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CONSTRUCTIVE APPROXIMATION

A Characterization of Best φ-Approximants with Applications to Multidimensional Isotonic Approximation

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Abstract. Some properties of best monotone approximants in several variables are obtained. We prove the following abstract characterization theorem. Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space and let $\mathcal{L} \subset \mathcal{A}$ be a σ-lattice. If f belongs to a Musielak–Orlicz space $L_\varphi(\Omega, \mathcal{A}, \mu)$, then there exists a σ-algebra $\mathcal{A}_f \subset \mathcal{A}$ such that g is a best φ-approximant to f from $L_\varphi(\mathcal{L})$ iff g is a best φ-approximant to f from $L_\varphi(\mathcal{A}_f)$. The σ-algebra $\mathcal{A}_f$ depends only on f. When $\Omega \subset \mathbf{R}^n$ and $L_\varphi(\mathcal{L})$ is the set of monotone functions in several variables, we give sufficient conditions on the geometry of Ω to obtain a uniqueness theorem. This result extends and unifies previous ones. Finally, we prove a coincidence relation between a function and its best φ-approximant. Our main results are new, even in the classical Lebesgue spaces L_p.

1. Introduction

This work is a continuation of our previous paper [16]. Our primary goal is to characterize best φ-approximants in Musielak–Orlicz spaces L_φ by elements of the convex cone $L_\varphi(\mathcal{L})$, where $\mathcal{L}$ is a σ-lattice (see Definitions 2.1 and 2.4). We have resolved this problem in [16] in the special case $L_\varphi = L_1$ and $\mathcal{L}$ a totally ordered σ-lattice. Here we show that best φ-approximants to f from $L_\varphi(\mathcal{L})$ are best φ-approximants to f from $L_\varphi(\mathcal{A}_f)$, where $\mathcal{A}_f$ is a σ-algebra depending only on f (see Theorem 3.6). This result transforms a best approximation problem from a convex set into a best approximation problem from a subspace.

When $\mathcal{L}$ is a totally ordered σ-lattice, we show that the σ-algebra $\mathcal{A}_f$ coincides with the σ-algebra generated by $\mathcal{L}$ outside the atoms of $\mathcal{A}_f$ (see Lemma 4.1). This fact, applied to the cone of nondecreasing functions on [0, 1], leads to the following well-known result (see [7], [15]): a nondecreasing function g is a best φ-approximant to f by nondecreasing functions iff g is a constant best φ-approximant on each atom of $\mathcal{A}_f$ and $f = g$ outside the atoms of $\mathcal{A}_f$ (see Theorem 4.3 and Corollary 4.4). In addition, our results embrace the discrete case studied in [14]. Therefore, Theorem 3.6 unifies and extends several

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previously known characterizations (see [7], [15], [14], [16]). In the multidimensional case, it seems that Theorem 3.6 was unknown even in the Hilbert space L_2.

The second objective of this paper is to unify and to extend the uniqueness results of [4], [8], [15]. Under certain continuity conditions on f, uniqueness of best approximants holds (see [4] for an approximately continuous f in $L_1((0,1)^n)$; [8] for continuous f in the Orlicz space $L_\varphi([0,1]^n)$ and [15] for approximately continuous f in the Orlicz space $L_\varphi([0,1])$). We obtain uniqueness of best approximants (see Theorem 5.2) on the Musielak–Orlicz space $L_\varphi(\Omega)$ with $\Omega \subset \mathbf{R}^n$ under the following conditions:

(i) Ω is an open set;
(ii) $\Omega \cap R$ is connected for certain parallelepipeds R with faces parallel to the coordinate's hyperplanes;
(iii) f is an approximately continuous function.

The main idea in the proof of Theorem 5.2 is the following: if g_1 and g_2 are two best approximants to f, then it is possible to "separate" the connected components of the set $\{g_1 < a < b < g_2\}$ by sets in $\mathcal{L}$ (in the one-dimensional case that set has only one component). Moreover, we give an example of a connected and open set $\Omega \subset \mathbf{R}^2$ and a continuous function f such that uniqueness fails to hold. We point out that these techniques provide new results, even in the case $L_\varphi = L_1$, when Ω is not a cube.

We showed in [16] that in the multidimensional case it is not necessarily true that a best approximant to f coincides with f outside the atoms of $\mathcal{A}_f$. In this paper, we describe a set $B_1 \subset \Omega$, where the equality $f = g$ holds. This result generalizes [15, Theorem 4(b)] in four aspects:

(i) we consider Musielak–Orlicz spaces;
(ii) we deal with the multidimensional case;
(iii) in [15] the following condition is assumed: f is essentially bounded on a certain set or φ has a bounded right derivative;
(iv) in the one-dimensional case, our set B_1 may be bigger, in the almost everywhere (a.e.) sense, than the corresponding set B_2 in [15].

In fact, we give a trivial example where $\mu(B_2) = 0$ and $\mu(B_1) = \mu(\Omega)$, here μ is the Lebesgue measure. We believe that points (ii) and (iv) are the more important ones.

Other references concerning best approximants by monotone functions are [1], [5], [13].

The remaining part of this paper is organized as follows. Section 2 consists mainly of notations and preliminary results. In Section 3 we establish a relation between best approximants and generalized Lebesgue–Radon–Nikodym (LRN) derivatives (see Theorem 3.2). Generalized LRN derivatives were introduced in [2], [9] as a generalization of the notion of conditional expectation and conditional mean. In [2] it was proved that LRN derivatives are solutions of certain variational problems, which include the best approximation problem in Musielak–Orlicz spaces. Therefore, Theorem 3.2 is essentially known. It seems that paper [2] remained unknown approximation theorist to, as a consequence of the fact that the connection between LRN derivatives and best approximants is only implicit in [2]. Thus, several results in [2] were independently rediscovered in others papers (see [12], [7], [15], [14], [16]). The main result in Section 3 is our characterization theorem (Theorem 3.6). In Section 4 we present some consequences of Theorem 3.6

when $\mathcal{L}$ is totally ordered. In Section 5 we give new uniqueness theorems. Finally, in Section 6, we prove a coincidence theorem in several variables.

2. Notations and Preliminary Results

Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space. We denote by $M = M(\Omega, \mathcal{A}, \mu)$ the set of all $\mathcal{A}$-measurable real-valued functions.

Definition 2.1. A set $\mathcal{L} \subset \mathcal{A}$ is called a σ-lattice if $\emptyset, \Omega \in \mathcal{L}$, $\mathcal{L}$ is closed under countable intersections and countable unions. The set $\mathcal{L}$ is called a complete σ-lattice iff $\mathcal{L}$ is a σ-lattice and $C \in \mathcal{L}$, $\mu(C \triangle C') = 0$ imply $C' \in \mathcal{L}$. For $\mathcal{L}$ a σ-lattice, we denote by $\overline{\mathcal{L}}$ the σ-lattice $\{D : \Omega \backslash D \in \mathcal{L}\}$. A function f is called $\mathcal{L}$ measurable if $\{f > a\} \in \mathcal{L}$, for every $a \in \mathbf{R}$.

Henceforth we assume that $\mathcal{L}$ denotes a complete σ-lattice.

We give two classical examples of σ-lattices extensively studied in the literature. The first one embraces the usual class of nondecreasing functions on $[0, 1]^n$ (see, e.g., [3], [4], [7], [15]), and the second one deals with the discrete case (see, e.g., [1], [14], [18]).

Example 2.2. Let Ω be a measurable set in $\mathbf{R}^n$ and μ the Lebesgue measure on it. For $x, y \in \Omega$ we say that $x \leq y$ iff $x_i \leq y_i$, $i = 1, \ldots, n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. A set $C \subset \Omega \subset \mathbf{R}^n$ is called a *final set* iff $x \in C$ and $x \leq y$ imply $y \in C$. The next set $\mathcal{L}^n = \mathcal{L}^n(\Omega)$ is the standard complete σ-lattice. The σ-lattice $\mathcal{L}^n$ is the class of those sets C for which there exists a final set $\tilde{C}$ such that $\mu(C \triangle \tilde{C}) = 0$. As usual, a real function $g : \Omega \to \mathbf{R}$ is called nondecreasing iff $g(x) \leq g(y)$ when $x \leq y$. It is easy to check that f is $\mathcal{L}^n$-measurable iff there exists a nondecreasing function g with $g = f$ μ-a.e.

Example 2.3. Let $\Omega = \mathbf{N}_1$ where $\mathbf{N}_1$ is a subset of $\mathbf{N}$. Let $\mathcal{L}^* \subset \mathbf{2}^{\mathbf{N}_1}$ be the σ-lattice containing all sets of the form $\{m \in \mathbf{N}_1 \mid m > n\}$, with $n \in \mathbf{N}$. Now it is easy to see that $f : \mathbf{N}_1 \to \mathbf{R}$ is $\mathcal{L}^*$-measurable iff f is nondecreasing.

We consider a function $\varphi : \Omega \times \mathbf{R} \to \mathbf{R}^+$ with the following properties:

(i) $\varphi(\cdot, a)$ is a measurable function for every $a \in \mathbf{R}$;
(ii) $\varphi(\omega, a) = 0$ iff $a = 0$;
(iii) $\varphi(\omega, \cdot)$ is an even, nonnull, and convex function.

For φ satisfying (i)–(iii) we denote φ_+ (φ_-) as the right (left) derivative of φ with respect to the second variable.

Definition 2.4. Let φ be a function satisfying the conditions (i)–(iii). We define the Musielak–Orlicz space (or generalized Orlicz space) L_φ by

$$L_\varphi := \left\{ f \in M \mid \exists \lambda > 0 : \int_\Omega \varphi(\omega, \lambda f(\omega))\, d\mu < \infty \right\}.$$

The Musielak–Orlicz space L_φ becomes a Banach space endowed with the norm

$$\|f\|_\varphi := \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\omega, \frac{f(\omega)}{\lambda}\right) d\mu \le 1\right\}.$$

If φ does not depend on the first variable, then L_φ is called an Orlicz space.

For further information about these spaces the reader is referred to [17], [10].

Throughout this paper we assume that the function φ verifies the following two additional conditions:

(iv) $\varphi(\omega, \cdot)$ satisfies a uniformly Δ_2 condition. That is, there are positive constants M and A_0, independent of ω, such that, for all $\omega \in \Omega$ and $|a| \ge A_0$,

$$\varphi(\omega, 2a) \le M\varphi(\omega, a).$$

Under this condition, it is easy to check that $f \in L_\varphi$ iff

$$\int_\Omega \varphi(\omega, \lambda f(\omega))\, d\mu < \infty.$$

for every $\lambda > 0$.

(v) L_φ contains all constant functions.

We will often write $\int_\Omega \varphi(\omega, f)\, d\mu$ instead of $\int_\Omega \varphi(\omega, f(\omega))\, d\mu$.

Lemma 2.5. *If $f, g \in L_\varphi$, then $\varphi_+(\omega, f(\omega))g(\omega)$ is an integrable function.*

Proof. Using (v) the proof follows the same lines as [15, p. 1]. ∎

For a σ-lattice $\mathcal{L}$ we denote by $L_\varphi(\mathcal{L})$ the convex closed cone of all $\mathcal{L}$-measurable functions in L_φ. Now, as usual, we say that $g \in L_\varphi(\mathcal{L})$ is a best φ-approximant to $f \in L_\varphi$ from $L_\varphi(\mathcal{L})$ iff

$$\int_\Omega \varphi(\omega, f - g)\, d\mu = \min_{h \in L_\varphi(\mathcal{L})} \int \varphi(\omega, f - h)\, d\mu.$$

We point out that the existence of best φ-approximants was proved in [10]. We denote by $\mu(f, \mathcal{L})$ the set of all best φ-approximants to f from $L_\varphi(\mathcal{L})$. Following a similar argument given in [11, Theorem 14] we can show that $\mu(f, \mathcal{L})$ has a minimum and a maximum element, i.e., there exist $L(f, \mathcal{L}) \in \mu(f, \mathcal{L})$ and $U(f, \mathcal{L}) \in \mu(f, \mathcal{L})$ such that, for all $g \in \mu(f, \mathcal{L})$,

$$L(f, \mathcal{L}) \le g \le U(f, \mathcal{L}).$$

Now we recall some concepts from [2].

Definition 2.6. Let ν be a signed measure on $\mathcal{A}$. We say that $C \in \mathcal{L}$ is a ν-positive set, if for all $D \in \overline{\mathcal{L}}$ we have $\nu(C \cap D) \ge 0$. A set $D \in \overline{\mathcal{L}}$ is called ν-negative, if for all $C \in \mathcal{L}$ we have $\nu(C \cap D) \le 0$.

It is easy to prove the following lemma:

Lemma 2.7. *The class of all ν-positive (negative) sets is closed under countable unions.*

Let $f \in L_\varphi$. For each $g \in L_\varphi(\mathcal{L})$ and $a \in \mathbf{R}$ we define the measures

$$\mu_g^+(A) = \int_A \varphi_+(\omega, f-g)\, d\mu, \qquad \mu_g^-(A) = \int_A \varphi_-(\omega, f-g)\, d\mu, \tag{1}$$

and

$$\mu_a^+(A) = \int_A \varphi_+(\omega, f-a)\, d\mu, \qquad \mu_a^-(A) = \int_A \varphi_-(\omega, f-a)\, d\mu. \tag{2}$$

We will need the following result which is a consequence of the Cavallieri principle:

Lemma 2.8. *For all $f \in L_\varphi$ we have $\mu_a^+ = \mu_a^-$ for a.e. $a \in \mathbf{R}$.*

Proof. We consider the product measure $\mu \times dx$ on $\Omega \times \mathbf{R}$, where dx is the Lebesgue measure on $\mathbf{R}$. For $A \subset \Omega \times \mathbf{R}$ we denote $A_\omega := \{a : (\omega, a) \in A\}$ and $A_a := \{\omega : (\omega, a) \in A\}$. We define the map $T : \Omega \times \mathbf{R} \to \Omega \times \mathbf{R}$ by $T(\omega, a) = (\omega, f(\omega) - a)$. We will show that

$$\mu \times dx(A) = \mu \times dx(T(A)),$$

for all $\mu \times dx$-measurable sets A. From the Fubini theorem we get

$$\mu \times dx(A) = \int_\Omega |A_\omega|\, d\mu = \int_\Omega |T(A)_\omega|\, d\mu = \mu \times dx(T(A)).$$

Now we consider the set $A := \{(\omega, a) : \varphi_+(\omega, a) > \varphi_-(\omega, a)\}$. Then A is $\mu \times dx$-measurable and the section A_ω is at most countable, for every $\omega \in \Omega$. Therefore, by the Fubini theorem we have that $\mu \times dx(A) = 0$. Hence $\mu \times dx(T(A)) = 0$. Now, applying Fubini's theorem again, we have

$$0 = \mu \times dx(T(A)) = \int_{-\infty}^{+\infty} \mu(T(A)_a)\, da.$$

Thus, $\mu(T(A)_a) = 0$ for a.e. $a \in \mathbf{R}$. That is, for a.e. $a \in \mathbf{R}$ we have $\varphi_+(\omega, f(\omega) - a) = \varphi_-(\omega, f(\omega) - a)$ μ-a.e. ■

Definition 2.9. For $f \in L_\varphi$ set $C(f) := \{a : \mu_a^+ = \mu_a^-\}$.

Let us recall the following definition from [2]:

Definition 2.10. Let $\{\nu_a\}_{a \in \mathbf{R}}$ be a family of measures on Ω. A $\mathcal{L}$-measurable function g is called a Lebesgue–Radon–Nikodym function (LRN function) of $\{\nu_a\}$ iff:

(i) $\{g > a\}$ is ν_a-positive for all $a \in \mathbf{R}$.
(ii) $\{g < b\}$ is ν_b-negative for all $b \in \mathbf{R}$.

Remark 1. In Definition 2.10 the set $\mathbf{R}$ may be replaced for a dense subset Q (see [2, p. 588]). On the other hand, it is easy to check that (see [2, Theorem 1.8]) g is an LRN function of $\{\nu_a\}$ iff there exists a dense set $A \subset \mathbf{R}$ such that $\{g \geq a\}$ is a ν_a-positive set and $\{g \leq a\}$ is a ν_a-negative set for every $a \in A$.

3. Characterizations of Best φ-Approximants

In this section we give three characterizations of best φ-approximants. The first two are essentially known and, for the sake of completeness, we give a short proof of these results.

Lemma 3.1. *Let $f \in L_\varphi$, $\mathcal{L} \subset \mathcal{A}$ be a σ-lattice and let $g \in L_\varphi(\mathcal{L})$. Then the following statements are equivalent*:

(i) $g \in \mu(f, \mathcal{L})$.
(ii) *For every $h \in L_\varphi(\mathcal{L})$ we have*:
(a)

$$\int_{\{g>h\}} \varphi_+(\omega, f-g)(g-h)\, d\mu \geq 0.$$

(b)

$$\int_{\{g<h\}} \varphi_-(\omega, f-g)(g-h)\, d\mu \geq 0.$$

Proof. Let $h \in L_\varphi(\mathcal{L})$. We consider the function

$$M(t) = \int \varphi(\omega, f-g-t(h-g))\, d\mu.$$

Then M is a convex function and g is a best φ-approximant iff

$$M'_+(0) \geq 0,$$

where M'_+ denotes the right derivative of M. As a consequence of Lemma 2.5, the last inequality implies

$$0 \leq \int_{\{g>h\}} \varphi_+(\omega, f-g)(g-h)\, d\mu + \int_{\{g<h\}} \varphi_-(\omega, f-g)(g-h)\, d\mu.$$

The inequalities (a) and (b) are obtained by replacing (in the last inequality) h by $h \wedge g$ and $h \vee g$, respectively. On the other hand, (a) and (b) imply $M'_+(0) \geq 0$. ■

Theorem 3.2. *Let $f \in L_\varphi$ and let $\mathcal{L} \subset \mathcal{A}$ be a σ-lattice. Then the following facts are equivalent*:

(i) $g \in \mu(f, \mathcal{L})$.

(ii) *The set* $\{g > a\}$ *is* μ_g^+*-positive and the set* $\{g < a\}$ *is* μ_g^-*-negative for every* $a \in \mathbf{R}$.

(iii) g *is an LRN function for the families* $\{\mu_a^\pm\}_{a\in\mathbf{R}}$.

Proof. (i) $\Rightarrow$ (ii). Let $D \in \overline{\mathcal{L}}$. We define the sets $A := \{g > a\}$ and $A_n := \{g > a + 1/n\}$, with $n \in \mathbf{N}$. Set

$$g_n(w) := \begin{cases} g(w), & \text{if } w \notin A \cap D, \\ a, & \text{if } w \in (A - A_n) \cap D, \\ g(w) - 1/n, & \text{if } w \in A_n \cap D. \end{cases}$$

Then $g_n \in L_\varphi(\mathcal{L})$. Now, replacing g_n in formula (ii)(a) of Lemma 3.1, we get

$$0 \le \int_{(A\setminus A_n)\cap D} \varphi_+(\omega, f - g)(g - a)\, d\mu + \frac{1}{n}\int_{A_n\cap D} \varphi_+(\omega, f - g)\, d\mu.$$

Hence, multiplying by n in the above inequality and taking the limit as $n \to \infty$ we obtain

$$\int_{A\cap D} \varphi_+(\omega, f - g)\, d\mu \ge 0,$$

i.e., $\{g > a\}$ is a μ_g^+-positive set. A similar argument shows that $\{g < a\}$ is a μ_g^--negative set.

(ii) $\Rightarrow$ (iii). For $a \in C(f)$ the proof follows immediately from the monotonicity of $\varphi_\pm$. For the general case, we apply Remark 1 and Lemma 2.8.

(iii) $\Rightarrow$ (ii). For $a \in \mathbf{R}$, $D \in \overline{\mathcal{L}}$, $k \in \mathbf{N} \cup \{0\}$, and $n \in \mathbf{N}$ we define the sets $A := \{a < g\} \cap D$ and $A_{k,n} := \{a + k/n < g \le a + (k+1)/n\} \cap D$. Then

$$\mu_{a+k/n}^+(A_{k,n}) = \int_{A_{k,n}} \varphi_+\left(\omega, f - a - \frac{k}{n}\right) d\mu \ge 0.$$

Therefore, from the monotonicity of φ_+, we get

$$\int_{A_{k,n}} \varphi_+\left(\omega, f - g + \frac{1}{n}\right) d\mu \ge 0.$$

Now, summing over $k = 0, 1, \ldots,$ we obtain

$$\int_A \varphi_+\left(\omega, f - g + \frac{1}{n}\right) d\mu \ge 0.$$

Since φ_+ is a right continuous function, taking the limit $n \to \infty$ we obtain that $\mu_g^+(A) = \mu_g^+(D \cap \{a < g\}) \ge 0$, i.e., $\{a < g\}$ is a μ_g^+-positive set. A similar argument shows that $\{g < a\}$ is a μ_g^--negative set.

(ii) $\Rightarrow$ (i). Let $h \in L_\varphi(\mathcal{L})$. Integrating on a in the inequality

$$\int_{\{h<a\}\cap\{a<g\}} \varphi_+(\omega, f - g)\, d\mu \ge 0$$

and applying the Fubini theorem we get inequality (ii)(a) in Lemma 3.1. Inequality (ii)(b) follows in a similar way. ■

For $a \in \mathbf{R}$ set $\mathcal{L}_a$ for the class of all sets $C \in \mathcal{L}$ such that C is μ_a^+-positive and $\Omega\backslash C$ is μ_a^+-negative. Now we define the set

$$\tilde{\mathcal{L}}_f := \bigcup_{a \in C(f)} \mathcal{L}_a.$$

Henceforth, when $a \in C(f)$ we denote by μ_a the measure $\mu_a^+ = \mu_a^-$.

As a consequence of Theorem 3.2 and the above notations we have

Corollary 3.3. *$g \in \mu(f, \mathcal{L})$ iff for every $a \in C(f)$, we have $\{g > a\} \in \mathcal{L}_a$.*

Proof. If $g \in \mu(f, \mathcal{L})$, $a \in C(f)$, and $b \in \mathbf{R}$, with $b > a$, we have that $\{g > a\}$ is μ_a-positive and $\{g < b\}$ is μ_b^--negative. Hence

$$\int_{\{g<b\}\cap C} \varphi_-(\omega, f-b)\, d\mu \leq 0,$$

for every $C \in \mathcal{L}$. Taking limit for $b \downarrow a$ in the above inequality we get $\mu_a^-(\{g \leq a\} \cap C) \leq 0$. Since $a \in C(f)$ we obtain $\mu_a^+(\{g \leq a\} \cap C) \leq 0$. Therefore, the set $\{g > a\} \in \mathcal{L}_a$.

Now suppose that $\{g > a\} \in \mathcal{L}_a$, for every $a \in C(f)$. Therefore, the set $\{g > a\}$ is μ_a-positive. On the other hand, for $b < a$ and $b \in C(f)$, we have

$$\int_{\{g\leq b\}\cap C} \varphi_+(x, f-b)\, dx \leq 0,$$

for every $C \in \mathcal{L}$. Taking limit for $b \uparrow a$ in the last inequality we get $\mu_a(\{g < a\} \cap C) \leq 0$. Therefore, the function g is an LRN function (see Remark 1) of $\{\mu_a^+\}$. ∎

Remark 2. We observe that we can put $\{g \geq a\}$ instead of $\{g > a\} \in \mathcal{L}_a$ in Corollary 3.3.

The following lemma plays a central role in many of our considerations:

Lemma 3.4. *Let $a, b \in C(f)$, with $a \leq b$, $C_1 \in \mathcal{L}_a$, and $C_2 \in \mathcal{L}_b$. Then $C_1 \cap C_2 \in \mathcal{L}_b$ and $C_1 \cup C_2 \in \mathcal{L}_a$. In particular, the class $\tilde{\mathcal{L}}_f$ is closed under finite unions and finite intersections, i.e., $\tilde{\mathcal{L}}_f$ is a lattice.*

Proof. We have that $\Omega\backslash C_1$ is μ_a-negative. Therefore, from the inequality $\mu_a \geq \mu_b$, we have that $\Omega\backslash C_1$ is μ_b-negative. Since $\Omega\backslash C_2$ is also negative, the set $(\Omega\backslash C_1) \cup (\Omega\backslash C_2)$ is μ_b-negative. On the other hand, suppose that there exists $D \in \overline{\mathcal{L}}$ such that

$$\mu_b(C_1 \cap C_2 \cap D) < 0. \tag{3}$$

We consider the set $D' = (\Omega\backslash C_1) \cup (C_1 \cap D) = (\Omega\backslash C_1) \cup D \in \overline{\mathcal{L}}$. Since $\mu_b \leq \mu_a$ and $C_1 \in \mathcal{L}_a$ we get

$$\begin{aligned} 0 &\leq \mu_b(C_2 \cap D') \\ &= \mu_b(C_2\backslash C_1) + \mu_b(C_1 \cap C_2 \cap D) \\ &< \mu_b(C_2\backslash C_1) \\ &\leq \mu_a(C_2\backslash C_1) \leq 0. \end{aligned}$$

Therefore, inequality (3) is false. This proves $C_1 \cap C_2 \in \mathcal{L}_b$. The relation $C_1 \cup C_2 \in \mathcal{L}_a$ follows analogously.

Finally, the second part of the lemma follows as a direct consequence of the first one. ■

Definition 3.5. We denote by $\mathcal{L}_f$ ($\mathcal{A}_f$) the lower complete σ-lattice (σ-algebra) containing $\tilde{\mathcal{L}}_f$.

We note that $C \in \mathcal{L}_f$ iff, for every $\varepsilon > 0$, there exists $C^* \in \tilde{\mathcal{L}}_f$ such that

$$\mu(C \triangle C^*) < \varepsilon. \tag{4}$$

and $A \in \mathcal{A}_f$ iff for every $\varepsilon > 0$ there exist sets $C_i \in \tilde{\mathcal{L}}_f$, $D_i \in \overline{\tilde{\mathcal{L}}}_f$, $i = 1, \ldots, n$ such that

$$\mu\left(A \triangle \bigcup_{i=1}^{n} C_i \cap D_i\right) < \varepsilon. \tag{5}$$

Moreover, it is not hard to prove that we can suppose the sets $C_i \cap D_i$, $i = 1, \ldots, n$, are mutually disjoint.

Next we present the main result of this section.

Theorem 3.6. *Let $f \in L_\varphi$ and let $\mathcal{L}$ be a σ-lattice. Then the following statements are equivalent*:

(i) $g \in \mu(f, \mathcal{L})$.
(ii) $g \in \mu(f, \mathcal{A}_f) \cap L_\varphi(\mathcal{L})$.

Proof. We assume $g \in \mu(f, \mathcal{L})$ and $a \in C(f)$. Thus $\{g > a\} \in \mathcal{L}_a \subset \mathcal{L}_f$. Now, from the density of $C(f)$ in $\mathbf{R}$ we obtain that g is an $\mathcal{L}_f$-measurable function (thus g is $\mathcal{A}_f$-measurable). In order to prove that $g \in \mu(f, \mathcal{A}_f)$, we need to show that g is an LRN function (with respect to $\mathcal{A}_f$) of the family $\{\mu_a^+ : a \in \mathbf{R}\}$. That is, for every $A \in \mathcal{A}_f$ and $a \in \mathbf{R}$, we must prove

$$\mu_a^+(\{g > a\} \cap A) \geq 0 \qquad \text{and} \qquad \mu_a^+(\{g < a\} \cap A) \leq 0. \tag{6}$$

From (5) it is sufficient to prove the inequalities (6) for $a \in C(f)$ and $A = C \cap D$, with $C \in \tilde{\mathcal{L}}_f$ and $\Omega \backslash D \in \mathcal{L}_f$. Let $b \in C(f)$ be such that $C \in \mathcal{L}_b$ and suppose $b < a$. Then, from Lemma 3.4, we get $\{g > a\} \cap C \in \mathcal{L}_a$. Therefore, $\mu_a(\{g > a\} \cap C \cap D) \geq 0$. Next we suppose $a \leq b$. Since $C \cap \{g > a\}$ is a μ_b-positive set, we obtain $\mu_a(\{g > a\} \cap C \cap D) \geq \mu_b(\{g > a\} \cap C \cap D) \geq 0$. This concludes the proof of (i) $\Rightarrow$ (ii).

Finally, we assume that $g \in L_\varphi(\mathcal{L}) \cap \mu(f, \mathcal{A}_f)$. Let $\tilde{g} \in \mu(f, \mathcal{L})$. As a consequence of part (i) $\Rightarrow$ (ii) of this theorem we get

$$\int \varphi(\omega, f - g)\, d\mu = \int \varphi(\omega, f - \tilde{g})\, d\mu.$$

Therefore, $g \in \mu(f, \mathcal{L})$. ■

4. Totally Ordered σ-Lattices

In this section we will use Theorem 3.6 to obtain a sharper characterization of best φ-approximants when $\mathcal{L}$ is a totally ordered σ-lattice. We recall that a σ-lattice $\mathcal{L}$ is totally ordered if for any sets $C_1, C_2 \in \mathcal{L}$ we have $C_1 \subset C_2$, μ-a.e. or $C_2 \subset C_1$, μ-a.e.

As usual, we say that a set B is an atom of the σ-algebra $\mathcal{B} \subset \mathcal{A}$ iff $B \in \mathcal{B}$ and for every $\mathcal{B}$ measurable set C we have $\mu(C \cap B) = 0$ or $\mu(C \cap B) = \mu(B)$. We consider the set of all equivalence classes (sets which differ in a μ-null set are equivalent) of atoms of the σ-algebra $\mathcal{B}$. We denote by Atom($\mathcal{B}$) a complete set of representatives of the atoms of $\mathcal{B}$ (since μ is a finite measure, then Atom($\mathcal{B}$) is at most countable). On the other hand, if $\Omega' \subset \Omega$, we denote by $\mathcal{B}_{\Omega'}$ the σ-algebra induced by $\mathcal{B}$ on Ω', i.e., $\mathcal{B}_{\Omega'} := \{B \cap \Omega' : B \in \mathcal{B}\}$. For $\mathcal{L}'$ a sub-σ-lattice of $\mathcal{L}$, we denote by $\mathcal{A}(\mathcal{L}')$ the σ-algebra generated by $\mathcal{L}'$.

Lemma 4.1. *Suppose $\mathcal{L}$ is a totally ordered σ-lattice. Let $\mathcal{L}'$ be a sub-σ-lattice of $\mathcal{L}$. We define the set*

$$\Omega' := \Omega \setminus \bigcup \{A : A \in \text{Atom}(\mathcal{A}(\mathcal{L}'))\}. \tag{7}$$

Then $\mathcal{A}(\mathcal{L})_{\Omega'} = \mathcal{A}(\mathcal{L}')_{\Omega'}$.

Proof. Step 1. We show that $A \in \text{Atom}(\mathcal{A}(\mathcal{L}'))$ iff for every $C \in \mathcal{L}'$ we have $\mu(C \cap A) = 0$ or $\mu(C \cap A) = \mu(A)$. The necessary condition follows trivially. For the sufficient implication note that it is enough to prove that $\mu(B \cap A) = 0$ or $\mu(B \cap A) = \mu(A)$, when B is a set of the form $B = \bigcup C_i \cap D_i$, with $C_i, \Omega \setminus D_i \in \mathcal{L}'$, $i = 1, \dots, n$. This follows immediately from the conditions that we are assuming on B.

Step 2. Let C be an arbitrary set in $\mathcal{L}$. We consider the following numbers:

$$\alpha := \inf\{\mu(C') : C' \in \mathcal{L}' \text{ and } C \subset C'\}$$

and

$$\beta := \sup\{\mu(C') : C' \in \mathcal{L}' \text{ and } C' \subset C\}.$$

We can find two sequences in $\mathcal{L}'$, $C_n, C^n \in \mathcal{L}'$ such that $C_n \subset C$, $C^n \supset C$, $\mu(C_n) \uparrow \beta$, and $\mu(C^n) \downarrow \alpha$. We define the sets $C^* = \bigcap C^n$ and $C_* = \bigcup C_n$. It follows immediately that $C_*, C^* \in \mathcal{L}'$, $\mu(C^*) = \alpha$, $\mu(C_*) = \beta$, and $C_* \subset C \subset C^*$. We affirm that $C^* \setminus C_* \in \text{Atom}(\mathcal{A}(\mathcal{L}'))$. Otherwise, we could find $C' \in \mathcal{L}'$ such that $0 < \mu(C' \cap (C^* \setminus C_*)) < \mu(C^* \setminus C_*)$. Since $\mathcal{L}$ is totally ordered, we have $C_* \subset C' \subset C^*$. We can suppose (w.l.o.g.) that $C \subset C'$. In this case we get $\mu(C') < \alpha$, which is a contradiction.

Hence, we have that $C \cap \Omega'$ is equal to $C^* \cap \Omega'$ μ-a.e. or $C_* \cap \Omega'$ μ-a.e. This implies the statement of the lemma. ■

Given a function f and a set A we denote by $f_{|A}$ the restriction of f to A.

Lemma 4.2. *Let $f \in L_\varphi$ and let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{A}$. Suppose $\Omega_i \subset \Omega$, $i = 1, 2, \dots,$ is a countable partition of Ω by $\mathcal{B}$-measurable sets. Then $g \in \mu(f, \mathcal{B})$ iff $g_{|\Omega_i} \in \mu(f_{|\Omega_i}, \mathcal{B}_{\Omega_i})$, for every $i \in \mathbf{N}$.*

Proof. This follows immediately from the definitions. ■

The following is the main result in this section:

Theorem 4.3. *Let $f \in L_\varphi$ and let Ω' be defined by* (7) *with $\mathcal{L}' = \mathcal{L}_f$. Then, a function $g \in \mu(f, \mathcal{L})$ iff:*

(i) $g \in L_\varphi(\mathcal{L})$.
(ii) *g is constant on each set $A \in$ Atom$(\mathcal{A}(\mathcal{L}_f))$. Moreover, $g_{|A}$ is a best constant φ-approximant to $f_{|A}$ on each set $A \in$ Atom$(\mathcal{A}_f)$.*
(iii) $g_{|\Omega'} \in \mu(f_{|\Omega'}, \mathcal{A}(\mathcal{L})_{\Omega'})$.

Proof. The theorem follows from Theorem 3.6, Lemmas 4.1 and 4.2. ■

The following corollary is well-known in Orlicz spaces (see [15]):

Corollary 4.4. *Let $\Omega = [0, 1]$ and $g \in \mu(f, \mathcal{L}^1)$. Then there exists an open set V such that g is constant on each component of V and $g = f$ on $[0, 1]\backslash V$ μ-a.e.*

Proof. It is a consequence of this that $\mathcal{L}^1_{\Omega'}$ is the Lebesgue σ-algebra restricted to Ω'. ■

5. Uniqueness Theorems on Domains of $\mathbf{R}^n$

Throughout this section, $\Omega \subset \mathbf{R}^n$ denotes an open subset of $\mathbf{R}^n$, $\mathcal{A}$ is the Lebesgue σ-algebra, and μ is the Lebesgue measure. For $x, y \in \Omega$ we denote $R_{x,y} := \{z \in \Omega : x \leq z \leq y\}$. We observe that $R_{x,y}$ is a n-parallelepiped with faces parallel to the coordinate's hyperplanes.

We are going to prove a uniqueness theorem of best φ-approximants which generalize those established in [4], [8], [15] for f approximately continuous. In [4], R. Darst and Shunsheng Fu considered $\Omega = (0, 1)^n$ and the space $L_1(\Omega)$. Later, in [15], M. Marano and J. Quesada gave a uniqueness theorem for $\Omega = (0, 1)$ and the Orlicz space $L_\varphi(\Omega)$.

We are looking for sufficient conditions on the geometry of Ω and on the function f such that a uniqueness theorem remains true.

Henceforth, for $C \subset \Omega$, we denote by ∂C the boundary of C relative to Ω. It is well-known that a nondecreasing function $g : \Omega \to \mathbf{R}$ is continuous a.e. (see [4]). Since χ_C is nondecreasing, for all final sets C, we obtain that $\mu(\partial C) = 0$.

Let A be a subset of $\mathbf{R}^n$. The *density* of A at a point $x \in \mathbf{R}^n$ is defined by

$$D(A, x) := \lim_{r \to 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))},$$

if the limit exists. A function $f : \Omega \to \mathbf{R}$ is said to be *approximately continuous* at a point $x \in \Omega$ iff, for every open set G containing $f(x)$, we have $D(f^{-1}(G), x) = 1$. We say that f is approximately continuous on Ω when, for every $x \in \Omega$, f is approximately continuous at x. An approximately continuous function f takes open connected sets into connected sets (see [6]).

Lemma 5.1. *Let g be an $\mathcal{L}^n$-measurable function. Then there exists a nondecreasing function $\overline{g}$ ($\underline{g}$) such that $\overline{g} = g$ μ-a.e. ($\underline{g} = g$ μ-a.e.) and $\overline{g}$ is u.s.c ($\underline{g}$ is l.s.c.).*

Proof. For each $q \in \mathbf{Q}$ there exists a final set C_q such that $\mu(\{g > q\} \triangle C_q) = 0$. It is easy to check that the set $B_q := \operatorname{int}(C_q)$ is also a final set and that $\mu(\{g > q\} \triangle B_q) = 0$. Now, we define

$$\underline{g}(x) := \sup_{x \in B_q} q.$$

It is not hard to see that $\underline{g}$ is a nondecreasing function, l.s.c, and $\underline{g} = g$ μ-a.e. The function $\overline{g}$ is defined analogously. ∎

Theorem 5.2. *Let Ω be an open set in $\mathbf{R}^n$ with $R_{x,y}$ connected for every $x, y \in \Omega$. We assume that $f : \Omega \to \mathbf{R}$ is an approximately continuous function. Then there exists a unique best φ-approximant to f from $L_\varphi(\mathcal{L}^n)$.*

Proof. Let $g_i \in \mu(f, \mathcal{L}^n)$, $i = 1, 2$. As a consequence of Lemma 5.1, we can assume that g_1 (g_2) is u.s.c. (l.s.c.) and nondecreasing. In order to prove the theorem, it is sufficient to show that, for every $a, b \in C(f)$ with $a < b$, we have

$$\mu(\{g_1 < a < b < g_2\}) = 0. \tag{8}$$

We note that the set $A = \{g_1 < a < b < g_2\}$ is open. Therefore, $A = \bigcup_{k=1}^{m} A_k$ ($m \in \mathbf{N} \cup \{+\infty\}$) with A_k, $k = 1, \ldots,$ are open and connected sets. We will show that for every $k = 1, \ldots$ there exists $C_k \in \mathcal{L}^n$ ($D_k \in \overline{\mathcal{L}^n}$) such that $A_k = A \cap C_k$ ($A_k = A \cap D_k$). We define

$$C_k := \bigcup_{x \in A_k} \{y \in \Omega : x \leq y\}.$$

Then C_k is a final set, thus $C_k \in \mathcal{L}^n$, with $A_k \subset C_k$. Suppose that for some $x \in A_k$ and $p \neq k$ we have that $\{y \in \Omega : x \leq y\} \cap A_p \neq \emptyset$. Let z be a point in $A_p \cap \{y \in \Omega : x \leq y\}$. Then, from the monotonicity properties of g_1 and g_2, we have that $R_{x,z} \subset A$. Moreover, from the hypothesis, $R_{x,z}$ is a connected set. Therefore, $R_{x,z} \subset A_k$ and $R_{x,z} \subset A_p$ which is a contradiction. Hence, we have that $C_k \cap A = A_k$. Analogously, we can prove that there exists $D_k \in \overline{\mathcal{L}^n}$ such that $A_k = D_k \cap A$ for all k.

On the other hand, from Theorem 3.2 we obtain that, for $k = 1, 2, \ldots,$

$$0 \leq \mu_b(A \cap C_k) \leq \mu_a(A \cap D_k) \leq 0.$$

Therefore

$$0 = \int_{A_k} \varphi_+(\omega, f - a)\, d\mu = \int_{A_k} \varphi_+(\omega, f - b)\, d\mu. \tag{9}$$

We note that $\varphi_+(\omega, f - a) \geq \varphi_+(\omega, f - b)$ μ-a.e. Since $\varphi_+(\omega, a) > 0$ for $a > 0$, we have the strict inequality $\varphi_+(\omega, f - a) > \varphi_+(\omega, f - b)$ on $\{a < f < b\} \cap A_k$. So, using (9), we get $\mu(\{a < f < b\} \cap A_k) = 0$. Thus, if $x \in \{a < f < b\} \cap A_k$, then $D(f^{-1}((a, b)), x) = 0$. This fact implies that $\{a < f < b\} \cap A_k = \emptyset$. On the other hand, formula (9) implies that $\mu(\{b \leq f\} \cap A_k) > 0$ and $\mu(\{f \leq a\} \cap A_k) > 0$. Therefore f takes the open and connected set A_k into a disconnected set. This contradiction concludes the proof. ■

Remark 3. Theorem 5.2 generalizes the uniqueness results in [4], [8], [15] in two aspects. We have considered best approximants in Musielak–Orlicz spaces and a more general domain Ω.

Remark 4. We give an example of a domain $\Omega \subset \mathbf{R}^2$ and a continuous function $f : \Omega \to \mathbf{R}$ such that f fails to have a unique best φ-approximant. Here $\varphi(\omega, a) = |a|$ and $\Omega = (-1, 1) \times (-1, 1) \backslash \{(x, 0) : x \geq 0\}$. We define f by

$$f(x, y) = \begin{cases} x, & \text{if } x \geq 0 \text{ and } y \leq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Now we define the following functions $g^1 \equiv 0$ and $g^2(x, y) = \sup\{0, x\}$. For any function f we denote by f_x the function $f_x(y) := f(x, y)$. It is easy to check that, on the vertical sections $\Omega_x := \{y : (x, y) \in \Omega\}$, the functions g_x^1 and g_x^2 are best φ-approximants to f_x from the class of nondecreasing functions. So, we have, for every $g \in L_\varphi(\mathcal{L}^2)$,

$$\int_{-1}^{1} |f_x - g_x^i| \, d\mu \leq \int_{-1}^{1} |f_x - g_x| \, d\mu,$$

for $i = 1, 2$. Hence, an integration with respect to x and the Fubini theorem show that g^i, $i = 1, 2$, are best φ-approximants.

6. A Coincidence Theorem in Several Variables

In this section Ω will be an open set in $\mathbf{R}^n$ and $\mathcal{L}$ will be the σ-lattice $\mathcal{L}^n$. Let $C \in \mathcal{L}^n$ be a final set and let $x \in \partial C$. We note that C contains the set $C_0^+(x) := (x + C_0^+) \cap \Omega$, where $C_0^+ := \{y \in \mathbf{R}^n : y_i > 0, i = 1, \ldots, n\}$. Similarly $\Omega \backslash C$ contains the set $C_0^-(x) := (x + C_0^-) \cap \Omega$, with $C_0^- := -C_0^+$.

As usual, we denote by e_j the canonical unit vector and we define $\overline{e}_j := (1, 1, \ldots, 1) - e_j$. Now, for $1 > \delta > 0$, define the following vectors $v_j^\delta := e_j + \delta \overline{e}_j$. We consider the following cones: $C_\delta^+ := \{x \in \mathbf{R}^n : \langle x, v_j^\delta \rangle > 0, j = 1, \ldots, n\}$ (note that if $n = 1$, then $C_\delta^+ = C_0^+$). We define the sets $C_\delta^+(x) := (x + C_\delta^+) \cap \Omega$. Analogously we define the sets C_δ^- and $C_\delta^-(x)$. It is easy to check that $C_{\delta_1}^\pm \subset C_{\delta_2}^\pm$ for $0 \leq \delta_1 < \delta_2 < 1$.

Definition 6.1. Given $C \in \mathcal{L}^n$, we say that $x \in \partial C$ is an upper (lower) δ-regular point of ∂C iff $C_\delta^+(x) \subset C$ ($C_\delta^-(x) \subset \Omega \backslash C$). The point $x \in \partial C$ will be called an *upper* (*lower*) *regular point of* ∂C iff it is upper (lower) δ-regular for some $\delta > 0$ (see Figure 1).

We note that when ∂C is "smooth" at x then x is a regular (lower and upper) point of ∂C iff $\nu_i > 0$ with $\nu = (\nu_1, \ldots, \nu_n)$ the unit inward normal vector to C at x. In particular, if $C = \{g > a\}$ with g smooth and nondecreasing, then $x \in \{g = a\}$ is a

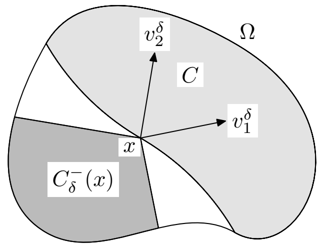

Fig. 1. A lower regular point.

regular point of $\{g = a\}$ iff $\partial g/\partial t_i|_{t=x} > 0$, for $i = 1, \ldots, n$. If $n = 1$ all points in ∂C are regular, for every $C \in \mathcal{L}^1$.

We denote by $R^+ = R^+(\mathcal{L}_f^n)$ $(R^- = R^-(\mathcal{L}_f^n))$ the set of all upper (lower) regular points of ∂C for some $C \in \mathcal{L}_f^n$.

Theorem 6.2. *Let $f \in L_\varphi$ and let $g \in \mu(f, \mathcal{L}^n)$. Then there exists a μ-null set $F \subset R^-$ $(F \subset R^+)$ such that $f(x) \geq g(x)$ $(f(x) \leq g(x))$ for $x \in R^+ \backslash F$ $(x \in R^- \backslash F)$.*

We need the following general lemma:

Lemma 6.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite complete measurable space and let $\mathcal{L} \subset \mathcal{A}$ be a complete σ-lattice. Suppose that $f \in L_\varphi$ and $g \in \mu(f, \mathcal{L})$ If $C \in \mathcal{L}_f$ then, for every $D \in \overline{\mathcal{L}}$,*

$$\int_{C \cap D} \varphi_+(x, f(x) - g(x))\, dx \geq 0. \tag{10}$$

That is, C is a μ_g^+-positive set.

Proof. Clearly, it is possible to assume that g is the maximum best φ-approximant. We start by supposing that $C \in \mathcal{L}_a$, for some $a \in C(f)$. We define

$$h(x) := \begin{cases} g(x), & x \notin C, \\ g(x) \vee a, & x \in C. \end{cases}$$

Then

$$\{h > b\} = \begin{cases} \{g > b\}, & \text{if } b \geq a, \\ \{g > b\} \cup C, & \text{if } b < a. \end{cases}$$

Therefore $h \in L_\varphi(\mathcal{L})$. Suppose that $b \in C(f)$. Now, from Corollary 3.3, we obtain that, for $b \geq a$, $\{h > b\} \in \mathcal{L}_b$. Further, from Lemma 3.4, we have that $\{g > b\} \cup C \in \mathcal{L}_b$, when $b < a$. Therefore, for every $b \in C(f)$, $\{h > b\} \in \mathcal{L}_b$. So $h \in \mu(f, \mathcal{L})$. Since g is the maximum best φ-approximant, we get $h \leq g$. Hence, $g \geq a$ on C.

Now, for every $n \in \mathbf{N}$, we choose a sequence $\{t_k^n\}$ such that:

(i) $a = t_1^n < t_2^n < \cdots$.
(ii) $t_k^n \in C(f)$ for every $k \in \mathbf{N}$.

(iii) There exist constants c and d independent of n and k such that

$$\frac{c}{n} \le t_k^n - t_{k-1}^n \le \frac{d}{n}.$$

for every $k > 1$.

We consider the sets $C_{kn} := C \cap \{g \ge t_k^n\}$ and $D_{kn} := D \cap \{g < t_{k+1}^n\}$. We have that $C \in \mathcal{L}_a$ and $\{g \ge t_k^n\} \in \mathcal{L}_{t_k^n}$. Therefore, we obtain $C \cap \{g \ge t_k^n\} \in \mathcal{L}_{t_k^n}$ (see Lemma 3.4 and Remark 2). Hence

$$\int_{C_{kn}\cap D_{kn}} \varphi_+(x, f(x) - t_k^n)\, dx \ge 0.$$

With a similar argument to the one used to prove (iii) $\Rightarrow$ (ii) of Theorem 3.2 and taking into account that $C \subset \{g \ge a\}$, we prove that inequality (10) holds for $C \in \mathcal{L}_a$ and $a \in C(f)$. A density argument proves (10) for every $C \in \mathcal{L}_f$. ■

Proof of Theorem 6.2. Let $E \subset \Omega$ be the set of all Lebesgue points of $\varphi_+(x, f(x) - g(x))$. That is, $x \in E$ iff

$$\lim_{\varepsilon\to 0} \frac{1}{\mu(B(x,\varepsilon))} \int_{B(x,\varepsilon)} |\varphi_+(t, f(t) - g(t)) - \varphi_+(x, f(x) - g(x))|\, dt = 0.$$

It is well-known that $F = \Omega\backslash E$ is a μ-null set. Let x be an arbitrary point in $R^-\backslash F$. We assume that $x \in \partial C$, with $C \in \mathcal{L}_f$. Let $C_\delta^-(x)$ be a set satisfying $C_\delta^-(x) \subset \Omega\backslash C$ and $0 < \delta < 1$. Now, for small $\varepsilon > 0$, we consider the set $D_\varepsilon := C_0^+(x + \varepsilon v)$, with $v = (1, \ldots, 1)$, and $S_\varepsilon := C \cap D_\varepsilon$ (see Figure 2).

From the relation $C_0^+(x) \cap D_\varepsilon \subset S_\varepsilon$, we obtain

$$c\varepsilon^n \le \mu(S_\varepsilon). \tag{11}$$

where c is independent of ε. Moreover, since $S_\varepsilon \subset (\Omega\backslash C_\delta^-(x)) \cap D_\varepsilon$, we get

$$\operatorname{diam} S_\varepsilon \le d\varepsilon \qquad \text{and} \qquad \mu(S_\varepsilon) \le d\varepsilon^n. \tag{12}$$

with d independent of ε.

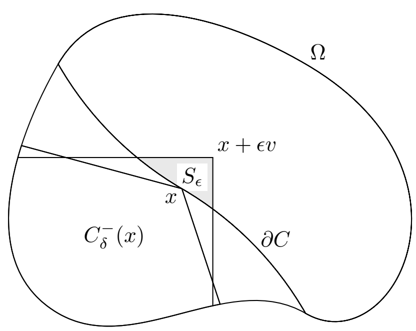

Fig. 2. The set S_ε.

Since x is a Lebesgue point of $\varphi_+(\cdot, f-g)$, from inequalities (11) and (12) and from $S_\varepsilon \subset B(x, d\varepsilon)$ we have

$$\lim_{\varepsilon\to 0} \frac{1}{\mu(S_\varepsilon)} \int_{S_\varepsilon} |\varphi_+(t, f(t)-g(t)) - \varphi_+(x, f(x)-g(x))|\, dt = 0.$$

From this equality and Lemma 6.3 we get $\varphi_+(x, f(x)-g(x)) \geq 0$. Therefore $f(x) \geq g(x)$.

The other case of the theorem follows analogously. ■

Corollary 6.4. *Let $f \in L_\varphi$ and let $g \in \mu(f, \mathcal{L}^n)$. Then there exists a μ-null set $F \subset \Omega$ such that $f(x) = g(x)$ for $x \in R^- \cap R^+ \backslash F$.*

Remark 5. In [15] there was proved a coincidence theorem in an Orlicz space $L_\varphi([0,1])$. More precisely, Marano and Quesada proved that:

(i) if f is approximately continuous at $x_0 \in [0,1]$;
(ii) if g is not constant at x_0 (i.e., $g(x) > g(x_0)$ for $x > x_0$ or $g(x) < g(x_0)$ for $x < x_0$);
(iii) if $\varphi_- \in L_\infty(\mathbf{R})$ or $f \in L_\infty(U)$, with U a neighborhood of x_0, then $f(x_0) = g(x_0)$.

They also proved that (i) and (iii) imply that g is continuous at x_0. We note that if g is not constant at x_0, then we have $\{x_0\} = \partial C \cap (0,1)$, with $C = \{g > g(x_0)\}$ or $C = \{g \geq g(x_0)\}$. Since g is $\mathcal{L}^n_f$-measurable, we have $C \in \mathcal{L}^n_f$. Moreover, in the one dimensional case, the unique point in ∂C is a regular point. The set of all Lebesgue points of $\varphi_\pm(\cdot, f-g)$ may not be the same set as the set of all points where f is approximately continuous. However, these sets are equal except possibly by a μ-null set. Therefore, our set $B_1 = R^+ \cap R^-$ contains μ-a.e. the set B_2 of all points satisfying (i)–(iii). On the other hand, it is possible that $\mu(B_1) > \mu(B_2)$. A simple example of that is a constant function f. In this case, we have $B_2 = \emptyset$ μ-a.e. and $B_1 = \Omega$ μ-a.e. In other words, the set B_1 also contains points where g is constant.

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References

1. J. Best, N. Chakravarti, V. Ubhaya (2000): *Minimizing separable convex functions subject to simple chain constraints*. SIAM J. Optim., **10**:658–672.
2. H. Brunk, S. Johansen (1970): *A generalized Radon–Nikodym derivative*. Pacific J. Math., **34**:585–617.
3. R. Darst, R. Huotari (1985): *Monotone L_1-approximation on the unit n-cube*. Proc. Amer. Math. Soc., **95**:425–428.
4. R. Darst, Shunsheng Fu (1986). *Best L_1-approximation of L_1-approximately continuous functions on* $(0,1)^n$ *by nondecreasing functions*. Proc. Amer. Math. Soc., **97**:262–264.
5. S. Favier, F. Zó (2001): *Extension of the best approximation operator in Orlicz spaces and weak-type inequalities*. Abstr. Appl. Anal., **6**:101–114.

6. C. GOFFMAN, D. WATERMAN (1961): *Approximately continuous transformation*. Proc. Amer. Math. Soc., **12**:116–121.
7. R. HUOTARI, A. MEYEROWITZ, M. SHEARD (1986): *Best monotone approximations in* $L_1[0, 1]$. J. Approx. Theory, **47**:85–91.
8. M. ITURRIETA, F. ZÓ (1998): *Best monotone* L_φ*-approximations in several variables*. Approx. Theory Appl., **14**:1–10.
9. S. JOHANSEN (1967): *The descriptive approach to the derivative of a set function with respect to a* σ*-lattice*. Pacific J. Math., **21**:49–58.
10. S. KILMER, W. KOZLOWSKI, G. LEWICKI (1990): *Best approximants in modular function spaces*. J. Approx. Theory, **63**:338–367.
11. D. LANDERS, L. ROGGE (1980): *Best approximants in* L_φ*-spaces*. Z. Wahrsch. Verw. Gabiete, **51**:215–237.
12. D. LANDERS, L. ROGGE (1981): *Isotonic approximation in* L_s. J. Approx. Theory, **31**:199–223.
13. M. MARANO (1996): *Structure of best radial monotone* Φ*-approximants*. J. Math. Anal. Appl., **199**:526–544.
14. M. MARANO (1997): *Monotone* l_φ*-approximation*. Approx. Theory Appl., **13**:51–57.
15. M. MARANO, J. QUESADA (1997): L_φ*-Approximation by non-decreasing functions on the interval*. Constr. Approx., **13**:177–186.
16. F. MAZZONE, H. CUENYA (2002): *Isotonic approximation in* L_1. J. Approx. Theory, **117**:279–300.
17. J. MUSIELAK (1983): Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, Vol. 1034. Berlin: Springer-Verlag.
18. T. ROBERTSON, F. WRIGHT, R. DYKSTRA (1988): Order Restricted Statistical Inference. London: Wiley.

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