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ORIGINAL ARTICLE

Kirillov structures and reduction of Hamiltonian systems by scaling and standard symmetries

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Abstract
In this paper, we discuss the reduction of symplectic Hamiltonian systems by scaling and standard symmetries which commute. We prove that such a reduction process produces a so-called Kirillov Hamiltonian system. Moreover, we show that if we reduce first by the scaling symmetries and then by the standard ones or in the opposite order, we obtain equivalent Kirillov Hamiltonian systems. In the particular case when the configuration space of the symplectic Hamiltonian system is a Lie group G, which coincides with the symmetry group, the reduced structure is an interesting Kirillov version of the Lie–Poisson structure on the dual space of the Lie algebra of G. We also discuss a reconstruction process for symplectic Hamiltonian systems which admit a scaling symmetry. All the previous results are illustrated in detail with some interesting examples.

KEYWORDS
contact structures, Hamiltonian systems, Kirillov structures, reconstruction process, reduction process, scaling symmetry, standard symmetry

1 | INTRODUCTION

1.1 | Physical motivation

The analysis of symmetries is one of the most important tools in theoretical physics. Usually, the formulation of a physical theory is given in terms of a variational principle and its associated symplectic Hamiltonian description. In this context, one typically looks for "standard symmetries," that is, symmetries which preserve the symplectic form and the Hamiltonian function. Among other things, this approach leads to a generalization of the Noether's theorem and the Marsden–Weinstein theory of reduction of the system by the action of a symmetry group (see the classical books and monographs by Marsden and collaborators,[1,2] Libermann and Marle,[3] or Olver[4]).

Recently, there has been a growing interest in the physical literature in considering "nonstandard symmetries," that is, symmetries of the physical system that do not necessarily preserve the symplectic structure. This is motivated mainly by the so-called scaling symmetries and by a well-known philosophical argument according to which any minimal description of the universe should avoid introducing a global scale into the picture, that is, it should be scale-invariant.[5,6] In this context, the theory of "shape dynamics" aims to rephrase our best description of the universe (general relativity) in a completely scale-invariant fashion.[7,8] This has led already to remarkable results that defy the way we understand the (classical) dynamics of the universe. For instance, the scale-reduced cosmological and black hole systems can be continued in some cases through the corresponding singularities.[9–11] Moreover, it has been further argued that the apparent dissipative nature of the scale-reduced systems may have important consequences for topics such as the origin of the arrow of time and the formulation of quantum mechanics through unitary operators.[6,12,13]

Interestingly, the reduction of a symplectic Hamiltonian system by a scaling symmetry produces a contact Hamiltonian system, which has been the subject of intensive study recently for their use in the description of, for example, dissipative, thermostatted and thermodynamic systems (see, e.g., Refs. 14–24 and the references therein). This intuition was first put forward in Ref. 25 and then formalized more precisely in the recent work,[26] where a thorough mathematical investigation of the role of scaling symmetries in symplectic Hamiltonian systems has been performed. Moreover, the relationship with the geometry of the blowups used in celestial mechanics has also been highlighted, together with the connection with other geometric structures.[27,28]

However, so far the study of the joint reduction by scaling and standard symmetries has not been considered in depth, at least from the mathematical perspective. Moreover, the case in which the reduced manifold is nonorientable, which seems to be the important case for the resolution of singularities in general relativity,[9–11] has been elusive of a fully fledged mathematical description (although, see Refs. 29–31). Finally, from the point of view of comparing the resulting physical theories, it is also crucial to highlight how to reconstruct the "original" symplectic system from the reduced one.

In this work, we perform a detailed mathematical analysis of all the above points. To give a feeling of the objects involved in our constructions, in the remainder of this introduction we provide a high-level description of the most important tools and results.

1.2 | Standard Lie symmetries for Kirillov Hamiltonian systems

A Kirillov structure on a real line bundle is a Lie algebra structure $[\cdot,\cdot]$ on the space of sections of the dual bundle such that, if we fix a section h on this line bundle, the operator $[\cdot,h]$ is a derivation. Thus, every section of the dual line bundle defines a vector field on the base manifold which is called the Hamiltonian vector field associated with the section. So, *a Kirillov Hamiltonian system* is a Kirillov structure on a real line bundle plus a section of the dual bundle (the Hamiltonian section).

Examples of Kirillov structures may be produced from symplectic, Poisson and Jacobi structures, contact 1-forms and contact structures (i.e., distributions of corank 1 which are maximally nonintegrable). Apart from the last case, in the other previous examples the real line bundle is trivial and the sections of the dual bundle are just C^∞ functions on the base manifold. Anyway, as we show in this paper, there exist interesting examples of Kirillov structures for which the real line bundle is not trivial. In particular, those in which the base space of the line bundle is the projective bundle associated with a vector bundle (for more details on Kirillov structures, see for instance, Refs. 31–35).

On the other hand, it is well-known that dynamical systems (in particular, mechanical systems), which are invariant under the action of a symmetry Lie group, have received a lot of attention from researchers in mathematics and physics. For this reason, in this paper we introduce the notion of *a standard Lie symmetry for a Kirillov Hamiltonian system*. It is a principal representation of a Lie group on the line bundle such that the dual representation preserves the Kirillov structure and the Hamiltonian section is equivariant. A Lie group of symplectic (respectively, Poisson, contact, or Jacobi) Hamiltonian symmetries is a particular example of a standard Lie symmetry for the corresponding Kirillov Hamiltonian system. Moreover, for a standard Lie symmetry on a Kirillov Hamiltonian system, the space of orbits of the action on the line bundle is again a line bundle. In fact, in the particular case when the Kirillov structure is Poisson (or Jacobi), we have a reduced Poisson (or Jacobi) structure. This is well-known in the theory of Poisson (or Jacobi) reduction (see, for instance, Refs. 36, 37). We remark that, very recently other (pre)contact reduction processes have been developed. So, in Ref. 38, the authors give an intrinsic geometric approach to reductions of contact manifolds. In fact, they discuss a precontact analog of the presymplectic reduction (i.e., precontact-to-contact reduction), a contact analog of the constant-rank reduction in symplectic geometry and a precontact analog of the Marsden–Weinstein reduction.

1.3 | Scaling symmetries for poisson Hamiltonian systems

In Ref. 26, the authors introduce the notion of a scaling symmetry for a symplectic Hamiltonian system and they exhibit several examples where such a symmetry is present (see also Refs. 25, 39, 40).

The previous notion may be extended for the more general class of Poisson Hamiltonian systems as follows. It is a principal action $\Phi : \mathbb{R}^\times \times P \to P$ of the Lie group $\mathbb{R}^\times$ (with $\mathbb{R}^\times = \mathbb{R}^+$ or

$\mathbb{R}^\times = \mathbb{R} - \{0\}$) on the Poisson manifold (P, Π) such that

$$\wedge^2 T\Phi \circ \Pi = s\Pi \circ \Phi, \qquad H \circ \Phi = sH$$

for all $s \in \mathbb{R}^\times$, where $H : P \to \mathbb{R}$ is the Hamiltonian function. In the particular case when P is a symplectic manifold S, it is proved in Refs. 29, 30 that the space of orbits $C = S/\mathbb{R}^\times$ admits a contact structure. In addition, the homogeneous function H on S induces a section of the dual bundle over C to the Kirillov line bundle in such a way that we have a reduced contact Hamiltonian system (see Refs. 30, 31).

1.4 | Our motivation

As we mentioned before, many symplectic Hamiltonian systems admit scaling symmetries. However, they do not only admit such symmetries, typically they also have standard Lie symmetries. In addition, the scaling and the standard Lie symmetries usually commute. So, one may reduce the dynamics by both types of symmetries, and some natural questions arise:

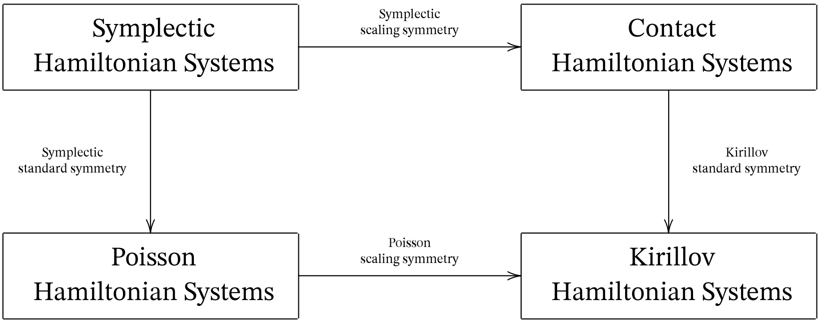
- What is the nature of the reduced system?
- If we reduce first by the scaling symmetries and then by the standard ones, is it the same as doing it the other way around?
- Is it possible to obtain the dynamics of the original symplectic Hamiltonian system from the dynamics of the reduced system via a suitable reconstruction process?

In this paper, we will provide answers to these questions.

1.5 | The results of the paper

For a symplectic Hamiltonian system with compatible scaling and standard Lie symmetries (i.e., they commute plus other natural topological conditions necessary to perform reduction by stages), we will develop two reduction processes:

- In the first reduction process, we start with the standard symmetry and then we apply the scaling symmetry. In this case, the first reduced system is a Poisson Hamiltonian system endowed with a scaling symmetry. The reduction of such a system by this scaling symmetry produces a Kirillov Hamiltonian system (see Theorem 2).
- In the second reduction process, we use the scaling symmetry and then the standard symmetry. In this case, the first reduced system is a contact Hamiltonian system endowed with a standard Lie symmetry. The reduction of the latter by this standard symmetry produces again a final Kirillov Hamiltonian system (see Theorem 4). In fact, the reduction of a general Kirillov Hamiltonian system by a standard Lie symmetry is again a Kirillov Hamiltonian system (see Theorem 3).
- We also prove that the final reduced Kirillov Hamiltonian systems obtained in both processes are Kirillov equivalent (see Theorem 5). The following diagram summarizes both reduction processes.



- Using more general ideas on reconstruction processes for dynamical systems in the presence of a symmetry Lie group, we present the reconstruction of the symplectic (respectively, Poisson) dynamics, for a system which admits a scaling symmetry, from the reduced contact and (respectively, Kirillov) Hamiltonian dynamics (see Section 7).
- All the previous constructions are applied to two examples of symplectic Hamiltonian systems which are interesting from the physical and mathematical point of view: The 2D harmonic oscillator and standard fiberwise-linear Hamiltonian systems on cotangent bundles induced by vector fields in the configuration space. For this last class of examples, when the cotangent bundle is that of a Lie group G, after the two reduction processes, we obtain an interesting Kirillov structure on the projective space associated with the dual space $\mathfrak{g}^*$ of the Lie algebra of G. This Kirillov structure may be considered as the Kirillov version of the Lie–Poisson structure on $\mathfrak{g}^*$. For this reason, it will be called the *Lie–Kirillov structure* (see the last part of Subsection 4.3). The geometric nature of this structure and its applications to Hamiltonian dynamics will be discussed in a next paper in progress. We remark that a holomorphic version of the Lie–Kirillov structure has been discussed in Ref. 41 (see Examples 54 in Ref. 41).

1.6 | Structure of the paper

The paper is structured as follows. In Section 2, we review some notions and properties of contact, Poisson, Jacobi, and Kirillov manifolds. At the end of the section, a diagram illustrates the relations between these kinds of structures. In Section 3, we show the scaling reduction process of a symplectic (Poisson) Hamiltonian system. This procedure is applied to two examples: The 2D harmonic oscillator and the standard fiberwise-linear Hamiltonian systems on cotangent bundles. In Section 4, we will discuss the reduction of symplectic Hamiltonian systems which are invariant under the action of a Lie group and, in addition, admit a scaling symmetry which is compatible with the standard symmetry. The reduction process starts by using first the standard symmetry and then the scaling symmetry. The process in the other direction (the first reduction is obtained by a scaling symmetry and the second one is done using the standard symmetry) is given in Section 5. Moreover, in this section we present a reduction process for general Kirillov Hamiltonian systems in the presence of a standard symmetry. In Sections 4 and 5, both processes are illustrated with the examples mentioned above. The equivalence between the reductions in both directions is proved in Section 6. Finally, in Section 7 we study the reconstruction process by focusing our attention on the case of symplectic Hamiltonian systems with scaling symmetries.

2 | CONTACT AND KIRILLOV HAMILTONIAN SYSTEMS

In this section, we recall some notions and properties of contact, Jacobi and Kirillov manifolds (for more details see, for instance, Refs. 3, 29, 31–35, 42–44).

A contact 1-form on a $(2n+1)$-dimensional manifold C is a 1-form η such that $\eta \wedge (d\eta)^n$ defines a volume form on C. We remark that a manifold with a contact 1-form is orientable and has a distinguished vector field $\mathcal{R} \in \mathfrak{X}(C)$, *the Reeb vector field*, characterized by the conditions

$$i_{\mathcal{R}} d\eta = 0 \qquad \text{and} \qquad i_{\mathcal{R}}\eta = 1.$$

The Reeb dynamics can be seen as the one induced by a Hamiltonian vector field on C. In fact, if $H : C \to \mathbb{R}$ is a smooth function on C, *the Hamiltonian vector field* $X_H^\eta \in \mathfrak{X}(C)$ of H is characterized by these two conditions

$$i_{X_H^\eta} d\eta = dH - \mathcal{R}(H)\eta \qquad \text{and} \qquad \eta(X_H^\eta) = H. \tag{1}$$

The Reeb vector field is just the Hamiltonian vector field for the constant function $H = 1$.

In the following example, we show a manifold endowed with a contact 1-form obtained by a reduction process.

Example 1 (The spherical cotangent bundle of a Riemannian manifold). Let (Q, g) be an n-dimensional Riemannian manifold and 0_Q the zero section of the cotangent bundle $\tau_Q^* : T^*Q \to Q$. On the open subset $T^*Q - 0_Q$ of T^*Q, we consider the action of the multiplicative group of the positive real numbers $\mathbb{R}^+$ given by

$$\phi : \mathbb{R}^+ \times (T^*Q - 0_Q) \to (T^*Q - 0_Q), \quad \phi(s, \alpha) = s\alpha, \tag{2}$$

which defines a principal bundle $\mathbf{p} : (T^*Q - 0_Q) \to (T^*Q - 0_Q)/\mathbb{R}^+$. The canonical symplectic structure ω_Q on $T^*Q - 0_Q$ is homogeneous with respect to this action, that is,

$$\phi_s^*(\omega_Q) = s\omega_Q, \qquad \text{for all } s \in \mathbb{R}^+, \tag{3}$$

or equivalently,

$$\mathcal{L}_{\Delta_Q}\omega_Q = \omega_Q,$$

where Δ_Q is the infinitesimal generator of the action ϕ, that is, Δ_Q is the Liouville vector field on T^*Q.

The quotient manifold $(T^*Q - \{0_Q\})/\mathbb{R}^+$ is diffeomorphic to the spherical cotangent bundle

$$\mathbb{S}(T^*Q) = \{\alpha \in T^*Q / \|\alpha\| = \sqrt{g(\alpha, \alpha)} = 1\},$$

where g denotes here the corresponding metric on T^*Q.

In the particular case when Q is $\mathbb{R}^{n+1}$, with the flat Riemannian metric, we have that the spherical cotangent bundle is

$$\mathbb{S}(T^*\mathbb{R}^{n+1}) \cong \mathbb{R}^{n+1} \times S^n\,, \tag{4}$$

with S^n the n-sphere in $\mathbb{R}^{n+1}$.

If λ_Q is the Liouville 1-form on T^*Q, that is,

$$\lambda_Q(\alpha)(v) = \alpha(T_\alpha \tau_Q^*(v)), \text{ for all } \alpha \in T^*Q,\ v \in T_\alpha(T^*Q),$$

and $i : \mathbb{S}(T^*Q) \to T^*Q$ is the inclusion map, then $\eta_Q = -i^*\lambda_Q$ is a contact 1-form on $\mathbb{S}(T^*Q)$ (see, for instance, Refs. 45–47).

We remark that the regular and singular Marsden–Weinstein reduction of the spherical cotangent bundle have been discussed some years ago.[48,49] In fact, this reduction process is a particular case of the more general Marsden–Weinstein contact reduction which has been intensively discussed by several authors.[38,50–53]

A contact 1-form is a particular case of a Jacobi structure. *A Jacobi manifold M* ([35,44]) is endowed with a pair $(\Pi, E) \in \mathcal{V}^2(M) \times \mathfrak{X}(M)$, where Π is a 2-vector field and E is a vector field on M such that

$$[\![\Pi, \Pi]\!] = 2E \wedge \Pi, \qquad [\![E, \Pi]\!] = 0,$$

$[\![\cdot, \cdot]\!]$ being the Schouten–Nijenhuis bracket on M. Associated with a Jacobi manifold $(M, (\Pi, E))$ we have a *Jacobi bracket*, given by

$$\{f_1, f_2\}_M = \Pi(df_1, df_2) + f_1 E(f_2) - f_2 E(f_1), \text{ for } f_1, f_2 \in C^\infty(M), \tag{5}$$

which is a Lie bracket on the space of functions on M such that

$$\{ff_1, f_2\}_M = f\{f_1, f_2\}_M + f_1\{f, f_2\}_M - f_1 f\{1, f_2\}_M$$

for $f, f_1, f_2 \in C^\infty(M)$. Reciprocally, a Jacobi bracket on the space of functions $C^\infty(M)$ defines a Jacobi structure (Π, E) satisfying (5).

Note that we have a vector field $X_{f_2}^{\{\cdot,\cdot\}_M}$ on M, *the Hamiltonian vector field associated with* f_2, such that

$$\{ff_1, f_2\}_M = f\{f_1, f_2\}_M + X_{f_2}^{\{\cdot,\cdot\}_M}(f)f_1. \tag{6}$$

In terms of the Jacobi structure, this vector field is given by

$$X_{f_2}^{\{\cdot,\cdot\}_M} = \Pi(\cdot, df_2) - f_2 E\,. \tag{7}$$

If $E = 0$, we recover the notion of a Poisson bracket on the space of functions on M and (M, Π) is a Poisson manifold.

For a manifold C with a contact 1-form η, the Jacobi structure is

$$\Pi_\eta(\alpha, \beta) = d\eta(\flat_\eta^{-1}(\alpha), \flat_\eta^{-1}(\beta)), \qquad E_\eta = -\mathcal{R}$$

for all $\alpha, \beta \in \Omega^1(C)$, where $\mathcal{R}$ is the Reeb vector field associated with η and $\flat_\eta : \mathfrak{X}(C) \to \Omega^1(C)$ is the isomorphism of $C^\infty(C)$-modules given by

$$\flat_\eta(X) = i_X d\eta + \langle \eta, X\rangle \eta, \quad \text{with } X \in \mathfrak{X}(C).$$

Moreover, the Hamiltonian vector field defined in (1) is just the corresponding Hamiltonian vector field $X_f^{\{\cdot,\cdot\}_M}$ associated with the Jacobi structure (Π_η, E_η) (see Ref. 44).

Example 2 (continuing Example 1). In the case of the spherical cotangent bundle of a Riemannian manifold (Q, g), we consider the differentiable function $\kappa_g : T^*Q - 0_Q \to \mathbb{R}$ defined by

$$\kappa_g(\alpha) = \frac{1}{2}\|\alpha\|^2, \text{ for } \alpha \in T^*Q.$$

If $X_{\kappa_g}^{\omega_Q} \in \mathfrak{X}(T^*Q - 0_Q)$ is the Hamiltonian vector field with respect to ω_Q of the function κ_g, that is, the vector field characterized by

$$i_{X_{\kappa_g}^{\omega_Q}} \omega_Q = d\kappa_g,$$

then the Jacobi structure $(\Pi_{\eta_Q}, E_{\eta_Q})$ on $(\mathbb{S}(T^*Q), \eta_Q)$ is just the restriction to $\mathbb{S}(T^*Q)$ of the Jacobi structure (Π, E) on T^*Q given by

$$\Pi = \Pi_{\omega_Q} - \Delta_Q \wedge X_{\kappa_g}^{\omega_Q}, \qquad E = X_{\kappa_g}^{\omega_Q},$$

where Π_{ω_Q} is the Poisson structure induced by the symplectic structure ω_Q on T^*Q.

On the other hand, contact 1-forms are also a particular kind of more general structures which are not, in general, Jacobi structures.

A contact structure on a $(2n+1)$-dimensional smooth manifold C is a distribution $\mathcal{D}$ on C of codimension 1 which is maximally nonintegrable, that is, for all $x \in C$, there is an open neighborhood U of x such that the distribution $\mathcal{D}$ on U is given by the annihilator $< \eta_U >^o$ of the vector subbundle of T^*C generated by a contact 1-form η_U on U, that is,

$$\mathcal{D}_U = \langle \eta_U \rangle^o = \{X \in TU / \eta_U(X) = 0\}.$$

In this case, the pair $(C, \mathcal{D})$ is *a contact manifold*.

It is clear that if C has a global contact 1-form, the pair $(C, \mathcal{D} =< \eta >^o)$ defines a contact manifold. But in general, a contact structure on C may not be defined by a global contact 1-form on C as the following example proves.

Example 3 (The projective cotangent bundle of a manifold). Let Q be an n-dimensional manifold and 0_Q the zero section of the cotangent bundle $\tau_Q^* : T^*Q \to Q$. On the open subset $T^*Q - 0_Q$ of T^*Q, we consider the action of the multiplicative group $\mathbb{R} - \{0\}$ given by

$$\phi : (\mathbb{R} - \{0\}) \times (T^*Q - 0_Q) \to (T^*Q - 0_Q), \qquad \phi(s, \alpha) = s\alpha. \tag{8}$$

Its infinitesimal generator Δ_Q is the Liouville vector field on $T^*Q - 0_Q$ and the reduced space $(T^*Q - 0_Q)/(\mathbb{R} - \{0\})$ is just the projective cotangent bundle $\mathbb{P}(T^*Q)$ of Q.

Remark 1. The notion of projective bundle $\mathbb{P}(V)$ may be defined for an arbitrary vector bundle $\tau : V \to Q$ as the quotient bundle induced by the action on $V - 0_Q$

$$(\mathbb{R} - \{0\}) \times (V - 0_Q) \to (V - 0_Q), \qquad (s, v) \to sv,$$

where 0_Q is the zero section of $\tau : V \to Q$.

A particular case is when Q is a point and V is the dual of a Lie algebra $\mathfrak{g}$. In this case, the base space of the projective bundle $p : \mathfrak{g}^* - \{0\} \to \mathbb{P}\mathfrak{g}^*$ is just the projective space $\mathbb{P}\mathfrak{g}^*$.

If λ_Q is the Liouville 1-form on T^*Q and $\mathbf{p} : (T^*Q - 0_Q) \to \mathbb{P}(T^*Q)$ is the quotient projection, using (3), one can prove that the distribution of corank 1

$$\widetilde{D} = \langle \lambda_Q \rangle^o$$

is $\mathbf{p}$-projectable. If D denotes its projection, then $(\mathbb{P}(T^*Q), D)$ is a contact manifold.

A simple example of this kind of contact manifolds is when Q is a Lie group G. In this case, the cotangent bundle T^*G may be left trivialized to the trivial vector bundle $G \times \mathfrak{g}^* \to G$, where $\mathfrak{g}$ is the Lie algebra of G. Under this identification, the action ϕ is just

$$\phi : (\mathbb{R} - \{0\}) \times (G \times (\mathfrak{g}^* - \{0\})) \to G \times (\mathfrak{g}^* - \{0\}), \qquad (s, (g, \mu)) \to (g, s\mu).$$

Then, the quotient bundle is $\mathbf{p} = Id_G \times p : G \times (\mathfrak{g}^* - \{0\}) \to G \times \mathbb{P}\mathfrak{g}^*$ and the contact structure is the distribution on $G \times \mathbb{P}\mathfrak{g}^*$ given by

$$D_{(g,p(\mu))} = \langle (T_g L_{g^{-1}})^*(\mu) \rangle^o \times T_{p(\mu)}(\mathbb{P}\mathfrak{g}^*)$$

for all $g \in G$ and $\mu \in \mathfrak{g}^* - \{0\}$. Here, $L : G \times G \to G$ denotes the left action of the Lie group G on itself.

In the particular case when $G = \mathbb{R}^{n+1}$, the projective cotangent bundle $\mathbb{P}(T^*\mathbb{R}^{n+1})$ can be identified with the Cartesian product $\mathbb{R}^{n+1} \times \mathbb{P}^n(\mathbb{R})$, where $\mathbb{P}^n(\mathbb{R})$ is the real projective space of dimension n. This space is nonorientable when n is even and therefore, $\mathbb{P}(T^*\mathbb{R}^{n+1})$ does not admit a global contact 1-form.

Contact and Jacobi structures are special examples of more general structures: *Kirillov structures* (see, Ref. 35 and also Refs. 29, 30, 32).

Definition 1. *A Kirillov structure* on a manifold K is a real line bundle $\pi_L : L \to K$ endowed with a Lie bracket $[\cdot,\cdot]_{L^*} : \Gamma(L^*) \times \Gamma(L^*) \to \Gamma(L^*)$ on the space $\Gamma(L^*)$ of sections of the dual line bundle $\pi_{L^*} : L^* \to K$ such that $[\cdot, h_2]_{L^*} : \Gamma(L^*) \to \Gamma(L^*)$ is a derivation for all $h_2 \in \Gamma(L^*)$, that is,

$$[fh_1, h_2]_{L^*} = f[h_1, h_2]_{L^*} + X_{h_2}^{[\cdot,\cdot]_{L^*}}(f)h_1, \text{ for all } h_1 \in \Gamma(L^*) \text{ and } f \in C^\infty(K), \tag{9}$$

with $X_{h_2}^{[\cdot,\cdot]_{L^*}}$ a vector field on K. The vector field $X_{h_2}^{[\cdot,\cdot]_{L^*}} \in \mathfrak{X}(K)$ is called *the symbol of* $[\cdot, h_2]_{L^*}$.

The line bundle (L^*, π_{L^*}, K) with the bracket $[\cdot,\cdot]_{L^*}$ on the space of sections of π_{L^*} is, in Marle's terminology,[31] a Jacobi bundle. This kind of structures are essentially equivalent to the conformal Jacobi structures studied in Ref. 43.

When the line bundle $\pi_L : L \to K$ is trivial, that is, $L \cong K \times \mathbb{R}$, the sections of π_{L^*} can be identified with smooth functions on K. Under this identification, the local Lie algebra $[\cdot,\cdot]_{L^*}$ is a Lie bracket

$$\{\cdot,\cdot\}_K : C^\infty(K) \times C^\infty(K) \to C^\infty(K)$$

satisfying that, for all $f \in C^\infty(K)$,

$$\{f f_1, f_2\}_K = f\{f_1, f_2\}_K + X_{f_2}^{\{\cdot,\cdot\}_K}(f) f_1$$

for all $f_1, f_2 \in C^\infty(K)$.

Note that if $f_1 = 1$ then $\{f, f_2\}_K = f\{1, f_2\}_K + X_{f_2}^{\{\cdot,\cdot\}_K}(f)$, which implies

$$\{f f_1, f_2\}_K = f\{f_1, f_2\}_K + f_1\{f, f_2\}_K - f f_1\{1, f_2\}_K.$$

This means that $\{\cdot,\cdot\}_K$ is a Jacobi bracket, whose associated Jacobi structure (Π, E) is given by

$$E(f_1) = \{1, f_1\}_K \qquad \text{and} \qquad \Pi(df_1, df_2) = \{f_1, f_2\}_K - f_1\{1, f_2\}_K + f_2\{1, f_1\}_K,$$

with $f_1, f_2 \in C^\infty(K)$. Conversely, every Jacobi manifold $(K, \{\cdot,\cdot\}_K)$ defines a Kirillov structure on the trivial line bundle $\pi : K \times \mathbb{R} \to K$. Therefore, Jacobi structures are just trivial Kirillov structures.

In the case of a contact manifold $(C, \mathcal{D})$, consider the line bundle with total space the annihilator bundle $\mathcal{D}^o$ of $\mathcal{D}$, $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$ of $\mathcal{D}$, which is, in general, not trivial. Using this line bundle and the representation $\mathbb{R}^\times \times \mathbb{R} \to \mathbb{R}$ of $\mathbb{R}^\times$ (with $\mathbb{R}^\times = \mathbb{R}^+$ or $\mathbb{R}^\times = \mathbb{R} - \{0\}$) over the vector space of real numbers given by

$$(s, t) \to \frac{t}{s},$$

we have a $\mathbb{R}^\times$-principal bundle $\mathbf{p} : S := (\mathcal{D}^o - 0_C) \to C \cong S/\mathbb{R}^\times$ (see the Appendix). Here, 0_C is the zero section of $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$. Moreover, we may consider the 1-form λ_S on S

$$\lambda_S(\alpha)(v) =< \alpha, T_\alpha \mathbf{p}(v) >, \quad \text{with } \alpha \in (\mathcal{D}^o - 0_C), \;\; v \in T_\alpha(\mathcal{D}^o - 0_C),$$

which defines the symplectic structure $\omega_S = -d\lambda_S$. This symplectic structure is homogeneous with respect to the $\mathbb{R}^\times$-action $\phi^S : \mathbb{R}^\times \times S \to S$ on S, that is,

$$(\phi_s^S)^*(\omega_S) = s\omega_S, \text{ for } s \in \mathbb{R}^\times.$$

Now, a Lie bracket $[\cdot,\cdot]_{(\mathcal{D}^o)^*}$ on the space of sections $\Gamma((\mathcal{D}^o)^*)$ of the line bundle $(\mathcal{D}^o)^* \to C$ can be constructed as follows.

There is a one-to-one correspondence between the sections of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to C$ and the homogeneous functions $H : S \to \mathbb{R}$ on S, that is, the functions satisfying

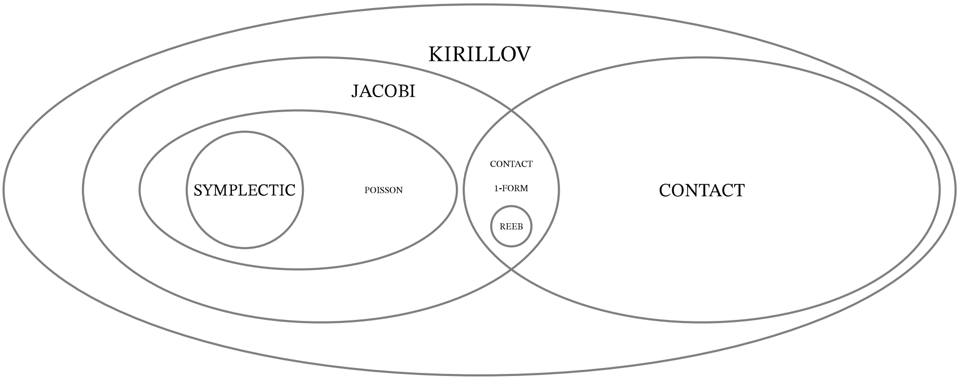
$$H \circ \phi_s^S = sH, \quad \text{for } s \in \mathbb{R}^\times$$

(see the Appendix). Using the homogeneous character of the symplectic structure ω_S, we deduce that the Poisson bracket $\{H_1, H_2\}_S$ induced by ω_S of two homogeneous functions $H_1, H_2 : S \to \mathbb{R}$ is again a homogeneous function. Taking into account this fact, we define the Kirillov bracket $[\cdot,\cdot]_{(\mathcal{D}^o)^*} : \Gamma((\mathcal{D}^o)^*) \times \Gamma((\mathcal{D}^o)^*) \to \Gamma((\mathcal{D}^o)^*)$ by the relation

$$\{H_1, H_2\}_S = -H_{[h_{H_1}, h_{H_2}]_{(\mathcal{D}^o)^*}}, \tag{10}$$

where h_{H_i} is the section of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to C$ associated with the homogeneous function H_i on S and $H_{[h_{H_1}, h_{H_2}]_{(\mathcal{D}^o)^*}}$ is the homogenous function associated with the section $[h_{H_1}, h_{H_2}]_{(\mathcal{D}^o)^*}$. In conclusion, every contact manifold $(C, \mathcal{D})$ admits a Kirillov structure on the line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$.

The following diagram illustrates the relations among all the previous geometric structures.



3 | SCALING SYMMETRIES AND SYMPLECTIC (POISSON) HAMILTONIAN SYSTEMS

In the previous examples, the reduction processes are the fundamental tool to obtain contact structures from symplectic structures. Now, we will show this process for a general symplectic Hamiltonian system, which was discussed in Ref. 30, and then we will present some examples. We begin by recalling the notion of scaling symmetries[26] for this kind of dynamical systems.

Definition 2. Let (S, ω) be a symplectic manifold and $H : S \to \mathbb{R}$ a function on S. A scaling symmetry for the dynamical system (S, ω, H) is a principal action $\phi : \mathbb{R}^\times \times S \to S$ of the multiplicative group $\mathbb{R}^\times$ (with $\mathbb{R}^\times = \mathbb{R}^+$ or $\mathbb{R}^\times = \mathbb{R} - \{0\}$) on S such that

$$\phi_s^* \omega = s\omega \qquad \text{and} \qquad \phi_s^* H = sH, \qquad \text{for all } s \in \mathbb{R}^\times.$$

Note that if $\Delta \in \mathfrak{X}(S)$ is the infinitesimal generator of the scaling symmetry, then

$$\mathcal{L}_\Delta \omega = \omega \qquad \text{and} \qquad \mathcal{L}_\Delta H = H.$$

In fact, if $\mathbb{R}^\times$ is connected (i.e., $\mathbb{R}^\times = \mathbb{R}^+$), then the previous conditions are equivalent to the fact that the principal action ϕ is a scaling symmetry.

An immediate consequence of the existence of a scaling symmetry is that the symplectic structure is exact, that is, $\omega = -d\lambda$ with $\lambda = -i_\Delta \omega$. Moreover, the 1-form λ is homogeneous, that is, $(\phi_s)^*\lambda = s\lambda$, and if Π_ω is the Poisson bivector induced by ω, then Π_ω satisfies the following relation:

$$\wedge^2 T\phi_s \circ \Pi_\omega = s\Pi_\omega \circ \phi_s, \tag{11}$$

where $\wedge^2 T\phi_s : \wedge^2 TS \to \wedge^2 TS$ is the vector bundle isomorphism induced by the diffeomorphism $\phi_s : S \to S$.

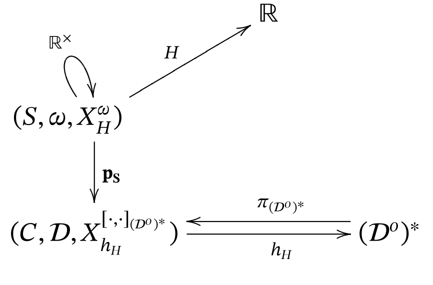
Now, we will develop the reduction process with the scaling symmetry ϕ.

Denote by $C := S/\mathbb{R}^\times$ the corresponding quotient manifold and by $\mathbf{p_S} : S \to C$ its quotient projection. Then, we may consider the distribution

$$\widetilde{\mathcal{D}} = \langle \lambda \rangle^o,$$

which is $\mathbf{p}$-projectable and the corresponding distribution $\mathcal{D}$ on C, which is a contact structure.

Denote by $[\cdot,\cdot]_{(\mathcal{D}^o)^*}$ the Kirillov bracket on the space of sections of the line bundle $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to C$ characterized by (10). On the other hand, from the homogeneity of $H : S \to \mathbb{R}$ with respect to the scaling symmetry, we have a section $h_H : C \to (\mathcal{D}^o)^*$ of $\pi_{(\mathcal{D}^o)^*}$. The corresponding symbol $X_{h_H}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*}}$ of h_H given as in (9) is just the $\mathbf{p}$-projection on C of the Hamilton vector field X_H^ω. The following diagram summarizes this reduction process (see Ref. 30 for more details on this reduction process).



Now, we will exhibit two examples of contact dynamical systems induced by a scaling reduction process.

Example 4 (The 2D harmonic oscillator and the spherical cotangent bundle). Consider the manifold $Q = \mathbb{R}^2 - \{(0,0)\}$, which is diffeomorphic to $\mathbb{R}^+ \times S^1$ via the map

$$\Psi : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^+ \times S^1, \qquad \Psi(q) = \left(\|q\|, \frac{q}{\|q\|} \right). \tag{12}$$

Then, under this identification the space $T^*Q - 0_Q \cong (\mathbb{R}^2 - \{(0,0)\}) \times (\mathbb{R}^2 - \{(0,0)\})$ is just $(\mathbb{R}^+ \times S^1) \times (\mathbb{R}^+ \times S^1)$. Moreover, if (r, θ) (respectively, (r, θ, r', θ')) are polar coordinates on $Q \cong \mathbb{R}^+ \times S^1$ (respectively, on $T^*Q - 0_Q \cong \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1$), we have that the local expression of the standard symplectic form ω_Q and the corresponding Poisson bivector Π_{ω_Q} on $\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1$ are, respectively,

$$\omega_Q = cos(\theta - \theta')dr \wedge dr' + r' \sin(\theta - \theta')dr \wedge d\theta' - r \sin(\theta - \theta')d\theta \wedge dr' + rr' cos(\theta - \theta')d\theta \wedge d\theta'$$

and

$$\Pi_{\omega_Q} = -\cos(\theta - \theta')\partial_r \wedge \partial_{r'} - \frac{\sin(\theta - \theta')}{r'}\partial_r \wedge \partial_{\theta'} + \frac{\sin(\theta - \theta')}{r}\partial_\theta \wedge \partial_{r'} - \frac{\cos(\theta - \theta')}{rr'}\partial_\theta \wedge \partial_{\theta'}. \tag{13}$$

Now, we consider the symplectic Hamiltonian system (T^*Q, ω_Q, H) of the harmonic oscillator where, under the identification (12), $H : T^*Q \to \mathbb{R}$ is the Hamiltonian function given by

$$H(r, \theta, r', \theta') = \frac{1}{2}\left(r^2 + (r')^2\right), \tag{14}$$

with $r, r' \in \mathbb{R}^+$. In this case, the dynamics is given by the Hamiltonian vector field

$$X_H^{\omega_Q} = r\cos(\theta - \theta')\partial_{r'} + r\frac{\sin(\theta - \theta')}{r'}\partial_{\theta'} - r'\cos(\theta - \theta')\partial_r + r'\frac{\sin(\theta - \theta')}{r}\partial_\theta.$$

We consider the action of $\mathbb{R}^+$ on $(\mathbb{R}^+ \times S^1) \times (\mathbb{R}^+ \times S^1)$, whose infinitesimal generator is

$$\Delta = \frac{1}{2}(r\partial_r + r'\partial_{r'}).$$

Note that it defines a scaling symmetry, since $\mathcal{L}_\Delta \omega_Q = \omega_Q$ and $\mathcal{L}_\Delta H = H$.

On the other hand, the diffeomorphism

$$\begin{array}{rcl} \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 & \to & \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 \\ (r, \theta, r', \theta') & \to & (\rho, \theta, \rho', \theta') = \left(r, \theta, \dfrac{r'}{r}, \theta'\right) \end{array}$$

transforms the generator Δ of the $\mathbb{R}^+$-action on $\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1$ into the vector field $\frac{1}{2}\rho\partial_\rho$. The inverse of this map is $(\rho, \theta, \rho', \theta') \to (\rho, \theta, \rho\rho', \theta')$. Then, we have that:

- The reduced space $\mathbb{S}(T^*(\mathbb{R}^+ \times S^1))$ (see Example 1) is diffeomorphic to $\mathbb{R}^+ \times S^1 \times S^1$. Under this identification, the quotient map $\mathbf{p} : \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 \to \mathbb{R}^+ \times S^1 \times S^1$ is just

$$\mathbf{p}(\rho, \theta, \rho', \theta') = (\rho', \theta, \theta')$$

- The contact 1-form under this identification is given by

$$\eta = \iota^*(i_\Delta \omega_Q) = \frac{1}{2}\left(\rho' \sin(\theta - \theta')(d\theta + d\theta') + \cos(\theta - \theta')d\rho'\right)$$

with $(\rho', \theta, \theta') \in \mathbb{R}^+ \times S^1 \times S^1$. Here $\iota : \mathbb{R}^+ \times S^1 \times S^1 \to \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1$ is the inclusion $\iota(\rho', \theta, \theta') = (1, \theta, \rho', \theta')$.

The Reeb vector field associated with this contact 1-form is

$$\mathcal{R} = 2\cos(\theta - \theta')\partial_{\rho'} + 2\frac{\sin(\theta - \theta')}{\rho'}\partial_{\theta'}.$$

From the homogeneity of the Poisson structure $\{\cdot, \cdot\}_{\omega_Q}$ with respect to the symplectic form ω_Q we deduce that

$$\{\rho^2 h, \rho^2 h'\}_{\omega_Q} = \frac{1}{2}\rho\partial_\rho\{\rho^2 h, \rho^2 h'\}_{\omega_Q},$$

with $h, h' \in C^\infty(\mathbb{R}^+ \times S^1 \times S^1)$. Therefore,

$$\{\rho^2 h, \rho^2 h'\}_{\omega_Q} = \rho^2\{h, h'\}_C,$$

where $\{\cdot, \cdot\}_C$ is the Jacobi bracket on $C = \mathbb{R}^+ \times S^1 \times S^1$ and $h, h' \in C^\infty(C)$.

From this fact and using the local expression of Π_{ω_Q} with respect to the coordinates $(\rho, \theta, \rho', \theta')$, we obtain the Jacobi bracket associated with the contact structure defined by η

$$\begin{aligned} \{h, h'\}_C &= -2\cos(\theta - \theta')(h\partial_{\rho'}h' - h'\partial_{\rho'}h) - 2\frac{\sin(\theta - \theta')}{\rho'}(h\partial_{\theta'}h' - h'\partial_{\theta'}h) \\ &\quad \sin(\theta - \theta')(\partial_{\rho'}h\partial_{\theta'}h' - \partial_{\rho'}h'\partial_{\theta'}h) + \sin(\theta - \theta')(\partial_\theta h\partial_{\rho'}h' - \partial_\theta h'\partial_{\rho'}h) \\ &\quad + \frac{\cos(\theta - \theta')}{\rho'}(\partial_\theta h\partial_{\theta'}h' - \partial_\theta h'\partial_{\theta'}h). \end{aligned} \tag{15}$$

Therefore, the Jacobi structure is given by

$$\begin{aligned} \Pi_C &= \sin(\theta - \theta')\partial_{\rho'} \wedge \partial_{\theta'} - \sin(\theta - \theta')\partial_{\rho'} \wedge \partial_\theta - \frac{\cos(\theta - \theta')}{\rho'}\partial_\theta \wedge \partial_{\theta'}, \\ E_C &= -2\cos(\theta - \theta')\partial_{\rho'} - 2\frac{\sin(\theta - \theta')}{\rho'}\partial_{\theta'}. \end{aligned} \tag{16}$$

- The reduced Hamiltonian function H is the function $H_{|\mathbb{R}^+ \times S^1 \times S^1}(\rho', \theta, \theta') = \frac{1}{2}((\rho')^2 + 1)$.
- The reduced vector field on $\mathbb{R}^+ \times S^1 \times S^1$ is

$$T\mathbf{p}(X_H^{\omega_Q}) = (1 + (\rho')^2)\cos(\theta - \theta')\partial_{\rho'} + \sin(\theta - \theta')\left(\frac{1}{\rho'}\partial_{\theta'} + \rho'\partial_\theta\right), \tag{17}$$

which is just the contact Hamiltonian vector field of the restriction $H_{|\mathbb{R}^+ \times S^1 \times S^1}$ with respect to the contact 1-form η or, equivalently, the Jacobi Hamiltonian vector field of $H_{|\mathbb{R}^+ \times S^1 \times S^1}$ with respect to (Π_C, E_C).

In the previous example, the Hamiltonian function H induces a function $H_{|C}$ on the reduced space C. However, in general, we do not necessarily have a function on the reduced space, as the following example proves.

Example 5. The projective cotangent Hamiltonian system deduced from a standard linear Hamiltonian system. Let $Y \in \mathfrak{X}(Q)$ be a vector field on the manifold Q of dimension n. We denote by $Y^\ell : T^*Q \to \mathbb{R}$ the fiberwise-linear function induced by Y, that is,

$$Y^\ell(\alpha) =< \alpha, Y(\tau_Q^*(\alpha)) >, \quad \forall \alpha \in T^*Q, \tag{18}$$

with $\tau_Q^* : T^*Q \to Q$ the canonical projection. If (q^i, p_i) are local coordinates of $T^*Q - 0_Q$, the local expression of Y^ℓ is

$$Y^\ell(q,p) = Y^i(q)p_i,$$

where $Y(q) = Y^i(q)\partial_{q^i}$. We remark that the linearity of Y^ℓ implies its homogeneity, that is,

$$Y^\ell(s\alpha) = sY^\ell(\alpha), \text{ for all } s \in \mathbb{R} - \{0\} \text{ and } \alpha \in T^*Q,$$

with respect to the action given in (8).

The local expression of the Hamiltonian vector field $X^{\omega_Q}_{Y^\ell} \in \mathfrak{X}(T^*Q)$ with respect to the canonical symplectic structure ω_Q on T^*Q is

$$X^{\omega_Q}_{Y^\ell} = Y^k\partial_{q^k} - p_j\partial_{q^k}Y^j\partial_{p_k}.$$

Moreover, if $\{\cdot,\cdot\}_{\omega_Q}$ is the Poisson bracket induced by ω_Q, then

$$\{Y^\ell, Z^\ell\}_{\omega_Q} = -[Y,Z]^\ell$$

for all $Y, Z \in \mathfrak{X}(Q)$.

Let U_{i_0} be the open subset of $T^*Q - 0_Q$ given by

$$U_{i_0} = \{(q^1, \dots, q^n, p_1, \dots, p_n) \in T^*Q - 0_Q / p_{i_0} \neq 0\}.$$

Then, if H is the restriction of Y^ℓ to $T^*Q - 0_Q$, after the reduction process of the symplectic Hamiltonian system $(T^*Q - 0_Q, \omega_Q, H)$ by the scaling symmetry, we have that:

- The corresponding reduced space is the projective cotangent bundle $\mathbf{p} : T^*Q - 0_Q \to \mathbb{P}(T^*Q)$ induced by the action (8). If we denote by $\widetilde{p} = (\widetilde{p}_1, \dots, \widetilde{p}_{i_0-1}, \widetilde{p}_{i_0+1}, \dots, \widetilde{p}_n)$ the standard coordinates on $\mathbf{p}(U_{i_0}) \subseteq \mathbb{P}(T^*Q)$, then the local expression of the projection $\mathbf{p}$ on U_{i_0} is

$$\mathbf{p}(q^1, \dots q^n, p_1, \dots p_n) = \left(q^1, \dots q^n, \frac{p_1}{p_{i_0}}, \dots, \frac{p_{i_0-1}}{p_{i_0}}, \frac{p_{i_0+1}}{p_{i_0}}, \dots, \frac{p_n}{p_{i_0}}\right) = (q, \widetilde{p}).$$

- The contact distribution $\mathcal{D}$ on $\mathbf{p}(U_{i_0})$ is just

$$\begin{aligned}(\mathcal{D}_{(q,\widetilde{p})})_{|\mathbf{p}(U_{i_0})} &= T_{(q,p)}\mathbf{p}(< p_i dq^i >^o) = T_{(q,p)}\mathbf{p} < X_1, \dots, X_{i_0-1}, X_{i_0+1}, \dots, X_n, \partial_{p_1}, \dots, \partial_{p_n} > \\ &= < \widetilde{X}_1, \dots, \widetilde{X}_{i_0-1}, \widetilde{X}_{i_0+1}, \dots, \widetilde{X}_n, \partial_{\widetilde{p}_1}, \dots, \partial_{\widetilde{p}_{i_0-1}}, \partial_{\widetilde{p}_{i_0+1}}, \dots, \partial_{\widetilde{p}_n} >\end{aligned}$$

with $X_i = p_i\partial_{q^{i_0}} - p_{i_0}\partial_{q^i}, \widetilde{X}_i = \widetilde{p}_i\partial_{q^{i_0}} - \widetilde{p}_{i_0}\partial_{q^i}$. Moreover, the local expression of the line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to \mathbb{P}(T^*Q)$ on $\mathbf{p}(U_{i_0})$ is

$$\pi_{\mathcal{D}^o}(q, \widetilde{p}, t) = (q, \widetilde{p}).$$

- The section $h_{Y^\ell} : \mathbb{P}(T^*Q) \to (\mathcal{D}^o)^*$ of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to \mathbb{P}(T^*Q)$ associated with Y^ℓ is defined locally by

$$h_{Y^\ell}(q, \widetilde{p})(q, \widetilde{p}, t) = Y^\ell(q, \widetilde{p}_1, \dots, \widetilde{p}_{i_0-1}, t, \widetilde{p}_{i_0+1} \cdots, \widetilde{p}_n) = Y^i(q)\widetilde{p}_i + Y^{i_0}(q)t. \tag{19}$$

- The Kirillov bracket $[\cdot,\cdot]_{(\mathcal{D}^o)^*}$ on the sections of the dual of the line bundle $\pi_{\mathcal{D}^o}$ satisfies the condition

$$[h_{X^\ell}, h_{Y^\ell}]_{(\mathcal{D}^o)^*} = -h_{\{X^\ell, Y^\ell\}_{\omega_Q}} = h_{[X,Y]^\ell}.$$

- The Hamiltonian vector field $X^{\omega_Q}_{Y^\ell} \in \mathfrak{X}(T^*Q)$ is $\mathbf{p}$-projectable to a vector field on $\mathbb{P}(T^*Q)$ whose local expression is

$$Y^i\partial_{q^i} + \left(\widetilde{p}_j(\widetilde{p}_i\partial_{q^{i_0}}Y^j - \partial_{q^i}Y^j) + \widetilde{p}_i\partial_{q^{i_0}}Y^{i_0} - \partial_{q^i}Y^{i_0}\right)\partial_{\widetilde{p}_i}.$$

The particular case of a Lie group. When Q is a Lie group G and the vector field Y on G is left-invariant, we have (see Example 3):

- The vector field Y is given by $Y(g) = T_eL_g(\xi)$, with ξ an element of the Lie algebra $\mathfrak{g}$ of G.
- The linear function $Y^\ell : G \times \mathfrak{g}^* \to \mathbb{R}$ is just $Y^\ell(g, \alpha) = \alpha(\xi)$.
- The reduced space is $G \times \mathbb{P}\mathfrak{g}^*$.
- The contact structure is the distribution on $G \times \mathbb{P}\mathfrak{g}^*$ given by

$$\mathcal{D}_{(g,p(\mu))} = \left\langle (T_gL_{g^{-1}})^*(\mu)\right\rangle^o \times T_{p(\mu)}(\mathbb{P}\mathfrak{g}^*) \text{ for all } g \in G \text{ and } \mu \in \mathfrak{g}^* - \{0\}.$$

  Here, $p : \mathfrak{g}^* - \{0\} \to \mathbb{P}\mathfrak{g}^*$ is the corresponding quotient map determined by the scaling symmetry on $\mathfrak{g}^* - \{0\}$.
- The fiber of the line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to G \times \mathbb{P}\mathfrak{g}^*$ at $(g, \mu) \in G \times \mathbb{P}\mathfrak{g}^*$ is just

$$\mathcal{D}^o{}_{(g,p(\mu))} = \left\langle (T_gL_{g^{-1}})^*(\mu)\right\rangle.$$

- The reduced Hamiltonian section of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to G \times \mathbb{P}\mathfrak{g}^*$ induced by Y^ℓ is

$$h_\xi(g, p(\mu))(t(T_gL_{g^{-1}})^*(\mu)) = t\mu(\xi)$$

  with $g \in G$, $\mu \in \mathfrak{g}^* - \{0\}$ and $\xi = Y(e)$.

- Under the identification $T^*G - 0_G \cong G \times (\mathfrak{g}^* - \{0\})$, the symplectic structure ω_G is given by

$$\omega_G(g,\mu)((v_1,\mu_1),(v_2,\mu_2)) = -\mu_1(T_gL_{g^{-1}}(v_2)) + \mu_2(T_gL_{g^{-1}}(v_1)) + \mu[T_gL_{g^{-1}}(v_1), T_gL_{g^{-1}}(v_2)]_{\mathfrak{g}}$$

for all $g \in G$, $\mu, \mu_1, \mu_2 \in \mathfrak{g}^*$ and $v_1, v_2 \in T_gG$ (see Ref. 1). Here, $[\cdot,\cdot]_{\mathfrak{g}}$ is the Lie algebra structure on $\mathfrak{g}$. Then, the Hamiltonian vector field $X^{\omega_G}_{Y^\ell} \in \mathfrak{X}(T^*G - 0_G)$ can be identified with the pair

$$(Y, \{\cdot, \xi^\ell\}_{\mathfrak{g}^*-\{0\}}) \in \mathfrak{X}(G) \times \mathfrak{X}(\mathfrak{g}^* - \{0\}),$$

where ξ^ℓ is the restriction to $\mathfrak{g}^* - \{0\}$ of the linear function $\xi^\ell : \mathfrak{g}^* \to \mathbb{R}$ induced by ξ and $\{\cdot,\cdot\}_{\mathfrak{g}^*-\{0\}}$ is the restriction to functions on $\mathfrak{g}^* - \{0\}$ of the Lie–Poisson bracket on $\mathfrak{g}^*$. We recall that this bracket is characterized by

$$\{\xi_1^\ell, \xi_2^\ell\}_{\mathfrak{g}^*}(\alpha) = -\alpha([\xi_1, \xi_2]_{\mathfrak{g}}), \tag{20}$$

with $\alpha \in \mathfrak{g}^*$ and $\xi_i \in \mathfrak{g}$ (for more details, see Ref. 1).

The reduced vector field after this reduction is just $(Y, X_{h_\xi}) \in \mathfrak{X}(G) \times \mathfrak{X}(\mathbb{P}\mathfrak{g}^*)$, such that

$$X_{h_\xi}(f) \circ p = \{f \circ p, \xi^\ell\}_{\mathfrak{g}^*-\{0\}}, \quad \forall f \in C^\infty(\mathbb{P}\mathfrak{g}^*), \tag{21}$$

which is the symbol of the derivation $[\cdot, h_\xi]_{(\mathcal{D}^o)^*}$.

A more explicit (local) expression of the vector field $X_{h_\xi} \in \mathfrak{X}(\mathbb{P}\mathfrak{g}^*)$ may be obtained as follows. For each $\nu \in \mathfrak{g} - \{0\}$, one can consider the coordinate open neighborhood $p(U)$ of $\mathbb{P}\mathfrak{g}^*$ with $U = \{\alpha \in \mathfrak{g}^* / \nu^\ell(\alpha) = \alpha(\nu) \neq 0\}$. On $p(U)$, the typical local coordinates in $\mathbb{P}\mathfrak{g}^*$ have the form $r(\zeta, \nu)$ characterized by

$$r(\zeta,\nu) \circ p = \frac{\zeta^\ell}{\nu^\ell}$$

with $\zeta \in \mathfrak{g} - \{0\}$. Moreover, using (20) and (21), we deduce that

$$X_{h_\xi}(r(\zeta,\nu)) \circ p = \frac{\zeta^\ell([\nu,\xi]_{\mathfrak{g}})^\ell - \nu^\ell([\zeta,\xi]_{\mathfrak{g}})^\ell}{(\nu^\ell)^2}. \tag{22}$$

Given the above facts, it is natural to ask if it is possible to extend the previous reduction to a Poisson Hamiltonian system, not necessarily symplectic. The following result gives an affirmative answer to this question. Before that, we introduce the notion of scaling symmetry for this kind of systems.

Definition 3. If (P, Π, H) is a Poisson Hamiltonian system on the Poisson manifold (P, Π), *a scaling symmetry for* (P, Π, H) is a principal action $\phi^P : \mathbb{R}^\times \times P \to P$ of the multiplicative group $\mathbb{R}^\times$ (with $\mathbb{R}^\times = \mathbb{R}^+$ or $\mathbb{R}^\times = \mathbb{R} - \{0\}$) on P such that the Poisson structure Π and the function H are homogeneous with respect to the action ϕ^P, that is,

$$\wedge^2 T\phi_s^P \circ \Pi = s\Pi \circ \phi_s^P \quad \text{and} \quad H \circ \phi_s^P = sH \quad \text{for } s \in \mathbb{R}^\times, \tag{23}$$

where $\wedge^2 T\phi_s^P : \wedge^2 TP \to \wedge^2 TP$ is the vector bundle isomorphism induced by $\phi_s^P : P \to P$.

The conditions in (23) are equivalent to the following ones:

$$\{F \circ \phi_s^P, G \circ \phi_s^P\}_P = s(\{F, G\}_P \circ \phi_s^P) \quad \text{and} \quad H \circ \phi_s^P = sH\,, \quad \text{for } F, G \in C^\infty(P) \text{ and } s \in \mathbb{R}^\times, \tag{24}$$

where $\{\cdot,\cdot\}_P$ is the Poisson bracket of functions on P.

We remark that (23) implies that the Poisson structure Π and the Hamiltonian function H satisfy (see Refs. 31, 43)

$$\mathcal{L}_{\Delta_P}\Pi = -\Pi \quad \text{and} \quad \mathcal{L}_{\Delta_P}H = H,$$

where Δ_P is the infinitesimal generator of ϕ^S. Moreover, if $\mathbb{R}^\times$ is connected (i.e., $\mathbb{R}^\times = \mathbb{R}^+$) the previous conditions are equivalent to (23). In addition, in the case of a symplectic manifold (S, ω), the condition

$$\wedge^2 T\phi_s^P \circ \Pi_\omega = s\Pi_\omega \circ \phi_s^P,$$

with Π_ω the Poisson structure induced by ω, is equivalent to $(\phi_s^P)^*\omega = s\omega$.

Now, we have the following result whose proof is similar to the proof in the symplectic case given in Ref. 30.

Theorem 1. *Let $\mathbf{p_P} : P \to K = P/\mathbb{R}^\times$ be a principal $\mathbb{R}^\times$-bundle with total space a homogeneous Poisson manifold (P, Π). If $\pi_L : L \to K$ is the line bundle associated with the principal bundle $\mathbf{p_P}$ (see the Appendix), then:*

(a) *There is a one-to-one correspondence between homogeneous functions $H : P \to \mathbb{R}$ and sections $h_H : L^* \to K$ of the dual line bundle $\pi_{L^*} : L^* \to K$ of π_L.*

(b) *On the space $\Gamma(L^*)$ of the sections of the line bundle $\pi_{L^*} : L^* \to K$, we have a Kirillov bracket*

$$[\cdot,\cdot]_{L^*} : \Gamma(L^*) \times \Gamma(L^*) \to \Gamma(L^*)$$

such that the Poisson bracket $\{H_1, H_2\}_P$ of two homogeneous functions $H_1, H_2 : P \to \mathbb{R}$ is just

$$\{H_1, H_2\}_P = -H_{[h_{H_1}, h_{H_2}]_{L^*}},$$

where $H_{[h_{H_1}, h_{H_2}]_{L^*}}$ is the homogeneous function on P associated with $[h_{H_1}, h_{H_2}]_{L^*}$.

(c) *The Hamiltonian vector field $X_H^{\{\cdot,\cdot\}_P} = -i(dH)\Pi \in \mathfrak{X}(P)$ of a homogeneous function H with respect to the Poisson bracket $\{\cdot,\cdot\}_P$ is $\mathbf{p_P}$-projectable and its projection is the symbol $X_{h_H}^{[\cdot,\cdot]_{L^*}} \in \mathfrak{X}(K)$ of the derivation $[\cdot, h_H]_{L^*}$, that is, the following diagram is commutative.*

$$\begin{array}{ccc} P & \xrightarrow{\mathbf{p_P}} & K \\ \Big\downarrow X_H^{\{\cdot,\cdot\}_P} & & \Big\downarrow X_{h_H}^{[\cdot,\cdot]_{L^*}} \\ TP & \xrightarrow{T\mathbf{p_P}} & TK \end{array}$$

(d) *We have that*

$$\left[X_{h_1}^{[\cdot,\cdot]_{L^*}}, X_{h_2}^{[\cdot,\cdot]_{L^*}}\right] = -X_{[h_1,h_2]_{L^*}}^{[\cdot,\cdot]_{L^*}}$$

for all $h_1, h_2 \in \Gamma(L^*)$.

Proof. For a proof of (a) see the Appendix.

If H_1, H_2 are two homogeneous functions then,

$$H_1 \circ \phi_s^P = sH_1 \text{ and } H_2 \circ \phi_s^P = sH_2 ,$$

and, using (24), we deduce that

$$\{H_1 \circ \phi_s^P, H_2 \circ \phi_s^P\}_P = s(\{H_1, H_2\}_P \circ \phi_s^P),$$

which implies that

$$s\{H_1, H_2\}_P = \{H_1, H_2\}_P \circ \phi_s^P,$$

that is, the function $\{H_1, H_2\}_P$ is homogeneous. Thus, the Poisson bracket $\{\cdot,\cdot\}_P$ is closed for homogeneous functions with respect to Δ_P.

Using this fact and Proposition A.1 (see the Appendix), we may define a bracket $[\cdot,\cdot]_{L^*} : \Gamma(L^*) \times \Gamma(L^*) \to \Gamma(L^*)$ on the space $\Gamma(L^*)$ of the sections of $\pi_{L^*} : L^* \to K$ which is characterized by

$$H_{[h_1,h_2]_{L^*}} = -\{H_{h_1}, H_{h_2}\}_P, \text{ with } h_1, h_2 \in \Gamma(L^*). \tag{25}$$

This bracket was described (up to the sign) in Ref. 29 (Theorem 3.2). Using the fact that the Poisson bracket $\{\cdot,\cdot\}_P$ defines a Lie algebra on the space of functions on P and (25), we deduce that $[\cdot,\cdot]_{L^*}$ is a Lie bracket. Moreover, for a C^∞ function $f : K \to \mathbb{R}$, from the properties of the Poisson bracket $\{\cdot,\cdot\}_P$, we have that

$$\begin{aligned} H_{[fh_1,h_2]_{L^*}} &= -\{H_{fh_1}, H_{h_2}\}_P = -\{(f\circ \mathbf{p_P})H_{h_1}, H_{h_2}\}_P \\ &= -(f\circ \mathbf{p_P})\{H_{h_1}, H_{h_2}\}_P - \{(f\circ \mathbf{p_P}), H_{h_2}\}_P H_{h_1} \\ &= (f\circ \mathbf{p_P})H_{[h_1,h_2]_{L^*}} + X_{H_{h_2}}^{\{\cdot,\cdot\}_P}(f\circ \mathbf{p_P})H_{h_1}. \end{aligned} \tag{26}$$

On the other hand, using the homogeneity of Π and H_{h_2}, we deduce that

$$\mathcal{L}_{\Delta_P} X_{H_{h_2}}^{\{\cdot,\cdot\}_P} = -\mathcal{L}_{\Delta_P} i_{dH_{h_2}} \Pi = -i_{dH_{h_2}} \mathcal{L}_{\Delta_P} \Pi - i(d(\Delta_P(H_{h_2})))\Pi = 0,$$

or, equivalently, $X_{H_{h_2}}^{\{\cdot,\cdot\}_P}$ is $\mathbf{p_P}$-projectable. Then, there is a vector field $X_{h_2}^{[\cdot,\cdot]_{L^*}}$ on K such that

$$X_{h_2}^{[\cdot,\cdot]_{L^*}} \circ \mathbf{p_P} = T\mathbf{p_P} \circ X_{H_{h_2}}^{\{\cdot,\cdot\}_P}. \tag{27}$$

From (26) and (27), we have that

$$H_{[fh_1,h_2]_{L^*}} = (f\circ \mathbf{p_P})H_{[h_1,h_2]_{L^*}} + (X_{h_2}^{[\cdot,\cdot]_{L^*}}(f)\circ \mathbf{p_P})H_{h_1},$$

and consequently (see (9)) we have a Kirillov structure on the space of sections of $\pi_{L^*} : L^* \to K$ and the symbol of $[\cdot, h]_{L^*}$ is just the $\mathbf{p_P}$-projection on K of the Hamiltonian vector field $X_{H_h}^{\{\cdot,\cdot\}_P}$. This proves (b) and (c).

Finally, from (27) and using that $\left[X^{\{\cdot,\cdot\}_P}_{H_{h_1}}, X^{\{\cdot,\cdot\}_P}_{H_{h_2}}\right] = -X^{\{\cdot,\cdot\}_P}_{\{H_{h_1},H_{h_2}\}_P}$, we have that

$$\left[X^{[\cdot,\cdot]_{L^*}}_{h_1}, X^{[\cdot,\cdot]_{L^*}}_{h_2}\right] = -X^{[\cdot,\cdot]_{L^*}}_{[h_1,h_2]_{L^*}}.$$

Therefore, we deduce (d). □

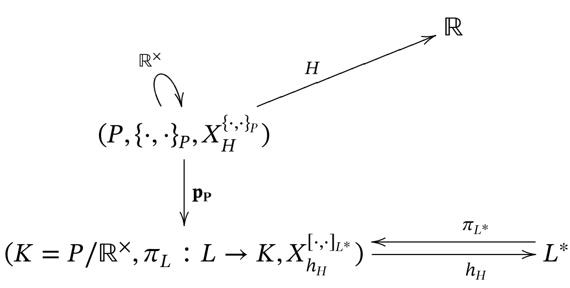
Remark 2. In Ref. 31, Marle proves that if $\pi_L : L \to K$ is a line bundle endowed with a Kirillov structure – (L^*, π_{L^*}, K) is a Jacobi bundle in his terminology—and $h : K \to L^*$ is a section of π_{L^*}, then one can induce a Poisson structure Π on L^* (which is homogeneous with respect to the negative of the Euler vector field Δ on L^*), a differentiable function $H : P := (L^* - 0_{L^*}) \to \mathbb{R}$ and a vector field X on L^* such that:

- The restriction of X to P is just the Hamiltonian vector field induced by Π and H.
- The vector field X projects on a vector field X_h on K

(see Theorem 4.3 and Proposition 4.7 in Ref. 31). Therefore, if the flow of Δ induces a principal action on P, then we have a Poisson Hamiltonian system (P, Π, H) with a scaling symmetry in such a way that the corresponding reduced Kirillov Hamiltonian system is just the original system. So, Marle's result may be considered as a converse of Theorem 1.

Remark 3. In Ref. 54 (see Theorem 2.2.6 of Ref. 54), the authors obtain a one-to-one correspondence between Atiyah (l, m)-tensors on a line bundle and homogeneous (l, m)-tensors on its slit dual bundle (the dual bundle with the zero section removed). Using this general result, one could prove that there exists a one-to-one correspondence between Kirillov structures on the line bundle and homogeneous Poisson structures on its slit dual bundle (see Example 2.4.2 in Ref. 54). Anyway, in order to have our paper more self-contained, we have included a direct and simple proof of the items *(a), (b), (c)*, and *(d)* of Theorem 1.

The following diagram summarizes Theorem 1.



4 | REDUCTION OF SYMPLECTIC HAMILTONIAN SYSTEMS USING FIRST STANDARD SYMMETRIES AND THEN SCALING SYMMETRIES

In this section, we will discuss the reduction of symplectic Hamiltonian systems which are invariant under the action of a symmetry Lie group and, in addition, admit a scaling symmetry. The standard and the scaling symmetries will be compatible in the following sense.

Definition 4. Let (S, ω, H) be a symplectic Hamiltonian system. Suppose that $\phi^S : \mathbb{R}^\times \times S \to S$ is a scaling symmetry on (S, ω, H). In addition, suppose that we have a Lie group G and a G-principal bundle $\wp_S : S \to S/G$ such that the corresponding action $\Phi^S : G \times S \to S$ on the symplectic manifold S satisfies:

(i) $(\Phi^S_g)^*(\omega) = \omega$, for $g \in G$, that is, the action Φ^S is symplectic.
(ii) $H : S \to \mathbb{R}$ is G-invariant, that is, $H(\Phi^S(g, x)) = H(x)$, for all $x \in S$ and $g \in G$.
(iii) The symplectic and the scaling actions commute, that is, $\Phi^S_g \circ \phi^S_s = \phi^S_s \circ \Phi^S_g$, for all $s \in \mathbb{R}^\times$ and $g \in G$.
(iv) $\mathbb{R}^\times x \cap Gx = \{x\}$, for all $x \in S$, where $\mathbb{R}^\times x$ is the $\mathbb{R}^\times$-orbit at x and Gx is the G-orbit at x.
(v) The quotient space $(S/\mathbb{R}^\times)/G$ induced by the action $\Phi^{S/\mathbb{R}^\times} : G \times S/\mathbb{R}^\times \to S/\mathbb{R}^\times$

$$\Phi^{S/\mathbb{R}^\times}_g [x] = [\Phi^S_g(x)]$$

is Hausdorff.

In this case, we say that the dynamical system (S, ω, H) admits a scaling symmetry $\phi^S : \mathbb{R}^\times \times S \to S$ and a symplectic G-symmetry $\Phi^S : G \times S \to S$ which are compatible.

Remark 4.

(i) Note that the previous conditions (i) and (ii) imply that

$$\mathcal{L}_{\xi_S} \Pi_\omega = 0 \quad \text{and} \quad \mathcal{L}_{\xi_S} H = 0, \tag{28}$$

where ξ_S is the infinitesimal generator of the action Φ^S associated with an element ξ of the Lie algebra $\mathfrak{g}$ of G and Π_ω is the Poisson bivector on S induced by the symplectic structure ω. If G is connected, then the conditions (i) and (ii) are equivalent to (28).

(ii) We remark that, from (iii) in Definition 4, the G-action $\Phi^{S/\mathbb{R}^\times}$ on the quotient manifold $S/\mathbb{R}^\times$ is well-defined. Moreover, using (iv), we have that this action is free and, thus,

$$Graph(\Phi^{S/\mathbb{R}^\times}) = \{([x], [\Phi^S_g(x)]) \in S/\mathbb{R}^\times \times S/\mathbb{R}^\times / x \in S, g \in G\}$$

is a submanifold of $S/\mathbb{R}^\times \times S/\mathbb{R}^\times$. In addition, condition (v) is equivalent to the fact that $Graph(\Phi^{S/\mathbb{R}^\times})$ is a closed submanifold of $S/\mathbb{R}^\times \times S/\mathbb{R}^\times$ and, therefore, $(S/\mathbb{R}^\times)/G$ is a quotient manifold (see Theorem 4.1.20 in Ref. 1). On the other hand, if $\mathbf{p_S} : S \to S/\mathbb{R}^\times$ is the canonical projection then, using *(iii)* in Definition 4, we have that

$$(\mathbf{p}_S \times \mathbf{p}_S)^{-1}(Graph(\Phi^{S/\mathbb{R}^\times})) = \{(x, \Phi^S_g(\phi^S_s(x))) \in S \times S / x \in S, s \in \mathbb{R}^\times \text{ and } g \in G\}.$$

So, condition (v) is equivalent to

$$Graph(\Phi^S, \phi^S) := \{(x, \Phi^S_g(\phi^S_s(x))) \in S \times S / x \in S, s \in \mathbb{R}^\times \text{ and } g \in G\}$$

is a closed subset of $S \times S$.

(iii) Using again (iii), we can induce an $\mathbb{R}^\times$-action $\Phi^{S/G}$ on the quotient manifold S/G and, as in the previous case, the quotient space $(S/G)/\mathbb{R}^\times$ is Hausdorff if and only if $Graph(\Phi^S, \phi^S)$ is a closed submanifold of $S \times S$. So, the spaces $(S/\mathbb{R}^\times)/G$ and $(S/G)/\mathbb{R}^\times$ are smooth manifolds and the canonical projections $S/G \to (S/G)/\mathbb{R}^\times$ and $S/\mathbb{R}^\times \to (S/\mathbb{R}^\times)/G$ are surjective submersions.

(iv) Using that the scaling action and the G-action commute, it is easy to prove that the condition (iv) in Definition 4 is equivalent to the fact that the actions $\Phi^{S/\mathbb{R}^\times}$ and $\phi^{S/G}$ are free. Thus, the conditions in Definition 4 imply that these actions are principal.

(v) From the previous comments, it follows that we have two chains of principal bundles

$$S \longrightarrow S/\mathbb{R}^\times \longrightarrow (S/\mathbb{R}^\times)/G \text{ and } S \longrightarrow S/G \longrightarrow (S/G)/\mathbb{R}^\times$$

(this will be the situation in our examples). So, one could consider a more general point of view using double principal bundles (see Ref. 55).

4.1 | The first step: Reduction by standard symmetries

It is well-known (see Ref. 36) that the symplectic structure on S induces a Poisson bracket $\{\cdot,\cdot\}_P$ on the quotient manifold $P := S/G$ characterized by

$$\{f_1 \circ \wp_S, f_2 \circ \wp_S\}_S = \{f_1, f_2\}_P \circ \wp_S \tag{29}$$

with $f_i \in C^\infty(P)$, where $\{\cdot,\cdot\}_S$ is the Poisson bracket induced by the symplectic structure ω on S. Consequently, the Poisson structure Π_P on P and the Poisson structure Π_ω induced by the symplectic structure ω are related as follows:

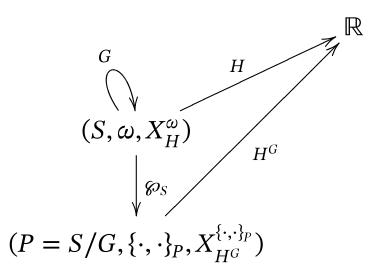
$$\wedge^2 T\wp_S \circ \Pi_\omega = \Pi_P \circ \wp_S. \tag{30}$$

In addition, from the G-invariance of H, there is a reduced Hamiltonian function $H^G : P \to \mathbb{R}$ such that

$$H^G \circ \wp_S = H. \tag{31}$$

Moreover, the Hamiltonian vector field $X_H^\omega \in \mathfrak{X}(S)$ is $\wp_S$-projectable and its projection is just the Hamiltonian vector field $X_{H^G}^{\{\cdot,\cdot\}_P} = \{\cdot, H^G\}_P \in \mathfrak{X}(P)$ associated with the Poisson structure Π_P.

The following diagram summarizes this first reduction process.



On the other hand, using that $\Phi^S_g \circ \phi^S_s = \phi^S_s \circ \Phi^S_g$, for all $s \in \mathbb{R}^\times$ and $g \in G$, the $\mathbb{R}^\times$-action ϕ^S induces an action $\phi^P : \mathbb{R}^\times \times P \to P$ characterized by

$$\phi^P_s(\wp_S(x)) = \wp_S(\phi^S_s(x)), \text{ for all } x \in S \text{ and } s \in \mathbb{R}^\times. \tag{32}$$

As we know, ϕ^P is a principal action (see Remark 4). In fact, we have

Proposition 1. *ϕ^P is a scaling symmetry for the Poisson Hamiltonian system (P, Π_P, H^G).*

Proof. Given $s \in \mathbb{R}^\times$, using (30) and (32), it follows that

$$\wedge^2 T\phi^P_s \circ \Pi_P \circ \wp_S = \wedge^2 T\phi^P_s \circ \wedge^2 T\wp_S \circ \Pi_\omega = \wedge^2 T\wp_S \circ \wedge^2 T\phi^S_s \circ \Pi_\omega.$$

Now, since ϕ^S is a scaling symmetry for the symplectic manifold (S, ω), we deduce that

$$\wedge^2 T\phi^P_s \circ \Pi_P \circ \wp_S = s \wedge^2 T\wp_S \circ \Pi_\omega \circ \phi^S_s$$

and, using again (30), we obtain that

$$\wedge^2 T\phi^P_s \circ \Pi_P \circ \wp_S = s\Pi_P \circ \phi^P_s \circ \wp_S.$$

This implies that

$$\wedge^2 T\phi^P_s \circ \Pi_P = s\Pi_P \circ \phi^P_s.$$

On the other hand, from (31) and (32), it follows that

$$H^G \circ \phi^P_s \circ \wp_S = H^G \circ \wp_S \circ \phi^S_s = H \circ \phi^S_s$$

and, since H is a homogeneous function for the action ϕ^S, we deduce that

$$H^G \circ \phi^P_s \circ \wp_S = sH = sH^G \circ \wp_S,$$

where for the last equality we use again (31). This implies that

$$H^G \circ \phi^P_s = sH = sH^G,$$

which ends the proof of the result. □

Now, we may apply the scaling reduction process.

4.2 | The second step: Reduction by scaling symmetry

Consider the Poisson Hamiltonian system (P, Π_P, H^G) obtained in the previous subsection by reduction from the symplectic Hamiltonian system (S, ω, H). In the second step of the reduction process, we will apply Theorem 1 to the Poisson Hamiltonian system (P, Π_P, H^G) and the scaling symmetry $\phi^P : \mathbb{R}^\times \times P \to P$.

The complete reduction process is described in the following theorem.

Theorem 2. *Let (S, ω, H) be a symplectic Hamiltonian system with compatible scaling symmetry $\phi^S : \mathbb{R}^\times \times S \to S$ and symplectic G-symmetry $\Phi^S : G \times S \to S$, G being a Lie group. Then:*

(1) The multiplicative group $\mathbb{R}^\times$ acts on the Poisson manifold $P = S/G$ such that the corresponding quotient map $\mathbf{p_P} : P \to P/\mathbb{R}^\times$ is a $\mathbb{R}^\times$-principal bundle. Moreover, if $\pi_L : L \to K = P/\mathbb{R}^\times$ is the line bundle associated with $\mathbf{p_P} : P \to K = P/\mathbb{R}^\times$, then the homogeneous function $H^G : P \to \mathbb{R}$ induces a section $h_{H^G} : K \to L^*$ of the dual line bundle $\pi_{L^*} : L^* \to K$ of π_L.

(2) On the space of sections $\Gamma(L^*)$ of $\pi_{L^*} : L^* \to K$, we have a Kirillov bracket

$$[\cdot,\cdot]_{L^*} : \Gamma(L^*) \times \Gamma(L^*) \to \Gamma(L^*)$$

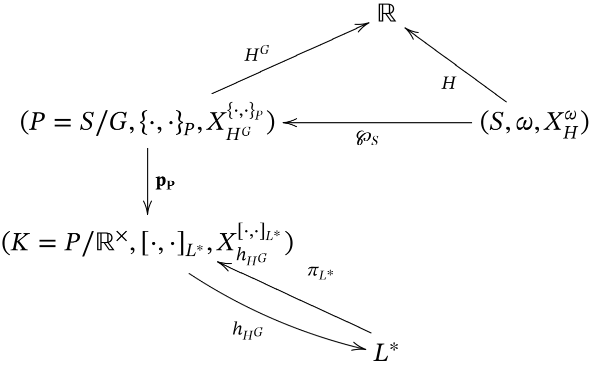
such that, if $\{\cdot,\cdot\}_P$ is the Poisson bracket on P,

$$[h_{H_1^G}, h_{H_2^G}]_{L^*} = -h_{\{H_1^G, H_2^G\}_P}$$

for $H_1^G, H_2^G \in C^\infty(P)$ homogeneous functions on P.

(3) The Hamiltonian vector field X_H^ω is $(\mathbf{p_P} \circ \wp_S)$-projectable on K and its projection is the symbol $X_{h_{H^G}}^{[\cdot,\cdot]_{L^*}} \in \mathfrak{X}(K)$ of $[\cdot, h_{H^G}]_{L^*}$.

The following diagram illustrates both reduction processes together.



4.3 | Examples

In this subsection, we will apply the previous reduction processes to Examples 4 and 5.

Example 6. Continuing Example 4: The 2D harmonic oscillator reduced first by a standard and then by a scaling symmetry. In this case, we have:

(1) A standard rotational S^1-symmetry, with infinitesimal generator $\xi_S = x\partial_y - y\partial_x + p_x\partial_{p_y} - p_y\partial_{p_x}$, where (x, y, p_x, p_y) are coordinates on $S = T^*(\mathbb{R}^2 - \{(0,0)\})$. Using the identification $\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 \cong T^*(\mathbb{R}^2 - \{(0,0)\}) - 0_{\mathbb{R}^2 - \{(0,0)\}}$, the local expression of ξ_S is

$$\xi_S = \partial_\theta + \partial_{\theta'},$$

where (r, θ, r', θ') are polar coordinates on $\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1$.

(2) A scaling $\mathbb{R}^+$-symmetry, with generator

$$\Delta_S = \frac{1}{2}(r\partial_r + r'\partial_{r'}).$$

One can also easily check that $[\xi_S, \Delta_S] = 0$ and thus, since the multiplicative group $\mathbb{R}^+$ and S^1 are connected, the two symmetries commute. Therefore, the corresponding actions satisfy (i), (ii), and (iii) of Definition 4. We shall see below that they also satisfy (iv) and (v), so we can apply Theorem 2. In order to highlight all the mechanisms involved, we will proceed by steps and indicate the main derivations.

In the first step, with the S^1-symmetry, the reduced objects are:

- **The reduced space**: We perform the reduction by the standard symmetry, obtaining the Poisson system (P, Π_P, H^G). First, we have that the symplectomorphism

$$\begin{array}{rcl} \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 & \to & S^1 \times (\mathbb{R}^+ \times \mathbb{R}^+ \times S^1) \\ ((r, \theta), (r', \theta')) & \to & (\theta, (r, r', \alpha)) = (\theta, (r, r', \theta - \theta')) \end{array}$$

transforms ξ_S into ∂_θ. Using this identification, the quotient manifold $(\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1)/S^1$ is just

$$P = \mathbb{R}^+ \times \mathbb{R}^+ \times S^1$$

and the reduced Poisson structure on P is given by (see (13))

$$\Pi_P(r, r', \alpha) = -\cos\alpha\ \partial_r \wedge \partial_{r'} + \frac{\sin\alpha}{r'}\partial_r \wedge \partial_\alpha - \frac{\sin\alpha}{r}\partial_{r'} \wedge \partial_\alpha, \tag{33}$$

where (r, r', α) are local coordinates on $\mathbb{R}^+ \times \mathbb{R}^+ \times S^1$.

- **The reduced Hamiltonian function**: The reduced Hamiltonian function is

$$H^{S^1}(r, r', \alpha) = \frac{1}{2}(r^2 + (r')^2). \tag{34}$$

- **The reduced dynamics**: The corresponding Hamiltonian vector field on P is just

$$X^{\{\cdot,\cdot\}_P}_{H^{S^1}} = r\cos\alpha\ \partial_{r'} - r'\cos\alpha\ \partial_r - \sin\alpha\left(\frac{r}{r'} - \frac{r'}{r}\right)\partial_\alpha.$$

- **The scaling symmetry on the reduced space**: The projection on P of the scaling symmetry Δ_S is

$$\Delta_P = \frac{1}{2}(r\partial_r + r'\partial_{r'}),$$

which generates the scaling action $\phi^P : \mathbb{R}^+ \times (\mathbb{R}^+ \times \mathbb{R}^+ \times S^1) \to (\mathbb{R}^+ \times \mathbb{R}^+ \times S^1)$ given by

$$\phi^P(s,(r,r',\theta)) = (\sqrt{s}r, \sqrt{s}r', \theta).$$

Note that this action is free.

Now, using Theorem 1, we can further reduce again the system (second step) with this last scaling symmetry. We obtain:

- **The reduced space**: Consider the diffeomorphim

$$\begin{array}{ccc} \mathbb{R}^+ \times \mathbb{R}^+ \times S^1 & \to & \mathbb{R}^+ \times (\mathbb{R}^+ \times S^1) \\ (r,r',\alpha) & \to & (\rho,\rho',\sigma) = \left(r, \dfrac{r'}{r}, \alpha\right), \end{array}$$

which transforms the generator Δ_P of the $\mathbb{R}^+$-action on $\mathbb{R}^+ \times \mathbb{R}^+ \times S^1$ into the vector field

$$\frac{1}{2}\rho\partial_\rho$$

with (ρ,ρ',σ) local coordinates on $\mathbb{R}^+ \times (\mathbb{R}^+ \times S^1)$.

Thus, the space of orbits of the reduced $\mathbb{R}^+$-action may be identified with

$$K = \mathbb{R}^+ \times S^1$$

and, under this identification, the canonical projection is

$$\mathbf{p_P} : P = \mathbb{R}^+ \times \mathbb{R}^+ \times S^1 \to K = \mathbb{R}^+ \times S^1, \qquad \mathbf{p_P}(r,r',\alpha) = \left(\frac{r'}{r}, \alpha\right).$$

In particular, the action ϕ^P of $\mathbb{R}^\times$ on P is a principal one and the standard S^1-symmetry and the scaling $\mathbb{R}^+$-symmetry are compatible.

The associated line bundle is trivial

$$\pi_L : L := \mathbb{R} \times \mathbb{R}^+ \times S^1 \to \mathbb{R}^+ \times S^1, \qquad \pi_L(t,\rho',\sigma) = (\rho',\sigma),$$

and therefore, we have a Jacobi bracket on the space of functions on K. In the sequel, we will describe this structure.

The expression of the reduced Poisson structure on P in terms of the new local coordinates (ρ,ρ',σ) is (see (33))

$$\Pi_P = -\frac{\cos\sigma}{\rho}\partial_\rho \wedge \partial_{\rho'} + \frac{\sin\sigma}{\rho\rho'}\partial_\rho \wedge \partial_\sigma - 2\frac{\sin\sigma}{\rho^2}\partial_{\rho'} \wedge \partial_\sigma. \tag{35}$$

Note that

$$\mathcal{L}_{\frac{1}{2}\rho\partial_\rho}\Pi_P = -\Pi_P.$$

Since the homogenous functions with respect to the vector field $\frac{1}{2}\rho\partial_\rho$ are of the form $\rho^2 h$, with $h \in C^\infty(\mathbb{R}^+ \times S^1)$, then we have that

$$\{\rho^2 h, \rho^2 h'\}_P = \frac{1}{2}\rho\partial_\rho\{\rho^2 h, \rho^2 h'\}_P$$

for all $h, h' \in C^\infty(\mathbb{R}^+ \times S^1)$. This implies that the Jacobi bracket $\{\cdot,\cdot\}_K$ on the space of functions on K satisfies

$$\{\rho^2 h, \rho^2 h'\}_P = \rho^2\{h, h'\}_K, \quad h, h' \in C^\infty(\mathbb{R}^+ \times S^1).$$

As a consequence (see (35)),

$$\{h, h'\}_K = -2\cos\sigma(h\partial_{\rho'}h' - h'\partial_{\rho'}h) + 2\frac{\sin\sigma}{\rho'}(h\partial_\sigma h' - h'\partial_\sigma h) - 2\sin\sigma(\partial_{\rho'}h\partial_\sigma h' - \partial_\sigma h\partial_{\rho'}h').$$

Therefore, the corresponding Jacobi structure (Π_K, E_K) is

$$\Pi_K = -2\sin\sigma\partial_{\rho'} \wedge \partial_\sigma, \qquad E_K = -2\cos\sigma\partial_{\rho'} + 2\frac{\sin\sigma}{\rho'}\partial_\sigma. \tag{36}$$

- **The reduced Hamiltonian function**: The Hamiltonian function H^{S^1} (see (34)), in terms of the local coordinates (ρ, ρ', σ), is

$$H^{S^1}(\rho, \rho', \sigma) = \frac{\rho^2}{2}(1 + (\rho')^2).$$

Since

$$\frac{1}{2}\rho\partial_\rho H^{S^1} = H^{S^1},$$

we deduce that $H^{S^1}(\rho^2, \rho', \sigma) = \rho^2 h_{H^{S^1}}(\rho', \sigma)$ and therefore

$$h_{H^{S^1}}(\rho', \sigma) = \frac{1}{2}(1 + (\rho')^2).$$

- **The reduced dynamics**: The Hamiltonian vector field induced by the previous Jacobi structure and the function $h_{H^{S^1}}$ is

$$X^{\{\cdot,\cdot\}_K}_{h_{H^{S^1}}} = -i(dh_{H^{S^1}})\Pi_K - h_{H^{S^1}}E_K = (1 + (\rho')^2)\cos\sigma\,\partial_{\rho'} + \frac{(\rho')^2 - 1}{\rho'}\sin\sigma\,\partial_\sigma, \tag{37}$$

which is the $\mathbf{p_P}$-projection of $X^{\{\cdot,\cdot\}_P}_{H^{S^1}}$.

Example 7 (Continuing Example 5: The linear Hamiltonian system reduced first by a standard and then by a scaling symmetry). Let $\Phi : G \times Q \to Q$ be a free and proper action of a Lie group G on a manifold Q. Denote by 0_Q the zero section of the cotangent bundle $\tau_Q^* : T^*Q \to Q$ and by $T^*\Phi : G \times (T^*Q - 0_Q) \to (T^*Q - 0_Q)$ the restriction to $T^*Q - 0_Q$ of the cotangent lift action, that is, the free and proper action given by

$$(T^*\Phi)_g(\alpha_q) = (T_{\Phi_g(q)}\Phi_{g^{-1}})^*(\alpha_q), \quad \forall g \in G \text{ and } \forall \alpha_q \in T_q^*Q - 0_q. \tag{38}$$

It is well-known that $(T^*\Phi)_g$ is a symplectomorphim with respect to the standard symplectic structure ω_Q on $T^*Q - 0_Q$.

Suppose that $Y \in \mathfrak{X}(Q)$ is a G-invariant vector field on Q, that is,

$$T_q\Phi_g(Y(q)) = Y(\Phi_g(q)), \quad g \in G \text{ and } q \in Q. \tag{39}$$

Moreover, let $\phi : \mathbb{R} - \{0\} \times (T^*Q - 0_Q) \to (T^*Q - 0_Q)$ be the action given by (8).

A direct computation, using (38) and (39), shows that the fiberwise-linear function $Y^\ell : T^*Q \to \mathbb{R}$ induced by Y is G-invariant, that is,

$$Y^\ell \circ (T^*\Phi)_g = Y^\ell.$$

The symplectic action $T^*\Phi$ is fiberwise linear. So,

$$(T^*\Phi)_g \circ \phi_s = \phi_s \circ (T^*\Phi)_g, \text{ for } g \in G \text{ and } s \in \mathbb{R} - \{0\}.$$

On the other hand, we have that $G\alpha_x \cap (\mathbb{R} - \{0\})\alpha_x = \{\alpha_x\}$. In fact, if $(T^*\Phi)_g(\alpha_x) = s\alpha_x$, for $g \in G$ and $s \in \mathbb{R}$, then $\Phi_g(x) = x$. Therefore, since Φ is free, $g = e$ and $s = 1$.

Moreover, the quotient space $((T^*Q - 0_Q)/G)/\mathbb{R}^\times$ can be identified with the projective bundle $\mathbb{P}(T^*Q/G)$ of the vector bundle $(\tau_Q^*)^G : (T^*Q - 0_G)/G \to Q/G$ (see Remark 1).

Thus, the previous comments imply that the actions $T^*\Phi$ and ϕ are compatible and the conditions of Theorem 2 hold. Now, we will reduce the Hamiltonian symplectic system $(T^*Q - 0_Q, \omega_Q, Y^\ell)$, first by $T^*\Phi$ and then by the scaling symmetry ϕ. The objets obtained after the G-reduction are:

- **The reduced space**: The restriction of the canonical projection $\tau_Q^* : T^*Q \to Q$ to $T^*Q - 0_Q$ is G-equivariant and therefore it induces a fibration

$$\tau_Q^G : P := (T^*Q - 0_Q)/G \to Q/G,$$

which is just the restriction of the Atiyah bundle $\tau_Q^G : T^*Q/G \to Q/G$ to $T^*Q/G - O$, with O the zero section of this vector bundle. The Poisson bracket $\{\cdot,\cdot\}_P$ on the space of functions $C^\infty((T^*Q - 0_Q)/G)$ is characterized by $\{f \circ \wp, g \circ \wp\}_{\omega_Q} = \{f, g\}_P \circ \wp$ with

$$\wp : (T^*Q - 0_Q) \to (T^*Q - 0_Q)/G$$

the quotient map.

- **The reduced Hamiltonian function**: The G-invariant function Y^ℓ induces a function $(Y^\ell)^G : (T^*Q - 0_Q)/G \to \mathbb{R}$ such that

$$(Y^\ell)^G(\wp(\alpha)) = Y^\ell(\alpha).$$

- **The reduced dynamics**: The Hamiltonian vector field $X^{\omega_Q}_{Y^\ell}$ is $\wp$-projectable and its projection is just

$$X^{\{\cdot,\cdot\}_P}_{(Y^\ell)^G} = \{\cdot, (Y^\ell)^G\}_P.$$

- **The scaling symmetry on the reduced space**: The scaling symmetry $\phi : (\mathbb{R} - \{0\}) \times (T^*Q - 0_Q) \to (T^*Q - 0_Q)$ induces a scaling symmetry $\phi^G : (\mathbb{R} - \{0\}) \times P \to P$ for the reduced Poisson Hamiltonian system $(P, \{\cdot,\cdot\}_P, (Y^\ell)^G)$ which is given by

$$\phi^G(s, \wp(\alpha)) = \wp(s\alpha), \text{ for } s \in \mathbb{R} - \{0\} \text{ and } \alpha \in T^*Q - 0_Q.$$

Now, we will apply the second reduction step to the Poisson Hamiltonian system $(P = (T^*Q - 0_Q)/G, \Pi_P, (Y^\ell)^G)$ with respect to the scaling symmetry $\phi^G : (\mathbb{R} - \{0\}) \times P \to P$. The reduced objects in this second reduction are:

- **The reduced space**: In this case, the reduced space is the projective bundle $\mathbb{P}(T^*Q/G) = ((T^*Q - 0_Q)/G)/(\mathbb{R} - \{0\})$ of the vector bundle $(\tau_Q^*)^G : (T^*Q - 0_G)/G \to Q/G$ (see Remark 1).
- **The reduced Hamiltonian section**: Denote by $\pi_L : L \to \mathbb{P}(T^*Q/G)$ the line bundle associated with $\mathbf{p_P} : (T^*Q - 0_Q)/G \to \mathbb{P}(T^*Q/G)$. The section of the dual bundle $\pi_{L^*} : L^* \to \mathbb{P}(T^*Q/G)$ induced by the homogeneous function $(Y^\ell)^G \in C^\infty((T^*Q - 0_Q)/G)$ is the reduced Hamiltonian section.
- **The reduced dynamics**: The Hamiltonian vector field $X^{\{\cdot,\cdot\}_P}_{(Y^\ell)^G}$ is $\mathbf{p_P}$-projectable and it determines the final reduced dynamics.

The particular case of a Lie group. In what follows, we will show the previous reduction process in the particular case when the initial manifold Q is a Lie group G. In such a case, one may use the left trivialization of the cotangent bundle T^*G in order to identify T^*G with the product manifold $G \times \mathfrak{g}^*$, where $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ is the Lie algebra of G, in such a way that the canonical projection $\tau_G^* : T^*G \to G$ is just the first projection $p_1 : G \times \mathfrak{g}^* \to G$.

The left action $\Phi : G \times G \to G$ on G is the one defined by the group operation of G. We take the left invariant vector field $Y = \overleftarrow{\xi}$ on G induced by an element ξ of $\mathfrak{g}$. In the first reduction with the cotangent lift of Φ, the reduced space is $(T^*G - 0_G)/G \cong \mathfrak{g}^* - \{0\}$ and the reduced function induced by Y is the restriction to $\mathfrak{g}^* - \{0\}$ of the linear map ξ^ℓ associated with $\xi \in \mathfrak{g}$, that is,

$$\xi^\ell : \mathfrak{g}^* - \{0\} \to \mathbb{R}, \quad \xi^\ell(\alpha) = \alpha(\xi).$$

On the other hand, the Lie–Poisson bracket $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ on $(T^*G - 0_G)/G \cong \mathfrak{g}^* - \{0\}$ is characterized by

$$\{{\xi_1}^\ell, {\xi_2}^\ell\}_{\mathfrak{g}^*} = -[\xi_1, \xi_2]_\mathfrak{g}^\ell, \quad \text{for all } \xi_1, \xi_2 \in \mathfrak{g}.$$

The scaling symmetry on $\mathfrak{g}^* - \{0\}$ is just

$$\phi^G : (\mathbb{R} - \{0\}) \times (\mathfrak{g}^* - \{0\}) \to (\mathfrak{g}^* - \{0\}), \qquad (s, \alpha) \to s\alpha. \tag{40}$$

Now, we apply the second reduction step to the (Lie)–Poisson Hamiltonian system $(\mathfrak{g}^* - \{0\}, \{\cdot,\cdot\}_{\mathfrak{g}^*}, \xi^\ell)$, with respect to the scaling symmetry ϕ^G. In this case, the reduced space is the projective space $\mathbb{P}\mathfrak{g}^*$. The corresponding line bundle $\pi_L : L := (\mathfrak{g}^* - \{0\} \times \mathbb{R})/(\mathbb{R} - \{0\}) \to \mathbb{P}\mathfrak{g}^*$ is defined by the action

$$\widetilde{\phi}^G : (\mathbb{R} - \{0\}) \times ((\mathfrak{g}^* - \{0\}) \times \mathbb{R}) \to (\mathfrak{g}^* - \{0\}) \times \mathbb{R}, \qquad \widetilde{\phi}^G_s(\alpha, t) = \left(s\alpha, \frac{t}{s}\right).$$

The section of the dual line bundle $\pi_{L^*} : L^* \to \mathbb{P}\mathfrak{g}^*$ associated with the linear map $\xi^\ell : \mathfrak{g}^* - \{0\} \to \mathbb{R}$ is

$$h_\xi(p(\alpha))([(\alpha, t)]) = t\alpha(\xi),$$

with $[(\alpha, t)] \in L$, where $p : (\mathfrak{g}^* - 0) \to \mathbb{P}\mathfrak{g}^*$ is the quotient projection.

The Kirillov bracket on the projective space $\mathbb{P}\mathfrak{g}^*$ is characterized by

$$\begin{aligned} [h_{\xi_1}, h_{\xi_2}]_{\mathbb{P}\mathfrak{g}^*}(p(\alpha))([(\alpha, t)]) &= -h_{\{\xi_1^\ell, \xi_2^\ell\}_{\mathfrak{g}^*}}(p(\alpha))([(\alpha, t)]) = -t\{\xi_1^\ell, \xi_2^\ell\}_{\mathfrak{g}^*}(\alpha) \\ &= t\alpha([\xi_1, \xi_2]_{\mathfrak{g}}) = h_{[\xi_1, \xi_2]_{\mathfrak{g}}}(p(\alpha))([(\alpha, t)]). \end{aligned}$$

This structure on the line bundle $L \to \mathbb{P}\mathfrak{g}^*$ may be considered as the Kirillov version of the Lie–Poisson structure on $\mathfrak{g}^*$ and for this reason we will use the terminology *the Lie–Kirillov structure on* $\mathbb{P}\mathfrak{g}^*$.

The reduced dynamics is determined by the p-projection of the Lie–Poisson Hamiltonian vector field associated with the linear function $\xi^\ell \in C^\infty(\mathfrak{g}^* - \{0\})$, that is,

$$X^{\{\cdot,\cdot\}_{\mathfrak{g}^*}}_{\xi^\ell} = \{\cdot, \xi^\ell\}_{\mathfrak{g}^*}.$$

Note however, that this p-projection of $X^{\{\cdot,\cdot\}_{\mathfrak{g}^*}}_{\xi^\ell}$ is just the vector field $X_{h_\xi} \in \mathfrak{X}(\mathbb{P}\mathfrak{g}^*)$, which is locally characterized by (22).

5 | REDUCTION OF SYMPLECTIC HAMILTONIAN SYSTEMS USING FIRST THE SCALING SYMMETRY AND THEN THE STANDARD SYMMETRIES

As in the previous section, we have a symplectic Hamiltonian system (S, ω, H) with a scaling symmetry $\phi^S : \mathbb{R}^\times \times S \to S$ and a symplectic G-symmetry $\Phi^S : G \times S \to S$ which are compatible. In what follows we describe the reduction process of the system (S, ω, H) in two steps, but in the following order: the first reduction is obtained by the scaling symmetry and the second step is done using the standard symmetry.

First of all, we will show a reduction process for Kirillov structures in the presence of a standard symmetry.

5.1 | Reduction of Kirillov structures by standard symmetries

Let $\pi_L : L \to K$ be a real line vector bundle with a Kirillov bracket

$$[\cdot,\cdot]_{L^*} : \Gamma(L^*) \times \Gamma(L^*) \to \Gamma(L^*)$$

on the space of the sections $\Gamma(L^*)$ of the dual vector bundle $\pi_{L^*} : L^* \to K$ of π_L. Denote by 0_L the zero section of π_L and by $\phi^{L-0_L} : \mathbb{R}^\times \times (L - 0_L) \to (L - 0_L)$ the $\mathbb{R}^\times$-action associated with the principal bundle $p_{L-0_L} : (L - 0_L) \to K$ whose line bundle is π_L (see the Appendix).

We suppose that $(\Phi^L : G \times L \to L, \Phi^K : G \times K \to K)$ is a representation of a Lie group G on the vector bundle $\pi_L : L \to K$. This means that (Φ^L_g, Φ^K_g) is a vector bundle isomorphism for every $g \in G$. So, we have a dual representation $(\Phi^{L^*} : G \times L^* \to L^*, \Phi^K : G \times K \to K)$ on the dual vector bundle $\pi_{L^*} : L^* \to K$. Here, $\Phi^{L^*} : G \times L^* \to L^*$ is the representation of G on L^* induced by Φ^L, given by

$$\langle \Phi^{L^*}_g(\alpha), x\rangle = \langle \alpha, \Phi^L_{g^{-1}}(x)\rangle, \quad \text{for all } \alpha \in L^* \text{ and } x \in L.$$

Note that $\pi_L \circ \Phi^L_g = \Phi^K_g \circ \pi_L$, which implies that $\pi_{L^*} \circ \Phi^{L^*}_g = \Phi^K_g \circ \pi_{L^*}$, for all $g \in G$.

Definition 5. If the local Lie algebra structure $[\cdot,\cdot]_{L^*}$ is closed for G-equivariant sections of $\Gamma(L^*)$, we say that the representation $(\Phi^L : G \times L \to L, \Phi^K : G \times K \to K)$ *is compatible with the Kirillov structure*.

We recall that a section $h : K \to L^*$ is *G-equivariant* if

$$\Phi^{L^*}_g \circ h = h \circ \Phi^K_g, \text{ for all } g \in G.$$

On the other hand, since the principal bundle associated with π_L is the restriction $p_{L-0_L} : L - 0_L \to K$ of π_L to $L - 0_L$ (see the Appendix), we deduce that

$$\Phi^K_g \circ p_{L-0_L} = p_{L-0_L} \circ \Phi^L_g. \tag{41}$$

In what follows, we suppose that the orbit space K/G of the action Φ^K of G on K is a smooth quotient manifold. As a consequence, the orbit space L/G is a real line bundle over K/G whose fibers are isomorphic to the fibers of $\pi_L : L \to K$.

Denote by $0_{L/G}$ the zero section of the line bundle $\pi_{L/G} : L/G \to K/G$. The $\mathbb{R}^\times$-principal bundle

$$p_{L/G-0_{L/G}} : (L/G - 0_{L/G}) \cong (L - 0_L)/G \to K/G$$

associated with $\pi_{L/G}$ is deduced from the G-equivariant principal bundle $p_{L-0_L} : (L - 0_L) \to K$.

Moreover, the principal actions ϕ^{L-0_L} and $\phi^{(L-0_L)/G}$ of $\mathbb{R}^\times$ on $L - 0_L$ and $(L - 0_L)/G$, respectively, are related by

$$\wp_{L-0_L} \circ \phi^{L-0_L}_s = \phi^{(L-0_L)/G}_s \circ \wp_{L-0_L}, \quad \text{for } s \in \mathbb{R}^\times, \tag{42}$$

where $\wp_{L-0_L} : L - 0_L \to (L - 0_L)/G$ is the quotient map.

On the other hand, the dual vector bundle $\pi^*_{L/G} : (L/G)^* \to K/G$ is isomorphic to the line bundle $\pi_{L^*/G} : L^*/G \to K/G$ deduced from the G-equivariant dual vector bundle $\pi_{L^*} : L^* \to K$ of π_L for the pair of actions (Φ^{L^*}, Φ^K). The following diagram summarizes the previous comments.

$$
\begin{array}{ccc}
\Phi^{L^*} \circlearrowright & & \circlearrowright \Phi^{K} \\
L^* & \xrightarrow{\;\pi_{L^*}\;} & K \\
\downarrow \wp_{L^*} & & \downarrow \wp_K \\
L^*/G & \xrightarrow{\;\pi_{L^*/G}\;} & K/G
\end{array}
$$

Now we can prove the following general result that will be used in the following.

Theorem 3. *Let $[\cdot,\cdot]_{L^*} : \Gamma(L^*) \times \Gamma(L^*) \to \Gamma(L^*)$ be a Kirillov structure on the real line bundle $\pi_L : L \to K$. Suppose that (Φ^L, Φ^K) is a compatible representation of G on L. Then:*

(1) There is a one-to-one correspondence between G-equivariant sections $h : K \to L^*$ of $\pi_{L^*} : L^* \to K$ with respect to (Φ^{L^*}, Φ^K) and sections $h^G : K/G \to L^*/G$ of the line bundle $\pi_{L^*/G} : L^*/G \to K/G$.

(2) On the space of sections of $\pi_{L^*/G} : L^*/G \to K/G$ there is a Kirillov structure $[\cdot,\cdot]_{L^*/G}$, characterized by

$$[h_1^G, h_2^G]_{L^*/G} = ([h_1, h_2]_{L^*})^G$$

for all G-equivariant sections h_1, h_2 of π_{L^*}.

(3) If $h : K \to L^*$ is a G-equivariant section of π_{L^*}, then the symbol $X_h^{[\cdot,\cdot]_{L^*}} \in \mathfrak{X}(K)$ associated with the derivation $[\cdot, h]_{L^*}$ is G-invariant with respect to Φ^K. Moreover, if $\wp_K : K \to K/G$ is the quotient map, the $\wp_K$-projection of $X_h^{[\cdot,\cdot]_{L^*}} \in \mathfrak{X}(K)$ is the symbol $X_{h^G}^{[\cdot,\cdot]_{L^*/G}} \in \mathfrak{X}(K/G)$ of the derivation $[\cdot, h^G]_{L^*/G}$.

Proof. From the general theory of representations of Lie groups, we have that there is a one-to-one correspondence between G-equivariant sections $h : K \to L^*$ of $\pi_{L^*} : L^* \to K$ with respect to (Φ^{L^*}, Φ^K) and sections $h^G : K/G \to L^*/G$ of the line bundle $\pi_{L^*/G} : L^*/G \to K/G$ such that

$$h^G(\wp_K(x)) = \wp_{L^*}(h(x)), \quad \text{for all } x \in K,$$

where $\wp_{L^*} : L^* \to L^*/G$ is the quotient map. Thus, we can induce a bracket $[\cdot,\cdot]_{L^*/G} : \Gamma(L^*/G) \times \Gamma(L^*/G) \to \Gamma(L^*/G)$ characterized by

$$[h_1^G, h_2^G]_{L^*/G} = ([h_1, h_2]_{L^*})^G, \tag{43}$$

where $h_1, h_2 : L^* \to K$ are G-equivariant sections of $\pi_{L^*} : L^* \to K$.

It is clear that $[\cdot,\cdot]_{L^*/G}$ is a Lie algebra structure. On the other hand, if $f \in C^\infty(K/G)$, then

$$[(f \circ \wp_K)h_1, h_2]_{L^*} = (f \circ \wp_K)[h_1, h_2]_{L^*} + X_{h_2}^{[\cdot,\cdot]_{L^*}}(f \circ \wp_K)h_1$$

for all $h_1, h_2 \in \Gamma(L^*)$.

Now, by hypothesis, the sections $[(f\circ\wp_K)h_1, h_2]_{L^*}$ and $(f\circ\wp_K)[h_1, h_2]_{L^*}$ are G-equivariant. Thus,

$$X_{h_2}^{[\cdot,\cdot]_{L^*}}(f\circ\wp_K)h_1$$

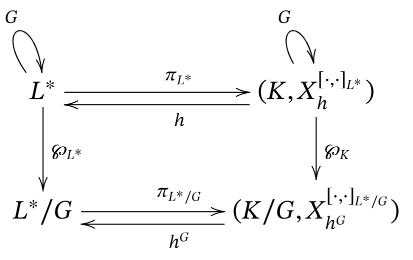
is G-equivariant too, which implies that the function $X_{h_2}^{[\cdot,\cdot]_{L^*}}(f\circ\wp_K)$ is $\wp_K$-basic.

So, we have proved that the vector field $X_{h_2}^{[\cdot,\cdot]_{L^*}}$ is $\wp_K$-projectable over a vector field $X_{h_2^G}^{[\cdot,\cdot]_{L^*/G}}$ on K/G and, in addition,

$$[fh_1^G, h_2^G]_{L^*/G} \quad = \quad f[h_1^G, h_2^G]_{L^*/G} + X_{h_2^G}^{[\cdot,\cdot]_{L^*/G}}(f)h_1^G.$$

Therefore, $[\cdot, h_2^G]_{L^*/G}$ is a derivation and its symbol is just the $\wp_K$-projection of the symbol of $[\cdot, h_2]_{L^*}$. This finishes the proof of the theorem. □

The following diagram summarizes this reduction process.



Remark 5. When the real line bundle $\pi_L : L \to K$ is trivial, the previous theorem is just the reduction process of Jacobi manifolds given in Ref. 37.

5.2 | The first step: Reduction by a scaling symmetry

Now, we start with the scaling reduction process of the symplectic Hamiltonian system (S, ω, H). In this case, we have (see Section 3):

- The reduced space $C = S/\mathbb{R}^\times$ admits a contact distribution $\mathcal{D}$.
- The principal bundle $\mathbf{p}_S : S \to C$ is isomorphic to the restriction of $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$ to $(\mathcal{D}^o - 0_C)$, where $\mathcal{D}^o$ is the annihilator of $\mathcal{D}$ and 0_C is its zero section. Therefore, the associated real line bundle, under this isomorphism, is $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$. Moreover, there is a one-to-one correspondence between the sections $h : C \to (\mathcal{D}^o)^*$ of the dual vector bundle of $\pi_{\mathcal{D}^o}$ and the homogeneous functions $H_h : S \to \mathbb{R}$ on the symplectic manifold S.
- On the space $\Gamma((\mathcal{D}^o)^*)$ of the sections of the dual vector bundle of $\pi_{\mathcal{D}^o}$, we have a Kirillov bracket

$$[\cdot,\cdot]_{(\mathcal{D}^o)^*} : \Gamma((\mathcal{D}^o)^*) \times \Gamma((\mathcal{D}^o)^*) \to \Gamma((\mathcal{D}^o)^*)$$

such that

$$H_{[h_1,h_2]_{(\mathcal{D}^o)^*}} = -\{H_{h_1}, H_{h_2}\}_S$$

for all $h_1, h_2 \in \Gamma((\mathcal{D}^o)^*)$, where $\{\cdot,\cdot\}_S$ is the Poisson bracket associated with the symplectic structure on S.

- The Hamiltonian vector field $X_H^\omega \in \mathfrak{X}(S)$ of H with respect to the symplectic structure ω is $\mathbf{p_S}$-projectable on C and its projection is the symbol $X_{h_H}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*}} \in \mathfrak{X}(C)$ of the derivation $[\cdot, h_H]_{(\mathcal{D}^o)^*}$.
- A G-action on C. In fact, the compatibility condition between the actions implies that $\Phi_g^S : S \to S$ is $\mathbb{R}^\times$-equivariant, for every $g \in G$, and it induces a principal action $\Phi^C : G \times C \to C$ such that

$$\Phi_g^C \circ \mathbf{p_S} = \mathbf{p_S} \circ \Phi_g^S. \tag{44}$$

Moreover,

$$T\Phi_g^S \circ \Delta = \Delta \circ \Phi_g^S \tag{45}$$

with Δ the infinitesimal generator of the scaling symmetry ϕ^S. Using this relation and that $(\Phi_g^S)^*\omega = \omega$, we conclude the G-invariance of the 1-form $\lambda = -i_\Delta\omega$, that is,

$$(\Phi_g^S)^*(\lambda) = -(\Phi_g^S)^*(i_\Delta\omega) = -i_\Delta\omega = \lambda. \tag{46}$$

Therefore,

$$T\Phi_g^S(\langle\lambda\rangle^o) = \left\langle \lambda \circ \Phi_g^S \right\rangle^o, \text{ for all } g \in G, \tag{47}$$

where $T\Phi^S : G \times TS \to TS$ is the tangent lift of the action of Φ^S. In other words, $\tilde{\mathcal{D}} = \langle\lambda\rangle^o$ is a G-invariant distribution. So, since $\Phi_g^S \circ \phi_s^S = \phi_s^S \circ \Phi_g^S$, we deduce that the contact distribution $\mathcal{D} = T\mathbf{p_S}(\tilde{\mathcal{D}})$ is G-invariant, that is,

$$T\Phi_g^C(\mathcal{D}) = \mathcal{D}.$$

This implies that the cotangent lift $T^*\Phi^C$ of the action Φ^C preserves the annihilator $\mathcal{D}^o$ of the contact distribution. Therefore, we have a representation $(\Phi^{\mathcal{D}^o} := (T^*\Phi^C)_{|\mathcal{D}^o}, \Phi^C)$ of G on the real line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$.

5.3 | The second step: Reduction by standard symmetries

Now, we apply the second reduction process with the representation $(\Phi^{\mathcal{D}^o} := (T^*\Phi^C)_{|\mathcal{D}^o}, \Phi^C)$. To do so, we will use Theorem 3 on the reduction of Kirillov structures.

Theorem 4. *Let (S, ω, H) be a symplectic Hamiltonian system with a scaling symmetry $\phi^S : \mathbb{R}^\times \times S \to S$, G a Lie group and $\Phi^S : G \times S \to S$ a symplectic G-symmetry which is compatible with ϕ^S. Then:*

(1) If $(C = S/\mathbb{R}^\times, \mathcal{D})$ is the contact manifold induced by the scaling symmetry ϕ^S, then we have a representation $(\Phi^{\mathcal{D}^o} : G \times \mathcal{D}^o \to \mathcal{D}^o, \Phi^C : G \times C \to C)$ on the line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$ such that the corresponding quotient vector bundle $\pi_{\mathcal{D}^o/G} : \mathcal{D}^o/G \to C/G$ is a real line bundle. Moreover, there is a one-to-one correspondence between the G-equivariant sections $h : C \to (\mathcal{D}^o)^*$ of the dual vector bundle of $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$ and sections $h^G : C/G \to (\mathcal{D}^o)^*/G$ of the dual vector bundle of $\pi_{\mathcal{D}^o/G}$.

(2) There is a Kirillov bracket $[\cdot,\cdot]_{(\mathcal{D}^o)^*/G} : \Gamma((\mathcal{D}^o)^*/G) \times \Gamma((\mathcal{D}^o)^*/G) \to \Gamma((\mathcal{D}^o)^*/G)$ *on the space* $\Gamma((\mathcal{D}^o)^*/G)$ *of the sections of the dual vector bundle* $\pi_{(\mathcal{D}^o)^*/G} : (\mathcal{D}^o)^*/G \to C/G$*, such that*

$$([h_1,h_2]_{(\mathcal{D}^o)^*})^G = [h_1^G, h_2^G]_{(\mathcal{D}^o)^*/G}$$

for $h_1, h_2 \in \Gamma((\mathcal{D}^o)^*)$ *G-invariant sections.*

(3) If $h_H : C \to (\mathcal{D}^o)^*$ *is the section of* $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to C$ *induced from H, the symbol* $X_{h_H}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*}} \in \mathfrak{X}(C)$ *of the derivation* $[\cdot, h_H]_{(\mathcal{D}^o)^*}$ *is G-invariant and the corresponding vector field on C/G is just the symbol* $X_{h_H^G}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*/G}} \in \mathfrak{X}(C/G)$ *of the derivation* $[\cdot, h_H^G]_{(\mathcal{D}^o)^*/G}$*. Thus, if* $\wp_C : C \to C/G$ *is the quotient map, the Hamiltonian vector field* $X_H^\omega \in \mathfrak{X}(S)$ *of H with respect to the symplectic structure* ω *is* $(\wp_C \circ \mathbf{p_S})$*-projectable on C/G and its projection is* $X_{h_H^G}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*/G}} \in \mathfrak{X}(C/G)$*.*

Proof. We have the representation $(\Phi^{\mathcal{D}^o}, \Phi^C)$ of G on the real line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$ defined previously.

Now, we will prove that if $h_1, h_2 : C \to (\mathcal{D}^o)^*$ are G-equivariant sections of $\pi_{(\mathcal{D}^o)^*}$, then the bracket $[h_1,h_2]_{(\mathcal{D}^o)^*}$ is also G-equivariant. From (A.2), (A.3) (see the Appendix), and the commutation of the actions Φ^S and ϕ^S, we deduce that $h : C \to (\mathcal{D}^o)^*$ is a G-equivariant section if and only if the corresponding homogeneous function $H_h : S \to \mathbb{R}$ is invariant with respect to the action Φ^S.

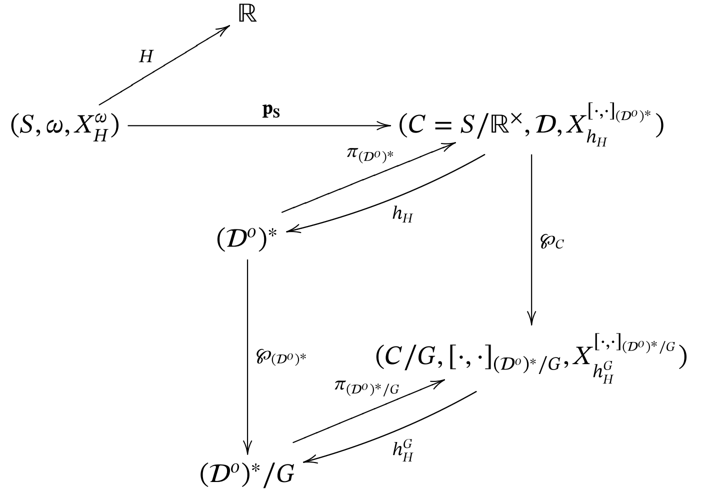
So, if $h_1, h_2 : C \to (\mathcal{D}^o)^*$ are G-equivariant, then H_{h_1} and H_{h_2} are G-invariant with respect to Φ^S and, since the action Φ^S is symplectic, we have that the function $\{H_{h_1}, H_{h_2}\}_S$ is G-invariant. Therefore,

$$H_{[h_1,h_2]_{(\mathcal{D}^o)^*}}(\Phi_g^S(x)) = -\{H_{h_1}, H_{h_2}\}_S(\Phi_g^S(x)) = -\{H_{h_1}, H_{h_2}\}_S(x) = H_{[h_1,h_2]_{(\mathcal{D}^o)^*}}(x)$$

for all $x \in S$. In conclusion, $[h_1,h_2]_{(\mathcal{D}^o)^*}$ is G-equivariant.

Now, applying Theorem 3, we deduce the result. □

The following diagram shows both reduction processes together.



Now we illustrate the reduction processes using the two examples considered above.

Example 8 (Continuing Example 6: The 2D harmonic oscillator reduced first by a scaling and then by a standard symmetry). We consider again the example of a two-dimensional harmonic oscillator (see Examples 4 and 6). In Example 6, we have shown how to apply the reduction process by first using the standard symmetry and then the scaling symmetry. Now, we take the reverse order.

We recall that in this example we have:

(1) A standard rotational S^1-symmetry, with infinitesimal generator $\xi_S = x\partial_y - y\partial_x + p_x\partial_{p_y} - p_y\partial_{p_x}$ where (x, y, p_x, p_y) are coordinates on $S = T^*(\mathbb{R}^2 - \{(0,0)\})$. Using the identification $\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 \cong T^*(\mathbb{R}^2 - \{(0,0)\}) - 0_{\mathbb{R}^2-\{(0,0)\}}$ the local expression of ξ_S is

$$\xi_S = \partial_\theta + \partial_{\theta'},$$

where (r, θ, r', θ') are polar coordinates on $\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1$.

(2) A scaling $\mathbb{R}^+$-symmetry, with generator

$$\Delta = \frac{1}{2}(r\partial_r + r'\partial_{r'}).$$

As seen in Example 4, by applying first the scaling $\mathbb{R}^+$-symmetry, we obtain:

- **The reduced space**: It is $\mathbb{R}^+ \times S^1 \times S^1$, with the quotient map

$$\mathbf{p} : \mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times S^1 \to \mathbb{R}^+ \times S^1 \times S^1, \quad \mathbf{p}(\rho, \theta, \rho', \theta') = (\rho', \theta, \theta').$$

  The Jacobi structure on $C = \mathbb{R}^+ \times S^1 \times S^1$ is given by (16).
- **The reduced Hamiltonian function**: The reduced Hamiltonian function is given by

$$H_{|\mathbb{R}^+\times S^1\times S^1}(\rho', \theta, \theta') = \frac{1}{2}((\rho')^2 + 1).$$

- **The reduced dynamics**: It is given by the vector field on $\mathbb{R}^+ \times S^1 \times S^1$ obtained by the $\mathbf{p}$-projection

$$\mathbf{p}_*(X_H^{\omega_Q}) = (1 + (\rho')^2)\cos(\theta - \theta')\partial_{\rho'} + \sin(\theta - \theta')\left(\frac{1}{\rho'}\partial_{\theta'} + \rho'\partial_\theta\right),$$

  which is just the contact Hamiltonian vector field $X^{\{\cdot,\cdot\}_C}_{H_{|\mathbb{R}^+\times S^1\times S^1}}$ of the function $H_{|\mathbb{R}^+\times S^1\times S^1}$ with respect to the Jacobi structure on $C = \mathbb{R}^+ \times S^1 \times S^1$ described in (16).
- **The standard symmetry on the reduced space**: We may induce an S^1-action on the reduced space $\mathbb{R}^+ \times S^1 \times S^1$ whose infinitesimal generator is

$$\xi_{\mathbb{R}^+\times S^1\times S^1} = \partial_\theta + \partial_{\theta'}.$$

Now, we apply the second step of the reduction process using this last symmetry, obtaining the reduction of the Kirillov structure by this standard symmetry. More precisely, the reduction of the Jacobi structure, because in this case the Kirillov line bundle is trivial.

- **The reduced space**: We consider the diffeomorphism

$$\begin{array}{rcl} \mathbb{R}^+ \times S^1 \times S^1 & \to & \mathbb{R}^+ \times S^1 \times S^1 \\ (\rho', \theta, \theta') & \to & (\rho', \theta, \theta - \theta'), \end{array}$$

which transforms $\xi_{\mathbb{R}^+\times S^1\times S^1} = \partial_\theta + \partial_{\theta'}$ into ∂_θ. Therefore, the quotient space $(\mathbb{R}^+ \times S^1 \times S^1)/S^1$ may be identified with

$$K = \mathbb{R}^+ \times S^1,$$

so that $\wp_K : \mathbb{R}^+ \times S^1 \times S^1 \to \mathbb{R}^+ \times S^1$ is the map $\wp_K(\rho', \theta, \theta') = (\rho', \theta - \theta')$.

In this case, the line bundle associated with $\wp_K$ is trivial and we obtained a Jacobi structure. From (16), we deduce that the Jacobi structure on $K = \mathbb{R}^+ \times S^1$ is

$$\Pi_K = -2 \sin\sigma \partial_{\rho'} \wedge \partial_\sigma, \qquad E_K = -2\cos\sigma \partial_{\rho'} + 2\frac{\sin\sigma}{\rho'}\partial_\sigma, \tag{48}$$

with (ρ', σ) polar coordinates on $\mathbb{R}^+ \times S^1$. Note that this Jacobi structure is just the one given in (36).

- **The reduced Hamiltonian function**: In this case, the reduced Hamiltonian is

$$H_{|\mathbb{R}^+\times S^1}(\rho', \sigma) = \frac{1}{2}((\rho')^2 + 1),$$

with (ρ', σ) polar coordinates of $\mathbb{R}^+ \times S^1$.

- **The reduced dynamics**: The reduced vector field is the $\wp_K$-projection

$$(\wp_K)_* X^{\{\cdot,\cdot\}_C}_{H_{|\mathbb{R}^+\times S^1}} = (1 + (\rho')^2)\cos\sigma\,\partial_{\rho'} - \frac{1-(\rho')^2}{\rho'}\sin\sigma\,\partial_\sigma,$$

which coincides precisely with the results obtained in Example 6, using the reverse reduction process (see (37)).

Example 9. Continuing Example 7: The linear Hamiltonian system reduced first by a scaling and then by a standard symmetry. We consider again the example of a free and proper action $\Phi : G \times Q \to Q$ of a Lie group G on a manifold Q with a G-invariant vector field $Y \in \mathfrak{X}(Q)$. Then, we have two symmetries on $T^*Q - 0_Q$:

- The restriction $T^*\Phi : G \times (T^*Q - 0_Q) \to (T^*Q - 0_Q)$ to $T^*Q - 0_Q$ of the cotangent lift of the action on Q.
- The scaling action $\phi : \mathbb{R} - \{0\} \times (T^*Q - 0_Q) \to (T^*Q - 0_Q)$ given by (8).

In Example 7, we have shown how to apply the reduction process by first using the standard symmetry and then the scaling symmetry. Now, we take the reverse order.

As seen in Example 5, by using first the scaling symmetry, we obtain the following reduced objects:

- **The reduced space**: It is the projective cotangent bundle $\mathbb{P}(T^*Q)$. Let $\mathcal{D}$ be the contact distribution on $\mathbb{P}(T^*Q)$ such that $\mathbf{p} : \mathcal{D}^o - 0_Q \to \mathbb{P}(T^*Q)$ is a principal bundle with real line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to \mathbb{P}(T^*Q)$. The Kirillov bracket on the sections of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to \mathbb{P}(T^*Q)$ satisfies

$$[h_{X^\ell}, h_{Z^\ell}]_{(\mathcal{D}^o)^*} = -h_{[X,Z]^\ell}$$

  for all $X, Z \in \mathfrak{X}(Q)$.
- **The reduced Hamiltonian section**: It is defined locally by (19).
- **The reduced dynamics**: The Hamiltonian vector field $X^{\omega_Q}_{Y^\ell} \in \mathfrak{X}(T^*Q - 0_Q)$ is $\mathbf{p}$-projectable and its projection is the symbol of the derivation $[\cdot, h_{Y^\ell}]_{(\mathcal{D}^o)^*}$.
- **The standard symmetry on the reduced space**: The action is defined by

$$G \times \mathbb{P}(T^*Q) \to \mathbb{P}(T^*Q), \qquad (g, \mathbf{p}(\alpha)) \to \mathbf{p}((T^*\Phi)_g(\alpha)).$$

Now, we can consider the second step of the reduction process. The standard symmetry on the reduced space satisfies the conditions of Theorem 3, and therefore, we have

- **The reduced space**: In this case, the reduced space is the quotient space $\mathbb{P}(T^*Q)/G$. Moreover, the projection $\mathbf{p} : \mathcal{D}^o - 0 \to \mathbb{P}(T^*Q)$ is G-invariant and it induces a reduced projection $\mathbf{p}^{\mathbf{G}} : (\mathcal{D}^o - 0)/G \to \mathbb{P}(T^*Q)/G$. The real line bundle $\boldsymbol{\pi}_{\mathcal{D}^o/\mathbf{G}} : L := \mathcal{D}^o/G \to K := \mathbb{P}(T^*Q)/G$ is deduced from the G-equivariant line bundle $\pi_{\mathcal{D}^o}$. On the space of sections of the dual of this real bundle, we have a Kirillov structure $[\cdot, \cdot]_{L^*}$ characterized by

$$[h^G_{X^\ell}, h^G_{Z^\ell}]_{L^*} = -h^G_{[X,Z]^\ell}$$

  for $X, Z \in \mathfrak{X}(Q)$ G-invariant vector fields on Q.
- **The reduced Hamiltonian section**: The section h_{Y^ℓ} of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to \mathbb{P}(T^*Q)$ is G-invariant and therefore it induces a section

$$h^G_{Y^\ell} : \mathbb{P}(T^*Q)/G \to (\mathcal{D}^o)^*/G.$$

- **The reduced dynamics**: The vector field $\mathbf{p}_*(X^{\omega_Q}_{Y^\ell})$ is G-invariant. Thus, it induces a vector field on $\mathbb{P}(T^*Q)/G$, which is just the symbol of $[\cdot, h^G_{Y^\ell}]_{L^*}$.

The particular case of a Lie group. When $Q = G$ is a Lie group, for the first reduction step with the scaling symmetry, we have (see Example 5):

- The reduced space is $G \times \mathbb{P}\mathfrak{g}^*$.
- The contact structure is the distribution on $G \times \mathbb{P}\mathfrak{g}^*$ given by

$$\mathcal{D}_{(g,p(\mu))} = \left\langle (T_g L_{g^{-1}})^*(\mu) \right\rangle^o \times T_{p(\mu)}(\mathbb{P}\mathfrak{g}^*)$$

  for all $g \in G$ and $\mu \in \mathfrak{g}^* - \{0\}$.

- The fiber of the real line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to G \times \mathbb{P}\mathfrak{g}^*$ at $(g, p(\mu)) \in G \times \mathbb{P}\mathfrak{g}^*$ is just

$$\mathcal{D}^o{}_{(g,p(\mu))} = \left\langle (T_g L_{g^{-1}})^*(\mu) \right\rangle.$$

- The reduced Hamiltonian section of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to G \times \mathbb{P}\mathfrak{g}^*$ induced by the function Y^ℓ is characterized by

$$h_\xi(g, p(\mu))((T_g L_{g^{-1}})^*(\mu)) = \mu(\xi),$$

with $g \in G$, $\mu \in \mathfrak{g}^* - \{0\}$, $\xi = Y(e)$ and $p : \mathfrak{g}^* - \{0\} \to \mathbb{P}\mathfrak{g}^*$ the corresponding quotient map determined by the scaling symmetry on $\mathfrak{g}^* - \{0\}$.
- The reduced vector field after this reduction is $(Y, X_{h_\xi}) \in \mathfrak{X}(G) \times \mathfrak{X}(\mathbb{P}\mathfrak{g}^*)$, such that

$$X_{h_\xi}(f) \circ p = \{f \circ p, \xi^\ell\}_{\mathfrak{g}^* - \{0\}}, \tag{49}$$

which is the symbol of the derivation $[\cdot, h_\xi]_{(\mathcal{D}^o)^*}$.

Now, if we perform the second reduction step associated with the induced G-action

$$G \times (G \times \mathbb{P}\mathfrak{g}^*) \to G \times \mathbb{P}\mathfrak{g}^*, \quad (g', (g, p(\mu))) \to (gg', p(\mu)),$$

the corresponding reduced elements are:

- The reduced space is the projective space $\mathbb{P}\mathfrak{g}^*$.
- The line vector bundle $\pi_L : L \to \mathbb{P}\mathfrak{g}^*$ is given by

$$L_{p(\mu)} = \langle \mu \rangle, \quad \mu \in \mathfrak{g}^*.$$

- The reduced section of $\pi_{L^*} : L^* \to \mathbb{P}\mathfrak{g}^*$ is just

$$h_\xi^G(p(\mu))(t\mu) = t\mu(\xi).$$

- The final reduced dynamics is the vector field X_{h_ξ} on $\mathbb{P}\mathfrak{g}^*$ described in (49), which is the symbol of $[\cdot, h_\xi^G]_{L^*}$ and whose local expression is (22).

So, also in this case, similarly to the two previous examples (see Examples 6, 7, 8, and 9), both reduction processes give rise to the same reduced dynamics. This fact motivates further analysis on the equivalence of the two reduction processes, which will be addressed in full generality in the following section.

6 | THE EQUIVALENCE OF THE TWO REDUCTION PROCESSES

Finally, we will prove that both processes considered in Sections 4 and 5 are equivalent. Let (S, ω, H) be a symplectic Hamiltonian system with a scaling symmetry $\phi^S : \mathbb{R}^\times \times S \to S$ and a symplectic G-symmetry $\Phi^S : G \times S \to S$ which are compatible, G being a Lie group.

Theorem 5. *Under the previous conditions we have that:*

(1) There exists a real line bundle isomorphism (Ψ, ψ) *between the line bundles* $\pi_L : L \to (S/G)/\mathbb{R}^\times$ *and* $\pi_{(\mathcal{D}^o)/G} : (\mathcal{D}^o)/G \to (S/\mathbb{R}^\times)/G$

$$\begin{array}{ccc} L & \xrightarrow{\ \Psi\ } & \mathcal{D}^o/G \\ \downarrow{\scriptstyle \pi_L} & & \downarrow{\scriptstyle \pi_{\mathcal{D}^o/G}} \\ (S/G)/\mathbb{R}^\times & \xrightarrow{\ \psi\ } & (S/\mathbb{R}^\times)/G \end{array}$$

(2) The sections $h_H^G \in \Gamma((\mathcal{D}^o)^*/G)$ *and* $h_{H^G} \in \Gamma(L^*)$ *induced by the Hamiltonian function* $H : S \to \mathbb{R}$ *and obtained in Theorems 4 and 2, respectively, are related as follows:*

$$h_{H^G} = \Psi^* \circ h_H^G \circ \psi, \tag{50}$$

where Ψ^* *is the dual isomorphism, between the line bundles* $\pi_{(\mathcal{D}^o)^*/G}$ *and* π_{L^*}*, deduced from* Ψ*.*

(3) The Kirillov structures $[\cdot,\cdot]_{L^*}$ *and* $[\cdot,\cdot]_{(\mathcal{D}^o)^*/G}$ *obtained in Theorems 2 and 4, respectively, are isomorphic. In fact, we have that*

$$[\Psi^* \circ h_1^G \circ \psi, \Psi^* \circ h_2^G \circ \psi]_{L^*} = \Psi^* \circ [h_1^G, h_2^G]_{(\mathcal{D}^o)^*/G} \circ \psi \tag{51}$$

for all h_1, h_2 *G-invariant sections of the line bundle* $\pi : \mathcal{D}^o \to S/\mathbb{R}^\times$*.*

(4) The vector fields $X_{h_{H^G}}^{[\cdot,\cdot]_{L^*}}$ *and* $X_{h_H^G}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*/G}}$ *given in Theorems 2 and 4, respectively, are* ψ*-related, that is, the following diagram is commutative.*

$$\begin{array}{ccc} (S/G)/\mathbb{R}^\times & \xrightarrow{\ \psi\ } & (S/\mathbb{R}^\times)/G \\ \downarrow{\scriptstyle X_{h_{H^G}}^{[\cdot,\cdot]_{L^*}}} & & \downarrow{\scriptstyle X_{h_H^G}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*/G}}} \\ T((S/G)/\mathbb{R}^\times) & \xrightarrow{\ T\psi\ } & T((S/\mathbb{R}^\times)/G) \end{array}$$

Proof.

(1) The diffeomorphism ψ is just

$$\psi : (S/G)/\mathbb{R}^\times \to (S/\mathbb{R}^\times)/G, \qquad \psi(\mathbf{p_P}(\wp_S(x))) = \wp_C(\mathbf{p}_S(x)), \qquad \text{for all } x \in S, \tag{52}$$

that is,

$$\begin{array}{ccc} S & \xrightarrow{\ Id_S\ } & S \\ \downarrow{\scriptstyle \wp_S} & & \downarrow{\scriptstyle \mathbf{p_S}} \\ P = S/G & & C = S/\mathbb{R}^\times \\ \downarrow{\scriptstyle \mathbf{p_P}} & & \downarrow{\scriptstyle \wp_C} \\ (S/G)/\mathbb{R}^\times & \xrightarrow{\ \psi\ } & (S/\mathbb{R}^\times)/G \end{array}$$

We remark that this map is a diffeomorphism from the equality $\Phi^S_g \circ \phi^S_s = \phi^S_s \circ \Phi^S_g$. Moreover, the diffeomorphism Ψ is characterized in this diagram

$$\begin{array}{ccc} S\times\mathbb{R} & \xrightarrow{\quad Id_{S\times\mathbb{R}}\quad} & S\times\mathbb{R} \\ \downarrow{\scriptstyle \wp_S\times Id_{\mathbb{R}}} & & \downarrow{\scriptstyle \mathbf{p}_{\mathbf{S}\times\mathbb{R}}} \\ P\times\mathbb{R}=S/G\times\mathbb{R} & & (S\times\mathbb{R})/\mathbb{R}^\times \\ \downarrow{\scriptstyle \mathbf{p}_{\mathbf{P}\times\mathbb{R}}} & & \downarrow{\scriptstyle \wp_{(S\times\mathbb{R})/\mathbb{R}^\times}} \\ L=((S/G)\times\mathbb{R})/\mathbb{R}^\times & \xrightarrow{\Psi} & ((S\times\mathbb{R})/\mathbb{R}^\times)/G\cong \mathcal{D}^o/G \end{array} \tag{53}$$

Here, $\mathbf{p}_{\mathbf{P}\times\mathbb{R}}$ is the quotient map deduced from the action

$$\mathbb{R}^\times \times (P\times\mathbb{R}) \to (P\times\mathbb{R}), \qquad (s,(\wp_S(x),t)) \to \left(\wp_S(sx),\frac{t}{s}\right),$$

and $\mathbf{p}_{\mathbf{S}\times\mathbb{R}}$ the quotient map deduced from the action

$$\mathbb{R}^\times \times (S\times\mathbb{R}) \to (S\times\mathbb{R}), \qquad (s,(x,t)) \to \left(sx,\frac{t}{s}\right).$$

(2) From (A.3) in the Appendix, we have

$$h_{H^G}(\mathbf{p}_\mathbf{P}(\wp_S(x)))(\mathbf{p}_{\mathbf{P}\times\mathbb{R}}(\wp_S(x),t)) = tH^G(\wp_S(x)) = tH(x)$$

for $x\in S$ and $t\in\mathbb{R}$.

On the other hand, using (52), the diagram (53) and again (A.3) in the Appendix, we obtain

$$(\Psi^*\circ h^G_H\circ\psi)(\mathbf{p}_\mathbf{P}(\wp_S(x)))(\mathbf{p}_{\mathbf{P}\times\mathbb{R}}(\wp_S(x),t)) = h_H(\mathbf{p}_\mathbf{S}(x))(\mathbf{p}_{\mathbf{S}\times\mathbb{R}}(x,t)) = tH(x).$$

(3) If h_1, h_2 are G-invariant sections of the line bundle $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to C = S/\mathbb{R}^+$, then from (2) in Theorem 4 and (50), we deduce

$$\begin{aligned} H_{\Psi^*\circ[h_1^G,h_2^G]_{(\mathcal{D}^o)^*/G}\circ\psi}\circ\wp_S &= H_{\Psi^*\circ[h_1,h_2]^G_{(\mathcal{D}^o)^*}\circ\psi}\circ\wp_S = H^G_{[h_1,h_2]_{(\mathcal{D}^o)^*}}\circ\wp_S \\ &= H_{[h_1,h_2]_{(\mathcal{D}^o)^*}} = -\{H_{h_1},H_{h_2}\}_S. \end{aligned}$$

On the other hand, using (b) in Theorem 1, (29) and (50), we have

$$\begin{aligned} H_{[\Psi^*\circ h_1^G\circ\psi,\Psi^*\circ h_2^G\circ\psi]_{L^*}}\circ\wp_S &= -\{H_{\Psi^*\circ h_1^G\circ\psi},H_{\Psi^*\circ h_2^G\circ\psi}\}_P\circ\wp_S \\ &= -\{H_{\Psi^*\circ h_1^G\circ\psi}\circ\wp_S,H_{\Psi^*\circ h_2^G\circ\psi}\circ\wp_S\}_S \\ &= -\{H^G_{h_1}\circ\wp_S,H^G_{h_2}\circ\wp_S\}_S = -\{H_{h_1},H_{h_2}\}_S. \end{aligned}$$

Therefore, we have (51).

(4) We consider the section $\Psi^* \circ h^G \circ \psi \in \Gamma(L^*)$, with h a G-invariant section on $\pi_{(\mathcal{D}^o)^*}$ and $f \in C^\infty((S/\mathbb{R}^\times)/G)$. From the properties of the Kirillov structure $[\cdot,\cdot]_{L^*}$, we have that

$$[(f\circ\psi)(\Psi^*\circ h^G\circ\psi), h_{H^G}]_{L^*} \;=\; (f\circ\psi)[\Psi^*\circ h^G\circ\psi, h_{H^G}]_{L^*} + X^{[\cdot,\cdot]_{L^*}}_{h_{H^G}}(f\circ\psi)(\Psi^*\circ h^G\circ\psi).$$

On the other hand, using (50) and (51), we obtain

$$[(f\circ\psi)(\Psi^*\circ h^G\circ\psi), h_{H^G}]_{L^*} = [\Psi^*\circ(fh^G)\circ\psi, \Psi^*\circ h^G_H\circ\psi]_{L^*} = \Psi^*\circ[fh^G, h^G_H]_{(\mathcal{D}^o)^*/G}\circ\psi,$$

$$(f\circ\psi)[\Psi^*\circ h^G\circ\psi, h_{H^G}]_{L^*} = \Psi^*\circ(f[h^G, h^G_H]_{(\mathcal{D}^o)^*/G})\circ\psi.$$

Replacing these relations in (6), we have that

$$\Psi^*\circ[fh^G, h^G_H]_{(\mathcal{D}^o)^*/G}\circ\psi = \Psi^*\circ(f[h^G, h^G_H]_{(\mathcal{D}^o)^*/G})\circ\psi + X^{[\cdot,\cdot]_{L^*}}_{h_{H^G}}(f\circ\psi)(\Psi^*\circ h^G\circ\psi). \tag{54}$$

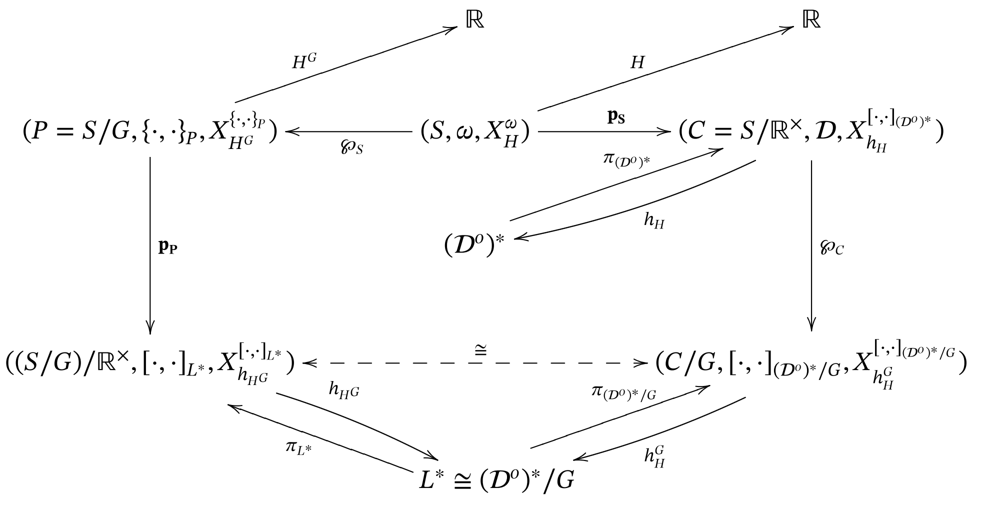
However, we know that

$$[fh^G, h^G_H]_{(\mathcal{D}^o)^*/G} = f[h^G, h^G_H]_{(\mathcal{D}^o)^*/G} + X^{[\cdot,\cdot]_{(\mathcal{D}^o)^*/G}}_{h^G_H}(f)h^G. \tag{55}$$

Comparing (54) and (55), we conclude (4).

□

Both reduction processes and the corresponding equivalence between them are summarized in the following diagram.



7 RECONSTRUCTION PROCESS FOR SCALING SYMMETRIES

In this section, we will study the inverse process of reduction: the *reconstruction process*. First, we shall introduce the general involved ideas, for arbitrary dynamical systems and Lie groups, and then we shall concentrate on the case of symplectic Hamiltonian systems with scaling symmetries.

7.1 | The general context

Let M be a manifold, $X \in \mathfrak{X}(M)$ a vector field on M and G a Lie group acting on M by an action $\phi^M : G \times M \to M$ such that X is G-invariant. Assume that ϕ^M defines a principal fiber bundle $p_M : M \to M/G$. In such a case, the G-invariance of X ensures that there exists a vector field $X^G \in \mathfrak{X}(M/G)$ such that $X^G \circ p_M = Tp_M \circ X$. The question is: how can we get the integral curves of X from those of X^G? To do that, we can proceed as follows. If we want the integral curve $\Gamma : (-\epsilon, \epsilon) \to M$ of X such that $\Gamma(0) = x_0$, then:

(1) consider the integral curve $\gamma : (-\epsilon, \epsilon) \to M/G$ of X^G such that $\gamma(0) = p_M(x_0)$;
(2) fix a principal connection $A : TM \to \mathfrak{g}$ for p_M (where $\mathfrak{g}$ is the Lie algebra of G) and fix a curve $\varphi : (-\epsilon, \epsilon) \to M$ such that $\varphi(0) = x_0$,

$$A\big(\varphi'(t)\big) = 0 \quad \text{and} \quad p_M(\varphi(t)) = \gamma(t) \tag{56}$$

(in other words, $t \to \varphi(t)$ is the horizontal lift of the curve γ by the principal connection A);
(3) and find the curve $g : (-\epsilon, \epsilon) \to G$ such that

$$g'(t) = T_e L_{g(t)}[A(X(\varphi(t)))], \quad g(0) = e. \tag{57}$$

From now on, we shall take ϵ small enough in order to fulfill above conditions. Then, proceeding as in Ref. 1 (see pp. 304–305), one may prove that

$$\Gamma(t) = \phi^M(g(t), \varphi(t))$$

is the curve we are looking for. The above three-step procedure is usually known as *reconstruction*. The steps 2 and 3 are known as the *reconstruction problem* (see, e.g., Ref. 2).

Clearly, such a procedure can be used for the standard as well as for the scaling symmetries. In the following, we shall focus on the latter, since the reconstruction process for scaling symmetries, as far as the authors know, has not been studied in the literature so far.

7.2 | Application to scaling symmetries and symplectic Hamiltonian systems

Now, as in Section 3, let us suppose that we have a scaling symmetry $\phi : \mathbb{R}^\times \times S \to S$ on a symplectic Hamiltonian system (S, ω, H), with infinitesimal generator Δ. Then, assuming that $\mathbf{p_S} : S \to C = S/\mathbb{R}^\times$ is a principal bundle (see the first part of Section 3),

- we have a contact distribution $\mathcal{D}$ on C and a related real line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to C$ with a Kirillov structure $[\cdot,\cdot]_{(\mathcal{D}^o)^*}$, and
- we can ensure that the Hamiltonian vector field $X_H^\omega \in \mathfrak{X}(S)$ of H projects onto the symbol $X_{h_H}^{[\cdot,\cdot]_{(\mathcal{D}^o)^*}} \in \mathfrak{X}(C)$ of the derivation $[\cdot, h_H]_{(\mathcal{D}^o)^*}$.

Recall that $h_H : C \to (\mathcal{D}^o)^*$ denotes the section of $\Gamma((\mathcal{D}^o)^*)$ related to the homogeneous function H. So, we are in the situation of the previous subsection, with $M = S$, $X = X_H^\omega$, $G = \mathbb{R}^\times$,

$\mathfrak{g} = \mathbb{R}$ and $X^G = X_{h_H}^{[\cdot,\cdot](\mathcal{D}^o)^*}$. We shall apply the reconstruction procedure described above in this particular context.

7.2.1 | Existence of a flat connection

There is a case in which solving the reconstruction problem is especially simple (as we will show later). This case is when there is a nonvanishing homogeneous function $F : S \to \mathbb{R}^{\times}$. This kind of functions are called *scaling functions*.[26]

In such a case, the map

$$(F, \mathbf{p_S}) : S \to \mathbb{R}^{\times} \times C$$

is a diffeomorphism and defines a trivialization for $\mathbf{p_S}$. Its inverse is given by

$$(F, \mathbf{p_S})^{-1} : (s, \mathbf{p_S}(x)) \in \mathbb{R}^{\times} \times C \to \phi\left(\frac{s}{F(x)}, x\right) \in S$$

for all $s \in \mathbb{R}^{\times}$ and $x \in S$, and we have a global section $\sigma : C \to S$ of $\mathbf{p_S}$ which takes the values

$$\sigma(y) = (F, \mathbf{p_S})^{-1}(1, y), \quad \forall y \in C. \tag{58}$$

Conversely, if $\mathbf{p_S} : S \to C$ is trivial, that is, $S \cong \mathbb{R}^{\times} \times C$ and $\mathbf{p_S}$ is the second projection, then the function $F : S \cong \mathbb{R}^{\times} \times C \to \mathbb{R}$ given by $F(s, x) = s$ is a nonvanishing homogenous function, that is, a scaling function.

Therefore, the existence of a scaling function F on S is equivalent with the trivialization of the principal bundle $\mathbf{p_S} : S \to C$. This fact guarantees the local existence of this kind of functions F (see Ref. 26).

Moreover, if Δ is the infinitesimal generator of ϕ, since

$$dF(x)(\Delta(x)) = F(x) \neq 0, \quad \forall x \in S,$$

then we have that

$$TS = \langle \Delta \rangle \oplus \langle dF \rangle^o.$$

So, the map $A : TS \to \mathbb{R}$, characterized by

$$A(\Delta(x)) = \Delta(F)(x), \quad \forall x \in S, \tag{59}$$

and

$$\ker A = \langle dF \rangle^o \tag{60}$$

is a principal *flat connection* for $\mathbf{p_S}$ (because $\ker A$ is integrable).

On the other hand, the 1-form $\eta := \sigma^*(\lambda)$ is a global generator of $\mathcal{D}^o$ with $\lambda = -i_\Delta \omega$, which makes $\pi_{\mathcal{D}^o}$ trivial. In fact, using (58), we have that $\sigma \circ \mathbf{p_S} = \phi_{\frac{1}{F}}$. Therefore,

$$(\mathbf{p_S})^* \eta = \left(\phi_{\frac{1}{F}}\right)^* \lambda = \left(\phi \circ \left(\frac{1}{F}, Id\right)\right)^* \lambda.$$

Since $\lambda = -i_\Delta \omega$, then $T_s^* \phi_x(\lambda(\phi(x,s)) = 0$, for all $(s,x) \in \mathbb{R}^\times \times S$. Thus, from the homogeneity of λ, we have that

$$((\mathbf{p_S})^* \eta)(x) = \left(\frac{1}{F}, Id\right)^* \left(0, \left(\phi_{\frac{1}{F(x)}}\right)^* (\lambda(x)\right) = \frac{1}{F(x)}\lambda(x).$$

In conclusion, we deduce that

$$(\mathbf{p_S})^* \eta = \frac{1}{F}\lambda. \tag{61}$$

This implies that $\mathcal{D} = \langle \eta \rangle^o$, and η is a contact 1-form on C.

The one-to-one correspondence between homogeneous functions $H : S \to \mathbb{R}$ on S and functions $h_H : C \to \mathbb{R}$ on C (sections of the trivial line bundle $\pi_{(\mathcal{D}^o)^*}$) is defined by the relation

$$h_H \circ \mathbf{p_S} = \frac{1}{F} H.$$

Note that the function on C associated with F is just the constant function 1.

The Jacobi bracket of two functions h_1, h_2 on C defined by the contact 1-form η is given by

$$\{h_1, h_2\}_C \circ \mathbf{p_S} = -\frac{1}{F}\{F(h_1 \circ \mathbf{p_S}), F(h_2 \circ \mathbf{p_S})\}_S. \tag{62}$$

The relation between the Hamiltonian vector field X_H^ω of H with respect ω and the Hamiltonian vector field $X_{h_H}^\eta$ of h_H with respect to the contact structure η is

$$T\mathbf{p_S} \circ X_H^\omega = X_{h_H}^\eta \circ \mathbf{p_S}.$$

Remark 6. Since above equation is actually true for any homogeneous function H, for $H = F$ we have that

$$T\mathbf{p_S} \circ X_F^\omega = X_1^\eta \circ \mathbf{p_S} = \mathcal{R} \circ \mathbf{p_S}, \tag{63}$$

where $\mathcal{R}$ is the Reeb vector field of η.

Below, we shall use all these facts to address the reconstruction problem for the system (S, ω, H) and the action ϕ.

7.2.2 | Solving the reconstruction problem

Suppose that we want to find the integral curve $\Gamma : (-\epsilon, \epsilon) \to S$ of X_H^ω such that $\Gamma(0) = x_0$. Following the step 1 of the reconstruction procedure, let us fix the integral curve $\gamma : (-\epsilon, \epsilon) \to C = S/\mathbb{R}^\times$ of $X_{h_H}^\eta$ such that $\gamma(0) = \mathbf{p_S}(x_0)$. Now, we need to find the curves $\varphi(t)$ and $g(t)$ of steps 2 and 3. Consider the flat connection A given by (59) and (60). Define

$$\varphi(t) := (F, \mathbf{p_S})^{-1}(s_0, \gamma(t)), \quad \forall t \in (-\epsilon, \epsilon),$$

with $s_0 = F(x_0)$. Then, $\mathbf{p_S}(\varphi(t)) = \gamma(t)$ and $F(\varphi(t)) = s_0$. In particular, $\varphi(t)$ belongs to a level set of F (of value $s_0 \in \mathbb{R}^\times$), and consequently its tangent vector belongs to $\langle dF\rangle^o = \ker A$, that is,

$$A\big(\varphi'(t)\big) = 0 \quad \forall t \in (-\epsilon, \epsilon).$$

Then, the two parts of (56) are satisfied. Furthermore, since $\gamma(0) = \mathbf{p_S}(x_0)$,

$$\varphi(0) = (F, \mathbf{p_S})^{-1}(s_0, \gamma(0)) = (F, \mathbf{p_S})^{-1}(F(x_0), \mathbf{p_S}(x_0)) = x_0.$$

Thus, the step 2 is complete.

In order to find the curve $g(t)$, let us calculate $A\left(X_H^\omega(\varphi(t))\right)$. Using the decomposition $TS = \langle\Delta\rangle \oplus \langle dF\rangle^o$, we have that

$$X_H^\omega = f\,\Delta + Z,$$

with $f \in C^\infty(S)$ and Z a vector field on S such that $Z(F) = 0$. It follows that

$$\{F, H\}_S = X_H^\omega(F) = f\,\Delta(F) = f\,F,$$

and consequently

$$f = \frac{\{F, H\}_S}{F}.$$

Then,

$$A \circ X_H^\omega = f\,\Delta = \frac{\{F, H\}_S}{F}\,\Delta.$$

Writing $g(t) = \exp(\alpha(t))$, Equation (57) translates to

$$\alpha'(t) = \frac{\{H, F\}_S(\varphi(t))}{s_0}, \quad \alpha(0) = 0,$$

which has the solution

$$\alpha(t) = \frac{1}{s_0}\int_0^t \{H, F\}_S(\varphi(s))\,ds.$$

Summing up, the trajectory which we are looking for is

$$\Gamma(t) = \phi^S\left(\exp\left(\frac{1}{s_0}\int_0^t \{H, F\}_S(\varphi(s))\,ds\right), \varphi(t)\right), \quad \text{with} \quad \varphi(t) = (F, \mathbf{p_S})^{-1}(s_0, \gamma(t)). \tag{64}$$

Remark 7. If H itself is a scaling function (i.e., $H(x) \neq 0$ for all x), then we can take $F = H$. In such a case, $\{H, F\}_S = 0$ and consequently

$$\Gamma(t) = \varphi(t) = (H, \mathbf{p_S})^{-1}(s_0, \gamma(t)), \tag{65}$$

where $s_0 = H(x_0)$.

Now, we shall construct an alternative expression of $\Gamma(t)$, which involves the Reeb vector field $\mathcal{R}$ of (C,η). Using (63) and acting with the first and last members on the differential of the contact Hamiltonian function h_H (related to H), it easily follows that

$$\{H,F\}_S = -F\,(\mathcal{R}(h_H)\circ\mathbf{p_S}).$$

Then,

$$\frac{1}{s_0}\int_0^t \{H,F\}_S(\varphi(s))\,ds = -\int_0^t \mathcal{R}(h_H)(\gamma(s))\,ds,$$

and applying $(F,\mathbf{p_S})$ on (64) we have that

$$(F,\mathbf{p_S})(\Gamma(t)) = \left(s_0\,\exp\left(-\int_0^t \mathcal{R}(h_H)(\gamma(s))\,ds\right),\gamma(t)\right). \tag{66}$$

Thus, we have found, up to quadratures, the trajectories $\Gamma(t)$ of X_H^ω from the trajectories $\gamma(t)$ of $X_{h_H}^\eta$.

Remark 8. According to the local existence of scaling functions, if there is not a (global) scaling function for ϕ^S, then we can proceed as above around every point $x \in S$, just replacing S by an appropriate coordinate neighborhood U of x_0. In particular, we can obtain the result of Remark 7 along the open submanifold of S where $H \neq 0$.

To end this section, suppose that, instead of a symplectic Hamiltonian system, we have a Poisson Hamiltonian system (P,Π,H) with scaling symmetry for $\phi^P : \mathbb{R}^\times \times P \to P$ such that $\mathbf{p_P} : P \to K = P/\mathbb{R}^\times$ is a principal bundle. Assume that $F : P \to \mathbb{R}^\times$ is a scaling function for ϕ^P. Then, as we saw above, the related line bundle $\pi_L : L \to K$ is trivial (via a global section as that given by (58)), so the sections of π_{L^*} can be identified with the functions $h : K \to \mathbb{R}$, which in turn are in bijection with the homogeneous functions $H : P \to \mathbb{R}$ through the equation $h_H\circ\mathbf{p_P} = \frac{1}{F}H$. Also, the related Kirillov bracket $[\cdot,\cdot]_{L^*}$ can be identified with the Jacobi bracket $\{\cdot,\cdot\}_K$ given by

$$\left\{h_{H_1},h_{H_2}\right\}_K\circ\mathbf{p_P} = -\frac{1}{F}\{H_1,H_2\}_P.$$

Moreover, following the same calculations made along this section for the symplectic case, given $x_0 \in P$, we can construct the trajectory $\Gamma(t)$ of $X_H^{\{\cdot,\cdot\}_P}$ such that $\Gamma(0) = x_0$, in terms of the trajectory $\gamma(t)$ of $X_{h_H}^{[\cdot,\cdot]_{L^*}}$ such that $\gamma(0) = \mathbf{p_P}(x_0)$, through the equation

$$(F,\mathbf{p_P})(\Gamma(t)) = \left(s_0\,\exp\left(-\int_0^t E(h_H)(\gamma(s))\,ds\right),\gamma(t)\right),$$

with $s_0 = F(x_0)$ and $E \in \mathfrak{X}(K)$ such that $E(f) = \{1,f\}_K$.

Example 10. The 2D harmonic oscillator. If we consider the local coordinates $(\rho,\theta,\rho',\theta')$ defined at the end of Example 4 on $\mathbb{R}^+\times S^1\times\mathbb{R}^+\times S^1 \cong T^*(\mathbb{R}^2-\{(0,0)\}) - 0_{\mathbb{R}^2-\{(0,0)\}}$, then we have that the local expression of the Hamiltonian function is just

$$H(\rho,\theta,\rho',\theta') = \frac{1}{2}\rho^2(1+(\rho')^2),$$

which is a scaling function.

Moreover, the reduced space is $\mathbb{R}^+\times S^1\times S^1$ and the principal bundle $\mathbf{p}:\mathbb{R}^+\times S^1\times\mathbb{R}^+\times S^1\to\mathbb{R}^+\times S^1\times S^1$ is given by $\mathbf{p}(\rho,\theta,\rho',\theta')=(\rho',\theta,\theta')$. Now, we will describe the integral curve $\Gamma:(-\epsilon,\epsilon)\to\mathbb{R}^+\times S^1\times\mathbb{R}^+\times S^1$ of X_H^ω with $\Gamma(0)=(\rho_0,\theta_0,\rho_0',\theta_0')$. Note that the inverse of the diffeomophism

$$(H,\mathbf{p}):\mathbb{R}^+\times S^1\times\mathbb{R}^+\times S^1\to\mathbb{R}^+\times\mathbb{R}^+\times S^1\times S^1,\quad (H,\mathbf{p})(\rho,\theta,\rho',\theta')=\left(\frac{1}{2}\rho^2(1+(\rho')^2\right),\rho',\theta,\theta')$$

is

$$(H,\mathbf{p})^{-1}:\mathbb{R}^+\times\mathbb{R}^+\times S^1\times S^1\to\mathbb{R}^+\times S^1\times\mathbb{R}^+\times S^1,\quad (H,\mathbf{p})^{-1}(\rho,\rho',\theta,\theta')=\left(\sqrt{\frac{2\rho}{1+(\rho')^2}},\theta,\rho',\theta'\right).$$

Therefore, the integral curve of X_H^ω such that $\Gamma(0)=(\rho_0,\theta_0,\rho_0',\theta_0')$ is (see (65))

$$\Gamma(t)=(H,\mathbf{p})^{-1}\left(\frac{1}{2}\rho_0^2(1+(\rho_0')^2),\gamma(t)\right)=\left(\rho_0\sqrt{\frac{(1+(\rho_0')^2)}{(1+\rho'(t)^2)}},\gamma(t)\right),$$

where $\gamma(t)=(\theta(t),\rho'(t),\theta'(t))$ is the integral curve of the contact Hamiltonian vector field

$$X^\eta_{h_H}=(1+(\rho')^2)\cos(\theta-\theta')\partial_{\rho'}+\sin(\theta-\theta')\left(\frac{1}{\rho'}\partial_{\theta'}+\rho'\partial_\theta\right)$$

(see (17)).

Example 11. The projective cotangent Hamiltonian system deduced from a standard linear Hamiltonian system. We consider Example 5 with $Y\in\mathfrak{X}(Q)$ a vector field on the manifold Q of dimension n. Let U_{i_0} be the open subset of T^*Q-0_Q given by

$$U_{i_0}=\{(q^1,\dots q^n,p_1,\dots p_n)\in T^*Q-0_Q/p_{i_0}\neq 0\},$$

with (q^i,p_i) local coordinates on T^*Q. The local expressions of the linear function Y^ℓ and of the corresponding Hamiltonian vector field $X^{\omega_Q}_{Y^\ell}$ are

$$Y^\ell(q,p)=Y^i(q)p_i\qquad\text{and}\qquad X^{\omega_Q}_{Y^\ell}=Y^k\partial_{q^k}-p_j\partial_{q^k}Y^j\partial_{p_k},$$

with $Y(q)=Y^i(q)\partial_{q^i}$.

Note that Y^ℓ is a scaling function if and only if Y is a vector field without zeros. In any case, we have a scaling function on U_{i_0} given by

$$F:U_{i_0}\to\mathbb{R},\quad F(q^i,p_i)=p_{i_0}.$$

After the reduction process of the Hamiltonian symplectic system (T^*Q-0_Q,ω_Q,H) by the scaling symmetry (8), we have that the local expressions of the reduced elements are:

- The local expression of the projective bundle $\mathbf{p} : T^*Q - 0_Q \to \mathbb{P}(T^*Q)$ on U_{i_0} :

$$\mathbf{p}(q^1, \dots q^n, p_1, \dots p_n) = \left(q^1, \dots q^n, \frac{p_1}{p_{i_0}}, \dots, \frac{p_{i_0-1}}{p_{i_0}}, \frac{p_{i_0+1}}{p_{i_0}}, \dots, \frac{p_n}{p_{i_0}}\right).$$

- The contact distribution $\mathcal{D}$ on U_{i_0} :

$$\begin{aligned}(\mathcal{D}_{(q,\tilde{p})})_{|\mathbf{p}(U_{i_0})} = & \quad T_{(q,p)}\mathbf{p}(< p_i dq^i >^o) = T_{(q,p)}\mathbf{p} < X_1, \dots, X_{i_0-1}, X_{i_0+1}, \dots, X_n, \partial_{p_1}, \dots, \partial_{p_n} > \\ = & \quad < \widetilde{X}_1, \dots, \widetilde{X}_{i_0-1}, \widetilde{X}_{i_0+1}, \dots, \widetilde{X}_n, \partial_{\widetilde{p}_1}, \dots, \partial_{\widetilde{p}_{i_0-1}}, \partial_{\widetilde{p}_{i_0+1}}, \dots, \partial_{\widetilde{p}_n} >,\end{aligned}$$

with $X_i = p_i\partial_{q^{i_0}} - p_{i_0}\partial_{q^i}$, $\widetilde{X}_i = \widetilde{p}_i\partial_{q^{i_0}} - \widetilde{p}_{i_0}\partial_{q^i}$ and $(q, \widetilde{p}) = (q, \widetilde{p}_1, \dots \widetilde{p}_{i_0-1}, \widetilde{p}_{i_0+1}, \dots, \widetilde{p}_n)$ local coordinates on $\mathbb{P}(T^*Q)$.

The local expression of the line bundle $\pi_{\mathcal{D}^o} : \mathcal{D}^o \to \mathbb{P}(T^*Q)$ on U_{i_0} is

$$\pi_{\mathcal{D}^o}(q, \widetilde{p}_i, t) = (q, \widetilde{p}_i).$$

- The section $h_{Y^\ell} : \mathbb{P}(T^*Q) \to (\mathcal{D}^o)^*$ of $\pi_{(\mathcal{D}^o)^*} : (\mathcal{D}^o)^* \to \mathbb{P}(T^*Q)$ associated with Y^ℓ :

$$h_{Y^\ell}(q, \widetilde{p})(q, \widetilde{p}, t) = Y^\ell(q, \widetilde{p}_1, \dots, \widetilde{p}_{i_0-1}, t, \widetilde{p}_{i_0+1}, \dots, \widetilde{p}_n) = Y^i(q)\widetilde{p}_i + Y^{i_0}(q)t. \tag{67}$$

- The $\mathbf{p}$-projection of the Hamiltonian vector field $X^{\omega_Q}_{Y^\ell} \in \mathfrak{X}(T^*Q - 0_Q)$ to $\mathbb{P}(T^*Q)$:

$$Y^i\partial_{q^i} + \left(\widetilde{p}_j(\widetilde{p}_i\partial_{q^{i_0}}Y^j - \partial_{q^i}Y^j) + \widetilde{p}_i\partial_{q^{i_0}}Y^{i_0} - \partial_{q^i}Y^{i_0}\right)\partial_{\widetilde{p}_i}. \tag{68}$$

- The trivialization $(F, \mathbf{p}) : T^*Q - 0_Q \to \mathbb{R}^\times \times \mathbb{P}(T^*Q)$:

$$(F, \mathbf{p})\big(q^1, \dots, q^n, p_1, \dots, p_n\big) = \left(p_{i_0}, \left(q^1, \dots, q^n, \frac{p_1}{p_{i_0}}, \dots, \frac{p_{i_0-1}}{p_{i_0}}, \frac{p_{i_0+1}}{p_{i_0}}, \dots, \frac{p_n}{p_{i_0}}\right)\right)$$

and its inverse map

$$(F, \mathbf{p})^{-1}(s, (q^1, \dots, q^n, \widetilde{p}_1, \dots, \widetilde{p}_{i_0-1}, \widetilde{p}_{i_0+1}, \dots, \widetilde{p}_n)) = (q^1, \dots, q^n, s\widetilde{p}_1, \dots, s\widetilde{p}_{i_0-1}, s, s\widetilde{p}_{i_0+1}, \dots, s\widetilde{p}_n).$$

The integral curve $\Gamma : (-\epsilon, \epsilon) \to T^*Q - 0_Q$ of the Hamiltonian vector field $X^{\omega_Q}_{Y^\ell}$ such that $\Gamma(0) = (q^i_0, p^0_i)$ is (see (64))

$$\Gamma(t) = (q^i(t), \exp\left(\frac{1}{p^0_{i_0}}\int_0^t (p_j\partial_{q^{i_0}}Y^j(q(s))ds)\right)(p^0_{i_0}\widetilde{p}_1(t), \dots, p^0_{i_0}\widetilde{p}_{i_0-1}(t), p^0_{i_0}, p^0_{i_0}\widetilde{p}_{i_0+1}(t), \dots, p^0_{i_0}\widetilde{p}_n(t)),$$

where $\gamma(t) = (q^i(t), \widetilde{p}_1(t), \dots, \widetilde{p}_{i_0-1}(t), \widetilde{p}_{i_0+1}(t), \dots, \widetilde{p}_n(t))$ is an integral curve of the vector field given in (68) such that

$$\gamma(0) = \left(q^i_0, \frac{p^0_1}{p^0_{i_0}}, \dots, \frac{p^0_{i_0}}{p^0_{i_0}}, \frac{p^0_{i_0+1}}{p^0_{i_0}}, \dots, \frac{p^0_n}{p^0_{i_0}}\right).$$

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DATA AVAILABILITY STATEMENT

The data supporting the conclusions of this paper are included within the article.

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APPENDIX: LINE BUNDLES AND $\mathbb{R}^\times$-PRINCIPAL BUNDLES

Let $p_M : M \to K$ be the principal bundle associated with an action $\phi^M : \mathbb{R}^\times \times M \to M$ of the multiplicative group $\mathbb{R}^\times$ (with $\mathbb{R}^\times = \mathbb{R} - \{0\}$ or $\mathbb{R}^\times = \mathbb{R}^+$) on the manifold M. Consider the representation $\mathbb{R}^\times \times \mathbb{R} \to \mathbb{R}$ of $\mathbb{R}^\times$ over the vector space of real numbers given by

$$(s, t) \to \frac{t}{s}.$$

Let $\tilde{\phi}^M : \mathbb{R}^\times \times (M \times \mathbb{R}) \to (M \times \mathbb{R})$ be the action of $\mathbb{R}^\times$ on the cartesian product $M \times \mathbb{R}$ given by

$$\tilde{\phi}^M(s, (x, t)) = \left(\phi^M(s, x), \frac{t}{s}\right) \text{ with } (s, (x, t)) \in \mathbb{R}^\times \times (M \times \mathbb{R}). \tag{A.1}$$

Then, the first projection $p_1 : M \times \mathbb{R} \to M$ is an equivariant map with respect to the actions $\tilde{\phi}^M$ and ϕ^M and the map $\pi_L : L := (M \times \mathbb{R})/\mathbb{R}^\times \to K = M/\mathbb{R}^\times$ between the corresponding quotient spaces is a vector bundle with fiber $\mathbb{R}$. It is *the line bundle associated with $p_M : M \to K$ and the representation (A.1)*.

If 0_L is the zero section of the vector bundle $\pi_L : L \to K$ and $\pi : M \times \mathbb{R} \to L = (M \times \mathbb{R})/\mathbb{R}^\times$ is the quotient map, one can identify M with $L - 0_L$, via the isomorphism of principal bundles

$$M \to (L - 0_L), \qquad x \in M \to \pi(x, 1) \in L - 0_L.$$

Conversely, if $\pi_L : L \to K$ is a line bundle (vector bundle with fiber $\mathbb{R}$) and 0_L is the zero section of π_L, then $p_M : M := (L - 0_L) \to K$ is a $\mathbb{R}^\times$-principal bundle. The action associated with this principal bundle is given by

$$\phi^M : \mathbb{R}^\times \times (L - 0_L) \to (L - 0_L), \qquad \phi^M(s, x) = sx$$

for $x \in L - 0_L$, and the line bundle associated with this principal bundle is isomorphic to π_L. In fact, the $\mathbb{R}^\times$-invariant map

$$(L - 0_L) \times \mathbb{R} \to L, \quad (x, t) \to tx, \text{ with } (x, t) \in (L - 0_L) \times \mathbb{R}$$

induces an isomorphism between the line bundles $((L - 0_L) \times \mathbb{R})/\mathbb{R}^\times$ and L.

Proposition A.1. *Let $p_M : M \to K$ be a $\mathbb{R}^\times$-principal bundle and $\pi_L : L \to K$ its associated line bundle. Then, there is a one-to-one correspondence between the sections $h : K \to L^*$ on the dual vector bundle of $\pi_L : L \to K$ and the homogeneous functions on M, that is, functions $H : M \to \mathbb{R}$ satisfying the condition*

$$H(\phi^M(s, x)) = sH(x), \quad \textit{for all } s \in \mathbb{R}^\times, x \in P,$$

where $\phi^M : \mathbb{R}^\times \times M \to M$ is the corresponding principal action.

Proof. Indeed, if $h : K \to L^*$ is a section of $\pi_{L^*} : L^* \to K$ and $\pi : M \times \mathbb{R} \to L = (M \times \mathbb{R})/\mathbb{R}^\times$ is the canonical projection, one can define the function

$$H_h : M \to \mathbb{R}, \quad H_h(x) = h(p_M(x))(\pi(x,1)), \text{ for all } x \in M, \tag{A.2}$$

which satisfies that

$$H_h(\phi^M(s,x)) = h(p_M(x))(\pi(\phi^M(s,x),1)) = h(p_M(x))(\pi(x,s)) = h(p_M(x))(s\pi(x,1)) = sH_h(x)$$

for $(s,x) \in \mathbb{R}^\times \times M$. Therefore, H_h is homogenous with respect to ϕ^M.

Conversely, if $H : M \to \mathbb{R}$ is a homogenous function for the action ϕ^M, then we have a section $h_H : K \to L^*$ of π_{L^*} given by

$$h_H(p_M(x))(\pi(x,t)) = tH(x) \text{ for all } x \in M \text{ and } t \in \mathbb{R}, \tag{A.3}$$

which is well-defined by the homogeneity of H. □