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Dynamic allocation of industrial utilities as an optimal stochastic tracking problem

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ABSTRACT

A new dynamic optimization strategy is substantiated for allocating demands, in a typical process plant, to a set of service equipment working in parallel. It is a stochastic process in nature, but its optimal control is based on the solution to a related deterministic optimal tracking problem to minimize a quadratic cost objective restricted by linear dynamics. The main theoretical novelty, demonstrated here, is the separation theorem for the stochastic tracking problem. This means: the desired optimal stochastic solution can be calculated from the solution to the deterministic problem, by replacing the state variable with their optimal estimates, which can be generated online following a Kalman filter scheme. The set-points assigned to each conventional controlled device are allowed to be continuously changed while: (i) minimizing a combined cost, which is cumulative in time and takes into account the dynamics of all the individual utilities, and (ii) generating a feedback law that can cope with general disturbances, like changes in fuel composition and with noisy measurements, i.e. with differences between the predicted and the measured values of the variables.

1. Introduction

Optimal allocation problems have a long tradition in engineering practice. In chemical processes, dynamic optimization frequently deals with distributing global service demands of the plant into individual targets assigned to each member of a group of service equipment, while minimizing a predetermined generalized cost. Typical service equipment (or utilities) include sets of boilers/steam generators, heat-exchangers, pumps, air-compressors, and the like (Bujak, 2009; Collins and Lang, 1998; Muller and Craig, 2014; Teles et al., 2008; Zhang et al., 2013). In what follows, the individual components from the 'group' under consideration will be referred to as 'units'. Units operate in parallel to meet the total demand required to the group. Usually the individual demands translate into set-points communicated to controllers of the PID type, which are properly tuned and perform efficiently. The sum of the demands assigned to the units is always assumed to equal the total demand required from the group.

With environmental policies, rising energy costs, and a struggling global economy, there has been an increasing concern on efficiency improvement in the process industries. Energy is supplied to (or removed from) a plant mostly through utilities and a reduction in the consumption of these utilities results in a direct energy saving (Pillaia and Bandyopadhyay, 2007; Shide et al., 2009). In Fig Fig 1. a schematic diagram of a utility group is shown (Hwan and Han, 2001). They provide vapor to the rest of the plant, where in this case power energy is generated by means of a system of turbines in parallel. The power demand is decided by a supervisory controller (depicted in green), which in turn imposes a total vapor demand to the steam generators. In traditional engineering practice this total vapor demand enters to the boiler system as a global set point (constant under normal operation conditions). The global demand α needs to be distributed into the units, realized conventionally as a constant fraction of α. A novel scheme is introduced at this point (illustrated in blue), where the set points (u_1, u_2,...u_n) of each boiler are permitted to change in time, following a trajectory and decided by the Optimal Allocation Controller. In this paper two aspects of this routine will be discussed: (i) the methodology for deciding the individual set-points after a new total demand is required from a group, and (ii) the convenience of changing these orders continuously in time, by optimizing some combined cumulative cost during a fixed finite time-horizon.

This type of approach (although resorting just to time-constant set-point orders) has been applied to steam generation (Bujak, 2009; Havlena, 2009; Likins and LaSpisa, 1986) towards minimizing energy losses to the environment, or equivalently to maximize the efficiency of a set of units, defined from theoretical relations among the many physical variables involved. This approach gives rise to a static

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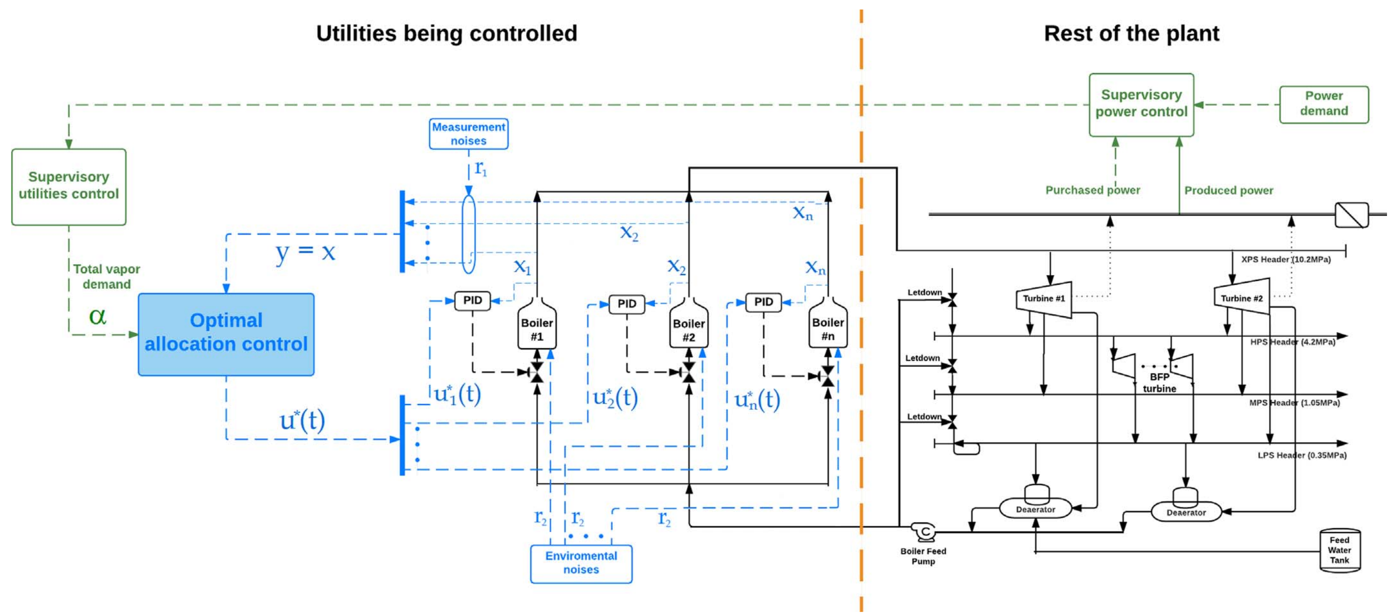


Fig. 1. Schematic diagram of supervisory control with optimal demand allocation of a utility plant. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

optimization problem managed by linear or nonlinear programming, exceptionally by Dynamic Programming (Hwan and Han, 2001; Mavromatisa and Kokossis, 1998; Pillaia and Bandyopadhyay, 2007).

They follow static optimization lines, common to research operation engineering (see for instance Hatzopoulos et al., 2011; Xu and Zeng, 2011). To the authors' knowledge, a dynamic point-of-view has only been applied to specialized derived situations, like redundant control and related problems (Härkegård and Glad, 2005).

An original 'dynamic feedback' strategy will be sought here, in the sense that the set-points to each unit will be allowed to change continuously while: (i) minimizing a combined cost, which is cumulative in time and takes into account the dynamics of all the individual units, and (ii) generating a stochastically optimal control that copes with general disturbances, like changes in fuel composition, noisy measurements, environmental interactions and the like.

With these objectives in mind, the dynamics for the responses of each unit to set-point indications will be assessed, directly from experimental data. Then the whole group of n units will be assembled into a general model with an $(n-1)$-dimensional control vector associated with the first $n-1$ set-points, the remaining one determined by the residue with respect to the global demand, which is constant during each optimization time-horizon. This new 'big' system, together with a typical quadratic cost functional conform an optimal control problem that has a close mathematical solution, leading to a linear feedback law with time-variant coefficients. Both the proportional coefficient and the feed-through term in the control law need to be calculated only once for a unitary global demand, and stored in memory, the updating procedure for another demand being straightforward. The nature of the modeling also admits a stochastically optimal handling of disturbances and systemic perturbations, and eventually a suboptimal online correction (Costanza, 2005; Costanza and Rivadeneira, 2014) of the feedback law due to hard restrictions on control values (Bujak, 2009; Havlena, 2009).

The rest of the article will be organized as follows: in Section 2 the modeling for the dynamics and the design of the cost objective are made explicit, and the optimal solution analytically found. Also the stochastic problem of estimation is posed for groups of units, and the 'Separation Principle' for tracking problems is demonstrated, which guarantees the optimality of the tandem filter-regulator. Section 3 is devoted to numerical calculations and validations, the stochastic aspects are substantiated, and all issues illustrated for boilers in a steam utility group. The last Section exposes the conclusions.

2. Theoretical setup

2.1. State space models for utility units

In what follows it will be assumed that a group of service equipment is in operation as part of an industrial plant, its units working in parallel, evolving within the admissible range of their main variables, and that each member is efficiently controlled, according to conventional engineering practice, to meet its assigned demand.

It is commonly accepted that the dynamics of each unit is in general nonlinear (Bujak, 2009; Havlena, 2009), of the form

$$\dot{x} = f(x, v), \tag{1}$$

where x denotes the relevant states and v some manipulated variable (for instance, the water inflow). As soon as a new set-point u for the 'production' state x_1 is received, then the manipulated variable will be assumed to follow some finely tuned control strategy

$$v = k(t, u), \tag{2}$$

which 'efficiently' drives x_1 towards u in due time. Eq. (2) represents the final form that the control trajectory (generated by a controller, typically a PID) will adopt after a set-point of magnitude u is assigned. This paper will not deal with the validity of the subjacent efficiency criterion nor with the design/tuning of the control strategies $k(t, u)$.

The 'production' state x_1 is attached to the 'service' required from the equipment. For instance, if the unit were a boiler, then the value $x_1(t)$ would reflect the amount of vapor produced by the boiler at time t. It follows that there will also be at least a main 'expense' variable x_2, necessary for the unit to actually realize the service. Again for a boiler, $x_2(t)$ could typically describe the amount of fuel that the unit is consuming at time t.

As a consequence of applying such efficient control strategies k, it can also be reasonably assumed that the dynamics of the variables x_1, x_2, u result approximately linear, i.e. that the finally controlled unit would perform well under proportionally different admissible set-points. This hypothesis has been corroborated by experimental data (Bujak, 2009; Costanza and Rivadeneira, 2015; Xu and Zeng, 2011; Havlena, 2009), and it amounts to propose a linear model for the new system, namely

$$\dot{x} = Ax + Bu, \tag{3}$$

where it is redefined as $x:=(x_1, x_2)'$, and then A is a 2×2 matrix, B is a 2-dimensional column vector and u works as the new control variable allowed to vary in the range of experimental values. The pair (A, B) will be assumed controllable, their coefficients identified from current registered data of the plant under study (see (Costanza and Rivadeneira, 2015) for details).

The demand now functions as a target communicated by the supervisory command to each service unit. In previous engineering literature it was kept constant (at a prescribed fixed value u, equal to a portion of the production required from the whole set of service units) during a period of time, until a new demand was decided, i.e. u has taken the form of a picewise-constant function of time. Since the ability to follow the individual demand is accurate and fast, there is no technical impediment to admit piecewise-continuous $u(\cdot)$ for the individual demands, seeking to improve the performance of the whole group. What is treated here is just how to continuously change the set-point $u(t)$ for each unit in operation, in order to meet the total demand assigned to the group while optimizing an economic cost criterion, to be explicitly designed below.

It must remain clear that, in trying to achieve this objective, the conventional control instrumentation already attached to each unit should be preserved and subject to standard maintenance regulations. These instruments usually are long-tested PID controllers implementing the strategies (2) alluded above, their tuning being a matter out of the scope of this paper. Therefore, in what follows the objects to optimize will be well instrumented service units, whose new inputs will be in each case a time-varying demand $u(\cdot)$ affecting the evolution of the new state variables $x(\cdot)$ as in linear control systems (3).

In the general case, n units in parallel will be optimized, whose identified linear models are denoted

$$\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = x_i, \quad i = 1, \ldots n; \tag{4}$$

where $x_i:=(x_{i1}, x_{i2})'$ is the state vector for unit i, x_{i1}: produced variable, x_{i2}: consumed variable, u_i: demanded value, and y_i: the output, equivalent to the state. For each boiler the matrices A's are 2×2 and 2×1, respectively.

When a total demand α is required to be supplied by the n boilers, then necessarily

$$u_n = \alpha - u_1 - \cdots - u_{n-1}. \tag{5}$$

Therefore, despite the facts that n set-points are to be ordered to the n utility units, there exist only $n-1$ degrees of freedom for treating the whole set. This leads to a possible setup for the optimal allocation problem to n units working in parallel. After redefining

$$x:=(x'_1 \vdots x'_2 \vdots \cdots \vdots x'_n)'; \quad u:=(u_1, \ldots, u_{n-1})'; \tag{6}$$

the dynamics for the set of boilers, under the restriction implied by Eq. (4), becomes a $2n$-dimensional linear-affine system with an $(n-1)$-dimensional control variable, namely

$$\dot{x} = \hat{A}x + \hat{B}u + \varphi. \tag{7}$$

with coefficient matrices

$$\hat{A}:=\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \cdots & A_{n-1} & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}, \quad \hat{B}:=\begin{pmatrix} B_1 & \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \cdots & B_{n-1} \\ -B_n & \cdots & -B_n \end{pmatrix}, \quad \varphi:=\alpha\begin{pmatrix} 0 \\ \vdots \\ 0 \\ B_n \end{pmatrix}, \tag{8}$$

Assuming that each model in Eq. (4) is in canonical form (controllable and observable), it follows from the form of matrix $\hat{A}$ that the global system will be uniformly bounded-input bounded-output (UBIBO) if and only if the same is true for each individual subsystem (utility unit). Therefore, under these assumptions, since the UBIBO property is equivalent to internal stability for each boiler, the global system will be externally stable when each identified matrix A_i is stable (its eigenvalues have negative real parts). It may then be concluded then that the stability properties of the $2n$-dimensional system will be the same as those of each one of the controlled units.

2.2. Optimal allocation as a LQR problem

An optimal control problem for the utility group described in the previous section will be posed here, aiming to minimize a quadratic objective cost functional of the form

$$\mathcal{J}(u) = \int_0^T [(x(t) - \bar{x})'Q(x(t) - \bar{x}) + u'(t)Ru(t)]dt + (x(T) - \bar{x})'S(x(T) - \bar{x}), \tag{9}$$

subject to the affine system of Eqs. (7)–(8). The reference value

$$\bar{x}:=(\bar{x}'_1 \vdots \bar{x}'_2 \vdots \cdots \vdots \bar{x}'_n)' \tag{10}$$

is a $2n$-dimensional design parameter vector, whose components may be interpreted as the average/standard desired values of production and expense, respectively, for units $i = 1, 2, \ldots, n$.

As usual, Q and S will be nonnegative $2n \times 2n$ symmetric matrices, and R will be $(n-1) \times (n-1)$ and positive definite. The coefficients of Q, R, S are usually decided according to the characteristics of each application. It seems most probable that Q and S be constructed from 2×2 symmetric submatrices displayed in diagonal, one such a submatrix related to each utility unit. In the same fashion, $R = diag(r_1, \ldots, r_{n-1})$, with all $r_i > 0$, would be a practical choice for weighting the control energy effort associated to the problem. Notice that, in order to take into account that there exist n units and only $n-1$ available coefficients r_i, a possible compromise would be to adopt

$$r_i = \tilde{r}_i + \frac{\tilde{r}_n}{n-1}, \quad i = 1, \ldots, n-1, \tag{11}$$

where $\tilde{r}_i$ is the weight of the control corresponding to the demand assigned to the individual unit i, for $i = 1, \ldots n$. Formula (11) would then satisfy

$$E_u := \sum_{i=1}^{n-1} r_i u_i^2 = \sum_{i=1}^{n} \tilde{r}_i u_i^2 := \tilde{E}_u \tag{12}$$

when $u_i \equiv \alpha/n$ (a typical choice for $\bar{x}_{i1}$), $i = 1, \ldots n$, and E_u would be a good approximation of $\tilde{E}_u$ for general values u_i under the following restrictions: (i) $u_i \in [0, \alpha]$, and (ii) $u_1 + \cdots + u_n = \alpha$, as it is the case here. As expected, this choice of r_i always verifies: $\lim_{n\to\infty} r_i = \tilde{r}_i$, for $i = 1, \ldots n$.

An interesting question concerning the optimal control design is how to choose the parameters Q, R and S involved in the cost function. The choosing of these matrices must lead to 'acceptable' levels of $x(t)$, $u(t)$, and $x(T)$. A classical approach (Bryson and Ho, 1975) initializes Q_{ii}, R_{ii} and S_{ii} tentatively, and then modify these values by trial and error, to reach a compromise among response time, damping and control effort. In more recent literature, there are several papers covering this subject, for instance (Das et al., 2013; Robinson, 1990). Here, simple diagonal weights were adopted for Q and S, so that they indicate how much each state and input deviation contribute to the overall cost, i.e.

$$Q = diag(q_1, q_2, \ldots, q_1, q_2), \quad S = diag(s_1, s_2, \ldots, s_1, s_2) \tag{13}$$

The following index

$$\eta = \frac{\sum_{i=1}^{n} (x_{i1}(T) - x_{i1}(0))}{\int_0^T \sum_{i=1}^{n} x_{i2}(t)dt} \tag{14}$$

will be used to tune the parameters Q, S, and R in order for η to be maximized. This index can be seen as a measure of the efficiency during a period T, since it relates the net production of the production variable and the total consumption of the combustible variable.

To treat the problem under a more convenient theoretical setup, the affine-linear system (7) describing the dynamics of the group will be transformed into a linear one by introducing the following variable vector z

$$z(t):=\int_0^t e^{\widehat{A}(t-\tau)}\varphi(\tau)d\tau, \quad (15)$$

which satisfies the initial-value problem

$$\dot{z}(t)=\widehat{A}z(t)+\varphi(t), \quad z(0)=0, \quad (16)$$

After a change of variables $x(t)\to x(t)-z(t)$, and $r(t)\to\overline{x}-z(t)$, the control dynamics becomes, for the 'new' state variable x

$$\dot{x}=\widehat{A}x+\widehat{B}u, \quad x(0)=x_0, \quad (17)$$

and the cost objective (9) can be written in the form:

$$\mathcal{J}(u)=\int_0^T[(x-r)'Q(x-r))+u'Ru]dt + (x(T)-r(T))'S(x(T)-r(T)), \quad (18)$$

Now, the problem defined by Eqs. (17), (18) is equivalent to the original one embodied in Eqs. (7), (9). The optimal tracking problem aims to drive the new state x, now governed by a linear dynamics, towards the newly defined reference trajectory $r(t)$. The performance of the control will be assessed by a quadratic cost with the same coefficients than in the original one.

The linear-quadratic tracking problem retains some features of the classical LQR problem (Costanza and Rivadeneira, 2014; Sontag, 1998). For instance, its Hamiltonian H

$$H(x,\lambda,u):=L+\lambda'f=(x-r)'Q(x-r)+u'Ru+\lambda'(\widehat{A}x+\widehat{B}u) \quad (19)$$

which is minimized by the same expression as in the LQR case, namely:

$$u^0(x,\lambda)=-\frac{1}{2}R^{-1}\widehat{B}'\lambda; \quad (20)$$

and the u-minimal Hamiltonian H^0 results

$$H^0(x,\lambda):=H(x,\lambda,u^0(x,\lambda))=(x-r)'Q(x-r)-\frac{1}{4}\lambda'\widehat{W}\lambda+\lambda'\widehat{A}x, \quad (21)$$

where the usual notation $\widehat{W}:=\widehat{B}R^{-1}\widehat{B}'$ has been introduced.

2.2.1. The optimal tracking solution

The appearance of $(x-r)$ instead of just x (regulator problem) proposes a complete quadratic dependence for the value function V

$$V(t,x):=x'P(t)x+2\xi'(t)x+\sigma(t), \quad (22)$$

with time-varying coefficients (P,ξ,σ); a (in principle symmetric) $2n\times 2n$ matrix P, an n-dimensional column vector ξ, and a scalar factor σ.

The Hamilton-Jacobi-Bellman (HJB)

$$\frac{\partial V}{\partial t}(t,x)=-H^0\left(x,\left(\frac{\partial V}{\partial x}\right)'(t,x)\right) \quad (23)$$

and its final condition

$$V(T,x)=(x(T)-r(T))'S(x(T)-r(T)) \quad (24)$$

must be satisfied by the proposed value function and its partial derivatives

$$\frac{\partial V}{\partial t}(t,x)=x'\dot{P}(t)x+2x'\dot{\xi}(t)+\dot{\sigma}(t), \quad (25)$$

$$\left(\frac{\partial V}{\partial x}\right)'(x(t))=2[P(t)x+\xi(t)], \quad (26)$$

which requires that for all admissible (t,x)

$$x'\dot{P}(t)x+2x'\dot{\xi}(t)+\dot{\sigma}(t)=-(x-r)'Q(x-r) + [P(t)x+\xi(t)]'\widehat{W}[P(t)x+\xi(t)] - 2[P(t)x+\xi(t)]'\widehat{A}x, \quad (27)$$

and at the end of the time-horizon

$$x'Sx-2Sr(T)+r(T)'Sr(T)=x'P(T)x+2\xi'(T)x+\sigma(T). \quad (28)$$

Since these equalities involve second order polynomials in x, their coefficients must also be equal, resulting in the following system of ODE's:

$$\dot{P}=P\widehat{W}P-\widehat{A}'P-P\widehat{A}-Q, \quad P(T)=S, \quad (29)$$

$$\dot{\xi}=-(\widehat{A}-\widehat{W}P)'\xi+Qr, \quad \xi(T)=-Sr(T), \quad (30)$$

$$\dot{\sigma}=\xi'\widehat{W}\xi-r'Qr, \quad \sigma(T)=r(T)'Sr(T), \quad (31)$$

Their solutions allow to express the optimal control u^* in the form

$$u^*(t)=u^0\left(x^*(t),\left(\frac{\partial V}{\partial x}\right)'(t,x^*(t))\right)=-R^{-1}\widehat{B}'[P(t)x^*(t)+\xi(t)], \quad (32)$$

where x^* denotes the optimal state, and also to calculate the optimal cost from

$$V(0,x_0)=x'_0P(0)x_0+2x'_0\xi(0)+\sigma(0). \quad (33)$$

Eq. (29) coincides with the Riccati Differential Equation for the LQR problem with coefficients $(\widehat{A},\widehat{B},Q,R,S)$, and results uncoupled from the remaining two ODEs. It should be noted that, even though the problem at hand does not possess the LQR structure, still the Eq. (32) can be interpreted as a linear-affine feedback law u_f, precisely

$$u_f(t,x):=-R^{-1}\widehat{B}'[P(t)x+\xi(t)], \quad (34)$$

and from Bellman's Principle it could be asserted that, notwithstanding that at some time t the actual state x may differ from the expected optimal state $x^*(t)$, yet the optimal control at that time, denoted u_x^*, can be computed as $u_x^*(t)=u_f(t,x)$. This property makes the previous results robust against sporadic state errors.

It is well known that, for infinite-horizon, constant-target, constant-coefficients LQR problems, the optimal feedback will stabilize the system provided that the pair (A,B) is controllable (which has already been assumed), and the state-penalty matrix Q, is positive definite (which will be assumed in what follows) as it is stated in Sontag (1998), Theorem 41. Therefore, the resulting strategies for such problems are asymptotically stable. Now, in our finite-horizon context, we can also guarantee that the state trajectories will be bounded during each optimization period of duration T, due to the compactness of $[0,T]$ and the continuity (actually differentiability) of the solutions to the closed-loop dynamics.

2.2.2. Handling changes in the total demand α

Eqs. (29), (30) have final (instead of initial) conditions and therefore can not be numerically integrated online with the process. They need to be solved offline and stored in the memory of the controller. This is an inconvenience common to the LQR, servo, and tracking problems, for which the feed-through terms and similar objects must be updated for the whole time-horizon in case the reference signal is modified. Fortunately, in the present case the calculation of the time-varying coefficient $\xi(\cdot)$ of the feedback law is required to be computed only once, namely for a unitary total demand ($\alpha=1$), and the same thing applies to the cost coefficient $\sigma(\cdot)$. These assertions are conveyed in precise terms by the next two equations:

$$u^*(t)=-R^{-1}\widehat{B}'[P(t)x(t)+\alpha\tilde{\xi}(t)], \quad (35)$$

$$J^*=V(0,x_0)=x'_0P(0)x_0+2\alpha x'_0\tilde{\xi}(0)+\alpha^2\tilde{\sigma}(0), \quad (36)$$

where $\tilde{\xi},\tilde{\sigma}$ denote the coefficients calculated for $\alpha=1$, or equivalently for $\tilde{z}(t):=z(t)/\alpha$, $\tilde{r}(t):=r(t)/\alpha$, and $\tilde{x}:=\overline{x}/\alpha=(\check{x}_1,0,\check{x}_2,0,\ldots,\check{x}_n,0)'$, $\sum_{i=1}^n\check{x}_i=1$. Eqs. (35), (36) can be justified as follows:

(i) the solution to Eq. (30) with final condition $\xi(T)=-Sr(T)=-\alpha S\tilde{r}$ is

$$\xi(t)=\alpha\Psi(t,T)\left\{-S\tilde{r}(T)+\int_T^t\Psi(t,\tau)Q\tilde{r}(\tau)d\tau\right\}=\alpha\tilde{\xi}(t), \quad (37)$$

where $\Psi(t, T)$ is the fundamental matrix associated with the linear (time-varying) system $\dot{\psi} = -(\widehat{A} - \widehat{W}P(t))'\psi$;

(ii) and similarly, for Eq. (31),

$$\sigma(t) = \alpha^2\left\{\tilde{r}(T)'S\tilde{r}(T) + \int_T^t [\tilde{\xi}'(\tau)\widehat{W}\tilde{\xi}(\tau) - \tilde{r}'Q\tilde{r}]d\tau\right\} = \alpha^2\tilde{\sigma}(t). \quad (38)$$

Now it should be decided how to handle changes in the total demand when they occur in some interior point t of a period $[t_0, t_0 + T]$. Let us assume that, in such a case, the optimization of the system is desired to be continued, at least for another interval of duration T. Some 'receding-horizon' decision has to be made (see Costanza and Rivadeneira, 2015) for practical implementation of this result).

2.3. Optimal filtering

Disturbances in the parameters of each device, and in the measurement and transmission of signals, are known to occur during real process operation. The common set up for these influences over the deterministic models assumed in previous sections is the following

$$\dot{x}(t) = \widehat{A}x(t) + \widehat{B}\tilde{u}(t) + r_1, \quad (39)$$

$$y = \widehat{C}x + r_2, \quad (40)$$

where the notation $\dot{x}$ should be understood as the differential of a Brownian process associated with the state of the utility group, resulting from the existence of zero-mean white noise r_1 fluctuations on the environment conditions; $\tilde{u}$ denoting the input variable $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_{n-1})$, $\tilde{u}_n := \alpha - \sum_{i=1}^{n-1}\tilde{u}_i$; and where r_2 are the zero-mean white noises in the measurements of the outputs y, which in this case are conceptually the same thing as the states, i.e. it is assumed that each subsystem i is observable (which is clearly true in most modern plants). Notice that y is the output vector of the whole group.

In this context, the least-squares optimal filtering problem for the unit group is known to be solvable (Fleming and Rishel, 1975) through the following pair of equations

$$\dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}\tilde{u} + \widehat{L}(y - \widehat{C}\hat{x}(t)), \quad \hat{x}(0) = \mathbb{E}[x_0] \quad (41)$$

$$\dot{\Pi} = \widehat{A}\Pi + \Pi\widehat{A}' + \widehat{Q} - \Pi\widehat{D}\Pi; \quad \Pi(0) = Cov(x_0), \quad (42)$$

$$\widehat{D} := \widehat{C}'\widehat{R}^{-1}\widehat{C}, \quad \widehat{L} := \Pi\widehat{C}'\widehat{R}^{-1}, \quad (43)$$

where $\hat{x}$ is the best estimation of the state x; $\mathbb{E}[r_1] = 0$, $\mathbb{E}[r_1(t).\, r'_1(\tau)] = \widehat{Q}.\, \delta(t-\tau)$, with $\widehat{Q}$ the r_1 covariance matrix, and $\mathbb{E}[r_2] = 0$, $\mathbb{E}[r_2(t).\, r'_2(\tau)] = \widehat{R}.\, \delta(t-\tau)$, with $\widehat{R}$ the r_2 covariance (invertible) matrix; $\widehat{L}$ denotes the 'gain' of the filter, which works analogously to an observer; x_0 is now a random vector with mean $\mathbb{E}[x_0] = \hat{x}(0)$ and initial covariance $\mathbb{E}[(x_0 - \hat{x}(0))(x_0 - \hat{x}(0))'] = \Pi(0)$, and finally Π is the dynamical covariance, solution to the Riccati-type ODE (29). The matrices $\widehat{C}$, $\widehat{Q}$, $\widehat{R}$, $\widehat{L}$ and Π are $2n \times 2n$; and can be conveniently partitioned in the form

$$G = \begin{pmatrix} \widetilde{G}_{11} & \cdots & \widetilde{G}_{1n} \\ \vdots & \ddots & \vdots \\ \widetilde{G}_{n1} & \cdots & \widetilde{G}_{nn} \end{pmatrix}, \quad (44)$$

with $\widetilde{G}_{ij}$ a 2×2 block. Let us adopt, for simplicity, the notation $\widetilde{G}_{ii} := \widetilde{G}_i$ $i = 1, \ldots, n$.

The following features are naturally assumed for these matrices:

i) The observation matrix $\widehat{C}$ is

$$\widehat{C} := diag(\widetilde{C}_1, \ldots, \widetilde{C}_n) \quad (45)$$

and $\widetilde{C}_i = I_2$, $i = 1, \ldots, n$.

ii) The noises affecting output and state in each unit are independent from those affecting the other units, then the covariance of noises between units is zero (each unit works in parallel). This is expressed through the choice

$$\widehat{Q} := diag(\widetilde{Q}_1, \ldots, \widetilde{Q}_n), \quad \widehat{R} := diag(\widetilde{R}_1, \ldots, \widetilde{R}_n), \quad (46)$$

the matrices $\widehat{Q}_i$, and $\widehat{R}_i$ are the covariance matrix of r_{1i} and r_{2i} in each unit, respectively. Besides, as $\widehat{R}$ is now a diagonal matrix its inverse is $\widehat{R}^{-1} = diag(\widetilde{R}_1^{-1}, \ldots, \widetilde{R}_n^{-1})$.

iii) The fluctuations in the initial condition depends only on the enviroment of each unit, then the initial covariance of the estimation is:

$$\Pi(0) := diag(\widetilde{\Pi}_1(0), \ldots, \widetilde{\Pi}_n(0)) = diag(Cov(x_{0_1}), \ldots, Cov(x_{0_n})). \quad (47)$$

Notice that Eq. (42), in partitioned form, results

$$\dot{\widetilde{\Pi}}_{ij} = \widetilde{A}_{ik}\widetilde{\Pi}_{kj} + \widetilde{\Pi}_{ik}\widetilde{A}'_{kj} - \widetilde{\Pi}_{ik}\widetilde{D}_{kl}\widetilde{\Pi}_{lj} + \widetilde{Q}_{ij}, \quad \Pi_{ij}(0) = Cov(x_{0_i})\delta_{ij}, \quad (48)$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

and $i, j, k, l = 1, \ldots, n$.

The nondiagonal terms of Π are irrelevant, as shown next. Consider the case $i \neq j$, and recall that $\widehat{A}$, $\widehat{D}$, and $\widehat{Q}$ are 'diagonal' matrices (in the sense of Eq. (44)), then Eq. (48) becomes

$$\dot{\widetilde{\Pi}}_{ij} = \widetilde{A}_i\widetilde{\Pi}_{ij} + \widetilde{\Pi}_{ij}\widetilde{A}'_j - \widetilde{\Pi}_{ik}\widetilde{D}_k\widetilde{\Pi}_{kj}, \quad \widetilde{\Pi}_{ij}(0) = 0. \quad (49)$$

Let us show that the functions $\widetilde{\Pi}_{ij}(t) \equiv 0$ satisfy Eq. (49), which will allow us to discard the non-diagonal sub-equations of Eq. (48). Notice that the terms $\widetilde{A}_i\widetilde{\Pi}_{ij}$, and $\widetilde{\Pi}_{ij}\widetilde{A}_j$ vanish when applying this solution. The remaining term $Z_{ij} = \widetilde{\Pi}_{ik}\widetilde{D}_k\widetilde{\Pi}_{kj}$ is also zero, because making $k = i$, $\widetilde{\Pi}_i\widetilde{D}_i\widetilde{\Pi}_{ij} = 0$ due to the last $\widetilde{\Pi}_{ij}$, and for k=j, also $\widetilde{\Pi}_{ij}\widetilde{D}_j\widetilde{\Pi}_j = 0$ due to the first $\widetilde{\Pi}_{ij}$. Besides for $k \neq i, j$ the result is obvious. Therefore, for $i \neq j$, the conclusion is that $Z_{ij} = 0$.

Now, consider the case i=j. Eq. (48) reads

$$\dot{\widetilde{\Pi}}_i = \widetilde{A}_i\widetilde{\Pi}_i + \widetilde{\Pi}_i\widetilde{A}'_i + \widetilde{Q}_i - \widetilde{\Pi}_i\widetilde{D}_i\widetilde{\Pi}_i, \quad \widetilde{\Pi}_i(0) = Cov(x_{0_i}), \quad (50)$$

which depends only on i, therefore only the objects of the unit i are involved. As a consequence, $\Pi(t) = diag(\widetilde{\Pi}_1(t), \ldots, \widetilde{\Pi}_n(t))$ is the solution of Eq. (42) for the whole group.

Since the filter gain can be written as $\widehat{L} = diag(L_1, \ldots, L_n)$, where $L_i = \Pi_i(t)C'_i\widehat{R}_i^{-1}$ for $i = 1, \ldots, n$, then each state vector $\hat{x}_i(t)$ in Eq. (41) verifies

$$\dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + B_i\tilde{u}_i(t) + L_i(y_i(t) - C_i\hat{x}_i(t)); \quad \hat{x}_i(0) = \mathbb{E}[x_{0_i}], \quad (51)$$

independently from the other units.

The practical consequence of this set-up is that the filtering and estimation problems can be solved for each unit of the group, so avoiding the integration of $2n \times 2n$ coupled differential equations.

2.4. Stochastic control

In this section it is shown that the control law derived in the deterministic context is also 'optimal' under random perturbations in measurement devices and model parameters. This optimality of the control is understood in the stochastic sense, i.e. by taking into account that the random perturbations force the model, specially the state $x(\cdot)$, to be considered as a stochastic process.

The stochastic optimal control problem attempts to minimize the functional

$$\mathcal{J}_{sto}(u) = \mathbb{E}\left[\int_0^T \{(x_u - r)'Q(x_u - r) + u'Ru\}dt + (x_u(T) - r(T))'S(x_u(T) - r(T))\right] \quad (52)$$

with respect to the deterministic control $u(\cdot)$, where $\mathbb{E}$ denotes the usual expected-value, $r(t)$ is a given deterministic reference trajectory, and subject to the dynamical constraint

$$\dot{x} = \hat{A}x + \hat{B}u + r_1, \quad x(0) = x_0, \tag{53}$$

where the input variable $u = (u_1, \ldots, u_{n-1})$, $u_n := \alpha - \sum_{i=1}^{n-1} u_i$, and x_0 is a random vector with mean $\mathbb{E}[x_0] = \hat{x}(0)$ and initial covariance $\mathbb{E}[(x_0 - \hat{x}(0))(x_0 - \hat{x}(0))'] = \Lambda_0 = \Pi(0)$. The variables $\hat{x}$, and Π are the solution to the filtering and estimation problems discussed in the previous subsection.

For the deterministic case (in Section 2.2), it was shown that the optimal control is

$$u^* = -R^{-1}B'[P(t)x(t) + \xi(t)], \tag{54}$$

where P, ξ are the optimal deterministic coefficients described by Eqs. (29) and (30). By analogy to the regulation case, the separation hypothesis proposes that the optimal control for the stochastic set-up would be

$$\hat{u}^* := -R^{-1}\hat{B}'[P(t)\hat{x}(t) + \xi(t)]. \tag{55}$$

Next, the hypothesis will be proved, and furthermore it will be shown that the optimal cost is

$$\mathcal{J}^*_{sto}(\hat{u}^*) = \mathcal{J}^*_{\det}(x_0) + \int_0^T Tr(PQ)dt + Tr(P(0)\Lambda_0) + \int_0^T Tr(P\widehat{W}P\Pi)dt. \tag{56}$$

where $\mathcal{J}^*_{\det}(x_0) = V(0, x_0)$ (see Eq. (33)). The proof resorts to the following change of variables

$$u := \hat{u} + \tilde{u}, \tag{57}$$

and re-expresses the dynamics and the cost function. The dynamics results

$$\dot{x} = (A - \widehat{W}P)x + \widehat{W}P(x - \hat{x}) - \widehat{W}\xi + B\tilde{u} + r_1. \tag{58}$$

If $\widetilde{A} := (A - \widehat{W}P)$, and $\tilde{x} := (x - \hat{x})$, then Eq. (58) becomes

$$\dot{x} = \widetilde{A}x + \widehat{W}P\tilde{x} - \widehat{W}\xi + B\tilde{u} + r_1.$$

From Eq. (29) and the latter transformations, the next equalities are obtained:

$$Q + P\widehat{W}P = -\dot{P} - P\widetilde{A} - \widetilde{A}'P, \tag{59}$$

$$x'Qx = -x'(\dot{P} + P\widetilde{A} + \widetilde{A}'P)x - x'(P\widehat{W}P)x. \tag{60}$$

The following calculations involving the terms of the cost function will be needed:

$$(\hat{u} + \tilde{u})'R(\hat{u} + \tilde{u}) = \hat{u}'R\hat{u} + 2\hat{u}'R\tilde{u} + \tilde{u}'R\tilde{u}, \tag{61}$$

$$\hat{u}'R\hat{u} = \hat{x}'P\widehat{W}P\hat{x} + 2\hat{x}'P\widehat{W}\xi + \xi'\widehat{W}\xi, \tag{62}$$

$$\hat{x}'P\widehat{W}P\hat{x} = x'P\widehat{W}Px + \tilde{x}'P\widehat{W}P\tilde{x} - 2x'P\widehat{W}P\tilde{x}, \tag{63}$$

$$2\hat{x}'P\widehat{W}\xi = 2x'P\widehat{W}\xi - 2\tilde{x}'P\widehat{W}\xi, \tag{64}$$

$$2\hat{u}'R\tilde{u} = -2x'PB\tilde{u} - 2\tilde{x}'PB\tilde{u} - 2\xi'B\tilde{u}. \tag{65}$$

From Eq. (30), the expression $\dot{\xi} + \widetilde{A}'\xi = Qr$ is obtained, and then

$$(x - r)'Q(x - r) = x'Qx - 2x'\dot{\xi} - 2x'\widetilde{A}'\xi + r'Qr. \tag{66}$$

The cost function (52), after these algebraic manipulations, becomes $\mathcal{J}_{sto}(\tilde{u}) = \mathbb{E}[\mathcal{J}^1_{sto} + \mathcal{J}^2_{sto} + \mathcal{J}^3_{sto}]$, where

$$\mathcal{J}^1_{sto} = \int_0^T (-x'\dot{P}x - 2x'P\dot{x} + 2x'Pr_1 + \tilde{x}'P\widehat{W}P\tilde{x} - 2\tilde{x}'PB\tilde{u})dt \tag{67}$$

$$+x(T)'Sx(T), \quad \mathcal{J}^2_{sto} = -\int_0^T (2\tilde{x}'P\widehat{W}\xi + 2\xi'B\tilde{u} + 2\xi'\widetilde{A}x + 2x'\dot{\xi} - \xi'\widehat{W}\xi - r'\widehat{Q}r)dt, \tag{68}$$

$$\mathcal{J}^3_{sto} = \int_0^T \tilde{u}'R\tilde{u}dt + r(T)'Sr(T) - 2x(T)'Sr(T). \tag{69}$$

In $\mathcal{J}^1_{sto}$ by taking into account that the error $\tilde{x}$ is orthogonal to the measurements y, and to $\hat{x}$, it follows that (see (Oksendal, 2005)),

$$\mathbb{E}\left[\int_0^T \tilde{x}'PB\tilde{u}dt\right] = 0, \quad \mathbb{E}\left[\int_0^T x'Pr_1dt\right] = 0, \tag{70}$$

and by Ito's integral properties (Oksendal, 2005)

$$d(x'Px) = x'\dot{P}xdt + 2x'Pdx + Tr(P\widehat{Q})dt. \tag{71}$$

and

$$\mathbb{E}\left[\int_0^T \tilde{x}'P\widehat{W}P\tilde{x}dt\right] = \int_0^T Tr(P\widehat{W}P\Pi)dt. \tag{72}$$

Then the first term $\mathcal{J}^1_{sto}$ can be processed to obtain

$$\mathbb{E}[\mathcal{J}^1_{sto}] = \mathbb{E}\left[\int_0^T -\frac{d}{dt}(x'Px)dt + x(T)'Sx(T) + \int_0^T Tr(P\widehat{Q})dt + \int_0^T \tilde{x}'P\widehat{W}P\tilde{x}dt\right] = \mathbb{E}\left[x(0)'P(0)x(0) + \int_0^T Tr(P\widehat{Q})dt + \int_0^T \tilde{x}'P\widehat{W}P\tilde{x}dt\right] \tag{73}$$

$$= x(0)'P(0)x(0) + Tr(P(0)\Lambda_0) + \int_0^T Tr(P\widehat{Q})dt + \int_0^T Tr(P\widehat{W}P\Pi)dt.$$

Now, the term $\widetilde{A}x$ can be replaced by $\widetilde{A}x = \dot{x} - WP\tilde{x} - B\tilde{u} + W\xi - r_1$ in Eq. (68), and therefore the second term of the cost functional becomes

$$\mathcal{J}^2_{sto} = -\int_0^T (2\xi'(\dot{x} + W\xi - r_1) - \xi'W\xi + 2x'\dot{\xi} - r'\widehat{Q}r)dt \tag{74}$$

$$= -\int_0^T (2\xi'\dot{x} - 2\xi'r_1 + 2x'\dot{\xi} + \xi'W\xi - r'\widehat{Q}r)dt, \tag{75}$$

since from Eq. (31), and $\dot{\sigma} = \xi'W\xi - r'Qr$

$$\mathcal{J}^2_{sto} = -\int_0^T (2\xi'\dot{x} - 2\xi'r_1 + 2x'\dot{\xi} + \dot{\sigma})dt. \tag{76}$$

Now, from Ito's calculus (Oksendal, 2005)

$$\mathbb{E}\left[\int_0^T \xi'r_1dt\right] = 0, \tag{77}$$

$$d(2x'\xi) = 2x'\dot{\xi}dt + 2\xi'dx. \tag{78}$$

By introducing the latter implications and after adding $\mathcal{J}^2_{sto}$ to $\mathcal{J}^3_{sto}$, the following equation is obtained

$$\mathbb{E}[\mathcal{J}^2_{sto} + \mathcal{J}^3_{sto}] = \tag{79}$$

$$= \mathbb{E}\left[\int_0^T -\frac{d}{dt}(2x'\xi)dt - 2x(T)'Sr(T) - \int_0^T \frac{d}{dt}(\sigma)dt + r(T)'Sr(T) + \int_0^T \tilde{u}'R\tilde{u}dt\right] = 2x(0)'\xi(0) + \sigma(0) + \int_0^T \tilde{u}'R\tilde{u}dt. \tag{80}$$

Finally, if the last equation is added to $\mathcal{J}^1_{sto}$, the total cost function results

$$\mathcal{J}_{sto}(\tilde{u}) = \mathbb{E}\left[\int_0^T \tilde{u}'R\tilde{u}dt\right] + x(0)'P(0)x(0) + 2x(0)'\xi(0) + \sigma(0) + \int_0^T Tr(P\widehat{Q})dt + Tr(P(0)\Lambda_0) + \int_0^T Tr(P\widehat{W}P\Pi)dt, \tag{81}$$

which attains its minimal value if and only if $\tilde{u} = 0$. This proves the hypothesis and establishes the 'Separation Principle for the Stochastic LQ Tracking Problem'.

3. A case study: two boilers in parallel

Consider the following system of two boilers producing vapor in parallel:

$$A_1 = \begin{pmatrix} -0.112 & 0.05 \\ -0.2 & -0.09 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.2 & 0.05 \\ -0.2 & -0.155 \end{pmatrix}, \tag{82}$$

$$B_1 = \begin{pmatrix} 0.05 \\ 0.315 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.125 \\ 0.4 \end{pmatrix}, \quad C_1 = C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These realizations are canonical, the eigenvalues of A_1, A_2 are $-0.1013 \pm 0.0993i$ and $-0.1775 \pm 0.0974i$, respectively, having the first boiler a faster response than the second one. For each boiler, the stationary output y_∞ corresponding to a constant input $u(\cdot) \equiv k$ is $y_i^\infty = -A_i^{-1} B_i k$,

$$y_1^\infty = \begin{pmatrix} 1.00847 \\ 1.25896 \end{pmatrix} k, \quad y_2^\infty = \begin{pmatrix} 0.960366 \\ 1.34146 \end{pmatrix} k. \tag{83}$$

This shows that the slower (second) boiler will be subject to an offset (first component of y^∞) of around 4%, bigger than the offset of the first one. Therefore, the coefficients of each realization convey enough information about the performance expected from each PID-controlled boiler, for instance, a measure of the static efficiency of the unit can be calculated as

$$\eta_\infty = \frac{y_i^\infty(1)}{y_i^\infty(2)}. \tag{84}$$

3.1. Cost function and deterministic control

As it was announced in Section 2.2 the index η was maximized in order to find the best values of Q, R and S to be used in the cost function, and for simplicity Q=S. The optimization horizon is fixed at T=20. All numerical calculations were performed with standard MATLAB ODE integration tools. The maximum CPU time recorded was 7.5689 and was measured in an ASUS Machine with 2.5 GHz Intel Core i7-4710HQ, 8 GB RAM.

In Fig. 2 the index η defined by Eq. (14) is plotted. In the first figure the maximum is located at $q_1 = 3.6$ and $q_2 = 0.1$. In the second one, the maximun is reached for r=1.41.

These were used in a simulation run: firstly P, ξ, σ, were calculated, and then introduced in the dynamics to obtain the optimal control u^* shown in Fig. 2. The parameters for the simulation were

$$r = 1.41, \quad q_1 = s_1 = 3.6, \quad q_2 = s_2 = 0.1, \quad \tilde{x} = 0.5, \quad T = 20,$$
$$\alpha = 150.$$

During the early stages of the process, the resulting control puts a high demand on the second boiler. In between, the controls to both boilers evolve around $\alpha/2$ in a nontrivial pattern, and at the end the first boiler is preferred again. The resulting state evolutions are illustrated in Fig. 4. The optimal cost is $J^* = 201629$.

The optimal cost J^* was compared against the outcome of applying different constant set-points of magnitude $u_1 = k\alpha$, $u_2 = (1-k)\alpha$, and

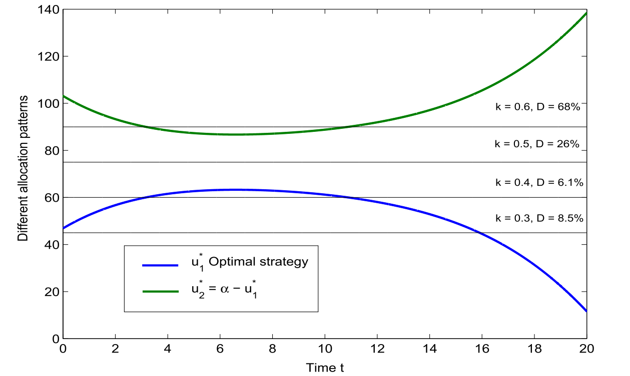


Fig. 3. Optimal control and different constant controls as in Table 1.

Table 1
Optimal cost compared against the outcome of applying a constant set-up of different magnitude, where k defines the magnitude of the control applied to the first boiler, and D_k shows the percentage of cost savings.

	k=0.3	k=0.35	k=0.4	k=0.45	k=0.5	k=0.6
D_k%	8.5	4.5	6.1	13.3	26.0	68.4

by calculating the corresponding costs J_k arising from Eq. (9) as it is illustrated in Fig. 3. Relative cost savings $D_k = 100(J^* - J_k)/J^*$ are reported in Table 1 .

3.2. Stochastic control simulations

Simulations for the two-boilers case, in presence of signal and environmental noise, are illustrated in Fig. 5. The measurement noise was simulated with zero-mean and the following covariances,

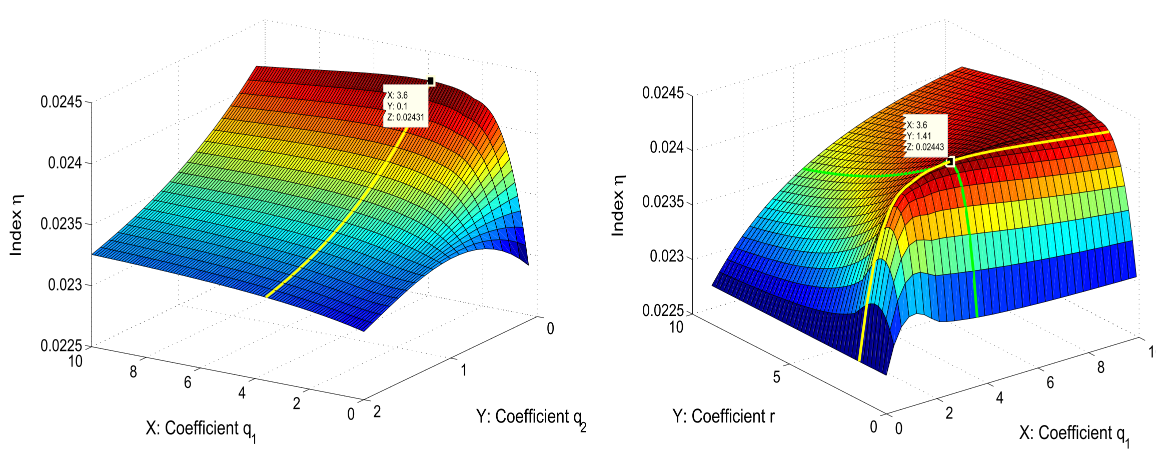
$$\hat{Q}_1 = \hat{Q}_2 = \begin{pmatrix} 1.34674 & 0.0318417 \\ 0.0318417 & 2.62 \end{pmatrix} \tag{85}$$

$$\hat{R}_1 = \hat{R}_2 = \begin{pmatrix} 2.76926 & 0.358896 \\ 0.358896 & 3.67155 \end{pmatrix}. \tag{86}$$

The meaning of $\hat{Q}_i$, $\hat{R}_i$, i=1,2 was explained in Section 2.3. Their numerical values were estimated from real data via standard least-squares algorithms.

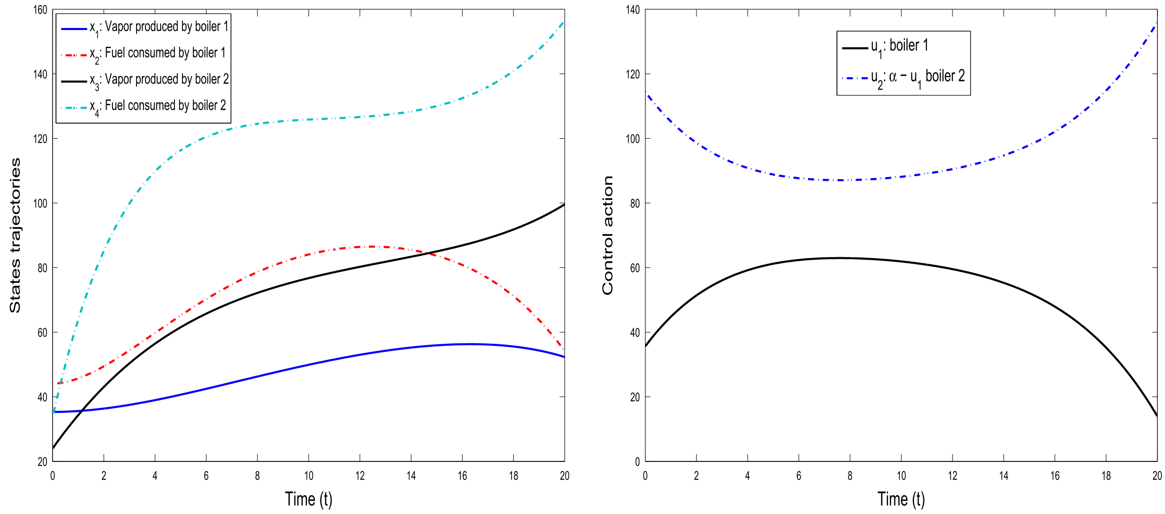
The deterministic coefficients used in the simulations were $q_1 = s_1 = 3.6$, $q_2 = s_2 = 0.1$, $R = 1.41$, $\tilde{x} = 0.5$, $T = 20$, $\alpha = 150$.

A numerical partial confirmation of the stochastic optimality of the feedback law in Eq. (55) was obtained by evaluating the costs



(a) η versus q_1 and q_2. (b) η versus q_1 and r.

Fig. 2. η surface for variations on q_1, q_2, and r. The maximum is attained for $q_1 = 3.6$, $q_2 = 0.1$, and r=1.41, (a) η versus q_1 and q_2. (b) η versus q_1 and r.



(a) State trajectories. (b) Control trajectories.

Fig. 4. Optimal trajectories resulting from the optimal control applied to two boilers in parallel. (a) State trajectories. (b) Control trajectories.

corresponding to appropriate combinations of the trajectories in Fig. 3; in precise terms,

$$J_1 := \int_0^T [(\hat{x}(t) - r(t))'Q(\hat{x}(t) - r(t)) + \hat{u}^{*'}(t)R\hat{u}^*(t)]dt + (\hat{x}(T) - r(T))'S(\hat{x}(T) - r(T)) \quad (87)$$

$$J_2 := \int_0^T [(\check{x}(t) - r(t))'Q(\check{x}(t) - r(t)) + \check{u}'(t)R\check{u}(t)]dt + (\check{x}(T) - r(T))'S(\check{x}(T) - r(T)) \quad (88)$$

where the meaning of the variables in J_1 are clear, and in J_2:

$$\check{u}(t) := -R^{-1}\hat{B}'[P(t)x(t) + \alpha\tilde{\xi}(t)], \quad (89)$$

where $x(\cdot)$ was a numerical zero-mean perturbation of the optimal $\hat{x}(\cdot)$, and $\check{x}(\cdot)$ denotes the numerical (deterministic) solution of the state Eq. (39) for inputs $\check{u}_i(\cdot)$, $r_{1i} \equiv 0$, $i = 1, \ldots, n$. The resulting values were:

$$\mathcal{J}_{sto}(\hat{u}^*) \cong J_1 = 209706 < 210487 = J_2 \cong \mathcal{J}_{sto}(\check{u}). \quad (90)$$

3.3. Additional relevant confirmations

3.3.1. Assessing optimality

A numerical experiment simulating the effect of perturbations over the optimal time-varying affine feedback law in Eq. (34) was assessed, through the convex combination of a family of variations covering both the optimal control and the nominal constant set-point $u(\cdot) \equiv \bar{x} = \alpha/n$

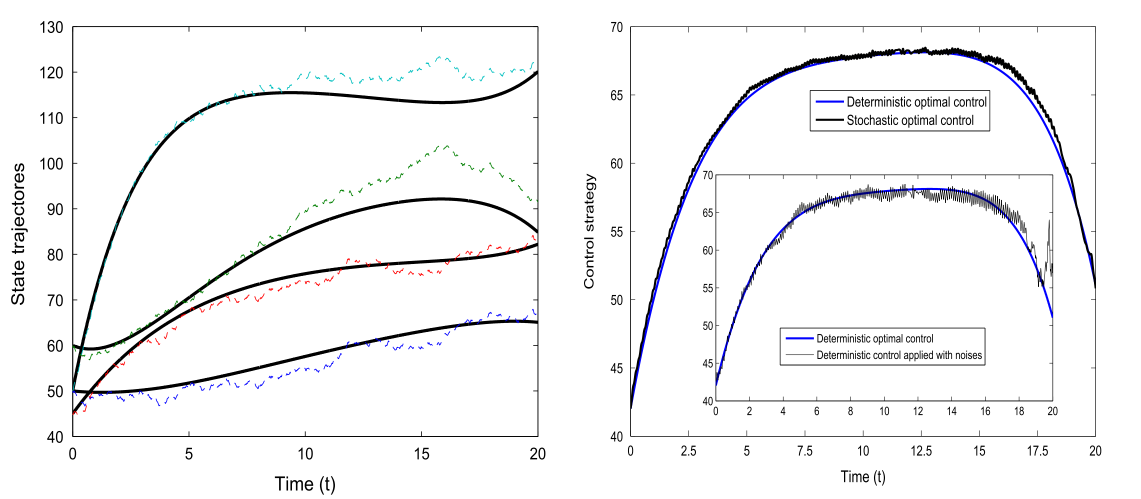
$$u = \beta u^* + (1 - \beta)\bar{x}, \quad (91)$$

where, for the case of two boilers, $u = u_1$, $u^* = u_1^*$, $u_2 = \alpha - u_1$, $\alpha = 150$, $\bar{x} = 75$, $\beta \in [-0.5, 2.5]$ and $Q = R = S = 0.5I$.

Some of the control variations are plotted in Fig. 6, and their corresponding cost values J_β compared against the optimal cost $J^* = J_1(u^*)$. Results validating the optimality of u^* are depicted in Fig. 6.

3.3.2. Dynamic efficiency

In engineering practice the 'efficiency' η^∞ (see Eq. (84)) of a boiler in stationary service roughly measures the ratio between the heat conveyed by the generated vapor versus the heat associated to the fuel



(a) State trajectories, stochastic with solid lines, and noisy trajectories in dash. (b) Controls for the deterministic and stochastic problems with same coefficients.

Fig. 5. Optimal stochastic trajectories in solid lines. Noisy trajectories in dashed lines, (a) State trajectories, stochastic with solid lines, and noisy trajectories in dash. (b) Controls for the deterministic and stochastic problems with same coefficients.

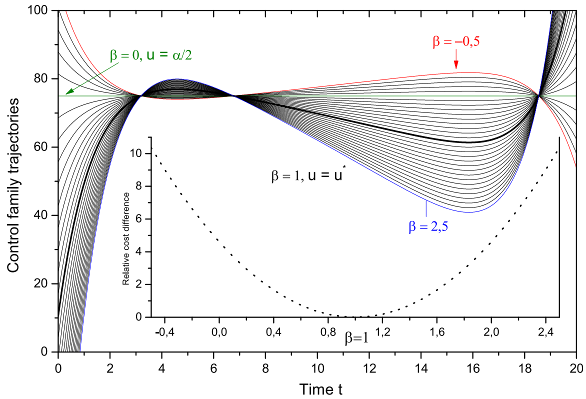


Fig. 6. Main frame: Variations around the theoretical optimal control. Small frame: Relative differences $(J_\beta - J^*)/J^*$ between the cost value generated by each control variation J_β and the optimal cost J^*. The parabolic behavior of the cost under linear combinations of controls reflects the linear-quadratic nature of the problem.

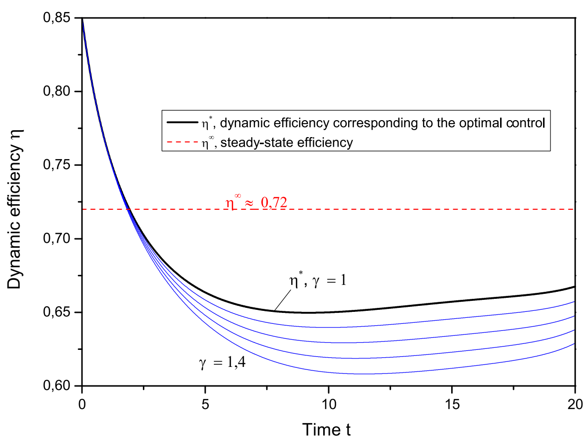


Fig. 7. Dynamic efficiencies corresponding to several members of the family of controls $\{u = \gamma u^*, 1 \le \gamma \le 1.4\}$ but maintaining a total demand $\alpha = 150$ for the two boilers.

supply. In the present context, a dynamic (transient) version of such a concept is represented by the following time-varying ratio, for each boiler $i = 1\ldots n$:

$$\eta_i(t) := \frac{x_{2i-1}(t)}{x_{2i}(t)} \simeq \frac{\text{Vapor produced at time } t}{\text{Fuel supply at time } t}, \tag{92}$$

where the variables are properly made dimensionless. For the group of boilers, the global dynamic efficiency may then be assessed from

$$\eta(t) := \frac{\sum_{i=1}^{n} x_{2i-1}(t)}{\sum_{i=1}^{n} x_{2i}(t)}. \tag{93}$$

If attempts are made to generate more vapor than the optimal allocation, then, correspondingly, more fuel should be supplied. This leads to smaller dynamic efficiencies, as reflected in Fig. 7, showing that LQR-optimal results are also more 'efficient' when adapting the classical stationary definition to the transient analysis pursued in this paper.

4. Conclusions

An optimal control strategy for dynamically changing the set-points assigned to a group of service equipment working n units in parallel, was proved and illustrated.

The resulting optimal feedback law minimizes the sum of two competing cost objectives: the departure of the production from the target, and the consumption of fuel (or of the main expense) during the optimization period. This provides online time-varying allocation of demands to each one of the units in multilayer controlled operation. The new manipulated variable has $n - 1$ degrees of freedom, since all individual targets must sum up to the total demand signal coming from the supervisory control of the plant. This partial deficit in the degrees of freedom generates an affine-linear structure for the dynamics of the problem when posed for the whole group. As the next step, through a suitable change of variables, the problem is transformed into a linear tracking problem.

The combination of quadratic individual costs for each of the units results in a quadratic total cost for the group. At the end, the treatment for the linear enlarged dynamics and quadratic cost differs little from the usual LQR setup, resembling now the equations associated with a tracking problem. To make results meaningful under the conventional viewpoint, a heuristic method to choose the cost weights Q, R and S that maximizes a dynamic version of the classical efficiency criterion is introduced.

The solution for the deterministic case is found in terms of the solution of the Riccati equation plus a feed-through time-varying vector that can be stored in memory once and for all, the changes in total demand being simply handled by introducing the new value as a factor in the feedback law. This feedback form of the control implies robustness with respect to state perturbations. Control parameters are calculated offline and do not need to be recalculated after changes on the total demand.

The choice of linear models for each unit also allows to refine the deterministic result into a stochastic one, coping with general disturbances like changes in fuel composition and noisy measurements. This is possible by a rigorous application of the Tracking Separation Principle, show here, and after the addition of a (least-squares optimal) Kalman filter.

In some simulations, cost savings have shown to be significant with respect to the expenditures generated by classical piecewise-constant strategies, above all when the optimal control indicates a departure from the conventionally adopted equipartition of total demand. The dynamic efficiency $\eta^*(t)$ corresponding to the optimal control is always better than that corresponding to strategies responding to excess vapor targets $(u(t) > u^*(t))$. In conclusion, the results presented here allow to improve the dynamic behavior and the economic performance of groups of service units operating in parallel.

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