THE MODULI SPACE OF LEFT-INVARIANT METRICS ON SIX-DIMENSIONAL CHARACTERISTICALLY SOLVABLE NILMANIFOLDS

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Abstract. A real Lie algebra is said to be characteristically solvable if its derivation algebra is solvable. We explicitly determine the moduli space of left-invariant metrics, up to isometric automorphism, for 6-dimensional nilmanifolds whose associated Lie algebra is characteristically solvable. We also compute the corresponding full isometry groups. For each left-invariant metric on these nilmanifolds we compute the index and distribution of symmetry. In particular, we find the first known examples of Lie groups which do not admit a left-invariant metric with positive index of symmetry. As an application we study the index of symmetry of nilsoliton metrics on characteristically solvable Lie algebras. We prove that nilsoliton metrics detect the existence of left-invariant metrics with positive index of symmetry.

1. Introduction

Let H be a Lie group with Lie algebra $\mathfrak{h}$. A very natural problem is to study is: how many essentially different left-invariant geometries does H admit? More precisely, one wants to determine the moduli space $\mathcal{M}(H)/{\sim}$ of left-invariant metrics on H up to isometric isomorphism and understand its topological structure. Solving this problem is extremely hard and the answer is unknown in general, even for Lie groups of low dimension. Some partial results are present in the literature. For instance in [Lau03] and [KTT11], the Lie groups H with low dimensional $\mathcal{M}(H)/{\sim}$ are classified. In [HL09], $\mathcal{M}(H)/{\sim}$ is completely determined when $\dim H = 3$. In the nilpotent case we can mention the work [DS13] where the problem was solved for the Iwasawa manifold, and more recently in [RV20] for some 6-dimensional nilpotent Lie groups with first Betti number equal to 4 and [FN18, FF23] for a certain nilpotent 6-dimensional Lie algebras generalizing the filiform family. In [CFS05] a description of the moduli space of nilpotent metric Lie algebra in dimension ≤ 6 is given. This last result is closely related with the problem mentioned above, but the approach is different since we look for a precise description of $\mathcal{M}(H)/{\sim}$ for a fixed isomorphism class.

When a description of the moduli space of left-invariant metrics on H is available, we can study how different (invariant) geometric objects vary along different left-invariant geometries. Some examples of such objects are Hermitian structures (in even dimension), the signatures of the Ricci operator or the so-called index of symmetry. This last object is of particular interest to us. Let us briefly say that the index of symmetry $i_{\mathfrak{s}}(M)$ of a homogeneous Riemannian manifold M is a geometric invariant that measures how far M is from being a symmetric space. There is a strong structural theory regarding the index (or the co-index $\dim M - i_{\mathfrak{s}}(M)$) of

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symmetry for compact homogeneous spaces, and there is also a classification of compact homogeneous spaces with co-index of symmetry 3. We refer to [ORT14, BOR17, Reg21] for more details on this topic. Despite some progress made for left-invariant metrics on 3-dimensional solvable Lie groups and naturally reductive nilpotent Lie groups (see [Reg19, May21, CR22]), a general theory for the index of symmetry in the non-compact case is virtually unknown. Moreover, even the classificatory results for low co-index in the non-compact case are sparse.

In this paper we deal with characteristically solvable Lie algebras, CSLA for short, namely nilpotent Lie algebras with solvable derivation algebra. In dimension 6, the family of CSLAs has 20 of the 34 isomorphism classes of (real) nilpotent Lie algebras. Our main goal is to classify the left-invariant metrics up to isometric automorphism in every CSLA of dimension 6 as well as computing the index of symmetry of every metric in this classification.

In order to determine the moduli space $\mathcal{M}(H)/\sim$ when H is 6-dimensional with $\mathfrak{h}$ characteristically solvable, we compute in Section 3 the full automorphism group $\mathrm{Aut}(\mathfrak{h})$. This includes an explicit description of $D = \mathrm{Aut}(\mathfrak{h})/\mathrm{Aut}_0(\mathfrak{h})$ which can be realized as a finite subgroup of $\mathrm{Aut}(\mathfrak{h})$ intersecting every connected component exactly once. Moreover, it follows from Theorem 3.2 that D is always a 2-group and $\mathrm{Aut}(\mathfrak{h})$ is isomorphic to a subgroup of the lower triangular group T_6 if and only if D is abelian. We must notice that there is some previous work dealing with the computation of the automorphism group of nilpotent Lie algebras, see for instance [Mag07]. However, as far as we know, in the existing literature there is not an explicit computation of the automorphism group for all of the CSLA. We present here an algorithmic procedure to compute $\mathrm{Aut}(\mathfrak{h})$ with a key simplification process. From this we obtain a nice presentation of the automorphism group with respect to a suitable basis of $\mathfrak{h}$, which plays an important role in determining the moduli space of left invariant-metrics.

In Section 4 we address the problem of determining the moduli space of left-invariant metrics on H. We start by describing the moduli space $\mathcal{M}(H)/\sim_0$ of left-invariant metrics on H modulo an isometric automorphism in the connected component of $\mathrm{Aut}(\mathfrak{h})$. We prove that $\mathcal{M}(H)/\sim_0$ is a smooth manifold diffeomorphic to a certain embedded submanifold Σ of T_6^+ (the subgroup of T_6 with positive entries on the diagonal). Then we explain how to obtain $\mathcal{M}(H)/\sim$ from $\mathcal{M}(H)/\sim_0$ as a finite quotient. It turns out that in most cases, $\mathcal{M}(H)/\sim$ is homeomorphic to the quotient of Σ modulo a finite group acting like reflections on T_6^+, which leaves Σ invariant.

In Section 5 we compute the full isometry group $\mathrm{I}(H, g_\sigma)$, where g_σ is a left-invariant metric representing a $\sigma \in \Sigma$. It is known from [Wol63] that $\mathrm{I}(H, g_\sigma) \simeq H \rtimes K$, where $K = \mathrm{Aut}(\mathfrak{h}) \cap \mathrm{O}(g_\sigma)$. We prove that if D is abelian, then $K \subset D$. In particular, when D is abelian there is a finite number of subgroups of $\mathrm{Aut}(\mathfrak{h})$ that can serve as the full isotropy subgroup of g_σ, for all $\sigma \in \Sigma$. This does not hold if D is not abelian. Indeed, we provide examples of 1-parameter families of subgroups $K_r \subset \mathrm{Aut}(\mathfrak{h})$ and $\sigma_r \in \Sigma$ such that K_r is the full isotropy group of g_{σ_r}. The explicit determination of the full isotropy groups K for any left-invariant metric is also possible and it is treated in that section.

In Section 6 we compute the index of symmetry of left-invariant metrics in the context of CSLAs of dimension 6. We remark that for every Lie algebra isomorphism class, apart from for five distinguished cases, every left-invariant metric on H has trivial index of symmetry. This provides the first known examples of Lie groups which do not admit a left-invariant metric with positive index of symmetry. On the other side, among the CSLAs whose associated Lie groups do admit metrics with non-trivial index of symmetry we can find new examples of non-compact homogeneous spaces with co-index of symmetry 3 and 4. Lastly, we find some examples of left-invariant metrics on nilpotent Lie groups whose distribution of symmetry (i.e. the left-invariant distribution induced by the Killing field parallel at the identity element) is not contained in the center of the Lie algebra. Nilpotent Lie groups with that property were previously unknown to exist.

As an application of our results, in Section 7 we study the index and distribution of symmetry of nilsoliton metrics in CSLAs. Recall that a left-invariant metric on H is called a *nilsoliton* metric if its Ricci operator satisfies $\mathrm{Ric} = c\,\mathrm{id}_{\mathfrak{h}} + D$ for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{h})$. These metrics are particular cases of *solvsoliton* metrics (same definition but for solvable Lie groups), a family that exhausts the homogeneous expanding Ricci soliton metrics (see [BL23]). Solvsoliton metrics are proved to be unique (if they exist) up to scaling. Moreover, nilsoliton metrics are given by the nilradicals of Einstein solvmanifolds (see [Lau11]). A classification of nilsoliton metrics in dimension 6 can be found in [Wil03]. Since nilsoliton metrics are Ricci soliton metrics, they are nice enough so that they are not improved under the Ricci flow (that is, evolving only by scaling or pulling-back by a diffeomorphism). Thus it is expected that nilsoliton metrics have low co-index of symmetry whenever the underlying Lie algebra structure do not obstruct the existence of positive index of symmetry. We prove that this is indeed the case. More precisely, if H admits metrics with positive index of symmetry, then the nilsoliton metric has positive index of symmetry. It is interesting noticing that the index of symmetry of nilsoliton metrics is not always maximal among all the left-invariant metrics. However, the distribution of symmetry of nilsoliton metrics is well behaved, in the sense that it is contained in the center of the Lie algebra when this is not obstructed by the isomorphism class of $\mathfrak{h}$.

Finally, in Section 8 we include some tables summarizing the classificatory results obtained in the paper. This results often rely in heavy computations that we perform with the computer software SageMath. The source code of such computation is freely available and provided in the separate GitHub repository [CCR24]. We believe that some of this code could be of use in studying other geometric problems in nilpotent Lie groups of low dimension.

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2. Characteristically solvable Lie algebras

In this article we mainly deal with 6-dimensional real nilpotent Lie algebras. This is the greatest dimension with a finite number of isomorphism classes: there are 34 isomorphism classes of nilpotent real Lie algebras. Observe that every 6-dimensional nilpotent Lie algebra admits a basis $e_1, \dots, e_6$ for which the structure coefficients are -1, 0 or 1. So, it is convenient to denote a Lie algebra with a tuple summarizing its structure. We will follow the notation used in [Sal01]. For example, the notation $\mathfrak{h}_{11} = (0, 0, 12, 13, 14 + 23)$ means that $\mathfrak{h}_{11}$ is the nilpotent Lie algebra $(\mathbb{R}^6, [\cdot, \cdot])$ where the Lie bracket in the canonical basis is given by the non-trivial relations

$$[e_1, e_2] = -e_3 \qquad [e_1, e_3] = -e_4 \qquad [e_1, e_4] = [e_2, e_3] = -e_6.$$

In Table 1 we list all the 34 isomorphism classes of 6-dimensional nilpotent Lie algebras. Note that some of the structure coefficients are indicated with an asterisk. This means that further on the paper we shall consider a different basis to better serve our purposes. We also mention that the third column on the table shows the nilpotency step and the last column indicates the dimension of the commuting ideals for the decomposable Lie algebras.

TABLE 1. Nilpotent Lie algebras of dimension 6

Name	Structure coefficients	Step	$\oplus$
$\mathfrak{h}_1$	$(0,0,0,0,0,0)$	1	$1+\cdots+1$
$\mathfrak{h}_2$	$(0,0,0,0,12,34)$	2	$3+3$
$\mathfrak{h}_3$	$(0,0,0,0,0,12+34)$	2	$1+5$
$\mathfrak{h}_4$	$(0,0,0,0,12,14+23)$	2	
$\mathfrak{h}_5$	$(0,0,0,0,13+42,14+23)$	2	
$\mathfrak{h}_6$	$(0,0,0,0,12,13)$	2	$1+5$
$\mathfrak{h}_7$	$(0,0,0,12,13,23)$	2	
$\mathfrak{h}_8$	$(0,0,0,0,0,12)$	2	$1+1+1+3$
$\mathfrak{h}_9$	$(0,0,0,0,12,14+25)^*$	3	$1+5$
$\mathfrak{h}_{10}$	$(0,0,0,12,13,14)$	3	
$\mathfrak{h}_{11}$	$(0,0,0,12,13,14+23)$	3	
$\mathfrak{h}_{12}$	$(0,0,0,12,13,24)$	3	
$\mathfrak{h}_{13}$	$(0,0,0,12,13+14,24)$	3	
$\mathfrak{h}_{14}$	$(0,0,0,12,14,13+42)$	3	
$\mathfrak{h}_{15}$	$(0,0,0,12,13+42,14+23$	3	
$\mathfrak{h}_{16}$	$(0,0,0,12,14,24)$	3	$1+5$
$\mathfrak{h}_{17}$	$(0,0,0,0,12,15)$	3	$1+1+4$
$\mathfrak{h}_{18}$	$(0,0,0,12,13,14+35)^*$	3	
$\mathfrak{h}_{19}^-$	$(0,0,0,12,23,14-35)$	3	
$\mathfrak{h}_{19}^+$	$(0,0,0,12,23,14+35)^*$	3	
$\mathfrak{h}_{20}$	$(0,0,0,0,12,15+34)$	3	
$\mathfrak{h}_{21}$	$(0,0,0,12,14,15)$	4	$1+5$
$\mathfrak{h}_{22}$	$(0,0,0,12,14,15+24)$	4	$1+5$
$\mathfrak{h}_{23}$	$(0,0,12,13,23,14)$	4	
$\mathfrak{h}_{24}$	$(0,0,0,12,14,15+23+24)$	4	
$\mathfrak{h}_{25}$	$(0,0,0,12,14,15+23)$	4	
$\mathfrak{h}_{26}^-$	$(0,0,12,13,23,14-25)^*$	4	
$\mathfrak{h}_{26}^+$	$(0,0,12,13,23,14+25)$	4	
$\mathfrak{h}_{27}$	$(0,0,0,12,14-23,15+34)$	4	
$\mathfrak{h}_{28}$	$(0,0,12,13,14,15)$	5	
$\mathfrak{h}_{29}$	$(0,0,12,13,14,23+15)$	5	
$\mathfrak{h}_{30}$	$(0,0,12,13,14+23,24+15)$	5	
$\mathfrak{h}_{31}$	$(0,0,12,13,14,34+52)$	5	
$\mathfrak{h}_{32}$	$(0,0,12,13,14+23,34+52)$	5	

Let $\mathfrak{h}$ be a nilpotent Lie algebra. We say that $\mathfrak{h}$ is *characteristically solvable, CSLA* for short, if its derivation algebra $\mathrm{Der}(\mathfrak{h})$ is a solvable Lie algebra. The next results gives the classification of CSLAs in dimension 6.

Proposition 2.1. *Let $\mathfrak{h}$ be a real nilpotent Lie algebra of dimension 6. Then $\mathfrak{h}$ is a CSLA if and only if $\mathfrak{h}$ is isomorphic to one of the following Lie algebras:*

$$\mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{18}, \mathfrak{h}_{19}^+, \mathfrak{h}_{21}, \mathfrak{h}_{22},$$

$$\mathfrak{h}_{23}, \mathfrak{h}_{24}, \mathfrak{h}_{25}, \mathfrak{h}_{26}^-, \mathfrak{h}_{27}, \mathfrak{h}_{28}, \mathfrak{h}_{29}, \mathfrak{h}_{30}, \mathfrak{h}_{31}, \mathfrak{h}_{32}.$$

Proof. By a direct computation, we can see that the derivation Lie algebras for $\mathfrak{h}_{10}, \dots, \mathfrak{h}_{14}$, $\mathfrak{h}_{21}, \dots, \mathfrak{h}_{25}$, $\mathfrak{h}_{27}, \dots, \mathfrak{h}_{32}$ are represented by triangular matrices in the standard basis. Consider the following changes of basis,

$$e_1' = e_2, \quad e_2' = e_1, \quad e_3' = e_4, \quad e_4' = e_3, \quad e_5' = e_5, \quad e_6' = e_6$$
$$e_1' = e_1, \quad e_2' = e_3, \quad e_3' = e_2, \quad e_4' = e_5, \quad e_5' = e_4, \quad e_6' = e_6$$

for $\mathfrak{h}_9$ and $\mathfrak{h}_{18}$, respectively,

$$e_1' = e_1 + e_3, \quad e_2' = e_2, \quad e_3' = e_1 - e_3,$$
$$e_4' = e_4 + e_5, \quad e_5' = e_4 - e_5, \quad e_6' = 2e_6$$

for $\mathfrak{h}_{19}^+$ and

$$e_1' = e_1 + e_2, \quad e_2' = e_1 - e_2, \quad e_3' = -2e_3,$$
$$e_4' = 2(e_4 + e_5), \quad e_5' = 2(e_4 - e_5), \quad e_6' = 4e_6$$

for $\mathfrak{h}_{26}^-$. These changes of basis simultaneously triangularize all the derivations. Finally, one can see that all the remaining nilpotent Lie algebras of dimension 6 admit a skew-symmetric derivation (with respect to the inner product induced by the standard basis). Thus, these Lie algebras cannot be characteristically solvable. The detailed computations are long and tedious to be made by hand and were instead performed with the software SageMath. The corresponding Jupyter notebook can be found in [CCR24, Notebook 01], the GitHub repository supporting this article. □

In Table 2 we list all the CSLAs. From now on we will consider as the standard basis the one with the structure coefficients given in this table (i.e., the standard basis is the same as in Table 1 except for $\mathfrak{h}_9, \mathfrak{h}_{18}, \mathfrak{h}_{19}^+$ and $\mathfrak{h}_{26}^-$ which is changed according to the proof of Proposition 2.1).

Table 2. CSLAs of dimension 6

Name	Structure coefficients	Name	Structure coefficients
$\mathfrak{h}_9$	$(0,0,0,0,12,51+23)$	$\mathfrak{h}_{23}$	$(0,0,12,13,23,14)$
$\mathfrak{h}_{10}$	$(0,0,0,12,13,14)$	$\mathfrak{h}_{24}$	$(0,0,0,12,14,15+23+24)$
$\mathfrak{h}_{11}$	$(0,0,0,12,13,14+23)$	$\mathfrak{h}_{25}$	$(0,0,0,12,14,15+23)$
$\mathfrak{h}_{12}$	$(0,0,0,12,13,24)$	$\mathfrak{h}_{26}^-$	$(0,0,12,31,32,15+24)$
$\mathfrak{h}_{13}$	$(0,0,0,12,13+14,24)$	$\mathfrak{h}_{27}$	$(0,0,0,12,14-23,15+34)$
$\mathfrak{h}_{14}$	$(0,0,0,12,14,13+42)$	$\mathfrak{h}_{28}$	$(0,0,12,13,14,15)$
$\mathfrak{h}_{18}$	$(0,0,0,12,13,15+24)$	$\mathfrak{h}_{29}$	$(0,0,12,13,14,23+15)$
$\mathfrak{h}_{19}^+$	$(0,0,0,23,21,14+35)$	$\mathfrak{h}_{30}$	$(0,0,12,13,14+23,24+15)$
$\mathfrak{h}_{21}$	$(0,0,0,12,14,15)$	$\mathfrak{h}_{31}$	$(0,0,12,13,14,34+52)$
$\mathfrak{h}_{22}$	$(0,0,0,12,14,15+24)$	$\mathfrak{h}_{32}$	$(0,0,12,13,14+23,34+52)$

Corollary 2.2. *Let $\mathfrak{h}$ be s-step nilpotent Lie algebra of dimension 6. If $\mathfrak{h}$ is a CSLA, then $s \geq 3$.*

3. The full automorphism group of a CSLA

Let us fix a CSLA $\mathfrak{h}$ of dimension 6. We denote by $\mathrm{T}_6 \subset \mathrm{GL}_6(\mathbb{R})$ the Lie subgroup of lower triangular matrices and by $\mathfrak{t}_6$ its Lie algebra. In order to compute the full automorphism group $\mathrm{Aut}(\mathfrak{h})$ of $\mathfrak{h}$ we proceed as follows. According to the proof of Proposition 2.1, up to a suitable change of basis, we can identify $\mathrm{Der}(\mathfrak{h})$ with a Lie subalgebra of $\mathfrak{t}_6$. Then $e^{\mathrm{Der}(\mathfrak{h})} = \mathrm{Aut}_0(\mathfrak{h})$ is the connected component of the automorphism group. Moreover, if we denote by $\mathrm{Der}_{\mathfrak{d}}(\mathfrak{h})$ the abelian subalgebra of (simultaneously) diagonalizable derivations, then we see that $\mathrm{Aut}(\mathfrak{h})$ has at least

2^k connected components, where $k = \dim \mathrm{Der}_{\mathfrak{d}}(\mathfrak{h})$. We will prove in Theorem 3.2 that this bound is almost always met.

Remark 3.1. Let $\mathfrak{h}$ be a CSLA of dimension 6. Since $\mathrm{Der}(\mathfrak{h}) \subset \mathfrak{t}_6$, a maximal compact subgroup D of $\mathrm{Aut}(\mathfrak{h})$ meets every connected component and so $\mathrm{Aut}(\mathfrak{h}) = \mathrm{Aut}_0(\mathfrak{h}) \rtimes D$. On the other hand, since $\mathrm{Aut}(\mathfrak{h})$ is an algebraic group, it has finitely many connected components hence D is a finite subgroup of automorphisms. Recall that D must be conjugated to a subgroup of $\mathrm{O}(n)$, however this does not imply that D consists of diagonal automorphisms in the standard basis.

Theorem 3.2. *Let $\mathfrak{h}$ be a CSLA of dimension 6 such that $\mathfrak{h} \not\simeq \mathfrak{h}_{13}$, $\mathfrak{h} \not\simeq \mathfrak{h}_{19}^+$ and $\mathfrak{h} \not\simeq \mathfrak{h}_{26}^-$. Then $\mathrm{Aut}(\mathfrak{h})$ is isomorphic to a subgroup of T_6. Moreover,*

(1) $\mathrm{Aut}(\mathfrak{h})/\mathrm{Aut}_0(\mathfrak{h}) \simeq (\mathbb{Z}_2)^k$ *where* $k = \dim \mathrm{Der}_{\mathfrak{d}}(\mathfrak{h})$.
(2) $\mathrm{Aut}(\mathfrak{h}_{13})/\mathrm{Aut}_0(\mathfrak{h}_{13}) \simeq \mathrm{G}_{16}^3 = (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_4$.
(3) $\mathrm{Aut}(\mathfrak{h}_{19}^+)/\mathrm{Aut}_0(\mathfrak{h}_{19}^+) \simeq \mathrm{Dih}_4 \times \mathbb{Z}_2$.
(4) $\mathrm{Aut}(\mathfrak{h}_{26}^-)/\mathrm{Aut}_0(\mathfrak{h}_{26}^-) \simeq \mathrm{Dih}_4$.

Proof. Let $\mathfrak{h}$ be a CSLA of dimension 6. It follows from Remark 3.1 that there exists a finite subgroup D of $\mathrm{Aut}(\mathfrak{h})$ which meets every connected component exactly once. So $\mathrm{Aut}(\mathfrak{h})/\mathrm{Aut}_0(\mathfrak{h}) \simeq D$ and as we observe before, D has a subgroup isomorphic to $(\mathbb{Z}_2)^k$, where $k = \dim \mathrm{Der}_{\mathfrak{d}}(\mathfrak{h})$. We will prove $D = (\mathbb{Z}_2)^k$ for all $\mathfrak{h} \neq \mathfrak{h}_{19}^+, \mathfrak{h}_{26}^-$. Let $e_1, \dots, e_6$ be the standard basis of $\mathfrak{h}$ (cfr. Table 2). Let s be the least natural number such that the Lie subalgebra generated by $e_1, \dots, e_s$ coincides with $\mathfrak{h}$. One can easily see that $s = 2, 3$ or 4. Let $\varphi \in \mathrm{Aut}(\mathfrak{h})$, then φ is lower triangular in the standard basis if and only if $e^i(\varphi(e_j)) = 0$ for all $1 \le i < j \le s$, where $e^1, \dots, e^6$ is the dual basis of the standard basis. We argue by cases according to the possible values of s. Let $X = \sum \alpha_i e_i$ be an arbitrary element of $\mathfrak{h}$.

Case 1: $s = 2$**.** In this case, $\mathfrak{h} \in \{\mathfrak{h}_{23}, \mathfrak{h}_{26}^-, \mathfrak{h}_{28}, \mathfrak{h}_{29}, \mathfrak{h}_{30}, \mathfrak{h}_{31}, \mathfrak{h}_{32}\}$. Then, for all $\mathfrak{h}$ except $\mathfrak{h}_{26}^-$ and $\mathfrak{h}_{32}$ we have that $\mathrm{rank}(\mathrm{ad}_{e_1}) > \mathrm{rank}(\mathrm{ad}_{e_2})$ and if $\alpha_1 \neq 0$ then $\mathrm{rank}(\mathrm{ad}_X) \ge \mathrm{rank}(\mathrm{ad}_{e_1})$. So, in these cases, we have $e^1(\varphi(e_2)) = 0$ and $\varphi \in \mathrm{T}_6$. Moreover, $D \simeq (\mathbb{Z}_2)^k$ where $k = 1$ for $\mathfrak{h} = \mathfrak{h}_{29}, \mathfrak{h}_{30}$ and $k = 2$ for $\mathfrak{h} = \mathfrak{h}_{23}, \mathfrak{h}_{28}, \mathfrak{h}_{31}$. In fact, it is easy to see that if $k = 1$, then $\varphi(e_2)$ is determined by $\varphi(e_1)$, hence D is isomorphic to the subgroup generated by the automorphism $\varphi_1 : e_1 \mapsto -e_1$. If $k = 2$ one can see that D is isomorphic to the subgroup generated by the automorphisms φ_1, φ_2 given by

$$\varphi_1(e_1) = -e_1 = -\varphi_2(e_1) \qquad \text{and} \qquad \varphi_1(e_2) = e_2 = -\varphi_2(e_2). \tag{3.1}$$

Let us look at the remaining cases.

Case 1.1: $\mathfrak{h} = \mathfrak{h}_{26}^-$**.** In this case $\mathrm{rank}(\mathrm{ad}_{e_1}) = \mathrm{rank}(\mathrm{ad}_{e_2}) = 3$. Moreover, if $\mathrm{rank}(\mathrm{ad}_X) = 3$, then $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Since $(\mathrm{ad}_{e_1})^3 = (\mathrm{ad}_{e_2})^3 = 0$ and $(\mathrm{ad}_X)^3 = 0$ if and only if $\alpha_1\alpha_2 = 0$, we have that

$$e^1(\varphi(e_1)) \neq 0 \implies e^2(\varphi(e_1)) = e^1(\varphi(e_2)) = 0 \neq e^2(\varphi(e_2))$$
$$e^2(\varphi(e_1)) \neq 0 \implies e^1(\varphi(e_1)) = e^2(\varphi(e_2)) = 0 \neq e^1(\varphi(e_2)).$$

So in addition to the automorphisms $\varphi_1, \varphi_2 \in D$ defined by the same formula as in (3.1) we can define an automorphism $\varphi_3 \in D$ by

$$\varphi_3(e_1) = e_2 \qquad\qquad \varphi_3(e_2) = e_1.$$

The above calculations show that D is generated by $\varphi_1, \varphi_2, \varphi_3$. Moreover, note that D is also generated by φ_1 and $\varphi_1 \circ \varphi_3$, and since $\varphi_1^2 = (\varphi_1 \circ \varphi_3)^4$ we conclude that $D \simeq \mathrm{Dih}_4$.

Case 1.2: $\mathfrak{h} = \mathfrak{h}_{32}$**.** Notice that $Z(\mathfrak{h}) = \mathbb{R}e_6$ and $\mathrm{im}(\mathrm{ad}_{e_2})^3 = Z(\mathfrak{h})$. This property is invariant under automorphisms and $\mathrm{im}(\mathrm{ad}_X)^3 = Z(\mathfrak{h})$ implies $\alpha_1 = 0$. So $e^1(\varphi(e_2)) = 0$ and reasoning as in some previous cases, $D \simeq \mathbb{Z}_2$.

Case 2: $s = 3$**.** In this case

$$\mathfrak{h} \in \{\mathfrak{h}_{10}, \mathfrak{h}_{11}, \mathfrak{h}_{12}, \mathfrak{h}_{13}, \mathfrak{h}_{14}, \mathfrak{h}_{18}, \mathfrak{h}_{19}^{+}, \mathfrak{h}_{21}, \mathfrak{h}_{22}, \mathfrak{h}_{24}, \mathfrak{h}_{25}, \mathfrak{h}_{27}\}.$$

Routine computations show that $\mathrm{rank}(\mathrm{ad}_{e_1}) > \mathrm{rank}(\mathrm{ad}_{e_i})$, $i = 2, 3$, for all $\mathfrak{h}$ except $\mathfrak{h}_{12}$, $\mathfrak{h}_{13}$ and $\mathfrak{h}_{19}^{+}$. Also, we have that $\mathrm{rank}(\mathrm{ad}_{e_2}) > \mathrm{rank}(\mathrm{ad}_{e_3})$ for all $\mathfrak{h}$ except $\mathfrak{h}_{10}$, $\mathfrak{h}_{11}$, $\mathfrak{h}_{19}^{+}$ and $\mathfrak{h}_{27}$.

Case 2.1: $\mathrm{rank}(\mathrm{ad}_{e_1}) > \mathrm{rank}(\mathrm{ad}_{e_2}) > \mathrm{rank}(\mathrm{ad}_{e_3})$**.** This case is analogous to the generic case with $s = 2$, since $\alpha_1 \neq 0$ implies $\mathrm{rank}(\mathrm{ad}_X) \geq \mathrm{rank}(\mathrm{ad}_{e_1})$ and $\alpha_2 \neq 0$ implies $\mathrm{rank}(\mathrm{ad}_X) \geq \mathrm{rank}(\mathrm{ad}_{e_2})$. This shows that every φ is triangular with respect to the standard basis, and $D \simeq (\mathbb{Z}_2)^3$ is generated by $\varphi_1, \varphi_2, \varphi_3$ defined by

$$\begin{array}{lll} \varphi_1(e_1) = -e_1 & \varphi_1(e_2) = e_2 & \varphi_1(e_3) = e_3 \\ \varphi_2(e_1) = e_1 & \varphi_2(e_2) = -e_2 & \varphi_2(e_3) = e_3 \\ \varphi_3(e_1) = e_1 & \varphi_3(e_2) = e_2 & \varphi_3(e_3) = -e_3 \end{array} \tag{3.2}$$

Case 2.2: $\mathrm{rank}(\mathrm{ad}_{e_1}) > \mathrm{rank}(\mathrm{ad}_{e_2}) = \mathrm{rank}(\mathrm{ad}_{e_3})$**.** Here we have three subcases.

Case 2.2.1: $\mathfrak{h} = \mathfrak{h}_{10}$**.** In this case we have $\mathrm{rank}(\mathrm{ad}_{e_1}) = 3$ and $\mathrm{rank}(\mathrm{ad}_{e_2}) = \mathrm{rank}(\mathrm{ad}_{e_3}) = 1$. Moreover, if $\alpha_1 \neq 0$ then $\mathrm{rank}(\mathrm{ad}_X) = 3$, and hence $e^1(\varphi(e_2)) = e^1(\varphi(e_3)) = 0$. Also notice that $\mathrm{im}(\mathrm{ad}_{e_3}) \subset Z(\mathfrak{h})$, but $\alpha_2 \neq 0$ implies $\mathrm{im}(\mathrm{ad}_X) \not\subset Z(\mathfrak{h})$, so $e^2(\varphi(e_3)) = 0$ and φ is triangular. One can check that D is generated by $\varphi_1, \varphi_2, \varphi_3$ defined with the same formulas as in (3.2).

Case 2.2.2: $\mathfrak{h} = \mathfrak{h}_{11}$**.** Here the same argument as in the above case works for proving that $\varphi \in T_6$. However, in this case $D \simeq (\mathbb{Z}_2)^2$ since $[e_2, e_3] = -[e_1, [e_1, e_2]]$ implies $e^3(\varphi(e_3)) = e^1(\varphi(e_1))^2$.

Case 2.2.3: $\mathfrak{h} = \mathfrak{h}_{27}$**.** In this case we can use the following variation of the above argument. Let $C^2(\mathfrak{h})$ be the third ideal in the lower central series of $\mathfrak{h}$. Then $\mathrm{im}(\mathrm{ad}_{e_3}) \subset C^2(\mathfrak{h})$ and $\alpha_2 \neq 0$ implies $\mathrm{im}(\mathrm{ad}_X) \not\subset C^2(\mathfrak{h})$. As in the previous case, we can see that $D \simeq (\mathbb{Z}_2)^2$.

Case 2.3: $\mathrm{rank}(\mathrm{ad}_{e_1}) = \mathrm{rank}(\mathrm{ad}_{e_2}) > \mathrm{rank}(\mathrm{ad}_{e_3})$**.** We have to consider the following sub-cases.

Case 2.3.1: $\mathfrak{h} = \mathfrak{h}_{12}$**.** Recall that $(\mathrm{ad}_{e_1})^2 = 0 \neq (\mathrm{ad}_{e_2})^2$ and $(\mathrm{ad}_X)^2 = 0$ if and only if $\alpha_2 = 0$. This shows $e^1(\varphi(e_2)) = 0$. In order to see that $e^1(\varphi(e_3)) = e^2(\varphi(e_3)) = 0$ notice that $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ imply $\mathrm{rank}(\mathrm{ad}_X) \geq 2$, but $\mathrm{rank}(\mathrm{ad}_{e_3}) = 1$. In this case $D = (\mathbb{Z}_2)^3$ with generators $\varphi_1, \varphi_2, \varphi_3$ defined as in (3.2).

Case 2.3.2: $\mathfrak{h} = \mathfrak{h}_{13}$**.** This case is more complicated than the previous ones since it cannot be solved using linear invariants. First notice that $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ implies $\mathrm{rank}(\mathrm{ad}_X) \geq 2$. This shows $e^1(\varphi(e_3)) = e^2(\varphi(e_3)) = 0$. Also since φ preserves the commutator and the center, one has $e^i(\varphi(e_4)) = 0$ for $i = 1, 2, 3$ and $e^i(\varphi(e_j)) = 0$ for $i = 1, 2, 3, 4$ and $j = 5, 6$. Now we use a brute force approach to recover the general form of the automorphism φ. When $e^1(\varphi(e_2)) = 0$ we obtain a subgroup of D isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which generated by the automorphism φ_1, φ_2 given by

$$\begin{array}{lll} \varphi_1(e_1) = -e_1, & \varphi_1(e_2) = e_2, & \varphi_1(e_3) = -e_3, \\ \varphi_2(e_1) = e_1, & \varphi_2(e_2) = -e_2, & \varphi_2(e_3) = -e_3. \end{array}$$

Finally, when $e^1(\varphi(e_2)) \neq$ we obtain a a subgroup of D isomorphic to $\mathbb{Z}_4$ with a generator φ_3 given by

$$\varphi_3(e_1) = e_2, \qquad \varphi_3(e_2) = -e_1, \qquad \varphi_3(e_3) = -e_3 + e_4.$$

Notice that $\varphi_3 \circ \varphi_1 \circ \varphi_3^{-1} = \varphi_2$. We refer to the corresponding SageMath notebook for the details.

Case 2.4: $\mathrm{rank}(\mathrm{ad}_{e_1}) = \mathrm{rank}(\mathrm{ad}_{e_2}) = \mathrm{rank}(\mathrm{ad}_{e_3})$**.** The only CSLA in this case is $\mathfrak{h} = \mathfrak{h}_{19}^{+}$. One can easily see that $\mathrm{Aut}(\mathfrak{h})$ has at least 8 connected components since

$\varphi_1, \varphi_2, \varphi_3$ defined as in (3.2) give a subgroup of D isomorphic to $(\mathbb{Z}_2)^3$. Moreover, define $\varphi_4 \in \operatorname{Aut}(\mathfrak{h})$ by

$$\varphi_4(e_1) = e_3, \qquad \varphi_4(e_2) = e_2, \qquad \varphi_4(e_3) = e_1.$$

Now $\varphi_1, \dots, \varphi_4$ generate a subgroup of order 16. Let us prove that this subgroup is isomorphic to D. In fact, note that if $\operatorname{im}(\operatorname{ad}_X) \cap Z(\mathfrak{h}) = 0$, then $\alpha_1 = \alpha_3 = 0$. This implies $e^1(\varphi(e_2)) = e^3(\varphi(e_2)) = 0$ and so $\varphi(e_2) = \gamma_2 e_2 + \gamma_4 e_4 + \gamma_5 e_5 + \gamma_6 e_6$. If $\varphi(e_1) = \beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3 + \beta_4 e_4 + \beta_5 e_5 + \beta_6 e_6$ then $\beta_1 \neq 0$ or $\beta_3 \neq 0$. Moreover, since $\operatorname{rank}(\operatorname{ad}_{e_1}) = 2$ then $\beta_2 = 0$. Now, since $0 = [e_1, e_5] = [e_1, [e_1, e_2]]$ we have

$$0 = [\varphi(e_1), [\varphi(e_1), \varphi(e_2)]] = -2\beta_1\beta_3\gamma_2 e_6.$$

So $\beta_1 \neq 0$ if and only if $\beta_3 = 0$. The above computations show that D equals the subgroup generated by $\varphi_1, \dots, \varphi_4$. Denote $a = \varphi_4$, $b = \varphi_2$ and $c = \varphi_1$. Recall that $\varphi_3 = \varphi_1 \circ (\varphi_4 \circ \varphi_1)^2$. Then it is not hard to see that

$$D \simeq \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^4 = e, ab = ba, bc = cb \rangle \simeq \operatorname{Dih}_4 \times \mathbb{Z}_2.$$

Case 3: $s = 4$. The only possibility here is $\mathfrak{h} = \mathfrak{h}_9$. This case was already treated in [RV20]. □

Remark 3.3. Supporting calculations for the proof of Theorem 3.2 can be found in [CCR24, Notebook 02] from the GitHub repository accompanying this paper. For the sake of completeness we have also included the computations for the case $\mathfrak{h} = \mathfrak{h}_9$.

It follows from Theorem 3.2 that the full automorphism group of $\mathfrak{h}$ can be computed essentially by taking exponential of derivations (and making appropriate changes of sign in the diagonal after that). Since this does not give a nice expression for a generic automorphism, we need to develop a simplification algorithm which works as follows. Let us write

$$\operatorname{Der}(\mathfrak{h}) = \operatorname{Der}_{\mathfrak{n}}(\mathfrak{h}) \oplus \operatorname{Der}_{\mathfrak{d}}(\mathfrak{h}) \qquad \text{(direct sum of subspaces)},$$

where $\operatorname{Der}_{\mathfrak{n}}(\mathfrak{h})$ consists of all the nilpotent derivations of $\mathfrak{h}$. Then we choose a basis $X_1, \dots, X_n$ of $\operatorname{Der}_{\mathfrak{n}}(\mathfrak{h})$ and a basis $Y_1, \dots, Y_m$ of $\operatorname{Der}_{\mathfrak{d}}(\mathfrak{h})$. In order to get better results, one must take sparse matrices for such basis. We then

- take the exponential of a generic element $X = \sum a_i X_i$ and simplify the expression for e^X;
- take the exponential of $b_j Y_j$ (separately) and simplify the expression for $e^{b_j Y_j}$;
- for every $k = 1, \dots, j$ define $\varphi_k = \varphi_{k-1} e^{b_k Y_k}$, where $\varphi_0 = e^X$, and simplify the expression for φ_k.

Note that a good simplification performed in every step is crucial and has the objective of preserving the original position of the coefficients a_i and b_j in the final expression for a generic automorphism. In Tables 4, 5 and 6 we present the full automorphism group for every CSLA of dimension 6. In the second column of these tables we give a description of the connected component $\operatorname{Aut}_0(\mathfrak{h})$ and in the third column we give the generators of the finite subgroup D of $\operatorname{Aut}(\mathfrak{h})$ such that $\operatorname{Aut}(\mathfrak{h}) = \operatorname{Aut}_0(\mathfrak{h}) \rtimes D$. Notice that we have renamed the free parameters to $a_0, a_1, a_2, \dots$

4. The moduli space of left-invariant metrics for CSLA

Let H be a simply connected Lie group with Lie algebra $\mathfrak{h}$. Every left-invariant metric on H is uniquely determined by an inner product on $\mathfrak{h}$. So, after a choice of a basis for $\mathfrak{h}$ we can identify the space $\mathcal{M}(H)$ of all left-invariant metrics on H with the symmetric space $\operatorname{Sym}_n^+ = \operatorname{GL}_n(\mathbb{R})/\operatorname{O}(n)$ of all the symmetric positive definite

$n \times n$ matrices. If we identify $\mathrm{Aut}(\mathfrak{h}) \subset \mathrm{GL}_n(\mathbb{R})$, then $\mathrm{Aut}(\mathfrak{h})$ acts on Sym_n^+ on the right by

$$g \cdot \varphi = \varphi^T g \varphi, \qquad g \in \mathrm{Sym}_n^+, \varphi \in \mathrm{Aut}(\mathfrak{h}).$$

We say that two left-invariant metrics on H are *equivalent* if they lie in the same orbit under the action of $\mathrm{Aut}(\mathfrak{h})$ (with the above identifications). The *moduli space of left-invariant metric up to isometric automorphism* is the quotient

$$\mathcal{M}(H)/\!\sim\, = \mathrm{Aut}(\mathfrak{h})\backslash \mathrm{Sym}_n^+ .$$

We will also consider the moduli space of left-invariant metrics up to an isometric automorphism in the connected component

$$\mathcal{M}(H)/\!\sim_0\, = \mathrm{Aut}_0(\mathfrak{h})\backslash \mathrm{Sym}_n^+ .$$

Assume that $\mathfrak{h}$ is a CSLA of dimension 6. Let T_6^+ be the connected component of T_6. Since T_6^+ acts simply transitively on Sym_6^+, the action of $\mathrm{Aut}_0(\mathfrak{h})$ on Sym_6^+ is equivalent to the left action of $\mathrm{Aut}_0(\mathfrak{h})$ on T_6^+.

Theorem 4.1. *Let $\mathfrak{h}$ be a CSLA of dimension 6 and let H be a simply connected Lie group with Lie algebra $\mathfrak{h}$. Then $\mathcal{M}(H)/\!\sim_0$ is a smooth manifold. Moreover, $\mathcal{M}(H)/\!\sim_0$ is diffeomorphic to a submanifold Σ of T_6^+ which is transversal to $\mathrm{Aut}_0(\mathfrak{h})$. The explicit description of the submanifolds Σ can be found in Tables 7, 8, 9 and 10.*

Proof. The same proof given in [RV20] for $\mathfrak{h}_9$ can be adapted to the rest of the CSLAs. Let us recall how the submanifold Σ is defined. Let $\varphi \in \mathrm{Aut}_0(\mathfrak{h})$ a generic element given as in Tables 4, 5 and 6. First we identify the places where the free parameters $a_0, a_1, a_2, \dots$ occur and set the corresponding non-diagonal entries of $\Sigma \in \mathrm{T}_6^+$ to 0 and the diagonal ones to 1. We fill the remaining places with free parameters $s_0, s_1, s_2, \dots$ where $s_i \in \mathbb{R}$ for non-diagonal entries and $s_i > 0$ for the diagonal ones. □

The algorithm for computing the Σ's can be found in [CCR24, Notebook 00] from our GitHub repository. We also define for future use the set $\mathrm{nd}(\Sigma)$ of non-diagonal free parameters of Σ.

Now we give a general description of $\mathcal{M}(H)/\!\sim$. For each $g \in \mathcal{M}(H)$, we denote the orbits of g under $\mathrm{Aut}(\mathfrak{h})$ and $\mathrm{Aut}_0(\mathfrak{h})$ by $[g]$ and $[g]_0$ respectively. Since $\mathrm{Aut}(\mathfrak{h}) = \mathrm{Aut}_0(\mathfrak{h}) \rtimes D$, we have

$$[g] = \bigcup_{\delta \in D} [\delta^T g \delta]_0.$$

Hence, every $\mathrm{Aut}(\mathfrak{h})$-orbit is a finite union of $\mathrm{Aut}_0(\mathfrak{h})$-orbits. This does not mean that D acts naturally on Σ.

However, for the CSLAs with $\mathrm{Aut}(\mathfrak{h}) \subset \mathrm{T}_6$, we have $D(\Sigma) \subset \Sigma$. Moreover, D acts on Σ as a finite subgroup of reflections of $\mathbb{R}^{n+1}$, where $s_0, s_1, \dots, s_n$ is the parametrization of Σ given in the proof of Theorem 4.1. Hence, there exists a submanifold $\Sigma_D \subset \Sigma$ where D acts trivially. For $\mathfrak{h}_{19}^+$ and $\mathfrak{h}_{26}^-$, we have $D \subset \mathrm{O}(6)$ and there exist a non-trivial submanifold $\Sigma_D \subset \Sigma$ as before where D acts trivially, although D is no longer a subgroup of reflections of Σ. For $\mathfrak{h}_{13}$ we have $D \not\subset \mathrm{O}(6)$ and one can easily see that $\Sigma_D = \varnothing$. In some cases, Σ_D completely determines the structure of $\mathcal{M}(H)/\!\sim$, but in general this information is not enough. Supporting computations for these facts and for the following examples are given in the attached attached notebook [CCR24, Notebook 03].

Example 4.2. Let $\mathfrak{h} = \mathfrak{h}_{24}$. Recall that $D \simeq \mathbb{Z}_2$. Here we have that Σ_D is given by elements of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & s_2 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_6 & 0 \\ 0 & 0 & 0 & s_7 & 0 & s_9 \end{pmatrix}$$

and $[g] = [g]_0$ if and only if $g = \sigma^T \sigma$ for some $\sigma \in \Sigma_D$.

Example 4.3. Let $\mathfrak{h} = \mathfrak{h}_{11}$. In this case $D \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by the automorphisms φ_1, φ_2 given in the proof of Theorem 3.2. If we denote by Σ_{φ_i} the submanifold of Σ where φ_i acts trivially, then

$$\Sigma_D = \Sigma_{\varphi_1} \cap \Sigma_{\varphi_2}.$$

For a given left-invariant metric g, we have that $[g]$ is the union of 1, 2, or 4 $\mathrm{Aut}_0(\mathfrak{h})$-orbits if and only if $[g]_0 = [\sigma^T \sigma]_0$, for $\sigma \in \Sigma_D$, $\sigma \in \Sigma_{\varphi_1} \triangle \Sigma_{\varphi_2}$ or $\sigma \in \Sigma - (\Sigma_{\varphi_1} \cup \Sigma_{\varphi_2})$, respectively.

Example 4.4. For $\mathfrak{h} = \mathfrak{h}_{19}^+$ we have Σ_D consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_9 \end{pmatrix}$$

and for $\mathfrak{h} = \mathfrak{h}_{26}^-$ we have that Σ_D consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_{10} \end{pmatrix}.$$

Example 4.5. Let $\mathfrak{h} = \mathfrak{h}_{13}$. Consider the metric

$$g_1 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \sigma_1^T \sigma_1 \text{ with } \sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \Sigma.$$

Now let $g_2 = \varphi_3^T g_1 \varphi_3$ where φ_3 is given as in the proof of Theorem 3.2. Then, there does not exist $\sigma \in \Sigma$ such that $g_2 = \sigma^T \sigma$. However, $g_2 \in [g_3]_0$ where

$$g_3 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \sigma_3^T \sigma_3 \text{ with } \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{2} \end{pmatrix} \in \Sigma.$$

5. The full isometry group of a CSLA

Let $\mathfrak{h}$ be a CSLA of dimension 6 and let H be the associated simply connected Lie group. We identify $\mathcal{M}(H)/{\sim_0}$ with $\Sigma \subset \mathrm{T}_6^+$ as in Theorem 4.1. Given $\sigma \in \Sigma$, define

$$g_\sigma = \sigma^T \sigma \in \mathrm{Sym}_6^+ .$$

We identify as usual g_σ with an inner product on $\mathfrak{h}$ and the corresponding left-invariant metric on H. In this section we address the problem of computing the full isometry group $\mathrm{I}(H, g_\sigma)$ of g_σ. According to [Wol63] we know that

$$\mathrm{I}(H, g_\sigma) \simeq H \rtimes K,$$

where H is identified as the subgroup of left translations of H and K is the full isotropy group. Moreover, via the isotropy representation we have the isomorphism

$$K \simeq \mathrm{Aut}(\mathfrak{h}) \cap \mathrm{O}(g_\sigma).$$

Recall from Remark 3.1 that $\mathrm{Aut}(\mathfrak{h}) = \mathrm{Aut}_0(\mathfrak{h}) \rtimes D$, where D is described in Theorem 3.2.

Theorem 5.1. *We keep the notation of this section. Assume that* $\mathfrak{h}$ *is not isomorphic to* $\mathfrak{h}_{13}, \mathfrak{h}_{19}^{+}$ *nor* $\mathfrak{h}_{26}^{-}$. *Then* $K \subset D$.

Proof. Notice that in these cases, under the usual identifications, $\mathrm{Aut}(\mathfrak{h}) \subset \mathrm{T}_6$ and $D \simeq \mathbb{Z}_2^k$ for some $k \in \{1, 2, 3\}$. Let $\varphi \in K = \mathrm{Aut}(\mathfrak{h}) \cap \mathrm{O}(g_\sigma)$ and let $\varphi = S + N$ its Chevalley decomposition, i.e., N is nilpotent, S is semisimple, $[N, S] = 0$ and N, S are polynomials in φ. Then φ and S have the same diagonal, and hence the elements on the diagonal are ± 1. We claim that $N = 0$. Indeed, first we assume that $\operatorname{diag} \varphi = (1, \dots, 1)$. So, $\varphi = N$ and the set $\{\varphi^n : n \in \mathbb{N}\}$ is unbounded in $\mathrm{Aut}(\mathfrak{h})$, which contradicts the fact that $\varphi \in \mathrm{O}(g_\sigma)$. In the general case we have that $\operatorname{diag} \varphi^2 = \operatorname{diag} S^2 = (1, \dots, 1)$ and so the nilpotent part of φ^2 is trivial. But $\varphi^2 = (S + N)^2 = S^2 + 2SN + N^2$ and thus $N^2 = -2SN$ which is impossible unless $N = 0$.

Now we have that $\varphi = S \in K$ is a semisimple automorphism such that $\varphi^2 = \mathrm{id}$ which satisfy the isometry condition

$$\varphi^T g_\sigma = g_\sigma \varphi. \tag{5.1}$$

Observe that equation (5.1) is linear in the coefficients of φ (although it is not necessarily linear in the free parameters $a_0, a_1, a_2, \dots$ describing φ). Also notice that if a free parameter a_ℓ appears for the first time (with the lexicographic order) in the entry φ_{ij}, then $\sigma_{ij} = 0$. From these facts it is not hard to conclude that all the entries bellow the diagonal of φ are zero. Hence $\varphi \in D$. One can also prove this by direct inspection. Such computations are easy (but long and tedious) and are provided in the accompanying notebook [CCR24, Notebook 04]. ☐

Remark 5.2. From Theorem 5.1, one can easily compute the full isometry group of any g_σ for all the $\mathfrak{h}$ such that $\mathrm{Aut}(\mathfrak{h}) \subset \mathrm{T}_6$. Such description is rather involved and it is not convenient to be included in the manuscript. However, we do include in Tables 7, 8 and 9 some relevant information on the metrics with nontrivial subgroup of isometric automorphisms. These tables must be read as follows. The first and second columns present the name of the Lie algebra and a generic representative of every left-invariant metric up to isometric automorphism (actually, here we keep it simple by considering $\mathcal{M}(H)/{\sim_0}$ instead of $\mathcal{M}(H)/{\sim}$). For each generic metric g_σ, one can equal to 0 a number p of non diagonal parameters. We indicate this in the third column with $\#\,\mathrm{nd}(\Sigma) = 0$. That is, if $\mathrm{nd}(\Sigma)$ has n elements, then there $\binom{n}{p}$ possible such substitutions. The last columns are for the isomorphism classes of subgroups of D (recall that K is a subgroup of D) and we present the amount of substitutions which have K isomorphic to one of these groups.

For example, if $\mathfrak{h} = \mathfrak{h}_{12}$, then $\mathrm{nd}(\Sigma) = \{s_0, s_1, s_3, s_5, s_6\}$ and we have $\binom{5}{2} = 10$ ways to choose two parameters equal to 0. For 8 of these choices we have $K = \{e\}$ and for the remaining 2 we have $K \simeq \mathbb{Z}_2$. The explicit substitutions and subgroups $K \hookrightarrow D$ can be found in [CCR24, Notebook 05].

Remark 5.3. According to Theorem 5.1, if $\mathfrak{h} \not\simeq \mathfrak{h}_{13}, \mathfrak{h}_{19}^{+}, \mathfrak{h}_{26}^{-}$, there exists a finite number of subgroups of $\mathrm{Aut}(\mathfrak{h})$ (and moreover, of D) that can be realized as the full isotropy group of $\mathrm{I}(H, g_\sigma)$ for any $\sigma \in \Sigma$. This is not true if $\mathfrak{h} = \mathfrak{h}_{13}, \mathfrak{h}_{19}^{+}$ or $\mathfrak{h}_{26}^{-}$. In fact, if $r \in (0, 1]$ then the automorphisms

$$\varphi_r = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r \\ 0 & 0 & 0 & 0 & -\frac{1}{r} & 0 \end{pmatrix}, \quad \varphi_r' = \begin{pmatrix} 0 & 0 & r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \varphi_r'' = \begin{pmatrix} 0 & r & 0 & 0 & 0 & 0 \\ \frac{1}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -\frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

of $\mathfrak{h}_{13}$, $\mathfrak{h}_{19}^{+}$ and $\mathfrak{h}_{26}^{-}$ respectively are isometric with respect to g_{σ_r}, $g_{\sigma'_r}$ and $g_{\sigma''_r}$ where

$$\sigma_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\sqrt{-r^2+1}}{r} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sigma'_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{-r^2+1}}{r} & 1 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{-r^2+1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma''_r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{-r^2+1}}{r} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

See [CCR24, Notebook 06] for verification. However, in any of these cases the subgroup K is conjugated to a subgroup of D, since D is a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{h})$.

6. The index of symmetry for CSLA

This section concerns the computation of the index of symmetry of a left-invariant metric on a Lie group associated with a CSLA. We refer to [ORT14] and [BOR17] for the general definitions and the structure theory related with the index of symmetry of general homogeneous Riemannian manifold.

Let $\mathfrak{h}$ be a CSLA of dimension 6 and let H be the associated simply connected Lie group. Endow H with a left-invariant Riemannian metric g and denote by $\mathrm{I}(H)$ its full isometry group. Since the isotropy subgroup of $\mathrm{I}(H)$ is discrete, every Killing field on H is right-invariant. Let $X, Y, Z \in \mathfrak{h}$ and denote by X^*, Y^*, Z^* the corresponding right-invariant vector fields. By definition, the *distribution of symmetry* at the identity e is given by

$$\mathfrak{s}_e = \{X_e : X \in \mathfrak{h} \text{ and } (\nabla X^*)_e = 0\}$$

Since $\mathfrak{s}$ is $\mathrm{I}(H)$-invariant, then it coincides with the distribution generated by the left-invariant fields $X \in \mathfrak{h}$ such that $X_e \in \mathfrak{s}_e$. The so-called *index of symmetry* of (H, g) is the rank of $\mathfrak{s}$. Recall the well-known Kozsul's formula for Killing fields: for all right-invariant fields X^*, Y^*, Z^*, we have

$$g(\nabla_{X^*} Y^*, Z^*) = \frac{1}{2}(g([X^*, Y^*], Z^*) + g([X^*, Z^*], Y^*) + g([Y^*, Z^*], X^*)).$$

So, $Y \in \mathfrak{s}$ if and only if for all $X, Z \in \mathfrak{h}$ holds

$$\begin{aligned} 0 &= 2g((\nabla_{X^*} Y^*)_e, (Z^*)_e) \\ &= g([X^*, Y^*]_e, (Z^*)_e) + g([X^*, Z^*]_e, (Y^*)_e) + g([Y^*, Z^*]_e, (X^*)_e) \\ &= -g([X, Y]_e, Z_e) - g([X, Z]_e, Y_e) - g([Y, Z]_e, X_e). \end{aligned}$$

since $[X^*, Z^*]_e = -[X, Z]^*_e$. So, the above equation is equivalent to

$$g([X, Y], Z) + g([X, Z], Y) + g([Y, Z], X) = 0. \tag{6.1}$$

for all $X, Z \in \mathfrak{h}$. Notice that we have identified with the same symbol g the Riemannian metric on H and the corresponding inner product on $\mathfrak{h}$. If $e_1, \dots, e_6$ is the standard basis of $\mathfrak{h}$ and we write $Y = \sum y_i e_i$, then the previous equations is linear in $y_1, \dots, y_6$. This equation is not linear if we also consider the metric coefficients. Since every g is isometric to a metric with the form $g_\sigma = \sigma^T \sigma$, for some $\sigma \in \Sigma$, in order to determine the existence of metrics with non-trivial index of symmetry it is enough to work with the metrics of this form.

Theorem 6.1. *Let $\mathfrak{h}$ be a CSLA of dimension 6 and let H its associated simply connected Lie group. Assume that $\mathfrak{h}$ is not isomorphic to $\mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{21}, \mathfrak{h}_{22}$ nor $\mathfrak{h}_{28}$. Then every left-invariant metric on H as trivial index of symmetry.*

As far as we know, these are the first examples in the literature of a Lie group all of whose left-invariant metric has trivial index of symmetry. Now we direct our attention to the Lie groups with characteristically solvable Lie algebra which admit left-invariant metrics with nontrivial index of symmetry. We use the notation of Section 4 for the left-invariant metrics under study. We begin with the cases where the distribution of symmetry is contained in the center of the Lie algebra.

Theorem 6.2. *The left-invariant metric $g_\sigma = \sigma^T\sigma$, $\sigma \in \Sigma$, on H_9 has nontrivial index of symmetry if and only if $s_2 = 0$. In this case, the index of symmetry is 1 and the distribution of symmetry is generated by the left-invariant field $Y = e_4$. In particular, $\mathfrak{s}$ is properly contained in the central distribution of H_9.*

Theorem 6.3. *The left-invariant metric $g_\sigma = \sigma^T\sigma$, $\sigma \in \Sigma$ on H_{22} has nontrivial index of symmetry if and only if $s_1 = s_3 = 0$. In this case, the index of symmetry is 1 and the distribution of symmetry is generated by the left-invariant field $Y = e_3$. In particular, $\mathfrak{s}$ is properly contained in the central distribution of H_{22}.*

Theorem 6.4. *The left-invariant metric $g_\sigma = \sigma^T\sigma$ on H_{10} has index nontrivial index of symmetry if and only if $s_3 = \frac{s_1 s_4}{s_2}$. In this case the index of symmetry is 1, the distribution of symmetry is generated by the left-invariant field*

$$Y = e_2 - \frac{s_1}{s_2}e_3 + \frac{s_0^2 s_4}{s_2^2 s_5}e_5 - \frac{s_0^2 s_2^2 + s_0^2 s_4^2}{s_2^2 s_5^2}e_6$$

which does not belong to the center of $\mathfrak{h}_{10}$.

Proof of Theorems 6.1, 6.2, 6.3 and 6.4. It follows from very long, but straightforward, computations. See the attached notebook [CCR24, Notebook 07]. □

The last two cases are more difficult to deal with, as the computations are lengthy and also more involved. A case by case analysis is needed to describe the general situation.

Theorem 6.5. *Let us consider the left-invariant metric $g_\sigma = \sigma^T\sigma$, $\sigma \in \Sigma$, on H_{21}. Then:*

(1) *if $s_2 \neq 0$, the metrics with nontrivial index of symmetry form an algebraic hypersurface of $\mathcal{M}(H_{21})/\sim_0$. For all these metrics, we have $i_{\mathfrak{s}}(H_{21}, g_\sigma) = 1$ and $\mathfrak{s} \not\subset Z(\mathfrak{h}_{21})$;*
(2) *if $s_2 = 0, s_0 \neq 0$ and $s_6 = \frac{s_3 s_7}{s_4}$, then $i_{\mathfrak{s}}(H_{21}, g_\sigma) = 1$ and $\mathfrak{s} \not\subset Z(\mathfrak{h}_{21})$;*
(3) *if $s_2 = 0, s_0 = 0$ and $s_6 \neq \frac{s_3 s_7}{s_4}$, the metrics with nontrivial index of symmetry form an algebraic submanifold of $\mathcal{M}(H_{21})/\sim_0$ of co-dimension 2. For these metrics we have $i_{\mathfrak{s}}(H_{21}, g_\sigma) = 2$ and $\mathfrak{s} \not\subset Z(\mathfrak{h}_{21})$;*
(4) *if $s_2 = 0, s_0 = 0, s_6 = \frac{s_3 s_7}{s_4}$ and $s_5 \neq \frac{s_1^2 s_7}{s_4^2}$ then $i_{\mathfrak{s}}(H_{21}, g_\sigma) = 1$ and $\mathfrak{s} \subset Z(\mathfrak{h}_{21})$;*
(5) *if $s_2 = 0, s_0 = 0, s_6 = \frac{s_3 s_7}{s_4}, s_5 = \frac{s_1^2 s_7}{s_4^2}$ and $s_3 \neq 0$ then $i_{\mathfrak{s}}(H_{21}, g_\sigma) = 2$ and $\mathfrak{s} \not\subset Z(\mathfrak{h}_{21})$;*
(6) *if $s_2 = 0, s_0 = 0, s_6 = \frac{s_3 s_7}{s_4}, s_5 = \frac{s_1^2 s_7}{s_4^2}$ and $s_3 = 0$, then $i_{\mathfrak{s}}(H_{21}, g_\sigma) = 3$ and $\mathfrak{s} \not\subset Z(\mathfrak{h}_{21})$.*[1]
(7) *In all the remaining cases the we have $i_{\mathfrak{s}}(H_{21}, g_\sigma) = 0$.*

Proof. See the attached notebook [CCR24, Notebook 08] for the supporting computations and the following remark for a precise description of distribution of symmetry. □

[1]In this case condition $s_6 = \frac{s_3 s_7}{s_4}$ just means $s_6 = 0$. We stated the theorem in this way in order to better visualize all the possible cases for $\sigma \in \Sigma$.

Remark 6.6. We keep the hypothesis from Theorem 6.5.

(1) If $s_2 \neq 0$, the algebraic hypersurface of Σ such that the associated left-invariant metrics have non-trivial index of symmetry is defined by

$$s_0 s_2 s_4^2 s_5 - s_0^2 s_1 s_4 s_6 - s_0 s_2 s_3 s_4 s_6 - s_0 s_1^2 s_2 s_7 + s_0^2 s_1 s_3 s_7 - s_1 s_2^2 s_3 s_7 + s_0 s_2 s_3^2 s_7 = 0.$$

In order to describe a left-invariant field Y generating $\mathfrak{s}$ we consider the case $s_0 \neq 0$, where

$$Y = e_2 - \frac{s_0^2 s_1 s_4 s_6 - \left(s_0^2 s_1 + s_1 s_2^2\right) s_3 s_7}{s_0 s_2^2 s_4 s_7} e_3 - \frac{s_0 s_1 + s_2 s_3}{s_2 s_4} e_4 - \frac{s_0 s_1^2 s_2 - s_0 s_2 s_3^2 - \left(s_0^2 s_1 - s_1 s_2^2\right) s_3}{s_0 s_2 s_4^2} e_5 + \alpha e_6 \notin Z(\mathfrak{h}_{21})$$

where

$$\alpha = \frac{s_0 s_1^3}{s_2 s_4^3} + \frac{2\, s_1^2 s_3}{s_4^3} - \frac{s_0^2 s_1^2 s_3}{s_2^2 s_4^3} - \frac{2\, s_0 s_1 s_3^2}{s_2 s_4^3} + \frac{s_1 s_2 s_3^2}{s_0 s_4^3} - \frac{s_3^3}{s_4^3} + \frac{s_0 s_1 s_4}{s_2 s_7^2} + \frac{s_1^2 s_6}{s_4^2 s_7} + \frac{s_0^2 s_1^2 s_6}{s_2^2 s_4^2 s_7} + \frac{s_0 s_1 s_3 s_6}{s_2 s_4^2 s_7} + \frac{s_1 s_2 s_3 s_6}{s_0 s_4^2 s_7};$$

and the case $s_0 = 0$, where

$$Y = e_2 + \frac{s_4^2 s_5 - s_1^2 s_7}{s_2 s_4 s_7} e_3 - \frac{s_5}{s_7} e_5 + \frac{s_5 s_6}{s_7^2} e_6 \notin Z(\mathfrak{h}_{21}).$$

(2) If $s_2 = 0, s_0 \neq 0, s_6 = \frac{s_3 s_7}{s_4}$, then $\mathfrak{s}$ is generated by

$$Y = \frac{s_4^2 s_5 - s_1^2 s_7}{s_0 s_1 s_7} e_3 + e_4 - \frac{s_3}{s_4} e_5 - \frac{s_4^4 + s_4^2 s_5 s_7 - s_3^2 s_7^2}{s_4^2 s_7^2} e_6 \notin Z(\mathfrak{h}_{21}).$$

(3) When $s_2 = 0$, $s_0 = 0$ and $s_6 \neq \frac{s_3 s_7}{s_4}$, the submanifold of Σ with associated to the metrics with nontrivial index of symmetry is described by the equation

$$s_4^4 s_5^2 - 2\, s_3 s_4^3 s_5 s_6 + s_3^2 s_4^2 s_6^2 + \left(s_1^4 + 2\, s_1^2 s_3^2 + s_3^4\right) s_7^2 - 2\left(\left(s_1^2 - s_3^2\right) s_4^2 s_5 + \left(s_1^2 s_3 + s_3^3\right) s_4 s_6\right) s_7 = 0$$

then $\mathfrak{s}$ is generated by $Y_1 = e_3 \in Z(\mathfrak{h}_{21})$ and

$$Y_2 = e_2 - \frac{s_4^2 s_5 + s_3 s_4 s_6 - \left(s_1^2 + s_3^2\right) s_7}{2\left(s_4^2 s_6 - s_3 s_4 s_7\right)} e_4 - \frac{s_4^2 s_5 s_6 - s_3 s_4 s_6^2 - \left(2\, s_3 s_4 s_5 - \left(s_1^2 + s_3^2\right) s_6\right) s_7}{2\left(s_4^2 s_6 s_7 - s_3 s_4 s_7^2\right)} e_5 + \alpha e_6 \notin Z(\mathfrak{h}_{21})$$

where

$$\alpha = \frac{1}{2\, s_4^2 s_6 s_7^2 - 2\, s_3 s_4 s_7^3} \left(s_4^4 s_5 - s_3 s_4^3 s_6 + s_4^2 s_5 s_6^2 - s_3 s_4 s_6^3 - \left(s_1^2 + s_3^2\right) s_5 s_7^2 + \left(s_4^2 s_5^2 - s_3 s_4 s_5 s_6 - \left(s_1^2 - s_3^2\right) s_4^2 + \left(s_1^2 + s_3^2\right) s_6^2\right) s_7\right).$$

(4) If $s_2 = 0$, $s_0 = 0$, $s_6 = \frac{s_3 s_7}{s_4}$ and $s_5 \neq \frac{s_1^2 s_7}{s_4^2}$ then $\mathfrak{s}$ is generated by $e_3 \in Z(\mathfrak{h}_{21})$.

(5) If $s_2 = 0$, $s_0 = 0$, $s_6 = \frac{s_3 s_7}{s_4}$, $s_5 = \frac{s_1^2 s_7}{s_4^2}$ and $s_3 \neq 0$, then $\mathfrak{s}$ is generated by $Y_1 = e_3 \in Z(\mathfrak{h}_{21})$ and

$$Y_2 = e_4 - \frac{s_3}{s_4} e_5 - \frac{s_4^4 + \left(s_1^2 - s_3^2\right) s_7^2}{s_4^2 s_7^2} e_6 \notin Z(\mathfrak{h}_{21}).$$

TABLE 3. Examples of left-invariant metrics on H_{28} with all the possible indexes of symmetry.

σ	g_σ	A	$\mathfrak{s}$
$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&\frac{1}{2}&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&\frac{5}{4}&\frac{1}{2}\\ 0&0&0&0&\frac{1}{2}&1 \end{pmatrix}$	$\begin{pmatrix} 0&-1&1\\ -1&0&-1\\ 0&-1&0 \end{pmatrix}$	0
$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&-1&0\\ -1&0&-1\\ 0&-1&0 \end{pmatrix}$	$\langle e_2 - e_4 + e_6 \rangle$
$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&2&1&2&0\\ 0&0&0&\frac{5}{4}&\frac{1}{2}&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&5&2&4&0\\ 0&0&2&\frac{57}{16}&\frac{21}{8}&\frac{5}{4}\\ 0&0&4&\frac{21}{8}&\frac{17}{4}&\frac{1}{2}\\ 0&0&0&\frac{5}{4}&\frac{1}{2}&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\ 0&1&0\\ 0&0&0 \end{pmatrix}$	$\langle e_2 - 4e_6, e_4 - \frac{1}{2}e_5 - 5e_6 \rangle$
$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&1&0\\ 0&0&0&2&0&1 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&2&0&1&0\\ 0&0&0&5&0&2\\ 0&0&1&0&1&0\\ 0&0&0&2&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$	$\langle e_2 - e_6, e_3 - 2e_5, e_4 - 3e_6 \rangle$

(6) If $s_2 = 0$, $s_0 = 0$, $s_6 = \frac{s_3 s_7}{s_4}$, $s_5 = \frac{s_1^2 s_7}{s_4^2}$ and $s_3 = 0$, then $\mathfrak{s}$ is generated by

$$Y_1 = e_2 - \tfrac{s_1^2}{s_4^2} e_5 \notin Z(\mathfrak{h}_{21}), \quad Y_2 = e_3 \in Z(\mathfrak{h}_{21}), \quad Y_3 = e_4 - \frac{\left(s_4^4 + s_1^2 s_7^2\right)}{s_4^2 s_7^2} e_6 \notin Z(\mathfrak{h}_{21}).$$

Theorem 6.7. *Let us consider the left-invariant metric $g_\sigma = \sigma^T \sigma$ on H_{28} and let us define*

$$A = \begin{pmatrix} a & b & \alpha \\ c & \beta & b \\ \gamma & c & a \end{pmatrix} \qquad \textit{where} \qquad \begin{array}{l} a = s_5^2 s_6 + s_3 s_5 s_8 - (s_1 s_2 + s_3 s_4) s_9, \\ b = s_5^2 s_7 + s_4 s_5 s_8 - \left(s_2^2 + s_4^2 + s_3 s_5\right) s_9, \\ c = s_4 s_5 s_6 + s_3 s_5 s_7 - \left(s_0^2 + s_1^2 + s_3^2\right) s_9, \\ \alpha = 2\, s_5^2 s_8 - 2\, s_4 s_5 s_9, \\ \beta = 2\, s_4 s_5 s_7 - 2\,(s_1 s_2 + s_3 s_4) s_9, \\ \gamma = 2\, s_3 s_5 s_6. \end{array}$$

Then

$$i_{\mathfrak{s}}(H_{28}, g_\sigma) = 3 - \operatorname{rank} A.$$

Proof. See the attached notebook [CCR24, Notebook 09]. □

Remark 6.8. The matrix A in Theorem 6.7 is a persymmetric matrix (i.e., it is symmetric with respect to its anti-diagonal). However this property has not a geometric or algebraic meaning as its form depends heavily on the basis of choice and even in the given order of the algorithm solving the equations for metrics with non-trivial index of symmetry. We present in Table 3 examples of different metrics g_σ, $\sigma \in \Sigma$, for which A has all the possible ranks. Notice that in these particular examples, $i_{\mathfrak{s}}(H_{28}, g_\sigma) > 0$ implies $\mathfrak{s} \not\subset Z(\mathfrak{h}_{28})$. Moreover, in the following result we prove that non-trivial distributions of symmetry are never central.

Proposition 6.9. *For any left-invariant metric on H_{28} we have that*

$$Z(\mathfrak{h}_{28}) \cap \mathfrak{s} = \{0\}.$$

Proof. Notice that $Z(\mathfrak{h}_{28}) = \mathbb{R}e_6$ and assume that $e_6 \in \mathfrak{s}$ where $\mathfrak{s}$ is the distribution of symmetry of some g_σ, $\sigma \in \Sigma$. Then from (6.1) we have that

$$0 = g_\sigma([e_1, e_6], e_5) + g_\sigma([e_1, e_5], e_6) + g_\sigma([e_6, e_5], e_1) = -s_9^2$$

which is a contradiction since g_σ is non-degenerate. □

7. Application to nilsoliton metrics on CSLA

Let H be a nilpotent simply connected Lie group endowed with a left-invariant metric g. Let us denote by $\mathfrak{h}$ the Lie algebra of H and with the same symbol g the inner product on $\mathfrak{h}$ induced by the metric. We say that g is a *nilsoliton* metric if

$$\mathrm{Ric}_g = c\,\mathrm{id}_{\mathfrak{h}} + D \tag{7.1}$$

for some $c \in \mathbb{R}$ and $D \in \mathrm{Der}(\mathfrak{h})$, where Ric_g is the Ricci operator of g (at the identity element of H). The classification of nilsoliton metrics is known in low dimensions. In particular the classification in dimension 6 is obtained in [Wil03] (see also [Wil11]). Since nilsoliton metrics are Ricci soliton metrics, they have distinguished geometric properties, so it is expected that these metrics possess non-trivial index of symmetry. In the next result we prove that this is indeed the case for CSLAs of dimension 6.

Theorem 7.1. *Let $\mathfrak{h}$ be a CSLA of dimension 6 and let H be the simply connected Lie group with Lie algebra $\mathfrak{h}$. Assume that H admits a left-invariant metric with nontrivial index of symmetry. If g is a nilsoliton metric on H, then $i_{\mathfrak{s}}(H, g) > 0$.*

Proof. From Theorem 6.1 we know that $\mathfrak{h} \in \{\mathfrak{h}_9, \mathfrak{h}_{10}, \mathfrak{h}_{21}, \mathfrak{h}_{22}, \mathfrak{h}_{28}\}$. While the classification of 6-dimensional nilsolitons exists, finding an explicit isometry between a nilsoliton in this classifications and a metric from Section 4 might be difficult. In fact, it is easier to solve equation 7.1 directly in $\Sigma \simeq \mathcal{M}(\mathfrak{h})/{\sim_0}$. In order to compute the Ricci operator we will use the well-known formula

$$g(\mathrm{Ric}_g\,\tilde{e}_j, \tilde{e}_h) = \frac{1}{2}\sum_{i,k} c_{iki}(c_{kjh} + c_{khj}) + \frac{1}{2}c_{ikh}c_{ikj} - c_{ijk}c_{khi} + c_{iki}c_{jhk} - c_{ijk}c_{ihk}$$

where $\tilde{e}_1, \dots, \tilde{e}_6$ is an orthonormal basis and

$$c_{ijk} = g([\tilde{e}_i, \tilde{e}_j], \tilde{e}_k).$$

Assume that $g = g_\sigma$ for some $\sigma \in \Sigma$ and that $\tilde{e}_1, \dots, \tilde{e}_6$ is obtained from the standard basis $e_1, \dots, e_6$ via the Gram-Schmidt process. Since the Ricci operator is symmetric and any derivation has triangular form in the standard basis, the equation $\mathrm{Ric}_g = c\,\mathrm{id}_{\mathfrak{h}} + D$, for $D \in \mathrm{Der}(\mathfrak{h})$ implies that Ric_g and D are diagonal in the standard basis. So we can assume further that σ, and hence g_σ are diagonal matrices (we will verify this fact a posteriori by solving the nilsoliton equation with σ diagonal, since the nilsoliton metric is unique up to scaling [Lau11]). In this case one has

$$g(\mathrm{Ric}_g\,\tilde{e}_j, \tilde{e}_h) = g(\mathrm{Ric}_g\,e_j, e_h).$$

With these simplifications it is not hard to solve equation 7.1 (see the attached notebook [CCR24, Notebook 10]) and we get that the nilsoliton metrics g_i on $\mathfrak{h}_i$ are given by

$$\begin{aligned}
g_9 &= \mathrm{diag}(1, 1, 2r, 1, r, r^2)\\
g_{10} &= \mathrm{diag}(1, 1, 1, r, \tfrac{1}{2}\,r, r^2)\\
g_{21} &= \mathrm{diag}(1, 1, 1, \tfrac{1}{2}\,3^{\frac{1}{3}}2^{\frac{1}{3}}r^{\frac{2}{3}}, \tfrac{1}{3}\,3^{\frac{2}{3}}2^{\frac{2}{3}}r^{\frac{4}{3}}, r^2)\\
g_{22} &= \mathrm{diag}(1, \tfrac{1}{2}\,3^{\frac{1}{4}}2^{\frac{3}{4}}\sqrt{r}, 1, \tfrac{1}{2}\,\sqrt{3}\sqrt{2}r, \tfrac{1}{2}\,3^{\frac{3}{4}}2^{\frac{1}{4}}r^{\frac{3}{2}}, r^2)\\
g_{28} &= \mathrm{diag}(1, 1, \tfrac{1}{3}\,\sqrt{3}\sqrt{2}\sqrt{r}, r, \tfrac{1}{2}\,\sqrt{3}\sqrt{2}r^{\frac{3}{2}}, r^2)
\end{aligned}$$

for $r > 0$. Now from Theorems 6.2, 6.3, 6.4, 6.5 and 6.7 we get that the index of symmetry of these metrics is nontrivial. □

Remark 7.2. Every nilsoliton metric in Theorem 7.1 has $i_{\mathfrak{s}}(H,g) = 1$. So it is quite surprising that the index of symmetry of a nilsoliton metric is not always maximal (since H_{22} and H_{28} admit metrics with index of symmetry 3). Notice however that the distribution of symmetry is well behaved with respect to the central distribution. Namely $\mathfrak{s} \subset Z(\mathfrak{h})$ when this property is not obstructed by the underlying Lie algebra structure (notice that no metric on H_{10} of H_{28} has well-behaved distribution of symmetry with respect to the central distribution).

Remark 7.3. For the sake of completeness, we will now identify the (normalized) nilsoliton metrics of Will's classification with their counterparts in the proof of Theorem 7.1. Following the notation from [Wil11], the 6-dimensional nilsoliton metrics with nontrivial index of symmetry are the one given by

$$\begin{aligned} \mathfrak{n}_9 &= (0,0,0,0,2^{\frac{1}{2}}\,\mathbf{12},\mathbf{14}+2^{\frac{1}{2}}\,\mathbf{25}) \\ \mathfrak{n}_{10} &= (0,0,0,2^{\frac{1}{2}}\mathbf{12},\mathbf{13},2^{\frac{1}{2}}\mathbf{14}) \\ \mathfrak{n}_{21} &= (0,0,0,3^{\frac{1}{2}}\,\mathbf{12},2\,\mathbf{14},3^{\frac{1}{2}}\,\mathbf{15}) \\ \mathfrak{n}_{22} &= (0,0,0,3^{\frac{1}{2}}\mathbf{12},3^{\frac{1}{2}}\mathbf{14},2^{\frac{1}{2}}\mathbf{15}+2^{\frac{1}{2}}\mathbf{24}) \\ \mathfrak{n}_{28} &= (0,0,2\,\mathbf{12},6^{\frac{1}{2}}\,\mathbf{13},6^{\frac{1}{2}}\,\mathbf{14},2\,\mathbf{15}) \end{aligned} \tag{7.2}$$

We briefly recall this notation since it slightly differs from ours. Let us consider, for example, the case of $\mathfrak{n}_{22}$. The above notation means that there exist an orthonormal basis $x_1,\dots,x_6$ of $\mathfrak{n}_{22}$ with nontrivial structure coefficients given by

$$[x_1,x_2] = \sqrt{2}\,x_4 \qquad [x_1,x_4] = \sqrt{3}\,x_5 \qquad [x_1,x_5] = [x_2,x_4] = \sqrt{2}\,x_6.$$

To continue with the example, if we want to find $r > 0$ such that $\mathfrak{n}_{22}$ is isometric to $(\mathfrak{h}_{22}, g_{22})$, we can define the Lie algebra isomorphism φ that maps

$$\begin{aligned} &\varphi(x_1) = e_1 && \varphi(x_2) = -\tfrac{1}{3}\sqrt{3}\,e_2 && \varphi(x_3) = e_3 \\ &\varphi(x_4) = \tfrac{1}{3}\,e_4 && \varphi(x_5) = -\tfrac{1}{9}\sqrt{3}\,e_5 && \varphi(x_6) = \tfrac{1}{18}\sqrt{6}\,e_6 \end{aligned}$$

and notice that φ is an isometry if and only if $r = 3\sqrt{6}$. With this very same idea we see that the nilsoliton metrics from 7.2 are isometric to

$$\begin{aligned} g_9|_{r=2} &= \mathrm{diag}(1,1,4,1,2,4) \\ g_{10}|_{r=2} &= \mathrm{diag}(1,1,1,2,1,4) \\ g_{21}|_{r=6} &= \mathrm{diag}(1,1,1,3,12,36) \\ g_{22}|_{r=3\sqrt{6}} &= \mathrm{diag}(1,3,1,9,27,54) \\ g_{28}|_{r=24} &= \mathrm{diag}(1,1,4,24,144,576) \end{aligned}$$

respectively.

8. Appendix: Tables

In this section we collect the tables with our main classificatory results.

References

[BL23] C. Böhm and R. Lafuente, *Non-compact Einstein manifolds with symmetry*, J. Amer. Math. Soc. **36** (2023), no. 3, 591–651.

[BOR17] J. Berndt, C. Olmos, and S. Reggiani, *Compact homogeneous Riemannian manifolds with low coindex of symmetry*, J. Eur. Math. Soc. **19** (2017), no. 1, 221–254.

[CCR24] I. Cardoso, A. Cosgaya, and S. Reggiani, *Auxiliary SageMath notebooks for the article The moduli space of left-invariant metrics on six-dimensional characteristically solvable nilmanifolds*, `https://github.com/silvioreggiani/csla-dim-6/`, 2024.

[CFS05] S. Console, A. Fino, and E. Samiou, *The Moduli Space of Six-Dimensional Two-Step Nilpotent Lie Algebras*, Ann. Global Anal. Geom. **27** (2005), no. 1, 17–32.

Table 4. Automorphism group of 3-step nilpotent CSLAs

$\mathfrak{h}$	$\mathrm{Aut}_0(\mathfrak{h})$	D
$\mathfrak{h}_9$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_0^2 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & 0 & 0 \\ a_9 & a_{10} & a_0a_1 & 0 & a_0a_2 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & -a_0a_{10}-a_2a_3+a_1a_4 & a_0^2a_2 \end{pmatrix}$, $a_0,a_2,a_8>0$	$\langle\mathrm{diag}(-1,1,1,1,-1,1)$, $\mathrm{diag}(1,-1,1,1,-1,-1)$, $\mathrm{diag}(1,1,1,-1,1,1)\rangle$
$\mathfrak{h}_{10}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & 0 & 0 & 0 \\ a_6 & a_7 & a_8 & a_0a_2 & 0 & 0 \\ a_9 & a_{10} & a_{11} & a_0a_4 & a_0a_5 & 0 \\ a_{12} & a_{13} & a_{14} & a_0a_7 & a_0a_8 & a_0^2a_2 \end{pmatrix}$, $a_0,a_2,a_5>0$	$\langle\mathrm{diag}(-1,1,1,-1,-1,1)$, $\mathrm{diag}(1,-1,1,-1,1,-1)$, $\mathrm{diag}(1,1,-1,1,-1,1)\rangle$
$\mathfrak{h}_{11}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_0^2 & 0 & 0 & 0 \\ a_5 & a_6 & a_7 & a_0a_2 & 0 & 0 \\ a_8 & a_9 & a_{10} & a_0a_4 & a_0^3 & 0 \\ a_{11} & a_{12} & a_{13} & -a_2a_3+a_1a_4+a_0a_6 & a_0^2a_1+a_0a_7 & a_0^2a_2 \end{pmatrix}$, $a_0,a_2>0$	$\langle\mathrm{diag}(-1,1,1,-1,-1,1)$, $\mathrm{diag}(1,-1,1,-1,1,-1)\rangle$
$\mathfrak{h}_{12}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_4 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & a_0a_1 & 0 & 0 \\ a_7 & a_8 & a_9 & a_0a_3 & a_0a_4 & 0 \\ a_{10} & a_{11} & a_{12} & -a_1a_5 & 0 & a_0a_1^2 \end{pmatrix}$, $a_0,a_1,a_4>0$	$\langle\mathrm{diag}(-1,1,1,-1,-1,-1)$, $\mathrm{diag}(1,-1,1,-1,1,1)$, $\mathrm{diag}(1,1,-1,1,-1,1)\rangle$
$\mathfrak{h}_{13}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_0a_1 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_0a_1 & 0 & 0 \\ a_6 & a_7 & a_8 & a_0a_3+a_0a_5 & a_0^2a_1 & 0 \\ a_9 & a_{10} & a_{11} & -a_1a_4 & 0 & a_0a_1^2 \end{pmatrix}$, $a_0,a_1>0$	$\langle\mathrm{diag}(-1,1,-1,-1,1,-1)$, $\mathrm{diag}(1,-1,-1,-1,-1,1)$, $\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}\rangle$
$\mathfrak{h}_{14}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_2^2 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & a_0a_2 & 0 & 0 \\ a_7 & a_8 & a_9 & a_0a_6 & a_0^2a_2 & 0 \\ a_{10} & a_{11} & a_{12} & a_0a_4+a_2a_5-a_1a_6 & -a_0a_1a_2 & a_0a_2^2 \end{pmatrix}$, $a_0,a_2>0$	$\langle\mathrm{diag}(-1,1,1,-1,1,-1)$, $\mathrm{diag}(1,-1,1,-1,-1,1)\rangle$
$\mathfrak{h}_{18}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & -\frac{a_1a_2}{a_0} & \frac{a_2^2}{a_0} & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_0a_2 & 0 & 0 \\ a_6 & a_7 & a_8 & -a_1a_2 & a_2^2 & 0 \\ a_9 & a_{10} & a_{11} & -a_2a_4+a_1a_5+a_0a_7 & a_0a_8 & a_0a_2^2 \end{pmatrix}$, $a_0,a_2>0$	$\langle\mathrm{diag}(-1,1,-1,-1,1,-1)$, $\mathrm{diag}(1,-1,1,-1,1,1)\rangle$
$\mathfrak{h}_{19}^+$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & a_1a_2 & 0 & 0 \\ \frac{a_0a_5}{a_2} & a_6 & a_7 & 0 & a_0a_1 & 0 \\ a_8 & a_9 & a_{10} & -a_2a_6 & -a_0a_4 & a_0a_1a_2 \end{pmatrix}$, $a_0,a_1,a_2>0$	$\langle\mathrm{diag}(-1,1,1,1,-1,-1)$, $\mathrm{diag}(1,-1,1,-1,-1,-1)$ $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\rangle$

[CR22] A. Cosgaya and S. Reggiani, *Isometry groups of three-dimensional Lie groups*, Ann. Global Anal. Geom. **61** (2022), no. 4, 831–845.

[DS13] A. J. Di Scala, *Invariant metrics on the Iwasawa manifold*, Quart. J. Math. **64** (2013), no. 2, 555–569.

[FF23] K. Ficzere and Á. Figula, *Isometry groups of six-dimensional nilmanifolds*, Aequat. Math. **97** (2023), no. 4, 725–752.

[FN18] Á. Figula and P. T. Nagy, *Isometry classes of simply connected nilmanifolds*, J. Geom. Phys. **132** (2018), 370–381.

[HL09] K. Y. Ha and J. B. Lee, *Left invariant metrics and curvatures on simply connected three-dimensional Lie groups: Left invariant metrics and curvatures*, Math. Nachr. **282** (2009), no. 6, 868–898.

TABLE 5. Automorphism group of 4-step nilpotent CSLAs

$\mathfrak{h}$	$\mathrm{Aut}_0(\mathfrak{h})$	D
$\mathfrak{h}_{21}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & 0 & 0 & 0 \\ a_6 & a_7 & 0 & a_0a_2 & 0 & 0 \\ a_8 & a_9 & 0 & a_0a_7 & a_0^2a_2 & 0 \\ a_{10} & a_{11} & a_{12} & a_0a_9 & a_0^2a_7 & a_0^3a_2 \end{pmatrix}$, $a_1,a_2,a_5>0$	$\langle\mathrm{diag}(-1,1,1,-1,1,-1)$, $\mathrm{diag}(1,-1,1,-1,-1,-1)$, $\mathrm{diag}(1,1,-1,1,1,1)\rangle$
$\mathfrak{h}_{22}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_0^2 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_4 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & a_0^3 & 0 & 0 \\ a_7 & a_8 & 0 & a_0a_6 & a_0^4 & 0 \\ a_9 & a_{10} & a_{11} & -a_0^2a_5+a_1a_6+a_0a_8 & a_0^3a_1+a_0^2a_6 & a_0^5 \end{pmatrix}$, $a_0,a_4>0$	$\langle\mathrm{diag}(-1,1,1,-1,1,-1)$, $\mathrm{diag}(1,1,-1,1,1,1)\rangle$
$\mathfrak{h}_{23}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_0a_2 & 0 & 0 & 0 \\ a_5 & a_6 & a_0a_4 & a_0^2a_2 & 0 & 0 \\ a_7 & a_8 & -a_2a_3+a_1a_4 & a_0a_1a_2 & a_0a_2^2 & 0 \\ a_9 & a_{10} & a_0a_6 & a_0^2a_4 & 0 & a_0^3a_2 \end{pmatrix}$, $a_0,a_2>0$	$\langle\mathrm{diag}(-1,1,-1,1,-1,-1)$, $\mathrm{diag}(1,-1,-1,-1,1,-1)\rangle$
$\mathfrak{h}_{24}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_0^2 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_0^3 & 0 & 0 & 0 \\ a_4 & a_5 & 0 & a_0^3 & 0 & 0 \\ a_6 & a_7 & -a_0^2a_1 & a_0a_5 & a_0^4 & 0 \\ a_8 & a_9 & a_{10} & \alpha & a_0^3a_1+a_0^2a_5 & a_0^5 \end{pmatrix}$, $a_0>0$, $\alpha=-a_0^2a_2-a_0^2a_4+a_1a_3+a_1a_5+a_0a_7$	$\langle\mathrm{diag}(-1,1,-1,-1,1,-1)\rangle$
$\mathfrak{h}_{25}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_0^3 & 0 & 0 & 0 \\ a_5 & a_6 & 0 & a_0a_2 & 0 & 0 \\ a_7 & a_8 & -a_0^2a_1 & a_0a_6 & a_0^2a_2 & 0 \\ a_9 & a_{10} & a_{11} & -a_2a_3+a_1a_4+a_0a_8 & a_0^2a_6 & a_0^3a_2 \end{pmatrix}$, $a_0,a_2>0$	$\langle\mathrm{diag}(-1,1,-1,-1,1,-1)$, $\mathrm{diag}(1,-1,1,-1,-1,-1)\rangle$
$\mathfrak{h}_{26}^{-}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_0a_1 & 0 & 0 & 0 \\ a_4 & a_5 & -a_0a_3 & a_0^2a_1 & 0 & 0 \\ a_6 & a_7 & a_1a_2 & 0 & a_0a_1^2 & 0 \\ a_8 & a_9 & -a_1a_4+a_0a_7 & -a_0a_1a_2 & a_0a_1a_3 & a_0^2a_1^2 \end{pmatrix}$, $a_0,a_1>0$	$\langle\mathrm{diag}(-1,1,-1,1,-1,1)$, $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\rangle$
$\mathfrak{h}_{27}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0^2 & 0 & 0 & 0 \\ a_3 & a_4 & a_0a_1 & a_0a_2 & 0 & 0 \\ a_5 & a_6 & a_0a_3 & a_0a_4 & a_0^2a_2 & 0 \\ a_7 & a_8 & a_9 & a_0a_6 & a_0^2a_4 & a_0^3a_2 \end{pmatrix}$, $a_0,a_2>0$	$\langle\mathrm{diag}(-1,1,1,-1,1,-1)$, $\mathrm{diag}(1,-1,1,-1,-1,-1)\rangle$

[KTT11] H. Kodama, A. Takahara, and H. Tamaru, *The space of left-invariant metrics on a Lie group up to isometry and scaling*, Manuscripta Math. **135** (2011), no. 1-2, 229–243.

[Lau03] J. Lauret, *Degenerations of Lie algebras and geometry of Lie groups*, Diff. Geom. Appl. **18** (2003), no. 2, 177–194.

[Lau11] ———, *Ricci soliton solvmanifolds*, J. Reine Angew. Math. **2011** (2011), no. 650, 1–21.

[Mag07] L. Magnin, *Complex structures on indecomposable 6-dimensional nilpotent real Lie algebras*, Int. J. Algebra Comput. **17** (2007), no. 01, 77–113.

[May21] R. May, *The index of symmetry for a left-invariant metric on a solvable three-dimensional Lie group*, 2021.

[ORT14] C. Olmos, S. Reggiani, and H. Tamaru, *The index of symmetry of compact naturally reductive spaces*, Math. Z. **277** (2014), no. 3-4, 611–628.

[Reg19] S. Reggiani, *The distribution of symmetry of a naturally reductive nilpotent Lie group*, Geom. Dedicata **200** (2019), no. 1, 61–65.

[Reg21] ———, *Manifolds admitting a metric with co-index of symmetry 4*, Manuscripta Math. **164** (2021), no. 3-4, 543–553.

[RV20] S. Reggiani and F. Vittone, *The moduli space of left-invariant metrics of a class of six-dimensional nilpotent Lie groups*, 2020.

TABLE 6. Automorphism group of 5-step nilpotent CSLAs

$\mathfrak{h}$	$\mathrm{Aut}_0(\mathfrak{h})$	D
$\mathfrak{h}_{28}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_0a_2 & 0 & 0 & 0 \\ a_5 & a_6 & a_0a_4 & a_0^2a_2 & 0 & 0 \\ a_7 & a_8 & a_0a_6 & a_0^2a_4 & a_0^3a_2 & 0 \\ a_9 & a_{10} & a_0a_8 & a_0^2a_6 & a_0^3a_4 & a_0^4a_2 \end{pmatrix}$, $a_0, a_2 > 0$	$\langle \mathrm{diag}(-1,1,-1,1,-1,1),$ $\mathrm{diag}(1,-1,-1,-1,-1,-1) \rangle$
$\mathfrak{h}_{29}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & a_0^3 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_0^4 & 0 & 0 & 0 \\ a_4 & a_5 & a_0a_3 & a_0^5 & 0 & 0 \\ a_6 & a_7 & a_0a_5 & a_0^2a_3 & a_0^6 & 0 \\ a_8 & a_9 & -a_0^3a_2+a_1a_3+a_0a_7 & a_0^4a_1+a_0^2a_5 & a_0^3a_3 & a_0^7 \end{pmatrix}$, $a_0 > 0$	$\langle \mathrm{diag}(-1,-1,1,-1,1,-1) \rangle$
$\mathfrak{h}_{30}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_0^2 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_0^3 & 0 & 0 & 0 \\ a_3 & a_4 & a_0a_2 & a_0^4 & 0 & 0 \\ a_5 & a_6 & -a_0^2a_1+a_0a_4 & a_0^2a_2 & a_0^5 & 0 \\ a_7 & a_8 & -a_0^2a_3+a_0a_6 & -a_0^3a_1+a_0^2a_4 & a_0^3a_2 & a_0^6 \end{pmatrix}$, $a_0 > 0$	$\langle \mathrm{diag}(-1,1,-1,1,-1,1) \rangle$
$\mathfrak{h}_{31}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & a_0a_1 & 0 & 0 & 0 \\ a_4 & \frac{a_3^2}{2a_1} & a_0a_3 & a_0^2a_1 & 0 & 0 \\ a_5 & a_6 & \frac{a_0a_3^2}{2a_1} & a_0^2a_3 & a_0^3a_1 & 0 \\ a_7 & a_8 & \alpha & a_0a_2a_3-a_0a_1a_4 & a_0^2a_1a_2 & a_0^3a_1^2 \end{pmatrix}$, $a_0, a_1 > 0$, $\alpha = \frac{a_2a_3^2-2a_1a_3a_4+2a_1^2a_5}{2a_1}$	$\langle \mathrm{diag}(-1,1,-1,1,-1,-1),$ $\mathrm{diag}(1,-1,-1,-1,-1,1) \rangle$
$\mathfrak{h}_{32}$	$\begin{pmatrix} a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_0^2 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_0^3 & 0 & 0 & 0 \\ a_3 & \frac{a_2^2}{2a_0^2} & a_0a_2 & a_0^4 & 0 & 0 \\ a_4 & a_5 & -\frac{2a_0^3a_1-a_2^2}{2a_0} & a_0^2a_2 & a_0^5 & 0 \\ a_6 & a_7 & \alpha & -a_0^3a_3+a_0a_1a_2 & a_0^4a_1 & a_0^7 \end{pmatrix}$, $a_0 > 0$, $\alpha = \frac{2a_0^4a_4-2a_0^2a_2a_3+a_1a_2^2}{2a_0^2}$	$\langle \mathrm{diag}(-1,1,-1,1,-1,-1) \rangle$

[Sal01] S. M. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 311–333.

[Wil03] C. Will, *Rank-one Einstein solvmanifolds of dimension 7*, Diff. Geom. Appl. **19** (2003), no. 3, 307–318.

[Wil11] ———, *The space of solvsolitons in low dimensions*, Ann. Glob. Anal. Geom. **40** (2011), no. 3, 291–309.

[Wol63] J. A. Wolf, *On locally symmetric spaces of non-negative curvature and certain other locally homogeneous spaces*, Comment. Math. Helv. **37** (1963), 266–295.

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TABLE 7. Moduli space of left-invariant metrics and isometric automorphisms for 3-step CSLAs except $\mathfrak{h}_{13}$ and $\mathfrak{h}_{19}^{+}$

$\mathfrak{h}$	Σ	$\#\,\mathrm{nd}(\Sigma) = 0$	$\{e\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$
$\mathfrak{h}_9$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&s_0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&s_1&s_2&s_3&0\\ 0&0&0&0&s_4&s_5 \end{pmatrix}$	0	1			
		1		3		
		2			3	
		3				1
$\mathfrak{h}_{10}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&s_0&0&0\\ 0&0&0&s_1&s_2&0\\ 0&0&0&s_3&s_4&s_5 \end{pmatrix}$	0		1		
		1		3		
		2			3	
		3				1
$\mathfrak{h}_{11}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&s_0&0&0&0\\ 0&0&0&s_1&0&0\\ 0&0&0&s_2&s_3&0\\ 0&0&0&s_4&s_5&s_6 \end{pmatrix}$	0	1			-
		1	3			-
		2		3		-
		3			1	-
$\mathfrak{h}_{12}$	$\begin{pmatrix} 1&0&0&0&0&0\\ s_0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&s_1&s_2&0&0\\ 0&0&0&s_3&s_4&0\\ 0&0&0&s_5&s_6&s_7 \end{pmatrix}$	0	1			
		1	5			
		2	8	2		
		3		10		
		4			5	
		5				1
$\mathfrak{h}_{14}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&s_0&0&0&0\\ 0&0&s_1&s_2&0&0\\ 0&0&0&s_3&s_4&0\\ 0&0&0&s_5&s_6&s_7 \end{pmatrix}$	0	1			-
		1	4			-
		2	5	1		-
		3		4		-
		4			1	-
$\mathfrak{h}_{18}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&s_0&s_1&0&0&0\\ 0&0&s_2&s_3&0&0\\ 0&0&0&s_4&s_5&0\\ 0&0&0&s_6&s_7&s_8 \end{pmatrix}$	0	1			-
		1	5			-
		2	10			-
		3	8	2		-
		4		5		-
		5			1	-

TABLE 8. Moduli space of left-invariant metrics and isometric automorphisms for 4-step CSLAs except $\mathfrak{h}_{26}^{-}$

$\mathfrak{h}$	Σ	$\#\,\mathrm{nd}(\Sigma)=0$	$\{e\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$
$\mathfrak{h}_{21}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&s_0&s_1&0&0\\ 0&0&s_2&s_3&s_4&0\\ 0&0&0&s_5&s_6&s_7 \end{pmatrix}$	0		1		
		1		5		
		2		9	1	
		3		5	5	
		4			4	1
		5				1
$\mathfrak{h}_{22}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&s_0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&s_1&s_2&0&0\\ 0&0&s_3&s_4&s_5&0\\ 0&0&0&s_6&s_7&s_8 \end{pmatrix}$	0	1			-
		1	5			-
		2	9	1		-
		3	5	5		-
		4		4	1	-
		5			1	-
$\mathfrak{h}_{23}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&s_0&0&0&0\\ 0&0&s_1&s_2&0&0\\ 0&0&s_3&s_4&s_5&0\\ 0&0&s_6&s_7&s_8&s_9 \end{pmatrix}$	0	1			-
		1	6			-
		2	15			-
		3	18	2		-
		4	8	7		-
		5		5	1	-
		6			1	-
$\mathfrak{h}_{24}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&s_0&0&0&0&0\\ 0&0&s_1&0&0&0\\ 0&0&s_2&s_3&0&0\\ 0&0&s_4&s_5&s_6&0\\ 0&0&0&s_7&s_8&s_9 \end{pmatrix}$	0	1		-	-
		1	5		-	-
		2	10		-	-
		3	9	1	-	-
		4	3	2	-	-
		5		1	-	-
$\mathfrak{h}_{25}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&s_0&0&0&0\\ 0&0&s_1&s_2&0&0\\ 0&0&s_3&s_4&s_5&0\\ 0&0&0&s_6&s_7&s_8 \end{pmatrix}$	0	1			-
		1	5			-
		2	9	1		-
		3	5	5		-
		4		4	1	-
		5			1	-
$\mathfrak{h}_{27}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ s_0&s_1&s_2&0&0&0\\ 0&0&s_3&s_4&0&0\\ 0&0&s_5&s_6&s_7&0\\ 0&0&0&s_8&s_9&s_{10} \end{pmatrix}$	0	1			-
		1	7			-
		2	21			-
		3	34	1		-
		4	30	5		-
		5	11	10		-
		6		6	1	-
		7			1	-

TABLE 9. Moduli space of left-invariant metrics and isometric automorphisms for 5-step CSLAs

$\mathfrak{h}$	Σ	$\#\,\mathrm{nd}(\Sigma)=0$	$\{e\}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathfrak{h}_{28}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&s_0&0&0&0\\ 0&0&s_1&s_2&0&0\\ 0&0&s_3&s_4&s_5&0\\ 0&0&s_6&s_7&s_8&s_9 \end{pmatrix}$	0		1	
		1		6	
		2		15	
		3		20	
		4		14	1
		5		4	2
		6			1
$\mathfrak{h}_{29}$	$\begin{pmatrix} 1&0&0&0&0&0\\ 0&s_0&0&0&0&0\\ 0&0&s_1&0&0&0\\ 0&0&s_2&s_3&0&0\\ 0&0&s_4&s_5&s_6&0\\ 0&0&s_7&s_8&s_9&s_{10} \end{pmatrix}$	0	1		-
		1	6		-
		2	15		-
		3	20		-
		4	14	1	-
		5	4	2	-
		6		1	-
$\mathfrak{h}_{30}$	$\begin{pmatrix} 1&0&0&0&0&0\\ s_0&s_1&0&0&0&0\\ 0&0&s_2&0&0&0\\ 0&0&s_3&s_4&0&0\\ 0&0&s_5&s_6&s_7&0\\ 0&0&s_8&s_9&s_{10}&s_{11} \end{pmatrix}$	0	1		-
		1	7		-
		2	21		-
		3	35		-
		4	35		-
		5	20	1	-
		6	5	2	-
		7		1	-
$\mathfrak{h}_{31}$	$\begin{pmatrix} 1&0&0&0&0&0\\ s_0&1&0&0&0&0\\ 0&0&s_1&0&0&0\\ 0&s_2&s_3&s_4&0&0\\ 0&0&s_5&s_6&s_7&0\\ 0&0&s_8&s_9&s_{10}&s_{11} \end{pmatrix}$	0	1		
		1	8		
		2	28		
		3	56		
		4	67	3	
		5	44	12	
		6	12	15	1
		7		6	2
		8			1
$\mathfrak{h}_{32}$	$\begin{pmatrix} 1&0&0&0&0&0\\ s_0&s_1&0&0&0&0\\ 0&0&s_2&0&0&0\\ 0&s_3&s_4&s_5&0&0\\ 0&0&s_6&s_7&s_8&0\\ 0&0&s_9&s_{10}&s_{11}&s_{12} \end{pmatrix}$	0	1		-
		1	8		-
		2	28		-
		3	56		-
		4	69	1	-
		5	52	4	-
		6	22	6	-
		7	4	4	-
		8		1	-

TABLE 10. Moduli space of left-invariant metrics for $\mathfrak{h}_{13}$, $\mathfrak{h}_{19}^{+}$ and $\mathfrak{h}_{26}^{-}$

$\mathfrak{h}$	Σ
$\mathfrak{h}_{13}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s_0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & s_2 & s_3 & 0 & 0 \\ 0 & 0 & 0 & s_4 & s_5 & 0 \\ 0 & 0 & 0 & s_6 & s_7 & s_8 \end{pmatrix}$
$\mathfrak{h}_{19}^{+}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s_0 & 1 & 0 & 0 & 0 & 0 \\ s_1 & s_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 & 0 & 0 \\ s_4 & 0 & 0 & s_5 & s_6 & 0 \\ 0 & 0 & 0 & s_7 & s_8 & s_9 \end{pmatrix}$
$\mathfrak{h}_{26}^{-}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ s_0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & s_2 & s_3 & 0 & 0 \\ 0 & 0 & s_4 & s_5 & s_6 & 0 \\ 0 & 0 & s_7 & s_8 & s_9 & s_{10} \end{pmatrix}$