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ON THE STUDY OF BIFURCATIONS IN DELAY-DIFFERENTIAL EQUATIONS: A FREQUENCY-DOMAIN APPROACH

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An improved version of a frequency-domain approach to study bifurcations in delay-differential equations is presented. The proposed methodology provides information about the frequency, amplitude, and stability of the orbit emerging from Hopf bifurcation. We apply this method to different schemes of the delayed van der Pol oscillator. The time-delay dependence can appear intrinsically because of the system dynamics or can be intentionally introduced in a feedback loop. Also, a discussion about system controllability and observability is given for a proper and rigorous application of the frequency domain technique. Collateral findings involving some types of static bifurcations are included for completeness.

Keywords: Time-delay systems, frequency-domain approach, bifurcations, van der Pol oscillator.

1. Introduction

After the original strategy introduced by Pyragas [Pyragas, 1992], many research efforts were dedicated to the area of time-delayed feedback control. Several schemes using this novel and practical technique have been proposed in order to achieve different control goals, such as the manipulation of orbit amplitude and frequency [Atay, 1998], chaos control [Pyragas, 2001] and stabilization of unstable limit cycles [Pyragas *et al.*, 2004; Just *et al.*, 2007], to name just a few.

Particularly, the classical van der Pol oscillator under the action of delayed feedback has been studied by several authors. In the motivating and timely article of Atay [Atay, 1998], the effect of two delayed control schemes has been compared very clearly with the proportional and derivative parts of a classical controller. The amplitude and frequency of periodic orbits are computed by means of an averaging method. Other researchers made complementary contributions from the perspective of bifurcation theory by using the normal form method [Wei & Jiang, 2005; Jiang & Wei, 2008; Wang & Jiang, 2010]. In addition, de Oliveira [2002] used a simplified version of the Lyapunov-Schmidt method for characterizing an existing limit cycle in another type of delayed van der Pol system.

In this paper, an improvement of the frequency-domain (FD) method for bifurcation analysis is shown in order to confirm some known results and present some new ones. The FD approach [Mees, 1981; Moiola &

Chen, 1996] is based on an input-output representation of the system, the harmonic-balance technique and the generalized Nyquist stability criterion [MacFarlane & Postlethwaite, 1977]. That approach is useful for detecting bifurcations in nonlinear autonomous systems giving, at the same time, the stability, direction and shape of the branch of periodic solutions. Although it was originally developed for finite dimensional systems [Mees & Chua, 1979], many interesting applications can be found in the terrain of time-delayed ones. For example, applications in control systems with delays are described in [Moiola & Chen, 1996; Pagano *et al.*, 1999]. Also, several authors have made progress in the field of neural networks by using this technique [Liao *et al.*, 2004; Yu & Cao, 2007; Yu *et al.*, 2008]. An extension of this method is presented in this paper, which allows to deal with a very general type of delay-differential equation (DDE). This extension arises naturally from the existing formulation, and does not increase significantly the computational effort. The proposed method has the advantage of enabling a reduction of the dimension of the problem, limiting the bifurcation analysis to the study of fewer eigenfunctions, usually only one. The key step for obtaining this reduction is to properly choose an input-output representation of the nonlinear system. This representation is not unique and can be made in many ways, taking into account that some criteria about controllability and observability must be fulfilled for a proper application of the FD approach [Agamennoni *et al.*, 2008]. Moreover, a clever realization can lead to straightforward calculations for the bifurcation conditions. For delayed systems, this election may be more involved than for non-delayed ones, because of the more elaborate controllability and observability criteria that they imply. However, the simplification by using a clever realization is a great asset of this methodology. Therefore, considerations respective to this subject for the case of delayed feedback are discussed as well as the defining conditions of some static bifurcations.

The article is organized as follows. In Section 2 we briefly describe the basic steps involved on the FD approach, and we show its application to two different schemes. In Section 3, three examples of van der Pol systems with delays are studied, two proposed originally in [Atay, 1998] and the last one presented in [de Oliveira, 2002]. We perform a bifurcation analysis of these systems, obtaining approximations of periodic orbit's amplitude and frequency when Hopf bifurcations are detected as well as some types of static bifurcations. Finally, the concluding remarks are included in Section 4.

2. Preliminaries and theoretical results

2.1. *The Frequency-Domain Approach*

In this subsection we briefly outline the basic steps for the utilization of the FD approach in systems described by ordinary differential equations. A more detailed description of this technique can be found in [Mees & Chua, 1979; Mees, 1981; Moiola & Chen, 1996]. The delayed case will be studied in the following subsections. Let us consider the autonomous parametric system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bg(y(t);\mu), \\ y(t) = -Cx(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{m\times n}$, $\mu \in \mathbf{R}$ is the key bifurcation parameter and $g : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^p$ is a smooth (at least $\mathcal{C}^4$) memoryless nonlinear function. Then, by applying the Laplace transform to the system above (with zero initial condition), we have

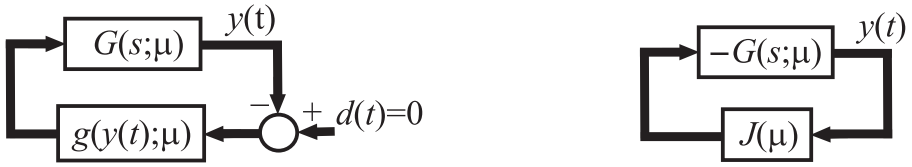
$$\mathfrak{L}\{x\} = (sI - A)^{-1}B\mathfrak{L}\{g(y;\mu)\},$$

and also

$$\mathfrak{L}\{y\} = -C\mathfrak{L}\{x\} = -C(sI - A)^{-1}B\mathfrak{L}\{g(y;\mu)\} := -G(s;\mu)\mathfrak{L}\{g(y;\mu)\},$$

where $G(s;\mu) = C(sI - A)^{-1}B$ is a transfer matrix representing the linear part of (1)[1] and s is the Laplace (complex) variable. Then, the system can be arranged in the feedback representation of Fig. 1a, in which the input $d(t)$ is assumed to be zero denoting that the system is autonomous. It is easy to see that the equilibrium points $\widehat{y}$ of the feedback system satisfy the equation $G(0;\mu)g(\widehat{y};\mu) = -\widehat{y}$. By supposing the

[1]By using the notation $G(s;\mu)$ we emphasize the usual dependence of matrices A, B and C on the parameter μ.



(a) (b)

Fig. 1. (a): Block representation of system (1); (b): Linearized feedback loop.

existence of at least one equilibrium point, and linearizing the feedback function around it, we obtain the linear loop shown in Fig. 1b, where matrix $J(\mu)$ denotes the Jacobian given by

$$J(\mu) = \frac{\partial g(y;\mu)}{\partial y}|_{y=\widehat{y}}.$$

With the aim of detecting bifurcations in the nonlinear system (1), it is useful to consider the following result:

Lemma 1 [Moiola & Chen, 1996]. *If an eigenvalue of the corresponding Jacobian of the nonlinear system (1), in the time domain, assumes a purely imaginary value $i\omega_0$ at a particular value $\mu = \mu_0$, then the corresponding eigenvalue of the constant matrix $G(i\omega_0;\mu_0)J(\mu_0)$ in the frequency-domain must assume the value $-1+i0$ at $\mu = \mu_0$.*

The eigenvalues of matrix $G(i\omega;\mu)J(\mu)$ are the roots of the characteristic polynomial

$$h(\lambda, s;\mu) := \det(\lambda I - GJ) = \lambda^m + a_{m-1}(s;\mu)\lambda^{m-1} + \ldots + a_0(s;\mu), \tag{2}$$

where $a_j(s;\mu)$, $j = 0, ..., m-1$ are rational functions in the variable s [MacFarlane & Postlethwaite, 1977]. Notice that if $p < m$, the order of polynomial (2) can be reduced provided by $(m-p)$ eigenvalues are zero. In addition, as usually $m < n$, the formulation in the FD generally involves a characteristic equation of lower order than its time domain analogue.
Let us suppose that the equilibrium of (1) is stable for $\mu < \mu_0$ and unstable for $\mu > \mu_0$, and that, for $\mu = \mu_0$, there is a simple root $\widehat{\lambda}(s;\mu)$ of the equation above that takes the value $-1+i0$ for a given $s = i\omega_0$. Hence, $\mu = \mu_0$ is a bifurcation point in the parameter space. If $\omega_0 \neq 0$ ($\omega_0 = 0$) the bifurcation is called *dynamic* or *Hopf* (*static*). Let us focalize on Hopf bifurcations, which are responsible for the appearance of smooth oscillations. If we consider the geometrical locus of $\widehat{\lambda}(i\omega;\mu)$, it can be viewed as a Nyquist curve (or *eigenlocus*) in the complex plane, parameterized on the variable ω, which describes a different contour for each fixed value of μ. Particularly, by picking $\mu = \mu_0$, this curve crosses the point $-1+i0$ at $\omega = \omega_0$. Now, let us consider the auxiliary vector

$$\xi(\omega;\mu) = -\frac{w^T G(i\omega;\mu)p(\omega;\mu)}{w^T v}, \tag{3}$$

where $p(\cdot)$ is defined later, and v and w are the right and left eigenvectors of GJ associated to $\widehat{\lambda}(\cdot)$ which satisfy

$$GJv = \widehat{\lambda}v, \quad w^T GJ = w^T\widehat{\lambda}.$$

Now, let μ vary slightly from μ_0, and consider the following theorem given in [Mees & Chua, 1979; Moiola & Chen, 1996], which for convenience to the reader is briefly stated as follows:

Theorem 1. *(Graphical Hopf Bifurcation Theorem) Suppose that when ω varies, the vector $\xi(\omega;\mu) \neq 0$, and that the half line starting from $-1+i0$ and pointing to the direction parallel to that of $\xi(\omega;\mu)$, first intersects the locus of $\widehat{\lambda}(i\omega;\mu)$ at the point*

$$\widehat{P} = \widehat{\lambda}(i\widehat{\omega};\widehat{\mu}) = -1 + \xi(\widehat{\omega};\widehat{\mu})\theta^2, \tag{4}$$

where constant $\theta = \theta(\widehat{\omega}) \geq 0$. Suppose, furthermore, that the above intersection is transversal, i.e., *$\widehat{\lambda}(i\widehat{\omega};\widehat{\mu})$ and $\xi(\widehat{\omega};\widehat{\mu})$ are not parallel. Then:*

(1) The nonlinear system (1) has a periodic solution which is unique in a ball of radius $\mathcal{O}(1)$ centered at $\widehat{y}$.
(2) If the total number of anticlockwise encirclements of the point $\widehat{P}+\delta\xi(\widehat{\omega};\widehat{\mu})$, for a small enough δ, is equal to the number of poles of $\widehat{\lambda}$ with positive real parts, then the limit cycle is stable.

It is worth mentioning that θ is a measure of the amplitude and $\widehat{\omega}$ is the approximate frequency of the periodic solution. If the hypotheses of Theorem 1 are fulfilled, that solution can be described accurately enough by the second-order Fourier expansion

$$y(t) \simeq \widehat{y} + \Re\left\{\sum_{k=0}^{2} \mathcal{Y}^k e^{ik\widehat{\omega}t}\right\}, \tag{5}$$

where coefficients $\mathcal{Y}^k$ are given below and $\Re(\cdot)$ means the real part. Vector $p(\cdot)$ in (3) represents the component of fundamental frequency of $g(y(t);\mu)$ when $y(t)$ is given by (5), and it is computed as

$$p(\omega_0;\mu) = (\mathbf{D}^2 g)v \otimes V_{02} + \frac{1}{2}(\mathbf{D}^2 g)\bar{v} \otimes V_{22} + \frac{1}{8}(\mathbf{D}^3 g)v \otimes v \otimes \bar{v}, \tag{6}$$

where $\otimes$ represents the tensor product operator, and $\mathbf{D}^2$ and $\mathbf{D}^3$ are the second and third order derivative operators, respectively, with respect to the first variable, evaluated at equilibrium. Tensors V_{02} and V_{22} represent the normalized zero and second harmonic components, respectively, of $y(t)$ in (5), and are computed as

$$V_{02} = -\frac{1}{4}(I+G(0;\mu)J(\mu))^{-1}G(0;\mu)(\mathbf{D}^2 g)v\otimes\bar{v}, \quad V_{22} = -\frac{1}{4}(I+G(i2\omega;\mu)J(\mu))^{-1}G(i2\omega;\mu)(\mathbf{D}^2 g)v\otimes v. \tag{7}$$

By using these vectors together with $\theta(\widehat{\mu})$ and $\widehat{\omega}(\widehat{\mu})$ obtained from (4), we find the $\mathcal{Y}^k$ coefficients in (5) as

$$\mathcal{Y}^0 = \theta(\widehat{\mu})^2 V_{02}, \quad \mathcal{Y}^1 = \theta(\widehat{\mu})v, \quad \mathcal{Y}_2 = \theta(\widehat{\mu})^2 V_{22}, \tag{8}$$

and finally compute the approximate periodic solution by using (5). The stability of this periodic orbit can also be determined algebraically via the *curvature coefficient*, which is given by

$$\sigma_0 = -\Re\left\{\frac{w^T G(i\omega_0;\mu)p(\omega_0;\mu)}{w^T G'(i\omega_0;\mu)J(\mu)v}\right\}, \tag{9}$$

where $G'(i\omega;\mu) := \partial G(s;\mu)/\partial s|_{s=i\omega}$. Then, if σ_0 is *negative* (*positive*), the Hopf bifurcation is *supercritical* (*subcritical*), and a *stable* (*unstable*) periodic solution exists when the equilibrium is *unstable* (*stable*).

2.2. *Relevant structures in time-delayed systems*

In this paper, we focalize on two relevant structures of time-delayed systems, which are specially interesting given their broad variety of applications. We show how the described methodology can be applied to each case.

2.2.1. *Case I: Pyragas control*

The following control scheme implements a feedback signal which is proportional to the difference between the current and delayed values of the output $y(t)$. This method was proposed originally and explored experimentally by Pyragas [1992]. An interesting discussion about that technique can be found in [Just *et al.*, 2003]. Consider a system described by an ordinary differential equation (ODE) as

$$\dot{x}(t) = f(x(t);\mu) + z(t), \tag{10}$$

where $x(t) \in \mathbf{R}^n$, $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is a nonlinear function, μ is a real parameter and $z(t)$ represents an accesible control input. Particularly, we are interested in a control of the form

$$y(t) = -Cx(t),$$
$$z(t) = K[y(t) - y(t-\tau)],$$

where $\tau > 0$ is a time delay, $K \in \mathbf{R}^{n\times m}$ is the control gain and matrix $C \in \mathbf{R}^{m\times n}$ selects the components of the state vector that are involved in the feedback and hence they are assumed measurable. Then, the closed-loop system is described by

$$\begin{cases} \dot{x}(t) = f(x(t);\mu) + K[y(t) - y(t-\tau)] \\ y(t) = -Cx(t). \end{cases} \tag{11}$$

Note that the delayed variable acts only *linearly* and appears as a consequence of the chosen control scheme, when some measured variables are delayed and re-injected at the input. Very often, this strategy is used with the aim of stabilizing an unstable limit cycle. Other common goal is to change the amplitude or frequency of the orbit, but not its stability. Equation (11) can be rewritten as

$$\begin{cases} \dot{x}(t) = A_0x(t) + A_1x(t-\tau) + Bg(y(t);\mu), \\ y(t) = -Cx(t), \end{cases} \tag{12}$$

for some matrices $A_0, A_1 \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times p}$. Hence, by choosing

$$G(s;\mu) = C(sI - A_0 - A_1e^{-s\tau})^{-1}B, \tag{13}$$

we obtain the same input-output representation of Fig. 1a, and the FD methodology can be applied as described in Section 2.1. A considerable degree of freedom exists for choosing an appropriate representation, and a clever choice may simplify the calculations remarkably. Although there are infinitely many ways to select matrices A_0, A_1, B and C [Mees & Chua, 1979; Mees, 1981], it is known that for a proper application of the FD approach, the aforementioned matrices must lead to a *controllable* and *observable* realization. If these properties fail, some dynamics can be lost due to zero-pole cancelations in the characteristic FD eigenvalues [Mees, 1981; Agamennoni *et al.*, 2008]. For time-delayed systems the situation is more complicated than for non-delayed ones, because the controllability and observability criteria become more elaborated, requiring an extra careful election. Different definitions about controllability and observability exist, but in the present paper we follow a result given in [Bhat & Koivo, 1976], which provides the simplest criterion and results useful for our purposes.

Theorem 2. *System (12) is controllable if*

$$rank\left(A_0 + A_1e^{-s\tau} - sI,\ B\right) = n, \quad s \in \Lambda,$$

and it is observable if

$$rank\begin{pmatrix} A_0 + A_1e^{-s\tau} - sI \\ C \end{pmatrix} = n, \quad s \in \Lambda,$$

where Λ *is the set of eigenvalues of* $A := A_0 + A_1e^{-s\tau}$.

It is worth mentioning that equation (10) is a special case of the systems studied in [Moiola & Chen, 1996; Moiola *et al.*, 1996], were the case of a nonlinearity depending on $y(t)$, but not simultaneously in $y(t-\tau)$, is studied. A more challenging problem arises when booth $y(t)$ and $y(t-\tau)$ are involved in the nonlinear part. This is the contribution of the following subsection.

2.2.2. *Case II: The general case*

The second description is more general than the previous one, and arises when the nonlinearity involves both delayed and non-delayed variables. Such system is described by an equation of the type

$$\dot{x}(t) = f(x(t), x(t-\tau);\mu), \tag{14}$$

where $f : \mathbf{R}^{2n} \times \mathbf{R} \to \mathbf{R}^n$ and the dimensionality of variables and parameters are similar to Case I. This equation constitutes a very general type of DDE with one delay, and represents a wide class of systems. The vector field $f(\cdot) = [f_1(\cdot)...f_n(\cdot)]^T$ may include linear and nonlinear terms, but usually some components f_i involve only linear ones. Thus we can consider a matrix $A \in \mathbf{R}^{n\times n}$ and $\tilde{f}(\cdot) := f(\cdot) - Ax(t)$, so that the product $Ax(t)$ contains only linear terms in $x(t)$ (A may be zero) and $\tilde{f}(\cdot)$ contains both linear and nonlinear terms in the delayed variables. Then, (14) can be arranged as

$$\dot{x}(t) = Ax(t) + \tilde{f}(x(t), x(t-\tau); \mu).$$

Moreover, many times only a subset of the state variables are involved in the nonlinearities, and $\tilde{f}(\cdot)$ can be expressed as

$$\tilde{f}(x(t), x(t-\tau); \mu) = Bg(-Cx(t), -Cx(t-\tau); \mu),$$

for some matrices B and C so that $g : \mathbf{R}^{2m} \times \mathbf{R} \to \mathbf{R}^p$ has not zero terms, matrix $B \in \mathbf{R}^{n\times m}$ selects the componentes in $g(\cdot)$ which take part in $\tilde{f}(\cdot)$ and matrix $C \in \mathbf{R}^{n\times p}$ selects the components of the vector $x(\cdot)$ involved in the nonlinear terms. As the reader may infer, the worst case occurs when all the f_i are strictly nonlinear and these nonlinearities involve all the state variables, leading to $C = B = I$ and $g(\cdot) = f(\cdot)$. However, in several occasions this is not the case, and m and p can be made smaller than n, giving a valuable reduction in the dimensionality of the problem. Finally, system (14) can be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bg(y(t), y(t-\tau); \mu), \\ y(t) = -Cx(t), \end{cases} \tag{15}$$

which is equivalent to the block scheme shown in Fig. 2. In this representation, the nonlinear feedback is a function of the output $y(t)$ and $y(t-\tau)$, and therefore, the analysis cannot be performed with the tools of the previous Subsection. To overcome this difficulty, we develop the following result.

Theorem 3. *System (15) can be properly analyzed via the frequency-domain approach by taking* $F(e^{-s\tau})G(s)$ *as the linear transfer function, where*

$$F(e^{-s\tau}) = \begin{pmatrix} I \\ Ie^{-s\tau} \end{pmatrix} \in \mathbf{R}^{2m\times m}, \quad G(s) = C(sI - A)^{-1}B,$$

and computing the Jacobian matrix as

$$J(\mu) = \begin{pmatrix} \dfrac{\partial g(\cdot)}{\partial y(t)} & \dfrac{\partial g(\cdot)}{\partial y(t-\tau)} \end{pmatrix} |_{(y(t),y(t-\tau))=(\hat{y},\hat{y})}.$$

The subsequent steps are analogous to the ones described in Section 2.1.

Proof. Matrix $F(e^{-s\tau})G(s)$ corresponds to a transfer function of a linear system whose output vector belongs to $\mathbf{R}^{2m}$ and is given by

$$y^*(t) = \mathfrak{L}^{-1}\begin{pmatrix} Y(s) \\ Y(s)e^{-s\tau} \end{pmatrix} = \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix}.$$

By a simple block manipulation, system (15) can be represented as shown in Fig. 3. Then, the linearized feedback loop is obtained by taking

$$J(\mu) = \begin{pmatrix} \dfrac{\partial g(\cdot)}{\partial y(t)} & \dfrac{\partial g(\cdot)}{\partial y(t-\tau)} \end{pmatrix} |_{(y(t),y(t-\tau))=(\hat{y},\hat{y})}.$$

Let us suppose that system shown in Fig. 3 has a solution of the form (5), then $y(t-\tau)$ reads

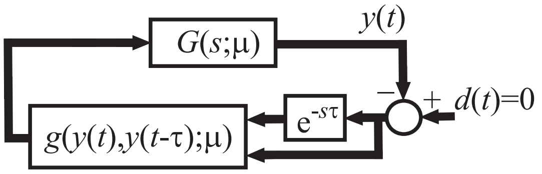


Fig. 2. Block representation of system (15).

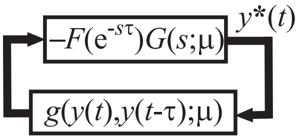


Fig. 3. Equivalent block representation of system (15).

$$y(t-\tau) \simeq \widehat{y} + \Re\left\{\sum_{k=0}^{2} \mathcal{Y}^k e^{-ik\omega\tau} e^{ik\omega t}\right\}. \tag{16}$$

Provided that $y(\cdot)$ has frequency ω, the same applies to $g(y(\cdot);\mu)$ and it is posible to compute its Fourier coefficients $\mathcal{G}^k$ in terms of $\mathcal{Y}^0$, $\mathcal{Y}^1$ and $\mathcal{Y}^2$. For the input and output of the linear part we have the relationship for equating each harmonic

$$\mathcal{Y}^k = -FG(i\omega k)\mathcal{G}^k, \qquad k = 0, 1, 2,$$

where the compact notation $FG(i\omega k) := F(e^{-i\omega k\tau})G(i\omega k)$ is used. Equation above is known as the second-order harmonic balance equation. By expanding the nonlinearity as a Taylor series, it is obtained

$$\begin{aligned} g(y(t), y(t-\tau);\mu) \simeq{}& g(\widehat{y},\widehat{y};\mu) + (D_1^1 g)e(t) + (D_2^1 g)e(t-\tau) + \frac{1}{2}(D_{11}^2 g)e(t)\otimes e(t) + (D_{12}^2 g)e(t)\otimes e(t-\tau) \\ &+\frac{1}{2}(D_{22}^2 g)e(t-\tau)\otimes e(t-\tau) + \frac{1}{3!}(D_{111}^3 g)e(t)\otimes e(t)\otimes e(t) \\ &+(D_{112}^3 g)e(t)\otimes e(t)\otimes e(t-\tau) + (D_{122}^3 g)e(t)\otimes e(t-\tau)\otimes e(t-\tau) \\ &+\frac{1}{3!}(D_{222}^3 g)e(t-\tau)\otimes e(t-\tau)\otimes e(t-\tau) + \mathcal{O}(e^4), \end{aligned}$$

where $e(t) := y(t) - \widehat{y}$, and $(D_{ij\ldots k}^l g) := \partial^l g(y_1, y_2;\mu)/\partial y_i y_j \ldots y_k$, where $i, j, \ldots k = 1$ or 2. Then, by substituting the trial expression for y, we obtain

$$\begin{aligned} \mathcal{Y}^0 &= -FG(0)\mathcal{G}^0 \\ &= -FG(0)\{(D_1^1 g)\mathcal{Y}^0 + (D_2^1 g)\mathcal{Y}^0 + \tfrac{1}{4}(D_{11}^2 g)\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1 + \tfrac{1}{4}(D_{12}^2 g)\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1 e^{i\omega\tau} \\ &\quad +\tfrac{1}{4}(D_{12}^2 g)\bar{\mathcal{Y}}^1\otimes\mathcal{Y}^1 e^{-i\omega\tau} + \tfrac{1}{4}(D_{22}^2 g)\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1\} + \mathcal{O}(|\mathcal{Y}^1|^4), \\ \mathcal{Y}^1 &= -FG(i\omega)\mathcal{G}^1 \\ &= -FG(i\omega)\{(D_1^1 g)\mathcal{Y}^1 + (D_2^1 g)\mathcal{Y}^1 e^{-i\omega\tau} + \tfrac{1}{2}(D_{11}^2 g)(2\mathcal{Y}^0\otimes\mathcal{Y}^1 + \bar{\mathcal{Y}}^1\otimes\mathcal{Y}^2) \\ &\quad +(D_{12}^2 g)(\mathcal{Y}^0\otimes\mathcal{Y}^1 e^{-i\omega\tau} + \mathcal{Y}^1\otimes\mathcal{Y}^0 + \tfrac{1}{2}\bar{\mathcal{Y}}^1\otimes\mathcal{Y}^2 e^{-i2\omega\tau} + \tfrac{1}{2}\mathcal{Y}^2\otimes\bar{\mathcal{Y}}^1 e^{i\omega\tau}) \\ &\quad +\tfrac{1}{2}(D_{22}^2 g)(2\mathcal{Y}^0\otimes\mathcal{Y}^1 e^{-i\omega\tau} + \bar{\mathcal{Y}}^1\otimes\mathcal{Y}^2 e^{-i\omega\tau}) + \tfrac{1}{8}(D_{111}^3 g)\mathcal{Y}^1\otimes\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1 \\ &\quad +(D_{112}^3 g)(\tfrac{1}{2}\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1\otimes\mathcal{Y}^1 e^{-i\omega\tau} + \tfrac{1}{4}\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1\otimes\bar{\mathcal{Y}}^1 e^{-i\omega\tau}) \\ &\quad +(D_{122}^3 g)(\tfrac{1}{2}\mathcal{Y}^1\otimes\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1 + \tfrac{1}{4}\bar{\mathcal{Y}}^1\otimes\mathcal{Y}^1\otimes\mathcal{Y}^1 e^{-i2\omega\tau}) \\ &\quad +\tfrac{1}{8}(D_{222}^3 g)\mathcal{Y}^1\otimes\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1 e^{-i\omega\tau}\} + \mathcal{O}(|\mathcal{Y}^1|^4), \\ \mathcal{Y}^2 &= -FG(i2\omega)\mathcal{G}^2 \\ &= -FG(i2\omega)\{(D_1^1 g)\mathcal{Y}^2 + (D_2^1 g)\mathcal{Y}^2 e^{-i2\omega\tau} + \tfrac{1}{4}(D_{11}^2 g)\mathcal{Y}^1\otimes\mathcal{Y}^1 \\ &\quad +\tfrac{1}{2}(D_{12}^2 g)\mathcal{Y}^1\otimes\mathcal{Y}^1 e^{-i\omega\tau} + \tfrac{1}{4}(D_{22}^2 g)\mathcal{Y}^1\otimes\mathcal{Y}^1 e^{-i2\omega\tau}\} + \mathcal{O}(|\mathcal{Y}^1|^4). \end{aligned}$$

It is convenient to solve $\mathcal{Y}^0$ and $\mathcal{Y}^2$ in terms of $\mathcal{Y}^1$ (see [Mees & Chua, 1979]). Then, neglecting the terms quartic in $|\mathcal{Y}^1|$, the expressions above can be arranged as

$$\left\{\begin{aligned} \mathcal{Y}^{0*} &= -\tfrac{1}{4}\left[I + FG(0)J\right]^{-1} FG(0)(\mathbf{D}^2 g)\mathcal{Y}^{1*}\otimes\bar{\mathcal{Y}}^{1*}, \\ \mathcal{Y}^{2*} &= -\tfrac{1}{4}\left[I + FG(i2\omega)J\right]^{-1} FG(i2\omega)(\mathbf{D}^2 g)\mathcal{Y}^{1*}\otimes\mathcal{Y}^{1*}, \end{aligned}\right. \tag{17}$$

$$\begin{aligned} &(I + FG(i\omega)J)\,\mathcal{Y}^{1*} = -FG(i\omega)p(i\omega;\mu), \\ &p(i\omega;\mu) = (\mathbf{D}^2 g)\mathcal{Y}^{1*}\otimes\mathcal{Y}^{0*} + \tfrac{1}{2}(\mathbf{D}^2 g)\bar{\mathcal{Y}}^{1*}\otimes\mathcal{Y}^{2*} + \tfrac{1}{8}(\mathbf{D}^3 g)\mathcal{Y}^1\otimes\mathcal{Y}^1\otimes\bar{\mathcal{Y}}^1 \end{aligned} \tag{18}$$

where

$$\mathcal{Y}^{k*} := \begin{pmatrix} \mathcal{Y}^k \\ \mathcal{Y}^k e^{-i\omega k\tau} \end{pmatrix}, \; J := \left[(D_1^1 g)\; (D_2^1 g)\right],$$
$$\mathbf{D}^2 g := \left[(D_{11}^2 g)\; (D_{12}^2 g)\; (D_{21}^2 g)\; (D_{22}^2 g)\right],$$
$$\mathbf{D}^3 g := \left[(D_{111}^3 g)\; (D_{112}^3 g)\; (D_{121}^3 g)\; (D_{122}^3 g)\; (D_{211}^3 g)\; (D_{212}^3 g)\; (D_{221}^3 g)\; (D_{222}^3 g)\right].$$

Notice that (17) is analogous to (7) and (8). The equation for $\mathcal{Y}^{1*}$ cannot be solved because matrix $(I + FG(i\omega)J)$ is not invertible at bifurcation. Instead, the component of fundamental frequency is approximated by the eigenvector v (see [Mees & Chua, 1979]). Thus, we arrived to formulae analogous to the described in Section 2.1. ■

3. Examples

In this section, three schemes of the van der Pol oscillator with delay are studied. In the first one, we examine the Hopf bifurcation arising in a scheme proposed by Atay [1998] by using the approach of Subsection 2.2.1. Then, the static bifurcations appearing in this system are investigated in Example 2. A comparison between the two approaches of Section 2.2 for the application of the FD method is provided. Finally, the formulation of Subsection 2.2.2 is used for the analysis of Hopf phenomenon in Examples 3 and 4.

Example 3.1. Consider the van der Pol oscillator under the action of a control force $f(t)$:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = f(t), \tag{19}$$

where $x \in \mathbf{R}$ and $\mu > 0$. In [Atay, 1998] it is proposed that $f(t) = \mu k x(t - \tau)$; in this paper this scheme will be studied by means of the FD approach. Note that if $\mu = 0$ in (19), the nonlinearity vanishes, leading to a degenerate bifurcation for this parameter value. Then, in similar fashion than Padín *et. al.* [2005], we scale the state variable as $u = \mu^{1/2}x$, yielding

$$\ddot{u} + u^2\dot{u} - \mu\dot{u} + u = \mu k u(t - \tau), \tag{20}$$

which is equivalent to

$$\begin{cases} \dot{u}_1 = \mu u_1 - u_2 - \frac{1}{3}u_1^3 + \mu k u_2(t - \tau), \\ \dot{u}_2 = u_1, \end{cases} \tag{21}$$

as can be verified by computing the time derivative of the first equation in (21), which results $\ddot{u}_1 = \mu\dot{u}_1 - \dot{u}_2 - u_1^2\dot{u}_1 + \mu k\dot{u}_2(t - \tau)$. Therefore, by taking $\dot{u}_2 = u_1$ and assigning $u_1 = u$, we finally obtain (20). Note that this example can be treated with the methodology described in Section 2.2.1 by choosing the realization

$$A_0 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \mu k \\ 0 & 0 \end{pmatrix}, \quad C^T = B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{22}$$

$$g(y; \mu) = -(\mu + 1)y + y^3/3.$$

Hence, the transfer function of the linear part is

$$G(s; \mu) = C(sI - A_0 - A_1 e^{-s\tau})^{-1}B = \frac{s}{s^2 + s + 1 - \mu k e^{-s\tau}}.$$

Also, the Jacobian is given by $J(\mu) = -(\mu + 1)$, and the unique FD eigenvalue of GJ results

$$\widehat{\lambda}(s; \mu) = \frac{-s(\mu + 1)}{s^2 + s + 1 - \mu k e^{-s\tau}}. \tag{23}$$

The Hopf bifurcation condition $\widehat{\lambda}(i\omega_0; \mu_0) = -1$ leads to the system

$$\begin{cases} 1 - \omega_0^2 = \mu_0 k \cos(\omega_0\tau), \\ \mu_0\omega_0 = \mu_0 k \sin(\omega_0\tau). \end{cases} \tag{24}$$

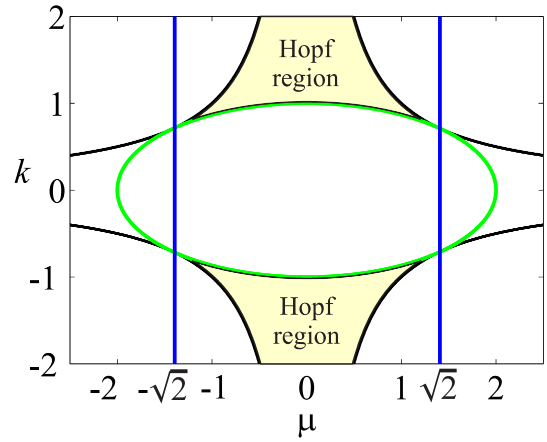


Fig. 4. Region of possible Hopf bifurcation determined by (26).

Taking squares and adding equations, we obtain

$$\omega_0^4 + (\mu_0^2 - 2)\omega_0^2 + 1 - \mu_0^2 k^2 = 0,$$

which can be solved for ω_0 as

$$\omega_0 = \sqrt{\frac{2-\mu_0^2}{2} \pm \frac{1}{2}|2-\mu_0^2|\sqrt{\eta}}\,, \tag{25}$$

where $\eta := 1 - 4(1-\mu_0^2k^2)/(2-\mu_0^2)^2$. Then, because $\omega_0 \in \mathbf{R}$, $2 - \mu^2 \geq 0$ and then $0 \leq \eta \leq 1$. From these inequalities, the necessary conditions for the existence of a Hopf bifurcation are stated as

$$\begin{cases} \quad |\mu| & \leq \sqrt{2}, \\ \quad |\mu k| & \leq 1, \\ k^2 + (\mu/2)^2 & \geq 1. \end{cases} \tag{26}$$

The region determined by these conditions in the parameter space is depicted in Fig. 4. The shaded area indicates the zone of possible Hopf bifurcations.
The eigenvectors associated to $\widehat{\lambda}(i\omega;\mu)$ are trivial, $w = v = 1$. Moreover, $\mathbf{D}^2 g = 0$ and $\mathbf{D}^3 g = 2$, causing that $V_{02} = V_{22} = 0$ and $p(i\omega) = 1/4$. By expressing $G(s;\mu) = s/\Delta(s;\mu)$, where $\Delta(s;\mu) := s^2 + s + 1 - \mu k e^{-s\tau}$, and taking the derivative it is obtained $G'(s;\mu) = N(s;\mu)/\Delta(s;\mu)^2$, where $N(s;\mu) = -s^2 + 1 - \mu k(1+s\tau)e^{-s\tau}$. Thus, after some simple calculations, the stability coefficient is obtained from (9) as

$$\sigma_0 = \frac{\omega_0}{4(\mu+1)} \Re \left\{ \frac{i\Delta(i\omega_0;\mu)}{N(i\omega_0;\mu)} \right\},$$

which from $\widehat{\lambda}(i\omega_0;\mu_0) = -1$, becomes

$$\sigma_0 = -\frac{\omega_0^2}{4|N(i\omega_0)|^2}[\omega_0^2 + 1 - \mu_0 k \cos(\omega_0\tau) - \mu_0 k \omega_0 \tau \sin(\omega_0\tau)],$$

and considering (24), we finally obtain

$$\sigma_0 = -\frac{\omega_0^4(2-\mu_0\tau)}{4|N(i\omega_0)|^2}. \tag{27}$$

Therefore, the Hopf bifurcation is *supercritical* if $\mu\tau < 2$ and *subcritical* if $\mu\tau > 2$. Also, we can determine the amplitude of the orbits from (3) and (4). By using (23) and taking into account that $p(i\omega) = 1/4$, the intersection point in (4) results in the $(\widehat{\theta}, \widehat{\omega})$ pair satisfying

$$\begin{cases} 1 - \widehat{\omega}^2 = \widehat{\mu} k \cos(\widehat{\omega}\tau) \\ \quad \widehat{\mu}\,\widehat{\omega} = \widehat{\mu} k \sin(\widehat{\omega}\tau) + \frac{1}{4}\widehat{\omega}\theta^2, \end{cases} \tag{28}$$

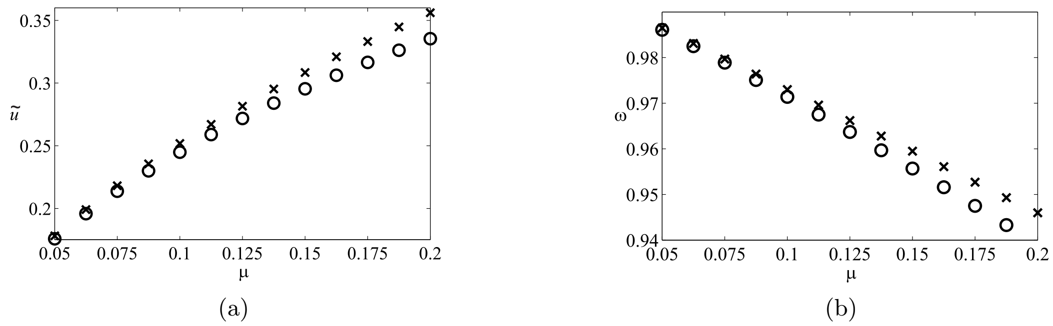


Fig. 5. Amplitude (a) and frequency (b) of the bifurcated limit cycle vs. μ for system (20), with $\tau = k = 1$. Crosses: Theoretical values ($\tilde{u} = 2\sqrt{\mu(1 - k\sin\tau)}$, $\omega = 1 - \mu k\cos\tau/2$); circles: simulations.

and from the second equality we can obtain θ as

$$\theta = 2\sqrt{\frac{\widehat{\mu}}{\widehat{\omega}}[\widehat{\omega} - k\sin(\widehat{\omega}\tau)]}.$$

For small μ values, the frequency is close to unity and the normalized amplitude is given by $\theta =\simeq 2\sqrt{\mu(1 - k\sin\tau)}$. Thus, it is clear that in addition to $\mu > 0$, for the existence of solution it must be certain that $k\sin\tau < 1$. Moreover, from (28) it can be easily obtained $\mu = f_1(\widehat{\omega}) := (1 - \widehat{\omega}^2)/[k\cos(\widehat{\omega}\tau)]$, and by expanding this function via a first-order Taylor approximation we have

$$f_1(\widehat{\omega}) \simeq \frac{-2}{k\cos\tau}(\widehat{\omega} - 1) + \mathcal{O}\left((\widehat{\omega} - 1)^2\right).$$

Thus, by equating this expression to μ, a corrected value of frequency can be obtained, given by $\omega \simeq 1 - \mu k\cos\tau/2$. Notice that because $V_{02} = V_{22} = 0$, the Fourier expansion (5) becomes $u(t) = y(t) = \Re\{Y_1 e^{i\widehat{\omega}t}\}$, *i.e.*, the shape of the orbit is determined uniquely by the fundamental frequency component. Moreover, as $v = 1$, the amplitude of the orbit for the system (20) is given by $\tilde{u} = Y_1 = 2[\mu(1 - k\sin\tau)]^{1/2}$. The dependence of amplitude and frequency on μ can be seen in Figs. 5a and 5b, respectively. These allow a comparison between the theoretical and simulation values. Notice that for system (19), the theoretical amplitude becomes $\tilde{x} = \mu^{-1/2}\tilde{u} = 2(1 - k\sin\tau)^{1/2}$, as found by Atay [1998].

Example 3.2. In addition to Hopf type, it is known that the system considered in the former example can exhibit several bifurcations. Singularities like Bogdanov-Takens, triple zero and zero-Hopf have been discovered [Wei & Jiang, 2005; Jiang & Wei, 2008; Wang & Jiang, 2010]. The classical technique used by these authors is the normal form reduction, which seems to be the most preferred tool in the literature. We show that the FD approach allows the detection of all those singularities, while its application is straightforward.

From (23) it can be seen that for $s = 0$, if $\mu k = 1$, the value of $\widehat{\lambda}(s;\mu)$ becomes undetermined. Moreover, Jiang & Wei [2008] found that for $\mu k = 1$, a static bifurcation takes place. Therefore, the question that arises is whether this bifurcation can be analyzed with the realization given by (22). Let us consider $\mu k = 1$, thus

$$A_0 + A_1 e^{-s\tau} - sI = \begin{pmatrix} -1-s & -1+e^{-s\tau} \\ 1 & -s \end{pmatrix},$$

then, $\mathrm{rank}(A_0 + A_1 e^{-s\tau}\ B) = 2$, but $\mathrm{rank}[(A_0 + A_1 e^{-s\tau})^T\ C^T]^T$ falls to one for $s = 0$, causing the failure of the observability condition according to Theorem 2. Sometimes, for fulfilling the conditions of controllability and observability, it is necessary to take a realization that is not the simplest one or that achieves a minimal dimension of matrices $G(\cdot)$ and $J(\cdot)$. For example, it can be easily verified that the following realization leads to a controllable and observable system

$$A_0 = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & \mu k \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For this choice, the linear transfer matrix results $G(s;\mu) = (s \quad 1)^T/\Delta(s;\mu)$, where $\Delta(s;\mu) := s^2 + s - \mu k e^{-s\tau}$. The Jacobian matrix becomes $J(\mu) = (-(\mu+1) \quad 1)$, and by computing $G(s;\mu)J(\mu)$ it can be found that the unique nonzero eigenvalue is given by

$$\widehat{\lambda}(s;\mu) = \frac{1 - s(\mu+1)}{s^2 + s - \mu k e^{-s\tau}}. \tag{29}$$

The poles of (23) and (29) are the solutions of trascendental equations. Then, even though we have obtained a unique characteristic function, the stability analysis has the same difficulty as in the time-domain formulation, because we must know the poles of $\widehat{\lambda}(s;\mu)$. Moreover, it is expected that this problem will always arise if the methodology described in Section 2.2.1 is used, because of the term $\det(sI - A_0 - A_1 e^{-s\tau})^{-1}$ involved in (13). Then, a better choice is to use the alternative of Section 2.2.2. For example, if we choose

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{30}$$

$$g(y^*;\mu) = -(\mu+1)y_1(t) + y_1(t)^3/3 - \mu k y_2(t-\tau),$$

then, we have

$$FG(s) = \frac{1}{s^2+s+1}\begin{pmatrix} s & 1 & se^{-s\tau} & e^{-s\tau} \end{pmatrix}, \; J(\mu) = \begin{pmatrix} -(\mu+1) & 0 & 0 & -\mu k \end{pmatrix}.$$

It is easy to verify that matrix $FG(s)J(\mu)$ has a unique nonzero eigenvalue or characteristic function given by

$$\widehat{\lambda}(s;\mu) = -\frac{(\mu+1)s + \mu k e^{-s\tau}}{s^2+s+1}. \tag{31}$$

In this case, the two poles of $\widehat{\lambda}(s;\mu)$ lie in the left half plane ($s_{1,2} = -1/2 \pm i\sqrt{3}/2$), then the stability of the equilibrium as well as the emerging cycle can be deduced from the number of encirclements of the critical point by the Nyquist plot, without dealing with the roots of a trascendental equation. Thus, the second approach is clearly advantageous over the first one and, even more, over the time-domain setting. In addition, the controllability and observability of the linear block can be checked in a classical way, avoiding the usage of the more cumbersome test for time-delayed systems. Notice that with this expression of $\widehat{\lambda}(\cdot)$, the bifurcation condition (24) remains unchanged, but now the static (ST) bifurcations can be properly analyzed. For this example, Eqn. (2) becomes

$$h(s,\lambda;\mu) = \lambda - \widehat{\lambda}(s;\mu) = \lambda + \frac{(\mu+1)s + \mu k e^{-s\tau}}{s^2+s+1},$$

and the bifurcation conditions can be stated in terms of this function. Particularly, for a ST bifurcation, the defining condition is $h(0,-1;\mu) = 0$ [Moiola & Chen, 1996], which yields $\mu k = 1$. For the following calculations, let us pick $\mu k = 1$. The expression of the critical frequency (25) reduces to

$$\omega_0 = \sqrt{\frac{2-\mu_0^2}{2} \pm \frac{1}{2}|2-\mu_0^2|},$$

then if $|\mu| < \sqrt{2}$, one solution is $\omega_{0,a} = 0$ and another is given by $\omega_{0,b} = \sqrt{2-\mu_0^2}$, meaning that it is possible to have simultaneously a single root at $s = 0$ plus a Hopf solution. We can look for multiple roots at $s = 0$, by taking successive derivatives of $h(s,-1;\mu)$. Further details about the defining conditions for multiplicity of roots can be found in [Moiola & Chen, 1996]. Consider the derivative $\partial h(\cdot)/\partial s$, which results in

$$\frac{\partial h(\cdot)}{\partial s} = \frac{\Gamma'(s;\mu)\Phi(s) - \Gamma(s;\mu)\Phi'(s)}{\Phi(s)^2},$$

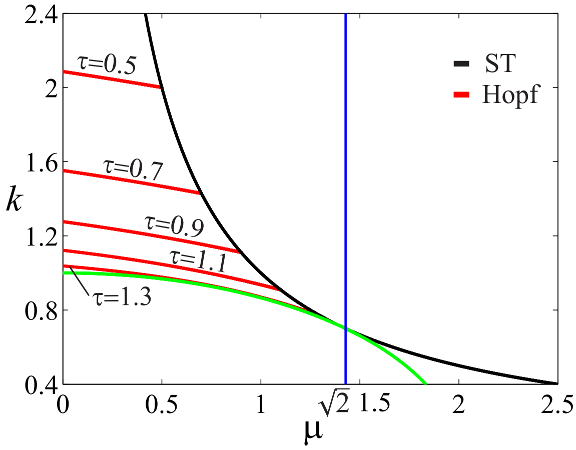


Fig. 6. Bifurcation curves in the (μ, k) parameter space. Static bifurcations (black), Hopf (red) and the border conditions given by (26) are indicated. The meeting point between ST and Hopf curves are BT bifurcations.

where $\Gamma(s;\mu) := s(\mu+1)+e^{-s\tau}$ and $\Phi(s) := s^2+s+1$. Then, for $s=0$ it is obtained $\partial h(\cdot)/\partial s|_{s=0} = \mu-\tau$. Then, $h(s,-1;\mu)$ has a double root at $s=0$ only if $\mu=\tau$. It is also called a Bogdanov-Takens (BT) bifurcation. Fixing also the restriction $\mu=\tau$ and taking the second derivative of $h(\cdot)$ it is obtained

$$\frac{\partial^2 h(\cdot)}{\partial s^2} = \frac{\Gamma''(s;\tau)\Phi(s) - \Gamma(s;\tau)\Phi''(s)}{\Phi(s)^4}.$$

For $s=0$ it results $\partial^2 h(\cdot)/\partial s^2|_{s=0} = -(2-\tau^2)$, thus the condition for a triple root at $s=0$ is $\tau=\sqrt{2}$.

By joining the results of Examples 3.1 and 3.2, we can obtain a bifurcation diagram in the (μ, k) parameter space. From (24), by expressing μ_0 and k in terms of ω_0, it is obtained

$$k = \frac{\omega_0}{\sin\omega_0\tau}, \qquad \mu_0 = \frac{1-\omega_0^2}{\omega_0}\tan\omega_0\tau,$$

which represents the Hopf bifurcation curve, parameterized on the critical frequency ω_0, where the type of bifurcation is deduced from the sign of the curvature coefficient (27). Figure 6 shows the ST and Hopf curves, where the last ones are drawn for some different values of τ. The green curve indicates the borderline given by $k^2+(\mu/2)^2=1$. Notice that all the shown Hopf curves correspond to supercritical bifurcations. The equilibrium is always unstable below the green curve or on the right of the ST one. On the left of ST, for a given fixed τ, the origin is stable if μ and k are such that the point (μ, k) lies above the Hopf curve; or, analogously, let us pick a pair (μ', k') verifying (26). Then, the origin will be stable if we choose a τ value such that the Hopf curve lies below the point (μ', k').
Some numerical results are in order. From (3), it can be checked that for realization (30), we have $\xi = -1/(\mu+1)$, thus this vector always points to the direction of the negative real axis. Figures 7 and 8 show the Nyquist diagram of $\widehat{\lambda}(i\omega;\mu)$ given by (31) and the phase portrait for $k=1.5$, $\mu=0.2$ and two different values of τ. In Fig. 7 ($\tau=0.5$) the Nyquist locus encircles once the critical point $-1+i0$, thus the equilibrium is unstable and, surrounding it, a stable limit cycle exists. In Fig. 8 ($\tau=1$) the locus of $\widehat{\lambda}(i\omega;\mu)$ does not encircle the critical point, thus the equilibrium is stable, and no limit cycle exists. Then, the delayed feedback has stabilized the origin via a Hopf bifurcation.

Example 3.3. Other system proposed in [Atay, 1998] is the following modified equation

$$\ddot{x} + \mu(x^2-1)x(t-\tau) + x = 0. \tag{32}$$

Applying the same transformation used before, $u=\mu^{1/2}x$, we have

$$\ddot{u} + (u^2-\mu)u(t-\tau) + u = 0. \tag{33}$$

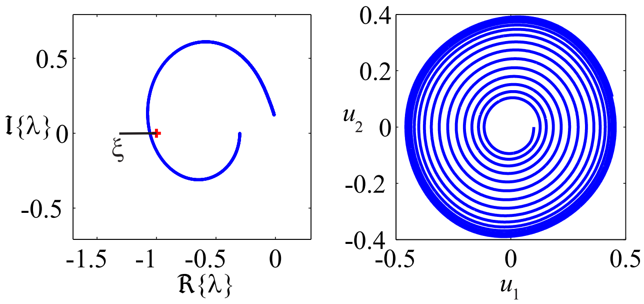


Fig. 7. Nyquist diagram (left) and phase portrait (right) for $k = 1.5$, $\mu = 0.2$ and $\tau = 0.5$. The locus encircles once the point $-1 + i0$, thus equilibrium is unstable and there exists a stable limit cycle surrounding it, since there is a solution of Eq. (4).

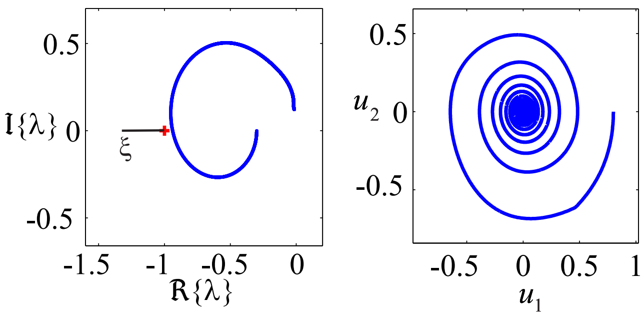


Fig. 8. Nyquist diagram (left) and phase portrait (right) for $k = 1.5$, $\mu = 0.2$ and $\tau = 1$. The locus does not encircle the point $-1 + i0$, and the equilibrium is stable. There is no solution of Eq. (4).

Defining the state variables as $u_2 = u$ and $u_1 = \dot{u}$, system above can be written as

$$\begin{cases} \dot{u}_1 = -(u_2^2 - \mu)u_2(t-\tau) - u_2, \\ \dot{u}_2 = u_1. \end{cases}$$

Let us select the realization

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \; 1 \end{pmatrix},$$
$$g(y^*; \mu) = (y(t)^2 - \mu)y(t-\tau) + y(t),$$

then the linear transfer function is $G(s) = C(sI - A)^{-1}B = 1/s^2$. Note that in this case the controllability and observability can be checked in a classical way, because the linear transfer matrix is delay-free. It can be easily verified that this realization leads to a controlable and observable system. Whereas in the former example the delayed variable acts as a linear term, now the dynamical system involves the product of the delayed and non-delayed variables. Then, the approach developed for Case II applies. For the system under study we have $F(e^{-s\tau})G(s) = (1 \;\; e^{-s\tau})^T/s^2$, and the Jacobian matrix is given by $J(\mu) = (1 \;\; -\mu)$, thus

$$F(e^{-s\tau})G(s)J(\mu) = \frac{1}{s^2} \times \begin{pmatrix} 1 & -\mu \\ e^{-s\tau} & -\mu e^{-s\tau} \end{pmatrix}.$$

As the rank of this matrix is one, there is a zero eigenvalue and the other equals the trace, $\widehat{\lambda}(s; \mu) = (1 - \mu e^{-s\tau})/s^2$, for which the Hopf bifurcation condition ($\widehat{\lambda}(i\omega_0; \mu_0) = -1$) leads to system

$$\begin{cases} \mu_0 \cos(\omega_0 \tau) = 1 - \omega_0^2, \\ \mu_0 \sin(\omega_0 \tau) = 0. \end{cases}$$

It is obvious that one solution is given by $\omega_0 = 1$ and $\mu_0 = 0$, the same to the non-delayed system (see [Padín *et al.*, 2005]). The right and left eigenvectors associated to $\widehat{\lambda}(i\omega; \mu)$ are $v = (1 \;\; e^{-i\omega\tau})^T$ and $w = (1 \;\; -\mu)^T$, and in addition $\mathbf{D}^2 g = 0$. Then, the vector $p(i\omega)$ is obtained as[2]

$$p(i\omega) = \frac{1}{8}(\mathbf{D}^3 g)v \otimes v \otimes \overline{v}$$
$$= \frac{1}{8}[0\ 2\ 2\ 0\ 2\ 0\ 0\ 0]\left[1\ e^{i\omega\tau}\ e^{-i\omega\tau}\ 1\ e^{-i\omega\tau}\ 1\ e^{-i2\omega\tau}\ e^{-i\omega\tau}\right]^T = e^{-i\omega\tau}/2 + e^{i\omega\tau}/4.$$

To compute the curvature coefficient we note that

$$w^T F(e^{-i\omega_0\tau})G(i\omega_0;\mu_0)p(i\omega_0) = -p(i\omega_0), \tag{34}$$

(because at criticality $\widehat{\lambda}(i\omega_0;\mu_0) = -(1-\mu_0 e^{-i\omega_0\tau})/\omega_0^2 = -1$). On the other hand, we have

$$w^T[F(e^{-i\omega\tau})G(i\omega)]' J(\mu)v = \frac{i}{\omega}[2 - \mu e^{-i\omega\tau}(2 + i\omega\tau)]. \tag{35}$$

Thus, after some algebraic calculations, we find

$$\sigma_0 = \frac{\omega}{4}\left(\frac{-2\mu\sin(2\omega\tau) + \mu\omega\tau\cos(2\omega\tau) - 2\sin(\omega\tau) + \mu\omega\tau}{4 + 4\mu^2 + \mu^2\omega^2\tau^2 - 8\mu\cos(\omega\tau) - 4\mu\omega\tau\sin(\omega\tau)}\right).$$

For example, for the solution $\omega_0 = 1$ and $\mu_0 = 0$, the curvature coefficient is

$$\sigma_0 = \frac{\sin(\tau)}{8}.$$

Then the Hopf bifurcation is supercritical if $\sin\tau < 0$ and subcritical if $\sin\tau > 0$. Analogously to the first example, it is possible to determine the amplitude of oscillations for small values of the parameter μ, by using (3) and (4). Taking into account that $\widehat{\lambda}(i\omega;\mu) = -(1-\mu e^{-i\omega\tau})/\omega^2$ and the expression of $p(i\omega)$, after separating into real and imaginary parts, (4) leads to

$$\begin{cases} 1 - \widehat{\mu}\cos(\widehat{\omega}\tau) = \widehat{\omega}^2 - \frac{3}{4}\cos(\widehat{\omega}\tau)\theta^2, \\ \widehat{\mu}\sin(\widehat{\omega}\tau) = \frac{1}{4}\sin(\widehat{\omega}\tau)\theta^2. \end{cases} \tag{36}$$

From the second equation it can be easily obtained $\theta = 2(\widehat{\mu})^{1/2}$, *i.e.* the amplitude of oscillations will be $\tilde{u} = 2(\widehat{\mu})^{1/2}$, that in the original system (32) corresponds to an oscillation of amplitude $\tilde{x} = 2$. Figure 9a shows the dependence of the amplitude of the bifurcated limit cycle on μ. It gives a comparison between the theoretical and simulation values. From the first equation in (36), it is obtained $\widehat{\mu} = f_2 := (\widehat{\omega}^2 - 1)/[2\cos(\widehat{\omega}\tau)]$, which in similar fashion to the former example, can be expressed as

$$\mu \simeq \frac{1}{\cos\tau}(\widehat{\omega} - 1) + \mathcal{O}\left((\widehat{\omega} - 1)^2\right),$$

then the corrected frequency results $\widehat{\omega} \simeq 1 + \mu\cos\tau$. Figure 9b provides a comparison between theoretical and simulation frequency values.

Example 3.4. A third type of van der Pol oscillator with delay was proposed by de Oliveira [2002], and the considered model is

$$\ddot{x} - \mu\dot{x} + x^2(t-\tau)\dot{x}(t-\tau) + x = 0, \tag{37}$$

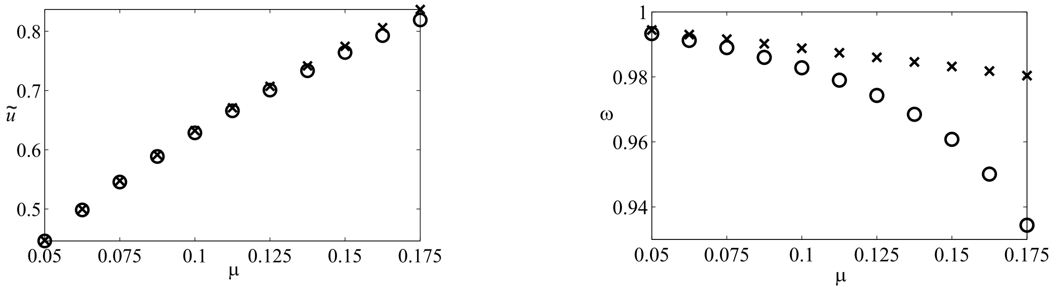
where $x \in \mathbf{R}$ and $\tau > 0$. Again, if $\mu = 0$, the nonlinearity vanishes leading to a degenerate bifurcation for this parameter value. Thus, scaling the variable as $u = \mu^{1/2}x$ it is obtained

$$\ddot{u} - \mu\dot{u} + u^2(t-\tau)\dot{u}(t-\tau) + u = 0. \tag{38}$$

The equation above is equivalent to the system

$$\begin{cases} \dot{u}_1 = \mu u_1 - u_2 - \frac{1}{3}u_1^3(t-\tau), \\ \dot{u}_2 = u_1. \end{cases} \tag{39}$$

[2]Where it is intended that $(y_1(t), y_2(t)) = (y(t), y(t-\tau))$ for calculating derivatives.



(a) (b)

Fig. 9. Amplitude (a) and frequency (b) of the bifurcated limit cycle vs. μ for system (33), with $\tau = 4.6$. Crosses: Theoretical values ($\tilde{u} = 2\sqrt{\mu}$, $\omega = 1 + \mu \cos\tau/2$); circles: simulations.

The nonlinear term involves only the delayed variable, but we cannot chose $g(y(t-\tau)) = \frac{1}{3}y^3(t-\tau)$ because the Jacobian must be nonzero for a proper application of the FD approach. Then, consider

$$A_0 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad C^T = B = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$g(y^*; \mu) = -(\mu+1)y(t) + y^3(t-\tau)/3, \tag{40}$$

thus $G(s) = s/(s^2 + s + 1)$. The controllability and observability conditions are satisfied and a proper application of the methodology is assured. For the linear part we have

$$F(e^{-s\tau})G(s) = \frac{s}{s^2+s+1} \times \begin{pmatrix} 1 \\ e^{-s\tau} \end{pmatrix},$$

and the Jacobian is given by $J(\mu) = (-(\mu+1)\ \ 0)$, then the nontrivial eigenvalue of FGJ results

$$\widehat{\lambda}(s;\mu) = \frac{-(\mu+1)s}{s^2+s+1}. \tag{41}$$

It is simple to see that the bifurcation condition $\widehat{\lambda}(i\omega;\mu) = -1$ again leads to the solution $\omega_0 = 1$ and $\mu_0 = 0$, the same to the non-delayed case. The reason is that the delay is involved only in the cubic term, and it does not affect the stability of equilibrium. However, it can modify the stability of the bifurcated limit cycle given its influence on σ_0. The right and left eigenvectors associated to FGJ are $v = (1\ \ e^{-i\omega\tau})^T$, $w = (1\ \ 0)^T$. All the second partial derivatives of $g(\cdot)$ are zero, and the unique nonzero third derivative is $(D^3_{222}g) = \partial^3 g(\cdot)/\partial y(t-\tau)^3 = 2$. Thus, we have $\mathbf{D}^2 g = \mathbf{0}$ and $\mathbf{D}^3 g = (0\,0\,0\,0\,0\,0\,0\,2)$. By computing the expressions involved in the curvature coefficient, we find

$$w^T F(e^{-i\omega\tau})G(i\omega;\mu)p(i\omega) = \frac{1}{4}\frac{i\omega e^{-i\omega\tau}}{1-\omega^2+i\omega},$$
$$w^T[F(e^{-i\omega\tau})G(i\omega;\mu)]' J(\mu)v = -\frac{(\mu+1)(1+\omega^2)}{(1-\omega^2+i\omega)^2},$$

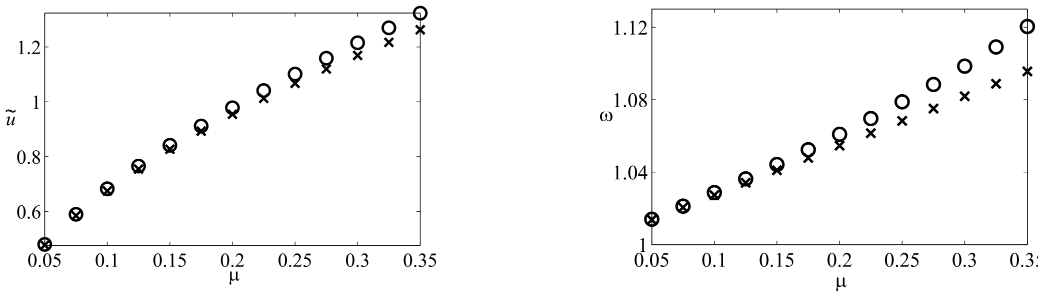
and finally

$$\sigma_0 = -\frac{\omega_0^2\cos\omega_0\tau - \omega_0(1-\omega_0^2)\sin\omega_0\tau}{4(1+\omega_0^2)(\mu+1)},$$

that for the critical frequency $\omega_0 = 1$ and $\mu_0 = 0$, results

$$\sigma_0 = -\frac{\cos\tau}{8}.$$

Therefore, the Hopf bifurcation is supercritical (subcritical) and the emerging limit cycle is stable (unstable) if $\cos\tau > 0$ ($\cos\tau < 0$) (see [de Oliveira, 2002]). Again, the amplitude of the periodic orbit can be obtained



(a) (b)

Fig. 10. Amplitude (a) and frequency (b) of the bifurcated limit cycle vs. μ for system (38), with $\tau = 0.5$. Crosses: Theoretical values ($\tilde{u} = 2\sqrt{\mu \sec \tau}$, $\omega = 1 + \mu \tan \tau/2$); circles: simulations.

by solving (4), where from (3) we have $\xi(\widehat{\omega}; \mu) = -G(i\widehat{\omega}; \mu)e^{-i\widehat{\omega}\tau}/4$. Then, (4) leads to

$$\begin{cases} 0 = 1 - \widehat{\omega}^2 + \frac{1}{4}\widehat{\omega}\sin(\widehat{\omega}\tau)\theta^2, \\ \widehat{\mu} = \frac{1}{4}\cos(\widehat{\omega}\tau)\theta^2, \end{cases}$$

from which it is obtained $\theta = 2[\widehat{\mu}\sec(\widehat{\omega}\tau)]^{1/2}$. For $\widehat{\omega}$ close to ω_0 (near bifurcation) the approximate amplitude of the orbit is $\tilde{u} \simeq \theta = 2\sqrt{\mu \sec \tau}$. A similar procedure to the former examples is performed for the calculation of the frequency. From the first equation above, we write $\widehat{\mu} = f_3(\widehat{\omega}) := (\widehat{\omega}^2 - 1)/[\widehat{\omega}\tan(\widehat{\omega}\tau)]$, which can be expanded for ω close to ω_0 as

$$\mu \simeq \frac{1}{\cos \tau}(\widehat{\omega} - 1) + \mathcal{O}\left((\widehat{\omega} - 1)^2\right).$$

Therefore, the corrected frequency is given by $\widehat{\omega} \simeq 1 + \mu \tan \tau/2$. Figures 10a and 10b show the modification of amplitude and frequency of the bifurcated cycle as μ varies, where $\tau = 0.5$ is fixed. The circles represent the values obtained by simulations, and the crosses the theoretical ones. Notice that for system (37), the predicted amplitude becomes $\tilde{x} = \mu^{-1/2}\tilde{u} \simeq 2\sqrt{\sec \tau}$, as obtained in [de Oliveira, 2002].

4. Conclusions

The FD approach provides a valuable tool for the detection and analysis of oscillations in nonlinear systems. In this article, this method has been extended for a wide variety of systems described by DDEs, allowing the treatment of several examples that previously were out of the scope. In addition with the expected reduction of the number of characteristic functions, the main advantage of this methodology is the ability to determine the stability of the equilibrium and the bifurcated orbits without dealing with a trascendental equation. On the other hand, the examples reveal a good behavior in the approximation of periodic solutions.
Taking into account that the methodology of Sec. 2.2.2 preserves the formulation given for ODEs, it is expected that the formulae developed for higher-order balances in [Moiola & Chen, 1996] could be applied for obtaining more accurate approximations.

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