschild cohomology group of A with coefficients in M.

Let us view the zero, first and second cohomologies in more detail. The case $n = 0$ deserves special attention. It is not explicitly written in any work. We will assume that $A^{\otimes 0} = \mathbb{F} = H^{\otimes 0}$. Then the 0-cochain $\varphi \in C^0(A, M)$ is a map

$$\varphi : \ \mathbb{F} \to \mathbb{F} \otimes_H M.$$

Hence, this map φ is fully determined by $\varphi(1) \in \mathbb{F} \otimes_H M \simeq M/H^+M$, where $H^+ = \{h \in H \mid \varepsilon(h) = 0\}$ is the augmentation ideal, and $\mathbb{F} \cdot h := \mathbb{F}\,\varepsilon(h)$. Therefore,

$$C^0(A, M) \simeq M/H^+M.$$

Observe that $C^1(A, M) = \mathrm{Hom}_H(A, M)$ and now we can see that the differential $d_0 \colon C^0(A, M) \to C^1(A, M)$ is defined by the following formula: if $\varphi \in C^0(A, M)$ and $u_\varphi := \varphi(1) \in M$, then $(d_0 \varphi)(a) = \sum_i (\mathrm{id} \otimes \varepsilon)(h_i) u_i - \sum_j (\varepsilon \otimes \mathrm{id})(l_j) v_j \in M$, where $a * u_\varphi = \sum_i h_i \otimes_H u_i \in H^{\otimes 2} \otimes_H M$ and $u_\varphi * a = \sum_j l_j \otimes_H v_j \in H^{\otimes 2} \otimes_H M$ for $a \in A$, or in a simpler form, we have

$$(d_0 \varphi)(a) = [(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](a \overset{M}{*} u_\varphi) - [(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u_\varphi \overset{M}{*} a). \tag{3.6}$$

It is clear that d_0 is well-defined: If $\varphi(1) = 1 \otimes_H hu$ with $\varepsilon(h) = 0$, then we simply have to use $a \overset{M}{*} hu = ((1 \otimes h) \otimes_H 1)(a \overset{M}{*} u)$ and $hu \overset{M}{*} a = ((h \otimes 1) \otimes_H 1)(u \overset{M}{*} a)$ in (3.6) to get the result. Similarly, it is easy to see that $d_0 \varphi \in C^1(A, M)$. Therefore, we obtain

$$\begin{aligned} H^0(A, M) = \big\{ &u \in M/H^+M \mid [(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](a \overset{M}{*} u) = [(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u \overset{M}{*} a) \\ &\text{for all } a \in A \big\}. \end{aligned}$$

Now, recall that $C^1(A, M) = \mathrm{Hom}_H(A, M)$ since we identified $H \otimes_H M \simeq M$. Observe that

$$\begin{aligned} C^2(A, M) = \big\{ &\varphi \colon A \otimes A \to H^{\otimes 2} \otimes_H M \mid \varphi(ha, gb) = ((h \otimes g) \otimes_H 1)\varphi(a, b) \\ &\forall\, a, b \in A \ \ \forall\, h, g \in H \big\} \end{aligned}$$

and the differential is given by $(d_1\varphi)(a,b) = a * \varphi(b) - \varphi(a * b) + \varphi(a) * b$. By the conventions (3.3) and (3.4), it is clear that

$$\big(d_1\varphi\big)(a,b) = a \overset{M}{*} \varphi(b) - \varphi(a*b) + \varphi(a) \overset{M}{*} b.$$

It remains to prove that the composition (3.5) means $\varphi(a*b) = (\mathrm{id}_{H^{\otimes 2}} \otimes_H \varphi)(a*b)$; that is, we have to consider the trivial extension of φ to a map from $H^{\otimes 2} \otimes_H A$ to M. In fact, if $a * b = \sum_j g_j \otimes_H c_j$ with $g_j \in H^{\otimes 2}$ and $c_j \in A$, then using (3.5), we have

$$\begin{aligned} \varphi(a*b) &= \sum_j \varphi(g_j \otimes_H c_j) = \sum_j (g_j \Delta \otimes_H \mathrm{id}_M)\, \varphi(c_j) \\ &= \sum_j g_j \otimes_H \varphi(c_j) = (\mathrm{id}_{H^{\otimes 2}} \otimes_H \varphi)(a*b), \end{aligned} \tag{3.7}$$

and in the middle of (3.7) we have used $\varphi(c_j) \in M$ since we identified $H \otimes_H M$ with M.

A map $f \in \mathrm{Hom}_H(A, M)$ is called a *derivation* from A to M if

$$f(a*b) = a \overset{M}{*} f(b) + f(a) \overset{M}{*} b$$

for all $a, b \in A$ and f extends trivially to a map from $H^{\otimes 2} \otimes_H A$ to M. We denote by $\mathrm{Der}(A, M)$ the set of all derivations from A to M. Then $\mathrm{Ker}\, d_1 = \mathrm{Der}(A, M)$.

Proposition 3.1. *For $u \in M$, we define $f_u \colon A \to M$ by*

$$f_u(a) = [(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](a * u) - [(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u * a).$$

Then f_u is H-linear and it is a derivation. Hence, $d_1 \circ d_0 = 0$.

Proof. First, we prove that f_u is H-linear:

$$\begin{aligned} f_u(ha) &= [(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M]((h \otimes 1) \otimes_H 1)(a * u) \\ &\quad - [(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M]((1 \otimes h) \otimes_H 1)(u * a) \\ &= (h \otimes_H \mathrm{id}_M)\big([(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](a*u) - [(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u*a)\big) \\ &= h\, f_u(a). \end{aligned}$$

In order to prove that f_u is a derivation, observe that

$$a * f_u(b) = a * \big([(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](b*u)\big) - a * \big([(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u*b)\big), \tag{3.8}$$

$$f_u(a) * b = \big([(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](a*u)\big) * b - \big([(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u*a)\big) * b, \tag{3.9}$$

$$f_u(a*b) = \sum_i (f_i \otimes g_i) \otimes_H \big([(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](e_i * u) - [(\varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u * e_i)\big), \tag{3.10}$$

where $a * b = \sum_i (f_i \otimes g_i) \otimes_H e_i$, and in (3.10) we used (3.7). Now, we see that the first term of (3.8) is equal to the first term of (3.10). If $b * u = \sum_i (h_i \otimes l_i) \otimes_H u_i$, then

$$[(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](b*u) = \sum_i h_i\, \varepsilon(l_i) u_i \in M.$$

Hence, using $a * u_i = \sum_j (h_{ij} \otimes l_{ij}) \otimes_H u_{ij}$, we see that the first term of (3.8) is equal to

$$\begin{aligned}
\sum_i a * (h_i\, \varepsilon(l_i)\, u_i) &= \sum_{i,j} (h_{ij} \otimes h_i\, \varepsilon(l_i)\, l_{ij}) \otimes_H u_{ij} \\
&= \sum_{i,j} (h_{ij} \otimes h_i\, l_{ij(1)}\, \varepsilon(l_{ij(2)})\, \varepsilon(l_i)) \otimes_H u_{ij} \\
&= \sum_{i,j} (\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(h_{ij} \otimes h_i\, l_{ij(1)} \otimes l_{ij(2)}\, l_i) \otimes_H u_{ij} \\
&= [(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](a * (b * u)).
\end{aligned} \tag{3.11}$$

If $e_i * u = \sum_j (f_{ij} \otimes g_{ij}) \otimes_H v_{ij}$, then $[(\mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](e_i * u) = \sum_j f_{ij}\, \varepsilon(g_{ij})\, v_{ij} \in M$. Hence, we observe that the first term of (3.10) is equal to

$$\begin{aligned}
\sum_{i,j} (f_i \otimes g_i)\, \Delta(f_{ij}\, \varepsilon(g_{ij})) \otimes_H v_{ij} &= \sum_{i,j} (f_i f_{ij(1)} \otimes g_i f_{ij(2)}\, \varepsilon(g_{ij})) \otimes_H v_{ij} \\
&= [(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M]((a * b) * u),
\end{aligned}$$

which is equal to (3.11), proving that the first term of (3.8) is equal to the first term of (3.10).

With the same ideas, one can prove that the second term of (3.8) is equal to the first term of (3.9), and the second term of (3.9) is equal to the second term of (3.10). More precisely, it is possible to prove

$$\begin{aligned}
a * f_u(b) &= [(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M](a * (b * u)) - [(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](a * (u * b)), \\
f_u(a) * b &= [(\mathrm{id} \otimes \varepsilon \otimes \mathrm{id}) \otimes_H \mathrm{id}_M]((a * u) * b) - [(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{id}_M]((u * a) * b), \\
f_u(a * b) &= [(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon) \otimes_H \mathrm{id}_M]((a * b) * u) - [(\varepsilon \otimes \mathrm{id} \otimes \mathrm{id}) \otimes_H \mathrm{id}_M](u * (a * b)),
\end{aligned}$$

and hence we see that f_u is a derivation. □

The derivations in Proposition 3.1 are called *inner derivations*, and we denote by $\mathrm{IDer}(A, M)$ the corresponding set. Therefore, $H^1(A, M) = \mathrm{Der}(A, M)/\,\mathrm{IDer}(A, M)$. If $\varphi \in C^2(A, M)$, then the definition of d_2 is clear:

$$\big(d_2\, \varphi\big)(a, b, c) = a * \varphi(b, c) - \varphi(a * b, c) + \varphi(a, b * c) - \varphi(a, b) * c,$$

as well as the cohomology group $H^2(A, M)$.

4 *H*-Pseudolinear Maps and Extensions of Modules over Associative *H*-Pseudoalgebras

In this section, we introduce the H-pseudoalgebra analog of the "Hom" functor, defined in [1, Section 10], and then we describe the extensions of modules over associative H-pseudoalgebras. The contents of this section are completely new.

Definition 4.1. Let M and N be two left H-modules. An *H-pseudolinear map* from M to N is an $\mathbb{F}$-linear map $\phi\colon M \to (H \otimes H) \otimes_H N$ such that

$$\phi(hu) = ((1 \otimes h) \otimes_H 1)\, \phi(u), \quad h \in H,\ u \in M.$$

We denote the space of all such ϕ by $\mathrm{Chom}(M,N)$. We define a left action of H on $\mathrm{Chom}(M,N)$ by

$$(h\phi)(u) = ((h\otimes 1)\otimes_H 1)\,\phi(u).$$

Consider the map $\rho\colon \mathrm{Chom}(M,N)\otimes M \to H^{\otimes 2}\otimes_H N$ given by $\rho(\phi\otimes u) := \phi(u)$. By definition, it is $H^{\otimes 2}$-linear, hence a polylinear map in $\mathcal{M}(H)$; see [1] for details. We will also use the notation $\phi * u = \phi(u)$ and consider this as a pseudoproduct or an action.

Proposition 4.2. *Let A be an associative H-pseudoalgebra, and let M and N be two finite left A-modules. Then we have the following statements:*
- (a) $\mathrm{Chom}(M,N)$ *is a left A-module with the action* $(a*\phi)(u) := a*(\phi * u)$ *for* $a\in A$, $\phi\in \mathrm{Chom}(M,N)$ *and* $u\in M$, *where the composition rules are those defined in* (2.1).
- (b) $\mathrm{Chom}(M,N)$ *is a right A-module with the action* $(\phi * a)(u) := \phi *(a*u)$.
- (c) $\mathrm{Chom}(M,N)$ *is a bimodule over A.*

The proof of this proposition follows immediately by the definitions of left and right modules over A, and the composition rules of polylinear maps.

Definition 4.3. Let M and N be two left A-modules. An *extension* E of N by M is an H-split exact sequence of left A-modules $0\to M\to E\to N\to 0$. Two extensions E_1 and E_2 are *equivalent* if there exists an isomorphism $h\colon E_1\to E_2$ of A-modules such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E_1 & \longrightarrow & N & \longrightarrow & 0 \\ & & {\scriptstyle 1_M}\big\downarrow & & {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle 1_N} & & \\ 0 & \longrightarrow & M & \longrightarrow & E_2 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

is commutative.

The following theorem is one of the main results of this work.

Theorem 4.4. *Given two finite left A-modules M and N, the equivalence classes of H-split extensions $0\to M\to E\to N\to 0$ of N by M are in one-to-one correspondence with the elements of* $H^1(A,\mathrm{Chom}(N,M))$.

Proof. Let $0\to M\xrightarrow{i} E\xrightarrow{p} N\to 0$ be an extension of A-modules, which is split over H. Choose a splitting $E = M\oplus N = \{(u,v)\mid u\in M,\ v\in N\}$ as H-modules. The fact that i and p are homomorphisms of left A-modules implies

$$a\overset{E}{*}u = a\overset{M}{*}u \quad\text{and}\quad a\overset{E}{*}v - a\overset{N}{*}v := \gamma(a)(v)\in H^{\otimes 2}\otimes_H M \tag{4.1}$$

for $a\in A$, $u\in M$ and $v\in N$. Using the $H^{\otimes 2}$-linearity of the action in the module E, we can easily see that $\gamma(a)\in \mathrm{Chom}(A,M)$ and $\gamma\colon A\to \mathrm{Chom}(N,M)$ is H-linear. In other words, we have $\gamma\in C^1(A,\mathrm{Chom}(N,M)) = \mathrm{Hom}_H(A,\mathrm{Chom}(N,M))$.

Using the associativity of E, we have

$$\begin{aligned}(a*b)*(u,v) &= ((a*b)*u+\gamma(a*b)(v),\,(a*b)*v),\\ a*(b*(u,v)) &= a*(b*u+\gamma(b)(v),\,b*v)\\ &= \big((a*(b*u))+a*(\gamma(b)(v))+\gamma(a)(b*v),\,a*(b*v)\big)\end{aligned}$$

for $a,b\in A$, $u\in M$ and $v\in N$. Subtracting these two equations and using (4.1), we have

$$\gamma(a*b)(v)=a*(\gamma(b)(v))+\gamma(a)(b*v),$$

and using the definition of the A-bimodule structure in $\mathrm{Chom}(N,M)$, we obtain $\gamma(a*b)(v)=(a*\gamma(b))(v)+((\gamma(a))*b)(v)$ for all $v\in N$. Therefore, the associativity in E is equivalent to $(d_1\gamma)(a,b)=a*\gamma(b)-\gamma(a*b)+(\gamma(a))*b=0$.

If we have two isomorphic extensions E and E' associated to the closed elements γ and γ', respectively, and we choose a compatible splitting over H, then the isomorphism $h\colon E\to E'$ is determined by an element $\beta\in\mathrm{Hom}_H(N,M)$, that is, $h\colon M\oplus N\to M\oplus N'$ with $h(u,v)=(u+\beta(v),\,v)'$. Using

$$\begin{aligned}h(a*(u,v)) &= (a*u+\gamma(a)(v)+\beta(a*v),\,a*v),\\ a*(h(u,v)) &= a*(u+\beta(v),\,v)'=(a*u+a*(\beta(v))+\gamma'(a)(v),\,a*v),\end{aligned}$$

we have

$$\gamma(a)(v)=\gamma'(a)(v)+a*(\beta(v))-\beta(a*v). \tag{4.2}$$

Now, using

$$\mathrm{Hom}_H(N,M)\simeq\mathbb{F}\otimes_H\mathrm{Chom}(N,M)\simeq C^0(A,\,\mathrm{Chom}(N,M)) \tag{4.3}$$

(see [1, Remark 10.1] for details), we need to prove that (4.2) is equivalent to $\gamma=\gamma'+(d_0\beta)$. In order to simplify the notation, recall that any element in $H^{\otimes 2}\otimes_H W$ can be written uniquely in the form $\sum_i(h_i\otimes 1)\otimes_H w_i$, where $\{h_i\}$ is a fixed $\mathbb{F}$-basis of H. In more detail, given $\phi\in\mathrm{Chom}(N,M)$, we define the map $\phi_1\colon N\to M$ as follows: if $\phi(v)=\sum_i(h_i\otimes 1)\otimes_H u_i$, then $\phi_1(v)=\sum_i\varepsilon(h_i)\,u_i$. The map ϕ_1 is H-linear and establishes the isomorphism in (4.3). Let $\phi\in\mathrm{Chom}(N,M)$ such that $\phi_1=\beta$. Observe that

$$(d_0\,\phi)(a)=[(\mathrm{id}\otimes\varepsilon)\otimes_H\mathrm{id}_{\mathrm{Chom}}](a*\phi)-[(\varepsilon\otimes\mathrm{id})\otimes_H\mathrm{id}_{\mathrm{Chom}}](\phi*a).$$

Hence, we need to prove

$$\big([(\mathrm{id}\otimes\varepsilon)\otimes_H\mathrm{id}_{\mathrm{Chom}}](a*\phi)\big)(v)=a*(\beta(v)), \tag{4.4}$$

$$\big([(\varepsilon\otimes\mathrm{id})\otimes_H\mathrm{id}_{\mathrm{Chom}}](\phi*a)\big)(v)=\beta(a*v) \tag{4.5}$$

for $v\in N$. Now, we will prove (4.4), and the proof of (4.5) is similar. First of all, we need to check that

$$\big([(\mathrm{id}\otimes\varepsilon)\otimes_H\mathrm{id}_{\mathrm{Chom}}](a*\phi)\big)(v)=[(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\otimes_H\mathrm{id}_M]((a*\phi)(v)). \tag{4.6}$$

Observe that $[(\mathrm{id}\otimes\varepsilon)\otimes_H \mathrm{id}_{\mathrm{Chom}}](a*\phi)=\sum_i \varepsilon(g_i)\, f_i\varphi_i$ if $a*\phi=\sum_i (f_i\otimes g_i)\otimes_H \varphi_i$. Hence, we have

$$\begin{aligned}
&\big([(\mathrm{id}\otimes\varepsilon)\otimes_H \mathrm{id}_{\mathrm{Chom}}](a*\phi)\big)(v)\\
&=\sum_i \varepsilon(g_i)\,(f_i\varphi_i)(v)=\sum_i \varepsilon(g_i)\,[(f_i\otimes 1)\otimes_H 1_M](\varphi_i(v))\\
&=\sum_{i,j} \varepsilon(g_i)\,(f_i f_{ij}\otimes g_{ij})\otimes_H u_{ij},
\end{aligned}\tag{4.7}$$

where $\varphi_i(v)=\sum_j (f_{ij}\otimes g_{ij})\otimes_H u_{ij}$. On the other hand, using the previous notation, we have $(a*\phi)(v)=\sum_{i,j}(f_i f_{ij(1)}\otimes g_i f_{ij(2)}\otimes g_{ij})\otimes_H u_{ij}$, thus obtaining

$$[(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\otimes_H \mathrm{id}_M]((a*\phi)(v))=\sum_{i,j}\varepsilon(g_i)\,(f_i f_{ij}\otimes g_{ij})\otimes_H u_{ij}. \tag{4.8}$$

Therefore, combining (4.7) and (4.8), we have proved (4.6).

If $\phi(v)=\sum_i (h_i\otimes 1)\otimes_H u_i$ and $a*u_i=\sum_j (h_j\otimes 1)\otimes_H u_{ij}$, then we observe that $a*(\phi(v))=\sum_{i,j}(h_j\otimes h_i\otimes 1)\otimes_H u_{ij}$, and by definition we get $(a*\phi)(v)=a*(\phi(v))$, so we have $[(\mathrm{id}\otimes\varepsilon\otimes\mathrm{id})\otimes_H \mathrm{id}_M]((a*\phi)(v))=\sum_{i,j}\varepsilon(h_i)(h_j\otimes 1)\otimes_H u_{ij}$. On the other hand, we see that $a*(\beta(v))=\sum_i \varepsilon(h_i)(a*u_i)=\sum_i \varepsilon(h_i)(h_j\otimes 1)\otimes_H u_{ij}$ since $\beta(v)=\phi_1(v)=\sum_i \varepsilon(h_i)\, u_i$, which finishes the proof of (4.4).

Conversely, given an element of $H^1(A,\,\mathrm{Chom}(A,M))$, we can choose a representative $\gamma\in C^1(A,\,\mathrm{Chom}(A,M))$ and define an action of A on $E=M\oplus N$ by (4.1), which will depend only on the cohomology class of γ, finishing the proof. □

5 Second Cohomology, Abelian Extensions and First-Order Deformations

In the first part of this section we describe the abelian extensions; see [12] for details.

Definition 5.1. An *abelian extension* of an associative H-pseudoalgebra A by an A-bimodule M is an associative H-pseudoalgebra E in a short exact sequence $0\to M\to E\to A\to 0$, where $M*M=0$ in E. Two abelian extensions E_1 and E_2 are *equivalent* if there exists an isomorphism $f\colon E_1\to E_2$ such that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0\\
 & & {\scriptstyle 1_M}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle 1_A} & & \\
0 & \longrightarrow & M & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

is commutative.

Theorem 5.2. (proved in [12]) *The equivalence classes of H-split abelian extensions of A by an A-bimodule M correspond bijectively to $H^2(A,M)$.*

The next part of this section is a new contribution.

Definition 5.3. (a) Let t be a formal variable and let $(A,*)$ be an associative H-pseudoalgebra. A *first-order deformation* of A is a family of H-pseudoproducts of the form $a \hat{*} b = a * b + t\, f(a,b)$ with $a, b \in A$, where $f\colon A \otimes A \to H^{\otimes 2} \otimes_H A$ is an $H^{\otimes 2}$-linear map (independent of t), such that $(A, \hat{*})$ is a family of associative H-pseudoalgebras up to the first-order in t (i.e., modulo t^2). More precisely, the H-pseudoproduct $\hat{*}$ is an $H^{\otimes 2}$-linear map and it satisfies

$$(a \hat{*} b) \hat{*} c = a \hat{*} (b \hat{*} c) \mod t^2, \tag{5.1}$$

where H acts trivially on t.

(b) Two first-order deformations $\overset{(1)}{*}$ and $\overset{(2)}{*}$ of A are *equivalent* if there exists a family of H-linear maps $\phi_t\colon A \to A[t]$ of the form $\phi_t = \mathrm{id}_A + t\, g$, where $g\colon A \to A$ is an H-linear map such that

$$\phi_t(a \overset{(1)}{*} b) = \phi_t(a) \overset{(2)}{*} \phi_t(b) \mod t^2 \qquad \text{for } a, b \in A. \tag{5.2}$$

The following theorem is the second main result of this work.

Theorem 5.4. *The equivalence classes of first-order deformations of an associative H-pseudoalgebra A (leaving the H-action intact) correspond bijectively to $H^2(A,A)$.*

Proof. Let $(A, *)$ be an associative H-pseudoalgebra and let $\hat{*}$ be given by

$$a \hat{*} b = a * b + t\, f(a,b) \quad \text{with } a, b \in A, \tag{5.3}$$

where $f\colon A \otimes A \to H^{\otimes 2} \otimes_H A$ is an $H^{\otimes 2}$-linear map. Then, using (5.3), we take the expansions in (5.1) mod t^2. By a direct computation, we see that the coefficient of t^0 corresponds exactly to the associativity property of $*$, and the coefficient of t^1 corresponds exactly to $f(a*b,c) + f(a,b)*c = f(a,b*c) + a*f(b,c)$. Therefore, we have seen that (5.3) is a first-order deformation of A if and only if $f \in Z^2(A,A)$.

Now, consider two first-order deformations of A given by $a \overset{(1)}{*} b = a*b + t\, f_1(a,b)$ and $a \overset{(2)}{*} b = a*b + t\, f_2(a,b)$. They are equivalent if and only if there exists g in $\mathrm{Hom}_H(A,A)$ such that $\phi_t := \mathrm{id}_A + t\, g$ satisfies (5.2). A direct computation shows that (5.2) is equivalent to $f_1(a,b) - f_2(a,b) = a * g(b) - g(a*b) + g(a)*b$ for all $a, b \in A$. Therefore, it is equivalent to $f_1 - f_2 = d_1 g$, finishing the proof. □

6 Cohomology of Associative Conformal Algebras

In this final section, we restrict the definitions and results of the previous sections to associative conformal algebras.

Conformal algebras are exactly H-pseudoalgebras over the polynomial Hopf algebra $H = \mathbb{C}[\partial]$, with coproduct $(\Delta f)(\partial) = f(\partial \otimes 1 + 1 \otimes \partial)$, counit $\varepsilon(f) = f(0)$, and antipode $(Sf)(\partial) = f(-\partial)$. The structure of a conformal algebra on a $\mathbb{C}[\partial]$-module A is given by a $\mathbb{C}$-linear map $A \otimes A \to A[\lambda]$, $a \otimes b \mapsto a_\lambda b$, called the λ-product. The relation between pseudoproduct and λ-product is given by

$$a * b = (a_\lambda b)_{|_{\lambda = -\partial \otimes 1}}.$$

The $H^{\otimes 2}$-linearity on $*$ corresponds to the *sesquilinearity*:

$$(\partial a)_\lambda b = -\lambda (a_\lambda b) \quad \text{and} \quad a_\lambda (\partial b) = (\lambda + \partial)(a_\lambda b). \tag{6.1}$$

The conformal algebra is called *associative* if $(a_\lambda b)_{\lambda+\mu}\, c = a_\lambda(b_\mu c)$, which is the restriction of the associative axiom of a pseudoproduct.

Definition 6.1. Let A be an associative conformal algebra.

(a) A *left conformal module* over A is a $\mathbb{C}[\partial]$-module M with a $\mathbb{C}$-linear map $A \otimes M \to \mathbb{C}[\lambda] \otimes M$ given by $a \otimes u \mapsto a_\lambda u$, called the λ-action, satisfying the following properties for $a, b \in A$ and $u \in M$:

$$(\partial a)_\lambda u = -\lambda\, a_\lambda u, \quad a_\lambda(\partial u) = (\lambda + \partial)\,(a_\lambda u), \quad a_\lambda(b_\mu u) = (a_\lambda b)_{\lambda+\mu} u.$$

(b) A *right conformal module* over A is a $\mathbb{C}[\partial]$-module M with a $\mathbb{C}$-linear map $M \otimes A \to \mathbb{C}[\lambda] \otimes M$ given by $u \otimes a \mapsto u_\lambda a$, called the λ-action, satisfying the corresponding sesquilinearity and $u_\lambda(a_\mu b) = (u_\lambda a)_{\lambda+\mu} b$.

(c) A *conformal bimodule* M over A is a left and right conformal module that satisfies $a_\lambda(u_\mu b) = (a_\lambda u)_{\lambda+\mu} b$.

The notion of conformal bimodule was introduced after Definition 1.4 in [4]. A conformal module is called *finite* if it is finitely generated over $\mathbb{C}[\partial]$.

Now, we describe $\mathrm{Chom}(M, N)$ in the conformal case, that is, $H = \mathbb{C}[\partial]$. Let M and N be two $\mathbb{C}[\partial]$-modules. A *conformal linear map* from M to N is a $\mathbb{C}$-linear map $f_\lambda \colon M \to N[\lambda]$ such that $f_\lambda(\partial u) = (\lambda + \partial)\, f_\lambda(u)$ for $u \in M$. We denote the vector space of all such maps by $\mathrm{Chom}(M, N)$. It has an structure of a $\mathbb{C}[\partial]$-module given by $(\partial f)_\lambda(u) := -\lambda\, f_\lambda(u)$. If M and N are finite left conformal A-modules, then $\mathrm{Chom}(M, N)$ is a left conformal A-module with the action

$$(a_\lambda f)_\mu u := a_\lambda(f_{\mu-\lambda} u)$$

for $a \in A$ and $u \in M$, and it is a right conformal A-module with the action

$$(f_\lambda a)_\mu u := f_\lambda(a_{\mu-\lambda} u)$$

for $a \in A$ and $u \in M$. With these structures, it is a conformal bimodule over A.

In [4], the Hochschild cohomology group was defined, for which the space of n-cochains has n variables, and it was necessary to take certain quotient. In [10], for the case of Lie conformal algebras, the definition was improved by taking $n - 1$ variables. Following this idea, we define the Hochschild cohomology for an associative conformal algebra A and a bimodule M over A. The space of n-cochains $C^n(A, M)$ consists of all maps

$$\varphi_{\lambda_1,\ldots,\lambda_{n-1}} :\ A^{\otimes n} \longrightarrow M[\lambda_1, \ldots, \lambda_{n-1}]$$

such that (here we use $H^{\otimes n} \otimes_H M \simeq H^{\otimes(n-1)} \otimes M$ and the $H^{\otimes n}$-linearity in (3.1) translates into the following sesquilinearity properties)

$$\varphi_{\lambda_1,\ldots,\lambda_{n-1}}(a_1, \ldots, \partial a_i, \ldots, a_n) = -\lambda_i\, \varphi_{\lambda_1,\ldots,\lambda_{n-1}}(a_1, \ldots, a_n), \quad i = 1, \ldots, n-1,$$
$$\varphi_{\lambda_1,\ldots,\lambda_{n-1}}(a_1, \ldots, \partial a_n) = (\partial + \lambda_1 + \cdots + \lambda_{n-1})\, \varphi_{\lambda_1,\ldots,\lambda_{n-1}}(a_1, \ldots, a_n).$$

The differential turns into

$$\begin{aligned}
&(d_n\,\varphi)_{\lambda_1,\dots,\lambda_n}(a_1,\dots,a_{n+1})\\
&= (a_1)_{\lambda_1}\varphi_{\lambda_2,\dots,\lambda_n}(a_2,\dots,a_{n+1})\\
&\quad+\sum_{i=1}^{n}(-1)^i\,\varphi_{\lambda_1,\,\dots,\,\lambda_i+\lambda_{i+1},\,\dots,\,\lambda_n}(a_1,\,\dots,\,(a_i)_{\lambda_i}(a_{i+1}),\,\dots,\,a_{n+1})\\
&\quad+(-1)^{n+1}\varphi_{\lambda_1,\dots,\lambda_{n-1}}(a_1,\dots,a_n)_{(\lambda_1+\cdots+\lambda_n)}a_{n+1}.
\end{aligned}$$

Now, we write the details of the lowest degree cohomologies. First of all, we have $C^0(A,M)\simeq M/\partial M$ and $C^1(A,M)=\mathrm{Hom}_{\mathbb{C}[\partial]}(A,M)$. In order to define the differential d_0, we need the following ideas. Choosing a set of generators $\{u_j\}$ of the $\mathbb{C}[\partial]$-module M, we can write $a_\lambda u=\sum_k Q_k(\lambda,\partial)\,u_k$ for $a\in A$ and $u\in M$, where Q_k are some polynomials in λ and ∂. Take $P_k(x,y):=Q_k(-x,x+y)$, and the corresponding left pseudoaction of A on M is given by the $\mathbb{C}[\partial]^{\otimes 2}$-linear map $*\colon A\otimes M\to (H\otimes H)\otimes_H M$ defined by

$$a*u=\sum_k P_k(\partial\otimes 1,\,1\otimes\partial)\otimes_H u_k.$$

We consider similar formulas for the right conformal pseudoactions. That is, if $u_\lambda a=\sum_i S_i(\lambda,\partial)\,u_i$, then we have $u*a=\sum_i R_i(\partial\otimes 1,\,1\otimes\partial)\otimes_H u_i$, where $R_i(x,y):=S_i(-x,x+y)$. Now, we apply the formula (3.6). If $\varphi(1)=u$, then we have $(d_0\,\varphi)(a)=\sum_k P_k(\partial,0)\,u_k-\sum_i R_i(0,\partial)\,u_i$. Thus, in the conformal case we obtain

$$(d_0\,\varphi)(a)=\sum_k Q_k(-\partial,\partial)\,u_k-\sum_i S_i(0,\partial)\,u_i=a_{{}_{-\partial}}u-u_{{}_0}a.$$

Therefore, $H^0(A,M)=\{u\in M/\partial M\mid a_{{}_{-\partial}}u=u_{{}_0}a \text{ for all } a\in A\}$.

A map $f\in\mathrm{Hom}_{\mathbb{C}[\partial]}(A,M)$ is called a *derivation* from A to M if for all $a,b\in A$, $f(a_\lambda b)=a_\lambda f(b)+f(a)_\lambda b$. Observe that

$$\begin{aligned}
C^2(A,M)=\{\varphi_\lambda\colon A\otimes A\to M[\lambda]\mid \varphi_\lambda(\partial a,b)=-\lambda\,\varphi_\lambda(a,b)\\
\text{and } \varphi_\lambda(a,\partial b)=(\lambda+\partial)\,\varphi_\lambda(a,b)\}
\end{aligned}$$

and the differential $d_1\colon C^1(A,M)\to C^2(A,M)$ is given by

$$(d_1\,\varphi)_\lambda(a,b)=a_\lambda\varphi(b)-\varphi(a_\lambda b)+\varphi(a)_\lambda b.$$

It is clear that $\mathrm{Ker}\,d_1=\mathrm{Der}(A,M)$. In addition, the maps $g_u\colon A\to M$ (for $u\in M$) defined by

$$g_u(a)=a_{{}_{-\partial}}u-u_{{}_0}a$$

correspond to the inner derivations or the image of d_0. By definition, we have

$$(d_2\,\varphi)_{\lambda,\mu}(a,b,c)=a_\lambda\varphi_\mu(b,c)-\varphi_{\lambda+\mu}(a_\lambda b,c)+\varphi_\lambda(a,b_\mu c)-\varphi_\lambda(a,b)_{\lambda+\mu}\,c.$$

Finally, Theorems 4.4, 5.2 and 5.4 hold for associative conformal algebras.

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