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IMPULSIVE CONTROL OF A SYMMETRIC BALL ROLLING WITHOUT SLIDING OR SPINNING

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Abstract. A ball having two of its three moments of inertia equal and whose center of mass coincides with its geometric center is called a *symmetric ball*. The free dynamics of a symmetric ball rolling without sliding or spinning on a horizontal plate has been studied in detail in a previous work by two of the authors, where it was shown that the equations of motion are equivalent to an ODE on the 3-manifold $S^2 \times S^1$. In this paper we present an approach to the impulsive control of the position and orientation of the ball and study the speed of convergence of the algorithm. As an example we apply this approach to the solutions of the isoparallel problem.

1. Dynamics of the symmetric ball.

We define a material ball of radius r and mass M to be *symmetric* if its center of mass coincides with its geometric center and its principal moments of inertia I_1, I_2, I_3 satisfy $I_1 = I_2$. Using a geometric approach, two of the authors perform in [5] a complete study of the dynamics of a symmetric ball rolling on a horizontal plate without sliding or spinning, including a precise description of all equilibrium points and closed orbits. In particular, it is shown that the system is equivalent to an ODE on $S^2 \times S^1$. All the trajectories on $S^2 \times S^1$ are *closed orbits* and all of them, including equilibrium points, are *stable*. From this, one can deduce that the trajectory $x(t)$ of the contact point

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on the horizontal plate is the superposition of a periodic motion and a uniform translation. In the next paragraphs we explain all this in detail.

Description of the model. We refer to [5] for a more detailed description. In the bibliography we give a list, which is by no means exhaustive, of general references on nonholonomic systems and control, including impulsive control, relevant to the present paper.

Let $A(t) \in \mathrm{SO}(3)$ be the time-dependent rotation describing the orientation of the ball at time t, which means, by definition, that $(A(t)\mathbf{e}_1, A(t)\mathbf{e}_2, A(t)\mathbf{e}_3)$ coincides with the moving frame formed by the principal axes of inertia of the ball for all t. Here $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is the canonical basis of $\mathbb{R}^3$, which we assume to represent an inertial frame. The ball rolls on the plane $(\mathbf{e}_1, \mathbf{e}_2)$, which is assumed to be horizontal. We denote $x(t) = (x_1(t), x_2(t))$ the contact point of the ball on the plate. We sometimes regard $x(t)$ as an element $(x_1(t), x_2(t), x_3(t))$ of $\mathbb{R}^3$ with null third component, $x_3(t) = 0$. Let $z(t) \in \mathrm{S}^2$ be given by $z(t) = A(t)\mathbf{e}_3$. The spatial angular velocity can be written as $\omega = v_0 z + z \times \dot{z}$, so $v_0 = \langle \omega, z \rangle$ is its component along z (see Figure 1). The *nonholonomic constraints* [8] are given by the *no sliding condition* $\omega \times r\mathbf{e}_3 = \dot{x}$, along with the *no spinning condition* [21] given by $\omega_3 \equiv \langle \omega, \mathbf{e}_3 \rangle = 0$. The reduced Lagrangian is given by $\left(I_1 \dot{z}^2 + I_3 v_0^2 + M\dot{x}^2\right)/2$. Using the nonholonomic constraints we can conclude that the kinetic energy of the actual motion of the symmetric ball is given by

$$E = \frac{1}{2}(I_1 + Mr^2)\dot{z}^2 + \frac{1}{2}(I_3 + Mr^2)v_0^2,$$

which is a preserved quantity.

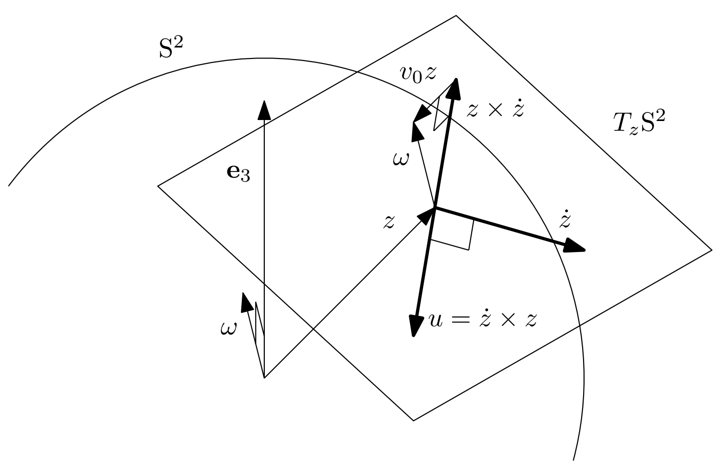

FIGURE 1. Some quantities involved in the model. Vectors drawn with a thick line are tangent to the sphere.

Equations of motion. We introduce the dimensionless quantities $\alpha = I_3/I_1$ and $\beta = Mr^2/I_1$. Moreover, we will normalize the problem so that $I_1 = I_2 = 1$ and $r = 1$, and thus α represents I_3 and β represents M.

The time derivative of the angular momentum, $A(I\dot{\Omega} - I\Omega \times \Omega)$, where $\Omega = A^{-1}\omega$ is the body angular velocity, must be compensated by the torque due to the forces of the constraints. For the case of the torque induced by the nonholonomic constraints described before we obtain the equation of motion

$$A(I\dot{\Omega} - I\Omega \times \Omega) \times \mathbf{e}_3 = -\beta(\dot{\omega} \times \mathbf{e}_3). \tag{1}$$

We refer the reader to [18] for further information on the dynamics of the rigid body and rolling constraints.

For a symmetric ball the condition $I_1 = I_2$ leads to a simplification of equation (1). First of all, the angular momentum vector is

$$I_1\Omega_1 A\mathbf{e}_1 + I_1\Omega_2 A\mathbf{e}_2 + I_3\Omega_3 A\mathbf{e}_3 = I_1\omega + (I_3 - I_1)\Omega_3 z.$$

Equating the time derivative of this quantity to the torque of the constraint forces one obtains, using the fact that $v_0 = \Omega_3$,

$$I_1\dot{\omega} + (I_3 - I_1)(v_0 z)^{\cdot} = -Mr^2\dot{\omega},$$

which is the equation of balance of forces of inertia and forces of the constraint, written in terms of the *spatial* angular velocity ω and its component along z, that is $v_0 = \langle \omega, z\rangle$. The normalized version of this equation is

$$\dot{\omega} + (\alpha - 1)(v_0 z)^{\cdot} = -\beta\dot{\omega}, \tag{2}$$

which can be written using the expression $\omega = v_0 z + z \times \dot{z}$ as

$$\alpha\dot{\omega} + (1 - \alpha)(z \times \ddot{z}) = -\beta\dot{\omega},$$

which only involves z and ω. Then equation (1) is equivalent to

$$\alpha(\dot{\omega} \times \mathbf{e}_3) + (1 - \alpha)(z \times \ddot{z}) \times \mathbf{e}_3 = -\beta(\dot{\omega} \times \mathbf{e}_3). \tag{3}$$

Remark. Another proof, more involved, using equation (1) directly, goes as follows. This time, we shall use the normalization of $I_1 = I_2 = 1$, $I_3 = \alpha$, $Mr^2 = \beta$ at the outset. First write $I\Omega = (\Omega_1, \Omega_2, \alpha\Omega_3) \equiv (\Omega_1, \Omega_2, \alpha v_0) = \Omega - (1 - \alpha)v_0\mathbf{e}_3$, from which we obtain $I\dot{\Omega} = \dot{\Omega} - (1 - \alpha)\dot{v}_0\mathbf{e}_3$. Then we have $A(I\dot{\Omega} - I\Omega \times \Omega) = \dot{\omega} - (1-\alpha)\dot{v}_0 z + (1-\alpha)v_0 z \times (v_0 z + z \times \dot{z})$, where we have used the equalities $\omega = A\Omega$, $\dot{\omega} = A\dot{\Omega}$, $\omega = v_0 z + z \times \dot{z}$. Finally, we get $A(I\dot{\Omega} - I\Omega \times \Omega) = \dot{\omega} - (1 - \alpha)(v_0 z)^{\cdot}$, and since $\dot{\omega} - (1 - \alpha)(v_0 z)^{\cdot} = \alpha\dot{\omega} + (1 - \alpha)(z \times \ddot{z})$ we obtain, again, equation (3).

Then the system of dynamical and constraint equations all together is equivalent to

$$\begin{aligned} (\alpha + \beta)(\dot{\omega} \times \mathbf{e}_3) + (1 - \alpha)(z \times \ddot{z}) \times \mathbf{e}_3 &= 0 \\ \omega &= v_0 z + z \times \dot{z} \\ \omega_3 &= 0. \end{aligned}$$

According to [5] we can write this system equivalently in the variables (z, u, v_0), calling $u = \dot{z} \times z$, as follows

$$\dot{z}_1 = z_2 u_3 - z_3 u_2 \tag{4}$$
$$\dot{z}_2 = z_3 u_1 - z_1 u_3 \tag{5}$$
$$\dot{z}_3 = z_1 u_2 - z_2 u_1 \tag{6}$$
$$z_2 \dot{u}_1 - z_1 \dot{u}_2 = \lambda v_0 u_3 \tag{7}$$
$$0 = u_3 - v_0 z_3 \tag{8}$$
$$0 = u_1^2 + u_2^2 + u_3^2 + \lambda v_0^2 - \mu \tag{9}$$
$$0 = z_1^2 + z_2^2 + z_3^2 - 1 \tag{10}$$
$$0 = z_1 u_1 + z_2 u_2 + z_3 u_3, \tag{11}$$

where $\lambda = (\alpha + \beta)/(1 + \beta)$ and equation (9) represents conservation of energy, $\mu = 2E/(I_1 + Mr^2)$.

Equations of motion on $\mathrm{S}^2 \times \mathrm{S}^1$. For each given value of the energy $\mu \geq 0$, the evolution of the system is given by the IDE (Implicit Differential Equation) (4)–(11), in which λ is a parameter solely related to the inertia of the ball. This IDE can be transformed into an equivalent ODE on a manifold, a task which is not always easy for a given IDE. In fact, according to [5], for each given, positive energy value $\mu > 0$, equations (8)–(11) define a submanifold N_μ of $\mathbb{R}^7$ on which the solution curves for that energy value are contained. Moreover, this submanifold N_μ is diffeomorphic to $\mathrm{S}^2 \times \mathrm{S}^1$ and can be parametrized by angles (θ, φ, ψ) as follows:

$$z_1 = \sin\theta \cos\varphi \tag{12}$$
$$z_2 = \sin\theta \sin\varphi \tag{13}$$
$$z_3 = \cos\theta \tag{14}$$
$$u_1 = -a\cos(\varphi - \psi)\cos^2\theta \cos\varphi - b\sin(\varphi - \psi)\sin\varphi \tag{15}$$
$$u_2 = -a\cos(\varphi - \psi)\cos^2\theta \sin\varphi + b\sin(\varphi - \psi)\cos\varphi \tag{16}$$
$$u_3 = a\cos(\varphi - \psi)\cos\theta \sin\theta \tag{17}$$
$$v_0 = a\cos(\varphi - \psi)\sin\theta, \tag{18}$$

where

$$a = \sqrt{\frac{\mu}{\lambda \sin^2\theta + \cos^2\theta}}, \quad b = \sqrt{\mu}.$$

We can check directly that the previous expression of $(z_1, z_2, z_3, u_1, u_2, u_3, v_0)$ in coordinates $(\theta,\ \varphi,\ \psi)$ satisfies (8)–(11). We can also see that equations (12)–(18) define a diffeomorphism $f\colon \mathrm{S}^2 \times \mathrm{S}^1 \to N_\mu,\ f(z, (\cos\psi, \sin\psi)) = (z, u, v_0)$. All this can be verified by standard (but lengthy) calculations, using the implicit function theorem.

Remark. A geometric interpretation of the angle ψ is the following. Equation (11) tells us that u is a vector tangent to the 2-sphere S^2 given by $z^2 - 1 = 0$. Heuristically, for each $z \in \mathrm{S}^2$ we consider the 3-dimensional space $T_z\mathrm{S}^2 \times R_z$, where R_z represents a line normal to the sphere at $z \in \mathrm{S}^2$, so the collection of all R_z is a trivial real line vector bundle with base S^2. We imagine that the variable v_0 is the coordinate of the axis R_z which is normal to $T_z\mathrm{S}^2$. Then, for each z, equation (8) is a plane in $T_z\mathrm{S}^2 \times R_z$ containing the origin $0 = 0_z$ since z_3 is fixed once z

is fixed. Equation (9) gives a 2-dimensional ellipsoid, since the variables u_1, u_2, u_3 are related by (11). The intersection of the plane with the ellipsoid is an ellipse. Therefore N_μ must be some fiber bundle with fiber S^1 and base S^2. Using all this and some imagination we can see that it is, in fact, the trivial bundle $\mathrm{S}^2 \times \mathrm{S}^1$, and we have the parametrization of N_μ in the variables (θ, φ, ψ) given by (12)–(18). Moreover, the interested reader can check that the quantities a and b defined above are precisely the semi axes of the ellipse.

It has been proven in [5] that the IDE (which is in fact a Differential Algebraic Equation (DAE)), (4)–(11) is equivalent to an analytic ODE on $\mathrm{S}^2 \times \mathrm{S}^1$, namely, the natural extension of

$$\dot{\theta} = -b \sin(\varphi - \psi) \tag{19}$$

$$\dot{\varphi} = -a \frac{\cos\theta}{\sin\theta} \cos(\varphi - \psi) \tag{20}$$

$$\dot{\psi} = (b - a) \frac{\cos\theta}{\sin\theta} \cos(\varphi - \psi), \tag{21}$$

where φ and θ parametrize $z \in \mathrm{S}^2$ as in (12)–(14), and $(\cos\psi, \sin\psi) \in \mathrm{S}^1$. Let us describe the proof briefly. First, (19)–(21) are obtained, assuming $\theta \neq 0, \pi$, by introducing (12)–(18) in (4)–(7). Let us call $X(z, (\cos\psi, \sin\psi))$ the analytic vector field defined on $\left(\mathrm{S}^2 - \{(0,0,1),(0,0,-1)\}\right) \times \mathrm{S}^1$, by equations (19)–(21) and let us show that it has a natural extension to an analytic vector field on $\mathrm{S}^2 \times \mathrm{S}^1$. For this purpose, we shall choose the parametrization of S^2 of a neighborhood of $(z_1, z_2, z_3) = (0, 0, 1)$ by the parameters (z_1, z_2) given by $z_3 = \sqrt{1 - (z_1^2 + z_2^2)}$, $(z_1, z_2) \in \{(z_1, z_2) \mid z_1^2 + z_2^2 < 1\}$. We are going to use the equalities $\sin^2\theta = z_1^2 + z_2^2$, $\cos^2\varphi = z_1^2/(z_1^2 + z_2^2)$, $\sin^2\varphi = z_2^2/(z_1^2 + z_2^2)$ and the series expansion

$$\begin{aligned} a - b &= \sqrt{\mu}\left(\Sigma_{k=1}^{\infty}(1-\lambda)^k \sin^{2k}\theta\right) \\ &= \sqrt{\mu}\left(\Sigma_{k=1}^{\infty}(1-\lambda)^k (z_1^2 + z_2^2)^k\right). \end{aligned}$$

By differentiating (12) with respect to time and using (19)–(21) we obtain

$$\dot{z}_1 = (a - b)\cos\theta\cos\varphi\sin\varphi\cos\psi + a\cos\theta\sin^2\varphi\sin\psi + b\cos\theta\cos^2\varphi\sin\psi,$$

and using the previous equalities and series expansion we can conclude that $\dot{z}_1$, which represents the first component of $X(z, (\cos\psi, \sin\psi))$, can be written as an analytic function of $(z_1, z_2, (\cos\psi, \sin\psi))$ on $\{(z_1, z_2) \mid z_1^2 + z_2^2 < 1\} \times \mathrm{S}^1$. We can proceed in a similar way with the other components of $X(z, (\cos\psi, \sin\psi))$, which proves that it has an analytic extension to $\left(\mathrm{S}^2 - \{(0,0,-1)\}\right) \times \mathrm{S}^1$. Finally, by an entirely similar procedure, we can show that $X(z, (\cos\psi, \sin\psi))$ can be extended further to an analytic vector field on $\mathrm{S}^2 \times \mathrm{S}^1$.

Remark. The dynamics of the system has been thoroughly studied in [5]. A trajectory $x(t)$ corresponding to a motion of the ball for the (apparently troublesome) initial condition $\theta = 0$ is easy to calculate, it is a straight line. In this case, the point z is initially on the top of the sphere and describes a circular motion on a vertical plane containing the center of the ball. For an initial condition with $\theta = \pi/2$ the solution $z(t)$ remains constant and $x(t)$ describes a straight line. Remarkably, it is possible to write the solutions of the equations of motion for this system explicitly in terms of elementary functions (see Appendix).

It can be easily shown that the equations of motion can be reduced to

$$\dot{\theta} = -b \sin w \tag{22}$$

$$\dot{w} = -b \cos\theta \cos w / \sin\theta, \tag{23}$$

where $w = \varphi - \psi$, which, in turn, leads to the separable equation

$$d\theta/dw = \tan\theta \tan w. \tag{24}$$

All the solutions $(\theta(t), \varphi(t), \psi(t))$ represent periodic motions and all are stable. Some solutions are shown in Figures 2 and 3. The equilibrium points are stable, even though from the linearized vector field nothing can be said about their stability.

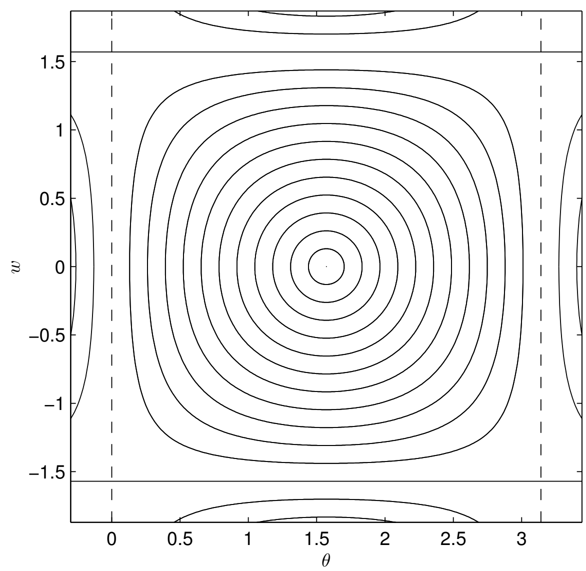

FIGURE 2. Solutions of the ODEs (22)–(23) for $\mu = 1$, $\theta(t_0) = m\pi/24$, $w(t_0) = n\pi$ $(n, m \in \mathbb{Z})$. The dashed lines correspond to the singularities $\theta = 0, \pi$ in the spherical coordinate system.

The periodicity of $(\theta(t), \varphi(t), \psi(t))$ implies that $z(t)$, $u(t)$ and $v_0(t)$ are periodic, and so is $\omega = v_0 z + z \times \dot{z}$. The solution $x(t)$ is determined by its initial value $x(t_0)$, the initial conditions $(\theta_0, \varphi_0, \psi_0)$ and the energy level. The no-sliding condition $\omega \times \mathbf{e}_3 = \dot{x}$ implies that $\dot{x}(t)$ is also periodic and therefore $x(t)$ is the superposition of a periodic motion and a uniform translation. A perturbation of the initial condition produces a deviation in the direction and speed of the uniform translation and also a perturbation of the periodic motion. As a result, it is clear that a perturbation of the initial condition propagates as a perturbation whose size increases at most linearly with time; more precisely, for two such given solutions $x(t)$, $\bar{x}(t)$, the condition $\|x(t) - \bar{x}(t)\| \leq a + bt$ is satisfied, for some $a, b > 0$.

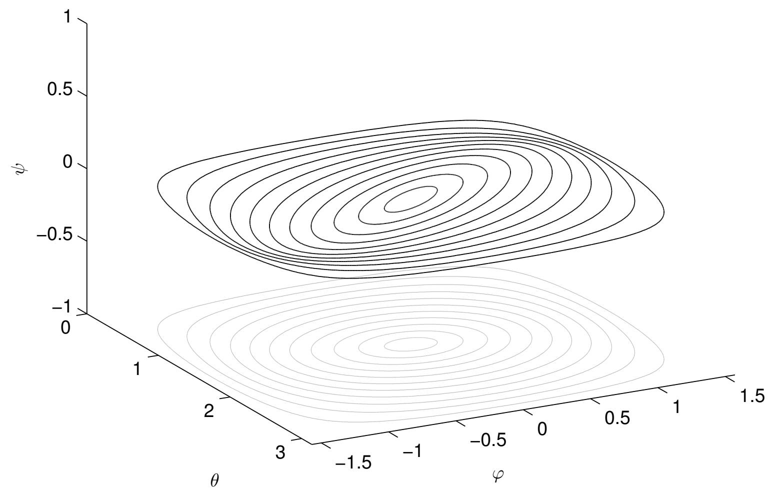

FIGURE 3. Solutions of the ODEs (19)–(21) for $\mu = 1$, $\lambda = 1/2$, $\psi(t_0) = 0$, $\varphi(t_0) = 0$, $\theta(t_0) = m\pi/24$ ($m = 1, 2, \ldots, 12$).

2. **Impulsive control of the symmetric ball.** A relatively new concept in robotics is that of graspless manipulation, where the manipulated object is not fixed onto the robot hand (see for instance [2]), but one of the problems with this kind of manipulation is stability [17]. Tapping and pushing are among the methods implemented to perform graspless manipulation [11, 12, 16]. In those references the manipulated object is supposed to slide with Coulomb friction law. A less studied problem in graspless control is the impulsive control of heavy rolling bodies. Taking advantage of nonholonomic rolling constraints has potential consequences for implementing graspless manipulation, for instance when the body is too heavy to be accurately controlled by grasping. For achieving this, a precise knowledge of the dynamics of the rolling object, including stability, seems necessary. In this section we show how to control a heavy symmetric ball rolling without sliding or spinning using impulses that do not change the (kinetic) energy of the ball, which means that each impulse represents an elastic collision between the ball and the robot hand with no transmission of energy.

2.1. **Description of the control system.** For doing experiments one may implement a *plate-ball system*. It consists of a material ball held between two horizontal plates, that rolls without sliding or spinning about its vertical axis (as in section 1). The lower plate is fixed while the upper one is actuated and can move horizontally without rotating, see Figure 4.

Consider a plate-ball system where the ball is symmetric, as described above, and the upper plate has mass M_{plate}. Since the upper plate moves with twice the velocity of the ball center, this is equivalent, as far as momentum is concerned, to having a symmetric ball with mass $M := M_{\text{ball}} + 2M_{\text{plate}}$ and no upper plate. The ball is controlled (via the upper plate) by impulsive horizontal forces. Between two such impulses it rolls freely and behaves as studied in the previous section.

Modeling an impulse. As we have explained before, for a fixed energy value μ equations (8)–(11) define a submanifold N_μ of $\mathbb{R}^7$ that can be parametrized with the variables (θ, φ, ψ) as in equations (12)–(18). If the ball is rolling and receives an impulse that does not change its energy μ, then its motion can be represented by a curve on N_μ with a jump discontinuity at the time of the impulse. We exclude, as a physical possibility, a discontinuity in the position variables θ and φ, so this impulse is represented by an instantaneous change $\Delta\psi$ of the parameter ψ.

Reciprocally, an arbitrary $\Delta\psi$ while keeping μ, θ and φ fixed represents an energy-preserving impulse. It represents an instantaneous change in u and v_0, which can be calculated using (15)–(18), and the preservation of μ is consistent with equation (9). This is, in a sense, an elastic collision, which can be implemented with the plate ball mechanism. More generally, one can implement with this mechanism an arbitrary instantaneous change $\Delta\omega$, which may involve an instantaneous change of μ.

Note that an impulse that does *not* preserve energy, taking, say, N_μ to $N_{\bar{\mu}}$, cannot be described by an instantaneous change in ψ. Such an impulse produces a jump from a solution curve $(z(t), u(t), v_0(t))$ contained in N_μ to a solution curve $(\bar{z}(t), \bar{u}(t), \bar{v}_0(t))$ contained in $N_{\bar{\mu}}$. On the other hand, the system (4)–(11) is invariant under a rescaling of time, if it is accompanied by a suitable rescaling of μ. This implies that there are solutions, having different kinetic energy, for which the ball rolls along the same trajectory and therefore produce the same reorientations along it, as explained in the paragraph *Rolling and reorientations*. In this case, passing instantaneously from one such solution to the other can be implemented by introducing an impulse which changes the value of $\|\omega\|$ while keeping its direction. This can be also implemented with the plate-ball mechanism.

Calculation of the instantaneous change of velocity of the upper plate. To implement $\Delta\psi$ physically in the plate-ball mechanism, one should find which horizontal instantaneous change of the velocity of the upper plate corresponds to $\Delta\psi$. To do this we first calculate, using equations (15)–(18), the increment Δu and Δv_0 due to the change of (θ, φ, ψ) to $(\theta, \varphi, \psi + \Delta\psi)$. Then one can calculate $\Delta\omega = (\Delta v_0)z - \Delta u$ and therefore the increase in horizontal velocity to be applied by the upper plate to the top of the ball will be $2\Delta\omega \times \mathbf{e}_3$. The plate-ball system should be designed to provide this instantaneous change in the velocity of the upper plate by simulating an elastic collision, with preservation of energy. This may require the

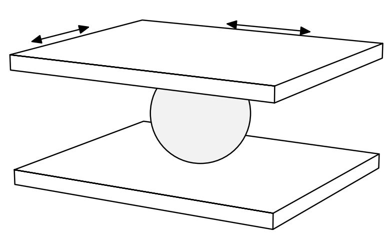

FIGURE 4. Plate-ball system

calculation of the instantaneous impulse, which can be done with the formulas that we have derived so far, but this is beyond the scope of this paper.

Joining two close points. We are going to show that there is a way to assign, in a unique way, for any two given points in $\mathbb{R}^2$ that are close enough, a trajectory (solution to equations of motion) $x(t)$ of the contact point of the rolling ball that joins them. This result will suffice for the purposes of the present paper. However, a generalization of this kind of question for the case of two arbitrary points in $\mathbb{R}^2$ is important for certain global control problems, and its study is planned for a future work.

Let $\bar{\sigma} = (\bar{\theta}, \bar{\varphi}, \bar{\psi})$ be given, and for each $s \in \mathbb{R}$ let $\bar{\sigma}_s = (\bar{\theta}, \bar{\varphi}, \bar{\psi}_s)$ where $\bar{\psi}_s = \bar{\psi} + s$. Then for each $s \in \mathbb{R}$, there is a uniquely defined solution $\sigma_s(t) = (\theta_s(t), \varphi_s(t), \psi_s(t))$ of (19)–(21) such that $\sigma_s(0) = \bar{\sigma}_s$. Using the general expressions $\omega = v_0 z - u$ and (12)–(18) we obtain

$$\omega_1 = a(\theta)\cos(\varphi - \psi)\cos\varphi + b\sin(\varphi - \psi)\sin\varphi$$
$$\omega_2 = a(\theta)\cos(\varphi - \psi)\sin\varphi - b\sin(\varphi - \psi)\cos\varphi.$$

Then, for each choice of t and s the vector $\sigma_s(t)$ determines $\omega_s(t)$, which, in turn, determines a solution $x_s(t)$ to the equations of motion. Using the general expression $\dot{x} = \omega \times \mathbf{e}_3$, and replacing ω by $\omega_s(t)$, we obtain the following expression for $\dot{x}_s(t)$,

$$\dot{x}_s(t)_1 = a(\theta_s(t))\cos(\varphi_s(t) - \psi_s(t))\sin\varphi_s(t) - b\sin(\varphi_s(t) - \psi_s(t))\cos\varphi_s(t) \quad (25)$$
$$\dot{x}_s(t)_2 = -a(\theta_s(t))\cos(\varphi_s(t) - \psi_s(t))\cos\varphi_s(t) - b\sin(\varphi_s(t) - \psi_s(t))\sin\varphi_s(t). \quad (26)$$

From the periodicity of $(\theta(t), \varphi(t), \psi(t))$ we deduce that $\omega(t)$ and $\dot{x}_s(t)$ are periodic. Then we obtain the trajectory $x_s(x_0, t)$ satisfying $x_s(x_0, 0) = x_0$, for a given initial condition $x_0 \in \mathbb{R}^2$ and a given $\bar{\sigma}_s$, as follows,

$$x_s(x_0, t) = x_0 + \int_0^t \omega_s(u) \times \mathbf{e}_3\, du,$$

which shows, in particular, that $x_s(x_0, t)$ is the superposition of a periodic motion and a uniform translation. From this it follows that each higher order partial derivative

$$\frac{\partial^{\alpha+\beta} x_s(x_0, t)}{\partial t^\alpha \partial s^\beta}$$

has an upper bound, depending solely on (α, β), say $K_{(\alpha,\beta)}$.

Now we are going to show that $0 \le t$ and $s \in [0, 2\pi)$ may be considered as a sort of local polar coordinates centered at the initial condition x_0. From (25) and (26) one deduces that the curve $s \mapsto x_0 + \dot{x}_s(0)$ is an ellipse centered at x_0, moreover, it is easy to see that, for each $s \in \mathbb{R}$,

$$\|\dot{x}_s(0)\|^2 = (a(\theta_s(0)))^2 \cos^2(\varphi_s(0) - \psi_s(0)) + b^2 \sin^2(\varphi_s(0) - \psi_s(0)) \quad (27)$$

and

$$\left\|\frac{\partial \dot{x}_s(0)}{\partial s}\right\|^2 = (a(\theta_s(0)))^2 \sin^2(\varphi_s(0) - \psi_s(0)) + b^2 \cos^2(\varphi_s(0) - \psi_s(0)). \quad (28)$$

From (27) one easily deduces the orientation and the modulus of the principal axes of the ellipse. On the other hand, (28) shows that the ellipse is parametrized by s as a regular curve. Then one may think of (t, s) with $t \ge 0$ as being polar coordinates, with some rescaling of the angle and the radius, centered at x_0, given by $(t, s) \mapsto y_s(x_0, t)$, where, by definition, $y_s(x_0, t) = x_0 + t\dot{x}_s(0)$. Now, using the Taylor expansion and the integral form of the remainder we have $x_s(x_0,t) = x_0 + t\dot{x}_s(0) + c(t,s)t^2$, with $c(t,s)$ a C^∞ function on $\mathbb{R}^2$ given by

$$c(t,s) = \int_0^1 (1-u)(\dot{x}_s(ut) - \dot{x}_s(0))\, du.$$

By differentiating under the integral we obtain the bounds

$$\|c(t,s)\| \le K_{(1,0)},\ \|(\partial c/\partial t)(t,s)\| \le K_{(2,0)},\ \|(\partial c/\partial t)(t,s)\| \le K_{(1,1)},$$

for all (t,s).

There is a map $P\colon \mathbb{R}^2 \to \mathbb{R}^2$, $x = P(y)$, defined by the composition of the two maps $y_s(x_0,t) \mapsto (t,s) \mapsto x_s(x_0,t)$, that is $x_s(x_0,t) = P(y_s(x_0,t))$, where $t \ge 0$. One can check immediately that $P(x_0) = x_0$ and that P is C^∞ on $\mathbb{R}^2 - \{0\}$. Let us show that P is at least C^1 and, moreover, $DP(x_0) = I$. From the definitions and the Taylor expansion we obtain $P(y) = y + \tilde{c}(y)\,\|y - x_0\|^2 / \|\dot{x}_s(0)\|^2$ where $\tilde{c}(y)$ is obtained as the composition of $c(t,s)$ with

$$s = h\,(\arctan(y_2 - x_{02})/(y_1 - x_{01}))$$

and

$$t = \frac{\|y - x_0\|}{\|\dot{x}_s(0)\|},$$

where h is a positive periodic function relating the parameter s with the angle described by $\dot{x}_s(0)$. After some calculations involving the bounds $K_{(1,0)}, K_{(2,0)}, K_{(1,1)}$, one can conclude that there is a universal constant K such that $\|DP(y) - I\| \le K\,\|y - x_0\|$. This shows that P is at least C^1 and that $DP(x_0) = I$. As we know from the inverse function theorem, if $r > 0$ satisfies $Kr < 1/2$ then the inverse function $y = P^{-1}(x)$ is defined and is C^1 on the ball $B_{r'}$ where $r' = r/2$. This means that one can also think of $(t,s) \mapsto x_s(x_0,t)$ as being a sort of (somewhat deformed) local polar coordinates. Using the previous considerations we can deduce the following lemma, to be used later on.

Lemma 2.1. *There exists a number r' (depending only on the dimensionless parameter λ) having the following property. Consider two points, say x_0 and x in $\mathbb{R}^2$, such that $\|x_0 - x\| < r'$, where the contact point is x_0 at $t = 0$. Consider also an arbitrary $\bar{\sigma} = (\bar{\theta}, \bar{\varphi}, \bar{\psi})$ (which implies an arbitrary initial orientation of the ball $(\bar{\theta}, \bar{\varphi})$). Then there exists (t,s), with $t \ge 0$, such that $x_s(x_0,t) = x$. The time $t \ge 0$ is uniquely determined while s is determined modulo 2π, and, moreover, there exists a universal constant K^1 (depending only on λ) such that $t \le K^1\|x - x_0\|$.*

Remark. A study of the conditions under which $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a C^∞ diffeomorphism is interesting for solving certain global control problems, and it is planned for a future paper.

Rolling and reorientations. If one knows the trajectory $x(t)$, $t \in [0,T]$ that the ball describes on the plate then its orientation $A(t) \in \mathrm{SO}(3)$ is uniquely determined, given an initial orientation $A(0)$. In fact, it is known that this can be seen as a horizontal lift with respect to a principal connection on a principal bundle $\mathbb{R}^2 \times \mathrm{SO}(3) \to \mathbb{R}^2$, with structure group $\mathrm{SO}(3)$, see for instance [6]. Then, if $x(t)$ is reparametrized, for example by rescaling time, then $A(t)$ becomes reparametrized in the same way. Note that the reorientation would not be uniquely determined if the ball were allowed to spin around its vertical axis.

The rotation $A(t)$ represents the transformation taking an orthonormal frame fixed on the ball to its position at time t, as in the description of the model at the beginning of this work. Therefore, if $[a,b] \subseteq [0,T]$, then we define the ***reorientation*** along the partial path $x(t)$, $t \in [a,b]$ as $A(b)A(a)^{-1}$.

We now state the problem concerning the impulsive control of the symmetric ball, and show an approach to its solution, and also how this could be implemented using a computer.

Statement of the problem. Given the initial position (x_0, A_0) of the ball and a smooth trajectory $p(\tau)$ on the plate, $\tau \in [0, \mathcal{T}]$, with $p(\tau_0) = x_0$, find a sequence of energy-preserving instantaneous impulses (so, in a sense, they will behave like elastic collisions) to be applied on the ball, such that the resulting trajectory approximates $p(\tau)$. This approximation is to be understood not with respect to $p(\tau)$ as a parametrized curve, but with respect to its image as a geometric curve on the lower plate. Also, estimate the difference between the final orientations of the ball corresponding to rolling along the planned trajectory $p(\tau)$ and along the actual trajectory resulting from the impulses.

A more general problem can be posed and solved in the case where $p(\tau)$ is only piecewise smooth, by approximating each one of the smooth portions.

Remark. The reorientation of a ball rolling along a path is a purely geometric outcome of the rolling process, and does not depend on the speed at which it rolls. Therefore, we will consider only energy-preserving impulses without loss of generality, since impulses that change the energy of the ball give rise to the same paths on the plate as energy-preserving ones. This means that energy-changing impulses do not allow additional control strategies that are better in terms of the final reorientation. However, from the engineering point of view it might be useful to consider such impulses, for example, for slowing the ball down before making a sharp turn. This can be done by introducing impulses that change $\|\omega\|$ without changing the direction of ω. This kind of impulses can be introduced freely in between the impulses of the computed energy-preserving control strategy, without affecting the final reorientation. Using a similar argument, we can show that the initial speed of the ball is not relevant for the reorientation problem.

2.2. **Example: The isoparallel problem for the rolling ball.** Take two points x_0 and x_1 on the lower plate, a smooth trajectory $x(t)$, $t \in [t_0, t_1]$, joining them, and assume, for simplicity, that $A(t_0) = \mathrm{I}$. As the ball rolls along the trajectory it undergoes a certain rotation $A(t) \in \mathrm{SO}(3)$, see Figure 5. The *isoparallel problem for the rolling ball* consists in finding the *shortest smooth trajectory* among those that induce the same final given reorientation $A(t_1)$ of the ball and join the same points on the lower plate, see [1, 10, 19]. This problem provides a setting where the impulsive control problem fits nicely and arises as a natural question, more precisely, we are interested in approximating the trajectory solution to the isoparallel problem by a sequence of impulses.

Using methods of geometric control theory [6, 10, 19], one obtains that such trajectories $x(t)$ satisfy

$$\dot{v} = (\mathbf{e}_3 \times \dot{x}) \times v$$
$$\ddot{x} = v^3 \mathbf{e}_3 \times \dot{x},$$

where $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$ is a Lagrange multiplier.

Remark. A delicate mathematical question that typically appears in this kind of control problems is that of *rigidity.* For example, the straight trajectories $x(t)$ are

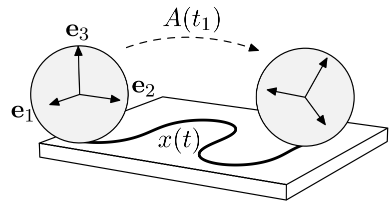

FIGURE 5. Following a trajectory $x(t)$ induces a rotation of an orthonormal frame fixed on the ball.

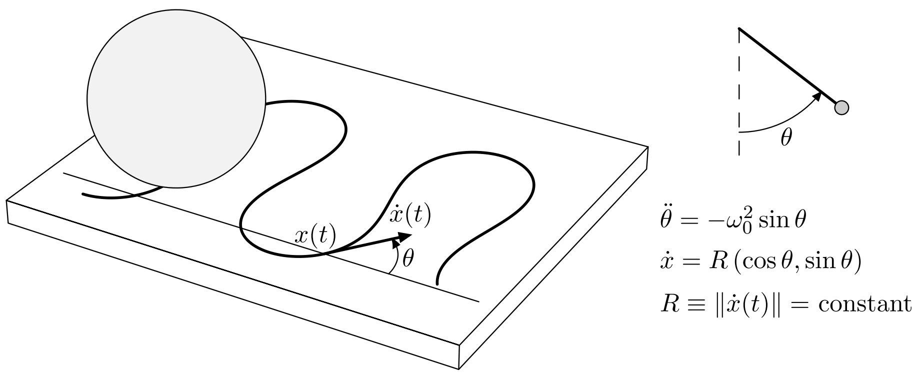

FIGURE 6. The velocity vector in the isoparallel problem behaves like a nonlinear pendulum. Here $\omega_0 = g/R$, where R and g are the length and the acceleration of gravity for the pendulum system, respectively.

C^1-*rigid* [4]. This means that they cannot be deformed as a C^1 curve while maintaining the same endpoints x_0, x_1, and final reorientation $A(t_1)$. However, there are Sobolev-class H^1 deformations satisfying these conditions. It can be proven that this kind of question is of no consequences for the problem treated in the present work.

The rolling ball and the pendulum. There is an interesting relationship between the plate-ball isoparallel problem and the dynamics of a simple nonlinear pendulum. The slope of the curve described by the contact point on the lower plate behaves exactly as the angle of a pendulum, see [1, 6] and Figure 6. In addition, $\|\dot{x}\|$ is a constant of the motion. Elliptic functions could therefore be employed to find the solutions.

Computation of the optimal trajectories. The isoparallel problem is a boundary value problem where the boundary conditions are (x_0, A_0), (x_1, A_1) in $\mathbb{R}^2 \times \mathrm{SO}(3)$. Let us take $A_0 = \mathrm{I}$ for simplicity, without loss of generality. Suppose that we have an optimal trajectory $\bar{x}(t)$ satisfying $\bar{x}(0) = x_0$ that reaches a certain point $\bar{x}_1$ on the circle with radius $\|x_1 - x_0\|$ centered at x_0, and denote the resulting reorientation of the ball by A. Let $B \in \mathrm{SO}(3)$ be the rotation of the space around the vertical axis through x_0 that takes $\bar{x}_1$ into x_1. Since the equations for the optimal curves are invariant under rotations in the plane, the trajectory obtained by applying this rotation to $\bar{x}(t)$ will reach x_1 with a resulting rotation of BAB^{-1}.

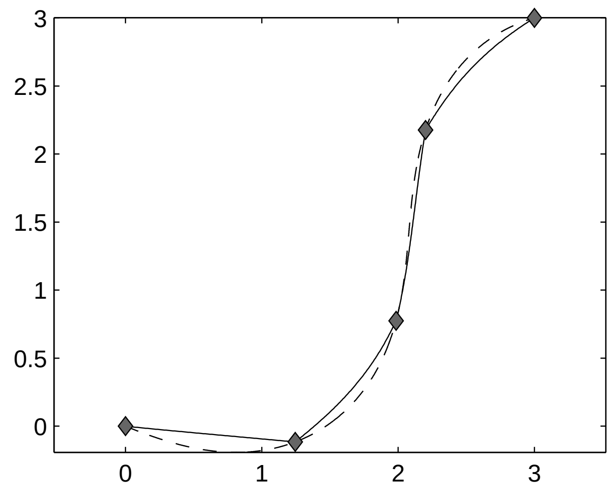

FIGURE 7. Approximating a solution to the isoparallel problem. The ball goes from $(0, 0)$ to $(3, 3)$, and the resulting rotation is the identity. The dashed line is the optimal curve, and the continuous one is an approximation using impulsive control and a coarse partition.

Therefore, a candidate trajectory can be tested by letting it run until it reaches the circle, calculating B and comparing BAB^{-1} to the desired A_1 (using, for example, Euler angles appropriately). This gives a strategy to solve the optimal control problem, by iteration. One should take into account that, using the analogy with the pendulum, there are three parameters to adjust: R (which affects the scaling of the trajectory), the energy of the pendulum, and the initial phase. The value of ω_0 can be normalized using a rescaling of time, which does not affect the resulting rotation (see [6] for further details). In any case, solving *very efficiently* the optimal control problem from the numerical point of view is not the purpose of the present paper.

2.3. **Control strategy.** Let $p(\tau)$, $\tau \in [0, \mathcal{T}]$ be the planned trajectory, which we assume is a smooth curve, meaning that there exists a smooth extension of p to an open interval $(-\epsilon, \mathcal{T} + \epsilon)$. A ***partition*** $\mathcal{P}$ is defined by a selection of $n + 1$ values $0 = \tau_0 < \tau_1 < \ldots < \tau_n = \mathcal{T}$, which determines n portions of the planned path. We define two ***norms*** of $\mathcal{P}$, namely, $\|\mathcal{P}\|_p$, which is the maximum of the numbers $\|p(\tau_k) - p(\tau_{k+1})\|$, $k = 0, \ldots, n-1$, and $\|\mathcal{P}\|_{[0,\mathcal{T}]}$, which is the maximum of the numbers $\tau_{k+1} - \tau_k$, $k = 0, \ldots, n-1$.

Given the state of the ball at time τ_0, which is described by θ_0, φ_0, ψ_0, and the position $p(\tau_0)$ of the contact point, we need to join $p(\tau_0)$ and $p(\tau_1)$ using a method that we will explain shortly. We do this repeatedly to join $p(\tau_k)$ and $p(\tau_{k+1})$, $k = 1, \ldots, n-1$, where the initial values θ_k, φ_k, ψ_k, $p(\tau_k)$, $k = 1, \ldots, n-1$, for each step follow from the previous one. By refining the partition, one obtains an arbitrarily close ***impulsive control approximation*** to the planned trajectory. Figure 7 shows an approximation to a solution of the isoparallel problem.

Joining consecutive points in the partition. Let $(\theta_0, \varphi_0, \psi_0)$ be an initial condition, which determines the energy μ, and consider the points $p(\tau_k)$ and $p(\tau_{k+1})$, $k = 0, \ldots, n-1$. First, we need to find $(\Delta\psi)_0$ such that the solution $(\theta(t), \varphi(t), \psi(t))$

with initial conditions $(\theta_0, \varphi_0, \psi_0 + (\Delta\psi)_0)$ is such that it induces, by rolling, a trajectory $x(t)$, $t \in [t_0, t_1]$ starting from $x(t_0) = p(\tau_0)$ and reaching $x(t_1) = p(\tau_1)$ for some $t_1 > t_0$. We have proven that this is possible in a unique way, with an energy-preserving trajectory, if the norm $\|\mathcal{P}\|_p$ of the partition is less than the number r' determined in the paragraph *Joining two close points* of subsection 2.1, which we will assume from now on. In a similar way, we want to introduce an impulsive change $(\Delta\psi)_k$ when the ball reaches the point $p(\tau_k)$, $k = 0, \ldots, n-1$. More precisely, one assumes that, at the point of contact $p(\tau_k)$, the ball is rolling with angular velocity ω_k and $(\theta(\tau_k), \varphi(\tau_k), \psi_k)$ is known, which determines its energy μ. An appropriate energy-preserving impulse is applied using for instance the plate ball system mechanism, which takes the value of ψ_k to $\psi_k + (\Delta\psi)_k$. Then the motion with the new initial conditions $(\theta(\tau_k), \varphi(\tau_k), \psi_k + (\Delta\psi)_k)$ is determined by the equations of motion, and we want to choose $(\Delta\psi)_k$ in such a way that, at a certain moment t_{k+1} to be determined, the point of contact reaches $p(\tau_{k+1})$. We also want to calculate the final values of all the variables.

Remark. In the present paper we are *not* interested in showing how to find $(\Delta\psi)_k$ such that the motion with initial condition $(\theta(\tau_k), \varphi(\tau_k), \psi_k)$, $p(\tau_k)$ reaches the contact point $p(\tau_{k+1})$ after a certain time $t_{k+1} - t_k$ to be determined has elapsed, *for points $p(\tau_k)$ and $p(\tau_{k+1})$ such that $\|p(\tau_k) - p(\tau_{k+1})\|$ is large.* This is because our control strategy is of a *local character* and requires only solving the problem of finding impulsive control approximations where the norm of the partition tends to zero.

However, the previous simple global impulsive control problem for the plate ball system for distant points $p(\tau_k)$ and $p(\tau_{k+1})$, can still be approached with the techniques described in this paper. A more extensive treatment of this and other global control problems should be the purpose of future work. Here is the idea of how to calculate $(\Delta\psi)_k$ for distant points $p(\tau_k)$ and $p(\tau_{k+1})$, using standard software packages. We take advantage of the fact that the motion $x(t)$ is the superposition of a periodic motion and a uniform translation. Because of this, we approach the problem by assuming first a uniform motion in the direction of the destination point. The instantaneous change in the direction of the uniform motion is our initial guess for $(\Delta\psi)_k$ (if the previous direction is not known, it is computed using a single period and $(\Delta\psi)_k = 0$). Then we solve the ODE with the new initial conditions and find the distance from the solution curve to $p(\tau_{k+1})$. We take advantage of the periodicity of the solutions when trying to reach distant points, since we can compute just one period and then repeat it until the trajectory passes near $p(\tau_{k+1})$. The value of $(\Delta\psi)_k$ is then adjusted by using, for example, MATLAB®'s `fsolve`, until the distance is zero within tolerance. See Figure 8 for examples.

Convergence. Let us now study the convergence of the method as the partition is refined. We are going to study an arbitrarily close impulsive approximation to a given smooth planned trajectory $p(\tau)$ on the plate and its resulting approximate reorientation, and we will estimate the difference.

Lemma 2.2. *Consider a family of curves C_ν on $\mathbb{R}^2$, for $\nu \in [0, 1]$, each curve given by $x_\nu(\tau)$, $\tau \in [a, b]$. We assume that $x_\nu(\tau)$ is smooth in both ν and τ, and that the endpoints are fixed with respect to ν. Regard each C_ν as a curve on $\mathbb{R}^3$ with null third component as in the description of the model. Let $R_\nu(\tau) \in \mathrm{SO}(3)$, $\tau \in [a, b]$*

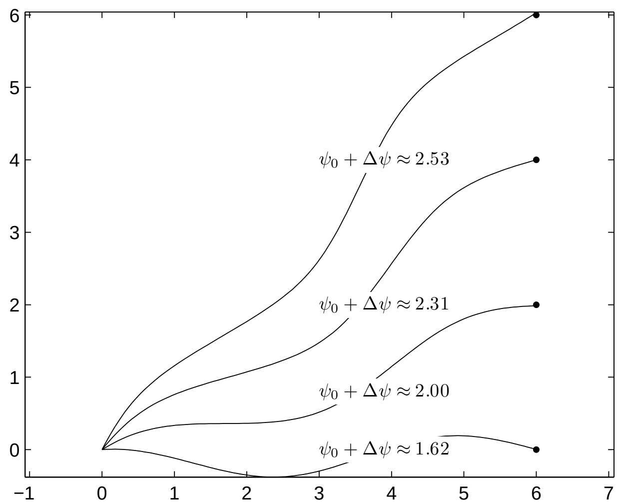

FIGURE 8. Reaching points by taking different values of $(\Delta\psi)_k$. Here $\mu = 1$, $\lambda = 2$, $\theta_0 = -\pi/3$, $\varphi_0 = \pi/2$, $\psi_0 = 0$.

be the reorientation that results from rolling the ball along C_ν from a to τ. Then $\nu \mapsto R_\nu(b)$ is a curve on SO(3) *with velocity vector $dR_\nu(b)/d\nu = \xi_\nu R_\nu(b)$, where*

$$\xi_\nu = \int_a^b (R_\nu(\tau))^{-1} \left(\frac{\partial x_\nu(\tau)}{\partial \nu} \times \frac{\partial x_\nu(\tau)}{\partial \tau} \right) d\tau$$

is in the Lie algebra $\mathfrak{so}(3) \equiv \mathbb{R}^3$ of SO(3).

Idea of the proof. One can regard the reorientation as being the holonomy of a curve with respect to a certain principal connection [6]. The integrand is essentially the curvature of the connection, and the formula for ξ_ν is a classical result for the derivative of the holonomy of a curve [10]. □

Notation. We are going to accept a slight abuse of notation, namely, for $a, b \in \mathbb{R}$, $a < b$, the interval $[a, b]$ will be sometimes denoted $[b, a]$.

Lemma 2.3. *Let $p(\tau)$, $\tau \in [0, \mathcal{T}]$ be a smooth curve in $\mathbb{R}^2$. For each $a, b \in [0, \mathcal{T}]$, let $p_{a,b}(\tau)$, $\tau \in [a, b]$, be the segment joining $p(a)$ and $p(b)$, that is,*

$$p_{a,b}(\tau) = p(a) + \frac{(\tau - a)}{b - a} \left(p(b) - p(a) \right).$$

Then there exists a constant K_p such that for all $a, b \in [0, \mathcal{T}]$, the distance between the curve and the segment satisfies $\|p(\tau) - p_{a,b}(\tau)\| \leq K_p \|b - a\|^2$, for all $\tau \in [a, b]$.

Proof. Let $p(a) + \dot{p}(a)(\tau - a) + c(\tau)(\tau - \tau_k)^2$ be the Taylor expansion of $p(\tau)$ at a. Then there exists a constant $K_p > 0$ such that $\|c(\tau)\| \leq K_p/2$, for all $\tau, a \in [0, \mathcal{T}]$. Then the following inequality can be proven in a straightforward way for all $\tau \in [a, b]$,

$$\|p(\tau) - p_{a,b}(\tau)\| = \| - (\tau - a)c(b)(b - a) + c(\tau)(\tau - a)^2\| \leq K_p \|b - a\|^2. \qquad \square$$

Lemma 2.4. *Assume the conditions and notation of lemma 2.3. Consider the family of curves $p_\nu(\tau) = p(\tau) + \nu(p_{a,b}(\tau) - p(\tau))$, representing a deformation of the straight segment $p_{a,b}(\tau)$ joining $p(a)$ and $p(b)$ into the portion of $p(\tau)$ between*

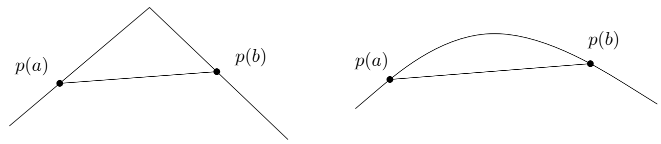

FIGURE 9. The enclosed area tends to zero like $\|p(a) - p(b)\|^2$ on the left, but like $\|p(a) - p(b)\|^3$ on the right.

the points $p(a)$ and $p(b)$. In particular, $p_0(\tau) = p(\tau)$ and $p_1(\tau) = p_{a,b}(\tau)$, for $\tau \in [a, b]$. If $R_\nu(\tau)$ represents the reorientation of the ball as it rolls along $p_\nu(\tau)$, then using the operator norm for matrices we have $\|R_1(b) - R_0(b)\| \le \bar{K}_p \|b - a\|^3$, where $\bar{K}_p = K_p M'_p$, with $M'_p = \max_{(\tau,\nu)\in[0,\mathcal{T}]\times[0,1]} \|\partial p_\nu(\tau)/\partial\tau\|$.

Proof. From lemma 2.2 we have

$$\|R_1(b) - R_0(b)\| \le \left\| \int_0^1 \xi_\nu \, d\nu \right\| \le \int_0^1 \int_a^b \left\| (R_\nu(\tau))^{-1} \left(\frac{\partial p_\nu(\tau)}{\partial \nu} \times \frac{\partial p_\nu(\tau)}{\partial \tau} \right) \right\| d\tau \, d\nu$$
$$\le \int_0^1 \int_a^b \left\| \frac{\partial p_\nu(\tau)}{\partial \nu} \times \frac{\partial p_\nu(\tau)}{\partial \tau} \right\| d\tau \, d\nu \le \int_0^1 \int_a^b \left\| \frac{\partial p_\nu(\tau)}{\partial \nu} \right\| \left\| \frac{\partial p_\nu(\tau)}{\partial \tau} \right\| d\tau \, d\nu.$$

The proof can be finished in a straightforward way taking into account the fact that $\partial x_\nu(\tau)/\partial\nu = p_{a,b}(\tau) - p(\tau)$ and using lemma 2.3. □

Corollary 1. *If $\partial p_\nu(\tau)/\partial\tau$ is nowhere zero for all $(\tau, \nu) \in [0, \mathcal{T}] \times [0, 1]$ then there are constants $C_p > 0$, $c_p > 0$ such that $c_p\|\mathcal{P}\|_p < \|\mathcal{P}\|_{[0,\mathcal{T}]} < C_p\|\mathcal{P}\|_p$. Then, under the hypotheses of lemma 2.4, the inequality $\|R_1(b) - R_0(b)\| \le \bar{K}_p C_p \|p(b) - p(a)\|^3$ holds.*

Remark. *(a)* It is clear that the reorientation $R(\tau)$ along a given path $p(\tau)$ does not depend on the parametrization of the path.

(b) There is an interesting geometric idea behind lemma 2.4. The integral

$$\int_0^1 \int_a^b \left\| \frac{\partial p_\nu(\tau)}{\partial \nu} \times \frac{\partial p_\nu(\tau)}{\partial \tau} \right\| d\tau \, d\nu$$

represents the area swept by the curve $p_\nu(\tau)$, $\tau \in [a, b]$ as ν varies from 0 to 1 (possibly counting some regions more than once, if $p_\nu(\tau)$ sweeps those regions several times). In fact, the integrand represents the absolute value of the Jacobian determinant of the map $[a, b] \times [0, 1] \mapsto \mathbb{R}^2$ given by $(\tau, \nu) \mapsto p_\nu(\tau)$. On the other hand, since $\ddot{p}(\tau)$ is bounded on $[a, b]$ one can conclude, by an elementary geometric argument, at least in the case in which the map $[a, b] \times [0, 1] \mapsto \mathbb{R}^2$ is injective and the Jacobian determinant is nonzero (except at most in a subset of measure zero) and besides $\dot{p}(\tau)$ is nowhere zero, $\tau \in [a, b]$, that this area is bounded by a constant times $\|p(a) - p(b)\|^3$. The idea behind the proof (which we do not think necessary to give here in detail) is depicted in Figure 9. In the diagram on the left, which shows a nonsmooth planned trajectory, the enclosed area tends to zero like $\|p(a) - p(b)\|^2$ as $p(a)$ and $p(b)$ move towards the vertex. However, for the one on the right, it tends to zero like $\|p(a) - p(b)\|^3$ as $p(a)$ and $p(b)$ move closer.

(c) We remark that lemma 2.4 can be applied to two cases that are of interest for this paper. Namely, to the trajectories described by the ball on the horizontal plate, rolling freely according to the equations of motion, and to the solutions to the isoparallel problem, since they are smooth.

Main results. Given a smooth trajectory $p(\tau)$, $\tau \in [0, \mathcal{T}]$ let us choose a partition $\mathcal{P}$ given by the selection of $0 = \tau_0 < \tau_1 < \cdots < \tau_n = \mathcal{T}$, which determines n portions of the trajectory. Now let $A(\tau) \in \mathrm{SO}(3)$ be the orientation of the ball at the value τ of the parameter when rolling along $p(\tau)$, and let $\bar{A}_k$, $k = 0, \dots, n$, be the orientation of the ball at the points $p(\tau_k)$ when rolling along the polyline formed by the straight segments joining the endpoints of the portions of $\mathcal{P}$. For instance, at the starting point $p(\tau_0)$ the orientation is $\bar{A}_0$, which we can assume to have some initial error with respect to $A(\tau_0)$. After rolling along the straight segment from $p(\tau_0)$ to $p(\tau_1)$, the orientation will be $\bar{A}_1$, and so on. Let $R_k = A(\tau_k)A(\tau_{k-1})^{-1}$, $k = 1, \dots, n$, be the reorientation corresponding to the portion of the planned trajectory between τ_{k-1} and τ_k, and let $S_k = \bar{A}_k \bar{A}_{k-1}^{-1}$ be the reorientation corresponding to the straight segment joining the endpoints of that portion. The following two theorems are our main results concerning convergence.

Theorem 2.5. *In the notation above, the error in the final orientation at time $\mathcal{T}$, assuming an initial error $\|A(0) - \bar{A}_0\|$, is bounded as follows*

$$\|A(\mathcal{T}) - \bar{A}_n\| \leq \|A(0) - \bar{A}_0\| + \bar{K}_p \left(\|\tau_0 - \tau_1\|^3 + \cdots + \|\tau_{n-1} - \tau_n\|^3 \right).$$

Proof. It is easy to show that if $A \in \mathrm{SO}(3)$ and X is a 3×3 matrix, then $\|AX\| = \|X\| = \|XA\|$. At time τ_k we have

$$\begin{aligned}\|A(\tau_k) - \bar{A}_k\| &= \|R_k A(\tau_{k-1}) - S_k \bar{A}_{k-1}\| = \|A(\tau_{k-1})\bar{A}_{k-1}^{-1} - R_k^{-1} S_k\| \\ &\leq \|A(\tau_{k-1})\bar{A}_{k-1}^{-1} - \mathrm{I}\| + \|\mathrm{I} - R_k^{-1} S_k\| = \|A(\tau_{k-1}) - \bar{A}_{k-1}\| + \|R_k - S_k\|.\end{aligned}$$

By lemma 2.4 we know that $\|R_k - S_k\| \leq \bar{K}_p \|\tau_k - \tau_{k-1}\|^3$, for all $k = 1, \dots, n$, so the lemma follows inductively. □

Corollary 2. *Since $\|\tau_k - \tau_{k-1}\| \leq \|\mathcal{P}\|_{[0,\mathcal{T}]}$ theorem 2.5 gives $\|A(\mathcal{T}) - \bar{A}_n\| \leq \|A(0) - \bar{A}_0\| + \bar{K}_p n \|\mathcal{P}\|^3_{[0,\mathcal{T}]}$. If the partition $\mathcal{P}$ divides the interval $[0, \mathcal{T}]$ in intervals of equal size $\mathcal{T}/n = \|\mathcal{P}\|_{[0,\mathcal{T}]}$ then $\|A(\mathcal{T}) - \bar{A}_n\| \leq \|A(0) - \bar{A}_0\| + \bar{K}_p \mathcal{T}^3/n^2$.*

The argument above compares the errors in reorientation between the planned path and a polyline joining selected points in the path via straight line segments.

For the next theorem we are going to work with partitions $\mathcal{P}$ such that $\|\mathcal{P}\|_p < r'$, where r' is defined in lemma 2.1. For such a partition we can consider three trajectories that pass through the points involved in it: the planned trajectory p, the polyline, and some trajectory arising from the free dynamical behavior of the ball between the consecutive impulses, that is, an impulsive control approximation of the planned path. As we know, such an impulsive control approximation is composed of the pieces $x_{s_k}(p(\tau_k), t)$, $t \in [t_k, t_{k+1}]$. Recall that S_{k+1} is the reorientation of the ball resulting from rolling along the segment joining $p(\tau_k)$ and $p(\tau_{k+1})$, this time parametrized with $t \in [t_k, t_{k+1}]$, which, as we know, does not change the reorientation. Denote by $\tilde{R}_{k+1}$ the reorientation resulting from rolling along the path $x_{s_k}(p(\tau_k), t)$, $t \in [t_k, t_{k+1}]$. Then, by theorem 2.5, we obtain $\|S_{k+1} - \tilde{R}_{k+1}\| \leq \tilde{K} \|t_{k+1} - t_k\|^3$, where $\tilde{K}$ is a constant depending only on the physical parameter λ and the chosen value of μ, which can be calculated using lemmas 2.3 and 2.4 in a similar way as we did with $\bar{K}_p$. Namely, we know that the function $c(t, s)$ calculated in (2.1) satisfies $\|c(t, s)\| \leq K_{(1,0)}$, so $\tilde{K} = 2K_{(1,0)} M'$, where M' is calculated in a similar way as we did with M'_p. Using this and an argument similar to the proof of theorem 2.5 we have

$$\|\bar{A} - \tilde{A}_n\| \leq \|\bar{A}_0 - \tilde{A}_0\| + \tilde{K} \left(\|t_0 - t_1\|^3 + \cdots + \|t_{n-1} - t_n\|^3 \right). \tag{29}$$

Using lemma 2.1 and (29) and taking into account that $\|p(\tau_k) - p(\tau_{k+1})\| < r'$, we obtain

$$\|\bar{A} - \tilde{A}_n\| \leq \|\bar{A}_0 - \tilde{A}_0\| + \tilde{K}K^1 \left(\|p(\tau_0) - p(\tau_1)\|^3 + \cdots + \|p(\tau_{n-1}) - p(\tau_n)\|^3\right).$$

Using the Taylor expansion of $p(\tau)$ to order 1 one can see that there is a constant, say D_p, depending on p, such that $\|p(\tau_k) - p(\tau_{k+1})\| \leq D_p\|\tau_k - \tau_{k+1}\|$, for $k = 0, \dots, n-1$. Let $\tilde{K}_p = \tilde{K}K^1 D_p$, then we deduce that

$$\|\bar{A} - \tilde{A}_n\| \leq \|\bar{A}_0 - \tilde{A}_0\| + \tilde{K}_p \left(\|\tau_0 - \tau_1\|^3 + \cdots + \|\tau_{n-1} - \tau_n\|^3\right).$$

From this, taking into account that $\|A(\mathcal{T}) - \tilde{A}_n\| \leq \|A(\mathcal{T}) - \bar{A}_n\| + \|\bar{A}_n - \tilde{A}_n\|$, we obtain the following theorem.

Theorem 2.6. *Let $p(\tau)$, $\tau \in [0, \mathcal{T}]$, be a given smooth path and let $\mathcal{P}$ be a given partition $0 = \tau_0 < \tau_1 < \ldots < \tau_n = \mathcal{T}$ such that $\|\mathcal{P}\|_p \leq r'$, with r' as in lemma 2.1. Consider the impulsive control approximation consisting of impulses given at the points $p(\tau_k)$, $k = 0, \dots, n-1$, and let $\tilde{A}_k$ be the orientation of the ball at the point $p(\tau_k)$, $k = 0, \dots, n$, when rolling along the trajectory resulting from these impulses. Then the error in the final orientation at value $\mathcal{T}$, assuming an initial error $\|A(0) - \tilde{A}_0\|$ at value τ_0, is*

$$\|A(\mathcal{T}) - \tilde{A}_n\| \leq \|A(0) - \tilde{A}_0\| + (\bar{K}_p + \tilde{K}_p) \left(\|\tau_0 - \tau_1\|^3 + \cdots + \|\tau_{n-1} - \tau_n\|^3\right).$$

We can also deduce the following corollary, which generalizes corollary 2.

Corollary 3. *Since $\|\tau_k - \tau_{k-1}\| \leq \|\mathcal{P}\|_{[0,\mathcal{T}]}$ we have $\|A(\mathcal{T}) - \tilde{A}_n\| \leq \|A(0) - \tilde{A}_0\| + \left(\bar{K}_p + \tilde{K}_p\right) n\|\mathcal{P}\|^3_{[0,\mathcal{T}]}$. If the partition $\mathcal{P}$ divides the interval $[0, \mathcal{T}]$ in intervals of equal size $\mathcal{T}/n = \|\mathcal{P}\|_{[0,\mathcal{T}]}$ then $\|A(\mathcal{T}) - \tilde{A}_n\| \leq \|A(0) - \tilde{A}_0\| + \left(\bar{K}_p + \tilde{K}_p\right) \mathcal{T}^3/n^2$.*

We should mention here, among others, two other articles regarding the control of the rolling ball, but these consider a continuous (non-impulsive) control. In addition, the dynamics of the ball does not play a role. A trajectory with exact reorientation is constructed in [15]. Also, [20] develops a robust planner, using certain maneuvers. However, in the case of the isoparalell problem, one of our goals is staying close to the optimal curve, precisely because its length is a minimum (or extremum) in the set of smooth curves, and the curves in these works do not guarantee this.

3. **Conclusions and final remarks.**

We have demonstrated that the position of a heavy symmetric ball rolling without sliding or spinning can be controlled by impulsive forces, which can be chosen to preserve the kinetic energy of the ball. This gives a graspless accurate control of the system. We have also shown how, for a given planned trajectory, by choosing a partition $\mathcal{P}$ such that $\|\mathcal{P}\|_{[0,\mathcal{T}]}$ is small enough, one can obtain an arbitrarily close impulsive control approximation to it. For doing this, we have used an accurate description of the dynamics of the ball as a nonholonomic system established in a previous work. More precisely, we have proven that the final error in the reorientation of the ball tends to zero as the partition is refined. In particular, if the intervals of the partition are, for each partition, of the same size, $\mathcal{T}/n$, it is $O(1/n^2)$, where n is the number of portions in which the planned path is partitioned. This is an estimation of the error of the speed of convergence of the method, without taking into account certain errors that would appear in a more realistic model, and that may lead to a slower convergence, roughly of order $1/n$. Studying this kind of errors is the purpose of a future work.

We remark that the problem of designing a single impulse for joining two given contact points on the plate, for instance two points of the partition of the planned trajectory, is relatively easy if the ball is homogeneous. This is in part because, in this case, the dynamical trajectory is a straight line joining the two points. However, it is more difficult and requires an accurate knowledge of the dynamics of the system if the ball has a nonhomogeneous mass distribution. For the case of a symmetric ball, studied in this paper, the dynamics is given by explicit and simple formulas. The dynamics for a general nonhomogeneous ball would be more complex, but questions similar to the ones treated in this paper can be approached and solved, numerically.

The control problem that we have considered involves the dynamics of the system in an essential way, and not only the geometric aspects. In this sense, it differs from the usual *plate-ball problem with a massless ball*. The latter appears, for instance, in [9], which includes a detailed study of theoretical and numerical solutions and their stability. Basically, the difference consists in that, in our problem, the trajectory of the system between two consecutive impulses cannot be chosen at will as in the case of the massless ball, but it is the natural trajectory dictated by the dynamical laws.

Now we would like to comment briefly on the continuous, as opposed to impulsive, control of our system. Let us consider a massless ball, then one can roll the ball along a path $p(\tau)$, using for instance the plate-ball mechanism. The impulsive control, however, does not make any sense, since it involves the dynamics of the ball. In any case $\omega(\tau)$ can be calculated readily. Now assuming that the ball has an arbitrary distribution of mass, and that $\tau = t$ represents the physical time, then the determination of the force to be applied to the upper plate of the plate-ball system at the instant of time t to keep the ball rolling along $p(t)$, is given by standard formulas in mechanics, involving the values, at time t, of the position of the center of mass, the inertia tensor and $\omega(t)$. This will be the continuous control of the heavy ball, at least from the perspective of this paper. There is no error of the method in this kind of continuous control. However, analyzing basic questions about the errors in more realistic models could be more difficult than in the impulsive control approach. For example, designing and implementing a robot arm controlling a heavy rolling ball in a continuous fashion could be more complex and costly than designing a control system by impulses that do not transfer energy to the ball.

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Appendix. Solutions to the dynamics. For the following computations, we assume $\mu = 1$. For arbitrary $\mu > 0$, the solutions are reparametrizations of the ones shown here. By integrating the equations of the planar system (22)–(23) we get the family of solutions

$$\theta(t) = \arctan\left(\frac{\sqrt{1 + (c_1 \sin t - c_2 \cos t)^2}}{c_1 \cos t + c_2 \sin t}\right) \tag{30}$$

$$w(t) = -\arctan(c_1 \sin t - c_2 \cos t). \tag{31}$$

Now we will find two constants of motion c and d, by finding functions such that the solutions lie on their level sets. Equation (24) can be easily integrated by separable variables, which yields

$$\theta(w) = \arcsin\left(\frac{c}{\cos w}\right),$$

so the solutions of the system must lie on the level surface

$$\sin\theta\,\cos w = c. \tag{32}$$

In order to get another level set for the solutions we first define

$$C_1(t) = c_1 \sin t - c_2 \cos t, \quad C_2(t) = c_1 \cos t + c_2 \sin t.$$

Note that $\bar{c} = \sqrt{1 + C_1^2(t) + C_2^2(t)} = \sqrt{1 + c_1^2 + c_2^2}$ is a constant. From (30)–(31) we can write

$$\tan\theta = \frac{\sqrt{1 + C_1^2(t)}}{C_2(t)}, \quad \cos^2 w = \frac{1}{1 + C_1^2(t)},$$

and

$$\sin^2\theta = \frac{1 + C_1^2(t)}{\bar{c}^2}, \quad \cos^2\theta = \frac{C_2^2(t)}{\bar{c}^2}.$$

Considering that

$$a = \sqrt{\frac{1}{\lambda \sin^2\theta + \cos^2\theta}},$$

we get its expression in terms of $C_1(t)$,

$$a = \bar{c}\sqrt{\frac{1}{(\lambda - 1)(1 + C_1^2(t)) + \bar{c}^2}}.$$

We can write

$$\begin{aligned}
\dot{\varphi} &= -a \cot\theta \cos w \\
&= -\bar{c}\sqrt{\frac{1}{(\lambda - 1)(1 + C_1^2(t)) + \bar{c}^2}}\,\frac{C_2(t)}{1 + C_1^2(t)} \\
&= -\bar{c}\sqrt{\frac{1}{(\lambda - 1)(1 + C_1^2(t)) + \bar{c}^2}}\,\frac{C_1'(t)}{1 + C_1^2(t)}
\end{aligned}$$

and integrate; in fact, if we call $x = C_1(t)$, we get

$$\varphi = -\bar{c}\int\sqrt{\frac{1}{(\lambda - 1)(1 + x^2) + \bar{c}^2}}\,\frac{dx}{1 + x^2} = -\bar{c}\int\sqrt{\frac{1}{(\lambda - 1)(1 + \tan^2 u) + \bar{c}^2}}\,du,$$

with the substitution $x = \tan u$ for the second integral. Considering that $1 + \tan^2 u = 1/(\cos^2 u)$, this yields

$$\varphi = -\bar{c}\int\sqrt{\frac{\cos^2 u}{(\lambda - 1) + \bar{c}^2 - \bar{c}^2\sin^2 u}}\,du = -\bar{c}\int\sqrt{\frac{1}{(\lambda - 1) + \bar{c}^2 - \bar{c}^2 v}}\,dv,$$

by writing $\sin u = v$. Finally, if we call $\bar{c}v = z$, then

$$\varphi = \int\sqrt{\frac{1}{m^2 - z^2}}dz = \arcsin\left(\frac{z}{m}\right) + d,$$

where $m^2 = \lambda - 1 + \bar{c}^2$. By some simple calculations we get

$$-\sin(\varphi - d) = \frac{\bar{c}\sin w}{m}. \tag{33}$$

Since $\cos^2 w = 1/(1 + C_1^2(t))$ and $\sin^2\theta = (1 + C_1^2(t))/\bar{c}^2$ we get

$$\sin\theta\cos w = \frac{1}{\bar{c}} = c,$$

which gives a relationship between c and $\bar{c}$. We can rewrite (33) as

$$\sqrt{(\lambda - 1)c^2 + 1}\,\sin(\varphi - d) + \sin w = 0. \tag{34}$$

Then the solutions must satisfy (32) and (34).

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