in which $f(X_k)$ is assumed to be observable. Based on this equation a recursive algorithm called GPOMDP is presented in [1].

If the recurrent state i^* is observable, we have the following algorithm (cf. Algorithm 3):

$$\frac{d\eta_\delta}{d\delta} = E\left\{ \frac{\frac{d}{d\theta}p_\theta(X_k, X_{k+1})}{p(X_k, X_{k+1})} \left\{ \sum_{l=k+1}^{u_{m+1}-1} [f(X_l) - \eta] \right\} \right\}.$$

IV. Remarks and Discussions

Early work on sample-path-based performance gradient estimation include the PA [18], [6], [13] and the likelihood ratio (LR) [also called the score function (SF)] methods [15], [16], [22], [23]. PA was first developed for queueing networks; efficient algorithms have been developed [17]. The main idea of PA, perturbation realization, was later extended to performance gradients of Markov systems [7], [8].

Policy gradient [1], [2] is a terminology used in recent years in RL community for sample-path-based performance gradient estimate of PA. However, there is a slight difference in their emphases. Most policy gradient papers focus on developing simulation/online algorithms for estimating performance gradients. PA, on the other hand, emphasizes two aspects: deriving performance gradient formulas (those similar to (2)), and developing estimation algorithms. With the concept of perturbation realization factors, we can flexibly derive sensitivity formulas for many problems; these formulas are otherwise difficult to conceive [11]. Sample-path-based algorithms can be developed/designed only after these performance gradient formulas are derived. The readers can find some examples of the performance gradient formulas for systems with special structures in [11]. The basic formula (7) and the general algorithm (8) presented in this note provide a direction for developing performance gradient algorithms using the performance gradient formulas.

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A Min-Plus Derivation of the Fundamental Car-Traffic Law

Pablo A. Lotito, Elina M. Mancinelli, and Jean-Pierre Quadrat

***Abstract*—We give deterministic and stochastic models of the traffic on a circular road without overtaking. From this model the mean speed is derived as an eigenvalue of the min-plus matrix describing the dynamics of the system in the deterministic case and as the Lyapunov exponent of a min-plus stochastic matrix in the stochastic case. The eigenvalue and the Lyapunov exponent are computed explicitly. From these formulas, we derive the fundamental law that links the flow to the density of vehicles on the road. Numerical experiments using the MaxPlus toolbox of Scilab confirm the theoretical results obtained.**

***Index Terms*—Cellular automata, fundamental diagram, Lyapunov exponent, max-plus algebra.**

I. Introduction

For simple traffic models a well known relation exists between the flow and the density of vehicles called *fundamental traffic law*. This law has been studied empirically and theoretically using exclusion processes (see, for example, [5]–[7], [3], [12], and [8]) and cellular automata (see [1]).

In this note, we analyze the simplest deterministic and stochastic traffic models using the so called *min-plus algebra*. Within this algebra the equations of the dynamics become linear and the eigenvalue or the Lyapunov exponent of the corresponding min-plus matrix gives the mean speed from which we easily derive the density-flow relation.

Manuscript received August 20, 2003; revised April 15, 2004 and July 6, 2004. Recommended by Associate Editor A. Giua.

P. A. Lotito is with the GRETIA-INRETS, 94114 Arccueil, France (e-mail: pablo.lotito@inria.fr).

E. M. Mancinelli is with the INRIA, 78153 Le Chesnay, Cedex, France, and also with the CONICET, Argentina (e-mail: elina.mancinelli@inria.fr).

J.-P. Quadrat is with the INRIA, 78153 Le Chesnay, Cedex, France (e-mail: jean-pierre.quadrat@inria.fr).

Digital Object Identifier 10.1109/TAC.2005.848336

The traffic model consists of N cars on a circular road of unitary length. In the deterministic case all cars want to move at a common desired given velocity ν, and must respect a safety distance of σ with respect to the car ahead. In the stochastic model the cars choose their velocities randomly and independently between two possible values η and ν (with $\eta < \nu$), respectively with probabilities (μ, λ). We consider here only the case where the cars are not allowed to overtake other cars.

First, in the deterministic case, the fundamental law is derived from the explicit computation of the min-plus eigenvalue of the matrix describing the dynamics of the system.

Next, we study the stochastic model showing that the average speed is the Lyapunov exponent of a stochastic min-plus matrix. In general, it is very difficult to compute a Lyapunov exponent. In our case, it is possible to characterize completely the stationary regime and from this characterization to obtain the Lyapunov exponent. The fundamental traffic law is then easily derived from this result.

The analysis of the deterministic model in terms of eigenvalues of a maxplus matrix is new, but the model and the results are very close[1] to [3]. In [11], more realistic deterministic models are studied. The state is defined by the vehicle position and its speed instead of only the position. Nevertheless, we obtain also a typical hat-shaped fundamental traffic law. This fact suggests that the acceleration is not fundamental in first order approximations. In fact, in Nagel–Hermann we see that in the stationary regime, for the parallel updating rule (the one used here), the system reaches the maximal allowed speed.

The stochastic model proposed is new. Its interest is mainly theoretical since the traffic law obtained is a smoothed version of the hat shape obtained in the deterministic case. The complete analysis can be done only in the simple case when the speed, which is random, can take only two values and when the size of the vehicles is zero. However, numerical experiments show that improving the model of speeds and giving a nonzero size to the cars has a negligible influence. Moreover, the analysis in the oversimplified but feasible case used here is qualitatively very informative. The more realistic stochastic model of Nagel–Schreckenberg [12] gives the same kind of traffic law, but the analysis can be done only by numerical experiments. Derrida, in [5]–[7], gives a complete theoretical analysis of an exclusion stochastic process modeling of traffic which is different from the one proposed here (a vehicle is characterized only by its position and can jump ahead with a given probability if the position ahead is free). In the Derrida model, the process is ergodic and the invariant probabilistic measure can be computed explicitly. Here, the process is not ergodic. However, we can characterize the stationary regimes and determine completely the invariant measures.

II. Deterministic Modelling

We consider N cars moving on a one-way circular road of length 1. Each car, indexed by $n = 1, \ldots, N$, has a desired speed ν, a size 0, and must respect a security distance σ with the car ahead $(N\sigma \leq 1)$. A discrete-time dynamic model is used where, at each unitary time step t, the driver tries to cover the distance ν taking into account that it cannot overtake the car ahead. The total distance covered at time t by car n is denoted x_n^t. In order to determine the dynamics of the system, we have to know at what precise instant the safety distances have to be verified. We consider two cases.

i) *The move of the driver ahead is anticipated* (at time t the driver n knows the position that will have the car ahead at time $t+1$). Having in mind that the road is circular and that its length is one,[2] the covered distances are given by

$$x_n^{t+1} = \begin{cases} \min\left(x_n^t + \nu, x_{n+1}^{t+1} - \sigma\right), & \text{if } n < N \\ \min\left(x_N^t + \nu, x_1^{t+1} + 1 - \sigma\right), & \text{if } n = N. \end{cases} \quad (1)$$

ii) *The move of the driver ahead is not anticipated*. The distances covered by the cars are

$$x_n^{t+1} = \begin{cases} \min\left(x_n^t + \nu, x_{n+1}^{t} - \sigma\right), & \text{if } n < N \\ \min\left(x_N^t + \nu, x_1^{t} + 1 - \sigma\right), & \text{if } n = N. \end{cases} \quad (2)$$

For these two models, we will derive a relation between the car density and the average car flow, that will correspond to the fundamental traffic law in traffic theory.

III. Min-Plus Algebra

To derive the fundamental traffic law we need to compute the eigenvalue of a min-plus matrix describing the dynamics of the traffic system. In this section, we present the principal definitions and properties of the min-plus algebra. The reader is referred to [2] for an in-depth treatment of the subject. A min-plus algebra is defined by the set $\mathbb{R} \cup \{+\infty\}$ together with the operations min (denoted by $\oplus$) and $+$ (denoted by $\otimes$). The element $\varepsilon = +\infty$ satisfies $\varepsilon \oplus x = x$ and $\varepsilon \otimes x = \varepsilon$ (ε acts as zero). The element $e = 0$ satisfies $e \otimes x = x$ (e is the identity). The main difference with respect to the conventional algebra is that $x \oplus x = x$ (idempotency). We denote $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$ this structure. $\mathbb{R}_{\min}$ is a special instance of dioid (semiring with idempotent addition).

This min-plus structure on scalars induces a dioid structure on square matrices with matrix product $A \otimes B$, for two compatible matrices with entries in $\mathbb{R}_{\min}$, defined by $(A \otimes X)_{ik} = \min_j (A_{ij} + B_{jk})$, where the unit matrix is denoted E. We associate to a square matrix A a precedence graph $\mathcal{G}(A)$ where the nodes correspond to the columns (or the rows) of the matrix A and the arcs to the nonzero entries (the weight of the arc (i,j) being the non zero entry A_{ji}). We define $|p|_w$ the weight of a path p in $\mathcal{G}(A)$ as the sum of the weights of the arcs composing the path. The arc number of the path p is denoted $|p|_l$. We will use the three fundamental results resumed in the following proposition (see [2]).

Proposition 1: Let A be a $(N \times N)$-$\mathbb{R}_{\min}$-matrix, and $\mathcal{C}$ the set of circuits of $\mathcal{G}(A)$. We have the following.

i) If the weights of all the circuits are positive, the equation $x = A \otimes x \oplus b$ admits a unique solution $x = A^* \otimes b$ where $A^* = E \oplus A \oplus \cdots \oplus A^{N-1} \oplus \cdots = E \oplus A \oplus \cdots \oplus A^{N-1}$.

ii) If $\mathcal{G}(A)$ is strongly connected, the matrix A admits a unique eigenvalue $\lambda \in \mathbb{R}_{\min}$:

$$\exists\, x \in \mathbb{R}_{\min}^N : A \otimes x = \lambda \otimes x \text{ with } \lambda = \min_{c \in \mathcal{C}} \frac{|c|_w}{|c|_l} \quad (3)$$

and the min-plus linear dynamic system $X^{t+1} = A \otimes X^t$ is asymptotically periodic

$$\exists T, K: \qquad \forall k \geq K : A^{k+T} = \lambda^T \otimes A^k.$$

IV. Fundamental Traffic Law in the Deterministic nonanticipative Case

Using the min-plus notation, the dynamics of the traffic in the non anticipative case given by (2) may be written in scalar form as follows:

$$x_n^{t+1} = \begin{cases} \nu \otimes x_n^t \oplus (-\sigma) \otimes x_{n+1}^t, & \text{if } n < N \\ \nu \otimes x_N^t \oplus (1-\sigma) \otimes x_1^t, & \text{if } n = N. \end{cases} \quad (4)$$

[1] In our model there are no cells, but the Blank's cell model is also maxplus linear and can be analyzed by the same method and gives the same traffic law.

[2] This explains the "+1" in (1) and (2).

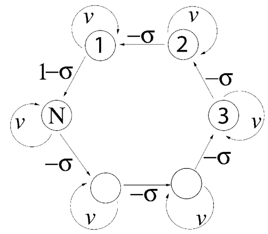

Fig. 1. Precedence graph of A.

In vectorial form, defining $X^t = [x_1^t, \ldots, x_N^t]'$, we have

$$X^{t+1} = A \otimes X^t \tag{5}$$

with

$$A = \begin{bmatrix} \nu & -\sigma & & \\ & \ddots & \ddots & \\ & & \ddots & -\sigma \\ 1-\sigma & & & \nu \end{bmatrix}$$

where the missing entries are ε. The precedence graph associated with A is given in Fig. 1.

In order to use the results given in the Proposition 1, we have to compute the circuits of the graph. The elementary circuits are the loops, of weight ν, and the complete circuit weighting $1 - N\sigma$. Using the eigenvalue formula (3), the eigenvalue of A is

$$\lambda = \min\left(\nu, \frac{1 - N\sigma}{N}\right). \tag{6}$$

Considering that the minimal space needed by a car on the road is σ, the car density d is $N\sigma$ divided by the length of the road, taken equal to 1, therefore, $d = N\sigma$. The average flow is equal to the car density times the average speed, that is $f = \lambda N\sigma$. Then, replacing in (6) we obtain the fundamental traffic law

$$f = \min\{\nu d, \sigma(1-d)\}.$$

Therefore, using this min-plus model, we find again the results presented in [3].

V. Fundamental Traffic Law in the Deterministic Anticipative Case

Using min-plus notation, the dynamics of the traffic in the anticipative case may be written

$$X^{t+1} = A \otimes X^{t+1} \oplus B \otimes X^t \tag{7}$$

where

$$A = \begin{bmatrix} \varepsilon & -\sigma & & \\ & \varepsilon & \ddots & \\ & & \ddots & -\sigma \\ 1-\sigma & & & \varepsilon \end{bmatrix} \quad B = \begin{bmatrix} \nu & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \nu \end{bmatrix}.$$

This is an implicit system, to obtain an explicit system we have to compute A^* (see Proposition 1-i or [2]). The existence of A^* is verified if and only if there is no circuit with negative weight in $\mathcal{G}(A)$, that is, if $1 - N\sigma \geq 0$, which is true by assumption. This condition means that there is enough place on the road for the N cars. The explicit form of the equation is

$$X^{t+1} = A^* \otimes B \otimes X^t. \tag{8}$$

The mean speed of the cars is the $\mathbb{R}_{\min}$ eigenvalue of $A^* \otimes B$ which can be easily verified to be equal to ν, therefore in this case, the fundamental traffic law is given by $f = \nu d$. This is an involved application of [2, Th. 3.28], nevertheless the result can be guessed without any computation. Indeed, in this deterministic anticipative case, all the cars can move with speed ν, (at the initial time the cars respect the security distance and they can move all together at speed ν respecting the safety distance).

VI. Stochastic Modelling

Now, we suppose that at each unitary time step t, each driver n chooses his desired speed v_n^t independently and randomly between $\{\eta, \nu\}$ with probabilities $\{\mu, \lambda\}$, $\eta \leq \nu$. That is, the random variables $\{v_n^t\}$, with $n = 1, \ldots, N$ and $t \in \mathbb{N}$, are i.i.d. Bernoulli random variables. We suppose that[3]: a) $\eta = 0$, b) that the safety distance is 0 (this means that two cars may be in the same position), c) the drivers may anticipate the move of the car ahead. Then, the dynamics of the system are given by

$$x_n^{t+1} = \begin{cases} \min\left(v_n^t + x_n^t, x_{n+1}^{t+1}\right), & \text{if } n < N \\ \min\left(v_N^t + x_N^t, 1 + x_1^{t+1}\right), & \text{if} n = N. \end{cases} \tag{9}$$

This system is still linear in the min-plus algebra but now it is stochastic. Within this algebra, (9) becomes

$$x_n^{t+1} = \begin{cases} v_n^t \otimes x_n^t \oplus x_{n+1}^{t+1}, & \text{if } n < N \\ v_N^t \otimes x_N^t \oplus 1 \otimes x_1^{t+1}, & \text{if } n = N. \end{cases} \tag{10}$$

Defining: $X^t = [x_1^t, \ldots, x_N^t]'$

$$A = \begin{bmatrix} \varepsilon & e & & \\ & \ddots & \ddots & \\ & & \ddots & e \\ 1 & & & \varepsilon \end{bmatrix} \quad B^t = B(v^t) = \begin{bmatrix} v_1^t & & & \\ & v_2^t & & \\ & & \ddots & \\ & & & v_N^t \end{bmatrix}$$

where the missing entries are ε, we can rewrite the equations more compactly as

$$X^{t+1} = A \otimes X^{t+1} \oplus B^t \otimes X^t. \tag{11}$$

In our case, A^* is easy to compute

$$A^* = \begin{bmatrix} e & e & \cdots & e \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & e \\ 1 & \cdots & 1 & e \end{bmatrix}.$$

Then

$$X^{t+1} = C^t \otimes X^t \tag{12}$$

with $C^t = A^* \otimes B^t$.

Using the fact that the matrices C^t are all irreducible (because there are no zero entries in C^t) we know by [2, Cor. 7.31] that

$$\lim_t x_n^t / t = \bar{v} \qquad \forall n$$

where $\bar{v}$ is called the Lyapunov exponent of the stochastic min-plus matrix C (with $(C^t)_{t\in\mathbb{N}}$, independent samples of C).

[3]Assumption a) is justified by the standard change of variables $x = x' + \eta t', t' = t$. Assumption b) allows us to obtain interesting mathematical results. The more general case ($\sigma \neq 0$) can be analogously modeled (see Section X) and numerical experiments have shown that the qualitative results are similar. Assumption c) is more realistic and can be analyzed mathematically but the non anticipative case is easier to analyze and gives the same kind of fundamental car-traffic laws.

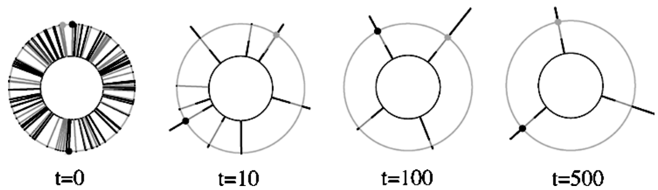

Fig. 2. Evolution of the system with 100 cars and $\nu = 1/3$.

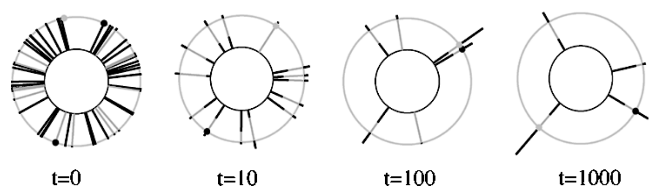

Fig. 3. Evolution of the system with 50 cars and $\nu = 0.3$.

In general, there is not a known method to compute explicitly the Lyapunov exponent. Explicit formulas involving computation of expectations are given in [10], but there is no way to compute explicitly these expectations. Nevertheless, here we are able to characterize the stationary regime of X^t. This allows us to compute explicitly the expectation appearing in $\bar{v}$.

VII. Jam Regime

In order to represent graphically the system state we use the diagrams shown on Figs. 2 and 3, where

i) each segment outside the outer ring has a length proportional to the number of cars in that position;

ii) the black (blue) [respectively, grey (green)] length of segments between the two rings are proportional to the number of cars with desired speed 0 [respectively, ν].

In Fig. 2, we show the evolution of the system for 100 cars with speeds 0 and $\nu = 1/3$.

In Fig. 3, we show the evolution of the system for 50 cars with speeds 0 and $\nu = 0.3$.

Definition 1: We call *state* the set of positions of the cars on the circle. We call *jam state*, a state where the cars are concentrated in k clusters, possibly empty, where $k = \lceil (1/\nu) \rceil$ is the upper round of $1/\nu$. The positions of the clusters are given by $\{\pi_1, \dots, \pi_k\}$ with $\pi_{i+1} - \pi_i = \nu$ for $i = 1, \dots, k-1$. In such a jam state, the distance between two clusters is ν except for at most one pair where the distance is $\gamma = 1-(k-1)\nu$. When $1/\nu \in \mathbb{N}$ we say that the jam state is *regular*, and the distance between all the clusters is ν.

Definition 2: When for all $t \geq T$ the system stays in a jam state we say that after T the system is in a *jam regime*.

Proposition 2: A jam state is characterized by $\delta(x) = 0$, where

$$\delta(x) = \min_{h=1,\dots,N} \left(\sum_{\substack{j=1,\dots,N \\ j \neq h}} \{x_{j+1} - x_j\} \right); \qquad (13)$$
$$\{x\} = x - \nu \left\lfloor \frac{x}{\nu} \right\rfloor .$$

For nonjam states, we have $\delta(x) > 0$. Moreover

$$\delta(X^T) = 0 \Rightarrow \delta(X^t) = 0 \qquad \forall t \geq T$$

that is, after reaching a jam state the system remains in a jam regime.

Proof: It is easy to see that $\delta(x) = 0$ for a jam state x. The question is then to show the converse. Let us suppose that $\delta(x) = 0$ by definition of δ there is an h^* such that

$$\sum_{j \neq h^*} \{x_{j+1} - x_j\} = 0$$

we can suppose without loss of generality that $h^* = N$, then for every $j \neq N, x_{j+1} - x_j$ is a multiple of ν. Defining $k = \lceil (1/\nu) \rceil$, and considering that

$$1 \geq x_N - x_1 = \sum_{j=1}^{N-1} x_{j+1} - x_j$$

we have that there are at most $k-1$ nonzero terms $j_1, \dots j_{k-1}$ which define the $k-1$ cluster in position $\pi_1 = x_{j_1}, \dots, \pi_{k-1} = x_{j_{k-1}}$ and the last cluster k is in position x_N. Therefore, the system is in a jam state.

Suppose the system has reached a jam state. Then, all the clusters are separated by ν except for the clusters h and $h+1$ which are separated by γ. As the cars try to move ν, it is easy to see that if the cluster $h+1$ is not empty, all the clusters will remain at the same relative position. If that cluster is empty then only the relative positions of clusters $h, h+1$ and $h+2$ will change from $\pi_{h+1} - \pi_h = \gamma, \pi_{h+2} - \pi_{h+1} = \nu$ to $\pi_{h+1} - \pi_h = \nu, \pi_{h+2} - \pi_{h+1} = \gamma$, remaining in a jam state. □

The function $\delta(x)$ can be seen as a sort of distance to a jam regime and it verifies the following property.

Theorem 1: The sequence $t \mapsto \delta(X^t)$ is nonincreasing.

The proof of the theorem is in Appendix 1.

Theorem 2: A jam regime is almost always reached, i.e., with probability one.

Proof: In order to prove that a jam regime is reachable, we construct a finite sequence of independent events with positive probability after which the system reaches a jam state. Then, this finite sequence will appear with probability one in an infinite sequence of events (Cantelli–Borel).

The dynamics of the system is given by the matrix $C(\omega) = A^* B(v(\omega))$, where B is the diagonal matrix of car desired-speeds chosen randomly and independently between 0 and ν. Let us consider the matrix C_j associated with the speed $(0 \cdots 0, \nu, 0 \cdots 0)$ with ν in position j. All the matrices $C_j, j = 1 \cdots N$ have a strictly positive probability of occurrence.

Consider the finite sequence of independent events associated to the following matrix product, where k is the number of clusters

$$C_1^k C_2^k, \dots, C_{N-2}^k C_{N-1}^k .$$

It is easy to understand why after these events, all the cars are together in only one cluster. The last car (N) stays at the same position, the previous car $(N-1)$ tries to move k times ν joining the car N and so on. At the end, all the cars will be together in only one cluster obtaining a jam state. □

The particular jam state used in the proof, has only the property of being easily characterized. Other jam states are reachable with a higher probability.

VIII. Stationary Car Distribution

Let us determine the stationary distribution of the population of cars $b = (b_1, \dots, b_k)$ in the k clusters.

Theorem 3: The stationary distribution of b is uniform on the simplex

$$B_{N,k} = \left\{ b \mid b.\mathbf{1} = N, b \in \mathbb{N}^k \right\}$$

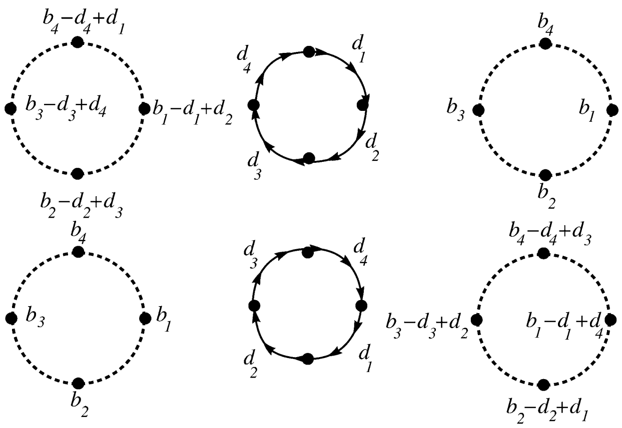

Fig. 4. Equally probable transitions.

where N is the total number of cars, k is the number of clusters in the stationary regime and $\mathbf{1}$ is a k-column vector of 1.

Proof: Let us consider the Markov chain where the states belong to $B_{N,k}$ having C_N^{N+k-1} nodes. Let us show that for each outgoing arc from a node there is an incoming arc with the same transition probability (which shows that the transition matrix is bistochastic).

Outgoing: Let us consider the transition from the state b to the state b'. This state can be written as $b' = b - d + \theta d$ where θ denotes the circular shift of a vector, $\theta : d = (d_1, \ldots, d_k) \mapsto (d_k, d_1, \ldots, d_{k-1})$ and d is the leaving cluster vector (this means that there are d_j cars that leave the cluster j to the cluster $j+1$), this is represented in the second line of Fig. 4. The probability of that event is

$$\lambda^{\sum d_j} \mu^{\sum \phi(b_j - d_j)} \text{ where } \phi(s) = \begin{cases} 0, & \text{if } s = 0 \\ 1, & \text{otherwise.} \end{cases}$$

Incoming: We consider now the state $b - d + \theta^{-1} d$ from which we can reach the state b with $\theta^{-1} d$ as leaving cluster vector (see the first line of Fig. 4). The probability of this event is

$$\lambda^{\sum d_j} \mu^{\sum \phi(b_j - d_j + d_{j+1} - d_{j+1})}$$

but $\phi(b_j - d_j + d_{j+1} - d_{j+1}) = \phi(b_j - d_j)$ and, thus, it has the same probability than the corresponding outgoing arc.

To complete the proof, we have to show that the map that associates to each output arc an input one, is bijective. For that, since the map is injective, let us show that the number of outgoing arcs from a particular state b is equal to the number of incoming arcs to this state. The number of outgoing arcs from b is the number of elements of the set $\{d \mid 0 \leq d \leq b\}$ where the order relation is considered componentwise. The incoming arcs to state b is given by d' such that there exists a state b' with $b' - d' + \theta d' = b$ and $0 \leq d \leq b'$, but this implies that $0 \leq d' \leq b$. Therefore, the set of incoming arcs to b is defined by $\{d' \mid 0 \leq d' \leq b\}$ which has the same cardinality that the set of outgoing arcs from b. □

IX. Mean Speed Computation

The knowledge of the distribution of probability of n allows the explicit computation of the mean speed. We do that in the following theorem.

Theorem 4: For the regular case the mean speed $\bar{v}_\lambda(N, k)$ can be obtained recursively as

$$\bar{v}_\lambda(N+1, k) = \frac{\lambda}{N+k}(1 + N\bar{v}_\lambda(N, k)) \tag{14}$$

where $\bar{v}_\lambda(1, k) = \lambda\nu$. Moreover, for large N, we have the asymptotic result

$$\bar{v}_\lambda(N, k) = \frac{\lambda}{N\mu} + o(1/N). \tag{15}$$

Proof: Let us compute the mean speed. Consider a cluster, the first car in the cluster leaves with probability λ increasing the mean speed in $\lambda\nu/N$, then the second car leaves this cluster with probability λ^2 increasing the mean speed in $\lambda^2\nu/N$ and so on. Then, the mean speed will be $\mathbb{E}(V)$ where

$$V = \sum_{s=1}^{k} \left(\sum_{j=1}^{b_s} \lambda^j \frac{\nu}{N} \right). \tag{16}$$

Developing (16), we obtain

$$V = \lambda \frac{\nu}{N} \sum_{s=1}^{k} \frac{1 - \lambda^{b_s}}{1 - \lambda} = \frac{\lambda}{\mu} \frac{\nu}{N} \left(k - \sum_{s=1}^{k} \lambda^{b_s} \right)$$

and by linearity

$$\bar{v}_\lambda(N, k) = \mathbb{E}(V) = \frac{\lambda}{\mu} \frac{\nu}{N} (k - S_k(N)) \tag{17}$$

where we have denoted

$$S_k(N) = \mathbb{E} \left(\sum_{s=1}^{k} \lambda^{b_s} \right). \tag{18}$$

Using the fact that the probability distribution of b is uniform, we have that

$$S_k(N) = \mathbb{E} \left(\sum_{s=1}^{k} \lambda^{b_s} \right) = \sum_{B_{N,k}} \pi_{N,k} \sum_{s=1}^{k} \lambda^{b_s} \tag{19}$$

where we have denoted $\pi_{N,k} = (\mathsf{C}_N^{N+k-1})^{-1}$. Interchanging the summation order we obtain

$$\begin{aligned} S_k(N) &= \pi_{N,k} \sum_{h=0}^{N} \sum_{s=1}^{k} \sum_{B_{N,k}} \{\lambda^h \mid b_s = h\} \\ &= \pi_{N,k} \sum_{h=0}^{N} \sum_{s=1}^{k} \mathsf{C}_{N-h}^{N-h+k-2} \lambda^h \\ &= k\pi_{N,k} \sum_{h=0}^{N} \mathsf{C}_{N-h}^{N-h+k-2} \lambda^h. \end{aligned}$$

Now, for $N+1$, we have

$$\begin{aligned} S_k(N+1) &= k\pi_{N+1,k} \sum_{h=0}^{N+1} \mathsf{C}_{N+1-h}^{N+1-h+k-2} \lambda^h \\ &= \frac{k(k-1)}{N+k} + k\pi_{N+1,k} \sum_{h=0}^{N} \mathsf{C}_{N-h}^{N-h+k-2} \lambda^{h+1} \\ &= \frac{k(k-1)}{N+k} + \frac{N+1}{N+k} \lambda S_k(N). \end{aligned}$$

Replacing the recursive formula of S_k in (17) we obtain (14). To find the asymptotic result (15) we remark that $S_k(N)$ goes to 0 when N goes to ∞. □

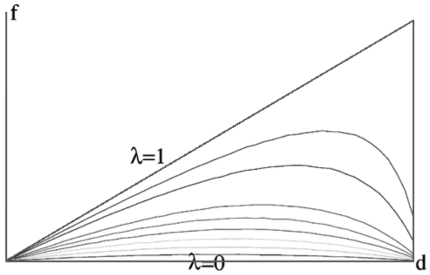

Fig. 5. Flow as a function of the density in the stochastic anticipative case for a continuation of λ when $\nu = \sigma$.

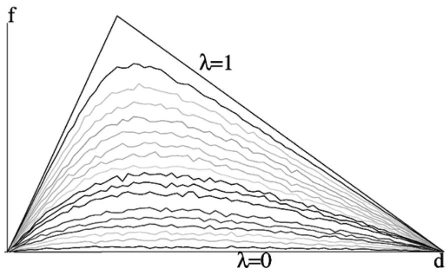

Fig. 6. Flow as a function of the density in the stochastic nonanticipative case for a continuation of λ when $\nu = 3\sigma$.

As an example we obtain for $N = 3$ and $k = 3$ that $\bar{v}_\lambda(3,3) = \nu(6\lambda + 3\lambda^2 + \lambda^3)/10$, and for $N = 4$ and $k = 4$ $\bar{v}_\lambda(4,4) = \nu(\lambda^4 + 4\lambda^3 + 10\lambda^2 + 20\lambda)/35$.

X. Extensions and Numerical Results

The previous analysis of the stochastic model may be done also in the nonanticipative case. It can be extended to the case where the cars have a non negligible size σ. The models are still stochastic min-plus linear. For example, in the latter case, we have

$$x_n^{t+1} = \begin{cases} v_n^t x_n^t \oplus (-\sigma) x_{n+1}^{t+1}, & \text{if } n < N \\ v_N^t x_N^t \oplus (1-\sigma) x_1^{t+1}, & \text{if } n = N. \end{cases}$$

Using the formula, obtained, or a simulator using the Max-Plus Scilab toolbox [14] we can plot the fundamental traffic law in the different cases; see Figs. 5 and 6.

XI. Conclusion

For traffic engineers, the main result is the obtainment of a realistic fundamental traffic law shape using a stochastic maxplus linear model defined by four parameters: Two possible desired speeds chosen randomly and a security length between the cars. This model can be still more simplified by taking only one desired speed. These models give the typical hat shape of the fundamental traffic law that can be adjusted by choosing these four parameters.

For system engineers, we have given an application of maxplus linear systems. The analysis of the deterministic case is a straightforward application of known results about maxplus algebra. On the other hand, the stochastic case is a rare example where a Lyapunov exponent can be computed explicitly.

Appendix
Proof of Theorem 1

Using the following notation for $0 \le j < l \le N$

$$\Delta_j^l(x) = \sum_{i=j}^{l-1} \{x_{i+1} - x_i\}$$

and

$$\hat{\Delta}_j^l(x) = \min_i \left(\Delta_j^i(x) + \Delta_{i+1}^l(x)\right)$$

the function δ can be written as

$$\delta(x) = \hat{\Delta}_1^N(x)$$

where the car numbered $N+1$ is identified with the car 1. If we call *hampered* at time t a car j such that $x_{j+1}^t - x_j^t < v_j^t$, Theorem 1 is an immediate consequence of the following lemma.

Lemma 1: At time t, for a sequence $(j+1, \ldots, l-1)$ of hampered cars and for unhampered cars j and l we have

$$\Delta_j^l(x^{t+1}) \le \Delta_j^l(x^t) \quad \hat{\Delta}_j^l(x^{t+1}) \le \hat{\Delta}_j^l(x^t).$$

Proof:

i) $\Delta_j^l(x^{t+1}) \le \Delta_j^l(x^t)$.

Indeed, in this case, we have

$$\Delta_j^l(x^t) = \{x_{j+1}^t - x_j^t\} + x_l^t - x_{j+1}^t.$$

If car l moves ν, the cars $j+1, \ldots, l-1$ move also ν and whatever is the desired speed of j we have $\Delta_i^j(x^{t+1}) = \Delta_i^j(x^t)$.

If car l does not move, as $\{a+b\} \le \{a\} + b$ and car j is unhampered, then

$$\Delta_j^l(x^{t+1}) \le \{x_{j+1}^t - x_j^t\} + x_{j+1}^{t+1} - x_{j+1}^t + \Delta_{j+1}^l(x^{t+1})$$

moreover, as car l does not move, we have

$$\begin{aligned}\Delta_{j+1}^l(x^{t+1}) &\le x_l^t - x_{j+1}^{t+1} \\ &= x_l^t - x_{j+1}^t - (x_{j+1}^{t+1} - x_{j+1}^t)\end{aligned}$$

and the result follows.

ii) Denoting i an index reaching the minimum in the definition of $\hat{\Delta}_j^l(x^t)$, we have

$$\begin{aligned}\hat{\Delta}_j^l(x^{t+1}) &\le \Delta_j^i(x^{t+1}) + \Delta_{i+1}^l(x^{t+1}) \\ &\le \Delta_j^i(x^t) + \Delta_{i+1}^l(x^t) = \hat{\Delta}_j^l(x^t).\end{aligned}$$

□

Acknowledgment

The authors would like to thank V. Malyshev for proposing the stochastic problem, all the Maxplus group for their useful comments, especially S. Gaubert who gave them the idea of Theorem 2, and the reviewers who helped them to improve the redaction.

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Stabilization of Oscillations Through Backstepping in High-Dimensional Systems

Javier Aracil, Francisco Gordillo, and Enrique Ponce

***Abstract*—This note introduces a method for obtaining stable and robust self-sustained oscillations in a class of single input nonlinear systems of dimension $n \geq 2$. The oscillations are associated to a limit cycle that is produced in a second-order subsystem by means of an appropriate feedback law. Then, the controller is extended to the full system by a backstepping procedure. It is shown that the closed-loop system turns out to be generalized Hamiltonian and that the limit cycle can be thought as born in a Hopf bifurcation after moving a parameter.**

***Index Terms*—Backstepping control, generalized Hamiltonian systems, Hopf bifurcation, limit cycle stabilization, nonlinear oscillations.**

I. Introduction and Statement of the Problem

Self-sustained oscillations are one of the distinctive behavioral characteristics of nonlinear systems. Whenever an oscillatory behavior is found or is to be built, there is or must be introduced an underlying nonlinearity. In this note, a procedure to obtain a nonlinear feedback law that renders a class of single input cascade systems oscillatory is introduced. The oscillation is associated with a stable limit cycle and therefore it is self-sustained and robust. The method is based on matching the open-loop system to a closed-loop one that displays such a stable limit cycle. The feedback law is obtained in two steps. In the first step, a second-order subsystem is controlled to yield a robust nonlinear oscillator. To this end, a fourth-degree polynomial Lyapunov function is introduced that guarantees the appropriate properties. Then, the cascade structure of the open-loop system allows us to apply backstepping to recursively obtain the feedback law for the full system. A very appealing byproduct is that the closed-loop system obtained has a generalized Hamiltonian structure [1].

The problem considered here is, therefore, the synthesis of limit cycles and belongs to the class of so-called inverse problems in dynamical systems. Several authors have considered this problem in the past (see, for instance, [2]–[4] and the references therein) by working with systems of moderate dimension. One of the interests of the algorithm proposed in this note is its ability to cope with arbitrary dimensions. Related material can be found in [5].

To set the problem under study in a precise form, consider the cascade systems for which the backstepping method is applicable. In particular, we will be concerned with the special class of strict-feedback systems [6] given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_0(x_1, x_2) + g_0(x_1, x_2)x_3 \\ \dot{x}_3 &= f_1(x_1, x_2, x_3) + g_1(x_1, x_2, x_3)x_4 \\ &\vdots \\ \dot{x}_{n-1} &= f_{m-1}(x_1, x_2, \ldots, x_{n-1}) + g_{m-1}(x_1, x_2, \ldots, x_{n-1})x_n \\ \dot{x}_n &= f_m(x_1, x_2, \ldots, x_n) + g_m(x_1, x_2, \ldots, x_n)u \end{aligned} \tag{1}$$

with $m = n - 2$ and $g_i \neq 0, \forall i$ in the domain of interest. The form for the first equation is quite usual, mainly in mechanical and electrical systems.

Our goal is to design a feedback law u for (1) that causes it to oscillate in a stable and robust way. This will be obtained through a controller such that the closed-loop system displays a limit cycle as a limit set. This limit cycle is responsible for the oscillatory behavior.

The note is organized as follows. In Section II, for $n = 2$, a control law that renders systems of the form (1) oscillatory is proposed. Next, in Section III the law is extended to arbitrary dimension n. The note closes with a section of conclusions and some technical details are relegated to the Appendix.

II. Oscillations in Two-Dimensional Systems

We start with the subsystem formed by the first two equations of (1)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_0(x_1, x_2) + g_0(x_1, x_2)x_3 \end{aligned} \tag{2}$$

where x_3 has to be interpreted as a virtual control $x_3 = \alpha_0(x_1, x_2)$. Now we design a feedback law to render this two-dimensional subsystem oscillatory. To that end, we adopt as reference behavior that of the nonlinear oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - k_0 x_2 \Gamma(x_1, x_2) \end{aligned} \tag{3}$$

Manuscript received July 12, 2002; revised June 13, 2003 and July 21, 2004. Recommended by Associate Editor J. M. A. Scherpen. This work under supported under MCyT-FEDER Grants DPI2003-00429 and DPI2001-2424-C02-01.

J. Aracil and F. Gordillo are with the Departamento de Ingeniería de Sistemas y Automática, Escuela Superior de Ingenieros, 41092-Seville, Spain (e-mail: aracil@esi.us.es).

E. Ponce is with the Departamento de Matemática Aplicada II, Escuela Superior de Ingenieros, 41092-Seville, Spain.

Digital Object Identifier 10.1109/TAC.2005.847038