A Cooperative Based Algorithm to Compute Solutions in the Assignment Game

Resumo: Com base em alguns resultados de jogos TU não balanceados (jogos com utilidades transferíveis) propomos um novo procedimento para obter designações ótimas para o jogo de designação de Shapley and Shubik (1972). O método possui alguns aspectos particulares que permite obter um algoritmo comparável altamente competitivo. O elemento principal para desenvolver este esquema é a forte relação entre alguns ciclos de pré-imputação, os quais aparecem em conexão com jogos não balanceados e os casamentos associados a designações ótimas. Nesta nota relacionamos as soluções de um jogo de designação com alguns tipos de ciclos utilizados previamente para caracterizar jogos TU não balanceados. Esta relação é logo utilizada para desenvolver um método prático para computar soluções de jogos de designação com um novo tratamento

Palavras - chaves: Jogos não balanceados, ciclos.

Abstract: Based on some recent results about non-balanced TU-games (games with transferable utilities) we propose a new procedure to get optimal assignments for the assignment game of Shapley and Shubik (1972). The method exhibits some particular features that could be exploited to obtain a highly parallelizable competitive algorithm. The key fact to develop the scheme is a strong relationship between some cycles of pre-imputation which appear in connection with non-balanced games, and the matching associated to optimal assignments. In this note we relate the solutions of an assignment game with some kind of cycles used previously to characterize non-balanced TU-games. This relationship is then used to develop a practical method to compute solutions of the assignment game with an approach which seems to be new.

Keywords: Non-balanced games, cycles, characterization.
JEL Classification: C71

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1. Introduction

The assignment game (Shapley and Shubik (1972)) is a two sided market where the agents are buyers and sellers and one good is present in indivisible units. Each seller owns a unit of the indivisible good and each buyer needs exactly one unit. Differentiation on the units is allowed and therefore a buyer might place different valuations on the units of different sellers. Besides the original paper, the reader is referred to Roth and Sotomayor (1990) for a general presentation.

Under the assumption that side payments among agents are allowed, and identifying utility with money, Shapley and Shubik (1972) proved that the core of the assignment game (that is to say the set of efficient outcomes that no coalition can improve upon, and which is a game-theoretic concept) is always non-empty and can be identified with the set of stable outcomes, which is a solution set based upon a linear programing formulation of the model. Several of the algorithm designed to compute points in the core of the assignment game have been developed from this latter approach. With a different auction flavor, Demange *et al.* (1986) presented another competitive procedure related to the assignment model embodied in the theory of general equilibrium theory. On the other hand, some very appealing transfer schemes has been introduced to compute point in the core of an n-person game with transferable utility (TU-game), for instance, those appearing in Wu (1977), Sengupta and Sengupta (1996) and Cesco (1998).

The main purpose of this paper is to lay down the fundamentals for developing an alternative procedure for computing optimal assignments by studying how a transfer scheme like that introduced in Cesco (1998) behaves in the framework of assignment games. That algorithm has proven to converge to a core imputation when the core of the game is non-empty (balanced game) and, in the case of non-balanced games, it approximates limit cycles of pre-imputations. Furthermore, each of these limit cycles has always associated a family of balanced coalitions (Cesco (2003)). This observation is the key fact in developing our algorithm. For instance, it allows us to restrict ourselves to run the algorithm by considering only the 1-player coalitions and 2-player coalitions containing one player of each type (buyer or seller), thus reducing the computational effort greatly, and making the proposal competitive. This is not a surprising fact however since the assignment game is a partitioning game (Kaneko and Wooders (1982)) in which the 2-person coalitions containing one player of each type, are the only playing an essential role. A graph theoretic approach to partitioning games is presented in Le Breton *et al.* (1992) and in Boros *et al.* (1995) among others.

In this note we show that, given an assignment game, an appropriate modification of the value of the grand coalition makes the modified game non-balanced, and then, the transfer scheme developed in Cesco (1998) can be applied to get limit cycles of imputations whose associated family of coalitions are a minimal balanced. An iteration procedure is used to reach a minimal balanced family having maximal worth, a key fact since such a family is always associated to an optimal assignment.

The note is organized as follows. Preliminaries about cycles in TU-games and some notation are set forth in the next section. Assignment games and some well-known facts about them are stated in Section 2. In Section 3 we present the algorithm and prove several basic facts. In the last section we elaborate some conclusions and set forth some open issues. We include an Appendix with numerical examples showing different features of the procedure presented in this note.

2. Preliminaries

A TU-game is a pair (N,v) where $N=\{1,2,\dots,n\}$ represents the *set of players* and v, the *characteristic function*. We assume that v is a real valued function defined on the family of subsets of $N, P(N)$ satisfying $v(\Phi)=0$. We will also assume that $v(\{i\})=0$ for each $i \in N$ (0-normalization) although this will represent no restriction at all since the concepts we study in this note are invariant under this type of transformations (invariance under strategic equivalence). The elements in $P(N)$ are called *coalitions*.

The set of *pre-imputations* is $E=\{x=(x_1,...,x_n)\in R^n:\sum_{i\in N}x_i=v(N)\}$ and the set of *imputations* is $A=\{x\in E: x_i\geq 0 \text{ for all } i\in N\}$.

Given a coalition $S\in P(N)$ and a pre-imputation x, the *excess* of the coalition S with respect to x is $e(S,x)=v(S)-x(S)$, where $x(S)=\sum_{i\in S}x_i$ if $S\neq\Phi$ and 0 otherwise. The excess of a coalition S represents the aggregate gain (or loss, if negative) of its members if they depart from an agreement that yields x in order to form their own coalition. The *core* of a game (N,v) is defined by $C=\{x\in E: e(S,x)\leq 0 \text{ for all } S\in P(N)\}$.

The core of a game may be an empty set. The Shapley-Bondareva theorem (Bondareva (1963), Shapley (1967)) characterizes the subclass of TU-games with non-empty core. A central role is played by balanced families of coalitions. A family $B\subseteq P(N)$ of non-empty coalitions is *balanced* if there exists a set of positive real numbers $(\lambda_S)_{S\in B}$, a set of *balancing weights*, satisfying $\sum_{S\in B, i\in S}\lambda_S=1$, for all $i\in N$. B is minimal balanced if there is no proper balanced subfamily of it. In this case, the set of balancing weights is unique. A well-known result establishes that

$$\sum_{S\in B}\lambda_S.x(S)=x(N), \qquad (2.1)$$

for any balanced family of coalitions B. Given a coalition S, $\chi_S\in R^n$ denotes its characteristic vector which is defined by $(\chi_S)_i=1$ if $i\in S$ and 0 if $i\in N\setminus S$. If B is a minimal balanced family of coalitions, the set of characteristic vectors ($\chi_S)_{S\in B}$ is linearly independent.

A game (N,v) is called balanced if

$$\sum_{SB}\lambda_S.v(S)\leq v(N), \qquad (2.2)$$

for all balanced family B with balancing weights $(\lambda_S)_{S\in B}$. Shapley- Bondareva's theorem states that a game (N,v) has non-empty core if and only if it is balanced.

An *objectionable* family is a balanced family not satisfying (2.2). For a clear exposition of the concepts and results about cooperative TU-games used in this note, the reader is referred to the recent book of Peleg and Sudhölter (2003).

In what follows, the notion of U-transfer will play a central role. Given $x\in E$ and a proper coalition S, we say that y results from x by the *U-transfer* from $N\setminus S$ to S (shortly, y is a U-transfer from x) if

$$y=x+e(S,x).\beta_S, \qquad (2.3)$$

with $e(S,x)>0$. Here $\beta_S=\frac{\chi_S}{|S|}-\frac{\chi_{\setminus S}}{|N\setminus S|}$ if S is a proper coalition and the zero vector of R^n otherwise. As usual, $|S|$ stands for the number of players in S. The vector β_S describes a uniform transfer of one unit of utility from the members of $N\setminus S$ to the members of S The U-transfer is *maximal* if $e(S,x)\geq e(T,x)$ for all $T\in P(N)$.

We now introduce some kinds of cycles of pre-imputations and state, without proof, several results proved in Cesco and Aguirre (2002) and Cesco (2003).

A *cycle* **c** in a TU-game (N,v) is a finite sequence of pre-imputations $(x^k)_{k=1}^m, m>1$, such that there exist associated sequences of positive real numbers $(\mu_k)_{k=1}^m$ and $(S_k)_{k=1}^m$ of non-empty, proper coalitions of N (not necessarily all different) satisfying the neighboring transfer properties

$$x^{k+1}=x^k+\mu_k.\beta_{S_k} \text{ for all } k=1,...,m, \qquad (2.4)$$

and

$$x^{m+1}=x^1 \qquad (2.5)$$

as well.

A cycle is *fundamental* if $\mu_k\leq e(S_k,x^k)$ for all $k=1,...,m.$

A cycle is a *U-cycle* if $\mu_k=e(S_k,x^k)$ for all $k=1,...,m.$

A U-cycle $(x^k)_{k=1}^m$ is *maximal* if for all $k=1,...,m, x^{k+1}$ is a maximal U-transfer from x^k, namely, if $e(S_k,x)\geq e(S,x)$ for all $S\in P(N)$.

Given a cycle $\mathbf{c}=(x^k)_{k=1}^m$, we denote the vector of coalitions $(S_k)_{k=1}^m$ by supp(**c**). {Let $B(\mathbf{c})=\{S:S=S_k$ for some entry of supp(**c**)}. Sometimes we will refer to supp(**c**) as the family of coalitions supporting **c.** Besides, we refer to the entries of the vector $(\mu_k)_{k=1}^m$ as the *transfer amounts*.

In Cesco (2003, Theorem 1) we proved that $B(\mathbf{c})$ the family supporting a cycle of pre-imputations **c,** is a balanced family of coalitions for every fundamental cycle **c**. We also showed there that the existence of a fundamental cycle implies the non-balancedness of the game and that every non-balanced TU-game has a fundamental cycle (Cesco (2003, Theorem 3 and Theorem 9)). These two results together provide a characterization for non-balanced TU-games (i.e., games with empty core) in terms of fundamental cycles. Later, in Cesco (2006, Theorem 6) that result was improved in the sense that a similar characterization was obtained in terms of the more restricted class of U-cycles. Indeed, it holds the following

Proposition 1 Let (N,v) be a TU-game and $B=\{S_1,S_2,...,S_k\}$ a minimal objectionable family of coalitions. Then there exists a U-cycle of pre-imputations $(x^k)_{k=1}^m$ such that

$$x^{k+1}=x^k+e(S_k,x^k).\beta_{S_k}, \quad (2.6)$$

for all $k=1,...,m$.

Remark 1 Proposition 1 implies that, with every ordering of the coalitions in U, there exists a U-cycle whose pre-imputations share the same cyclic ordering as the coalitions in B. These cycles are not, in general, maximal U-cycles (see Cesco (2006, Appendix)).

Related to the U-transfers given by (2.3) there exists a transfer scheme originally designed to converge to a point in the core of a balanced TU-game. A maximal U-*transfer schemes* (Cesco (1998)) is a sequence $(x^j)_{j=1}^{\infty}$ of pre-imputations such that, for all $j=1,2,...$, x^{j+1} is a maximal U-transfer from x^j. A maximal transfer scheme converges if and only if the core of the game is non-empty (Cesco 1998, Proposition 3.1). An algorithm for computing maximal transfer schemes has been implemented and, when applied to non-balanced games, the sequences of pre-imputations generated always have had a maximal U-cycle as a limit cycle. However, there is no general proof of this fact.

U-transfer schemes are strongly connected to a class of transfer schemes studied in Wu (1977). It also has close similarities with the scheme developed in Sengupta and Sengupta (1996). However, while the latter reaches a point in the core of a balanced game in a finite number of steps (see also Koczy (2004)), Wu's procedures and ours usually generate an infinite sequence of point converging to an imputation in the core. The fact that Sengupta's scheme always generates imputations and ours only pre-imputations establishes another important difference.

3. The assignment game

Assignment games were first introduced in Shapley and Shubik (1972), and since then, they have been extensively studied. The book of Roth and Sotomayor (1990) contains a very good presentation of the main results related to this model. Here we give a very concise exposition of the facts which we are going to use later.

An assignment game is an $(m+n)$-person TU-game where the set of players is divided into two disjoint sets P and Q with cardinality m and n respectively. We will consider $P=\{1,...,m\}$ and $Q=\{m+1,...,m+n\}$ It is assumed that, with each possible partnership $(i,j)\in P\times Q$ there is associated a non-negative number $\alpha_{i,j}$. Then, the characteristic function v for the game is given by

$$v(S)=\alpha_{i,j} \text{ if } S=\{i,j\} \text{ for } i\in P \text{ and } j\in Q \quad (3.1)$$

$$v(S)=0 \text{ if } S\subseteq P \text{ or } S\subseteq Q. \quad (3.2)$$

For any other non-empty coalition S

$$v(S)=\max(\alpha_{i_1,j_1}+...+\alpha_{i_k,j_k}) \quad (3.3)$$

with the maximum to be taken over all sets $\{(i_1,j_1),...,(i_k,j_k)\}$ of k distinct pairs in $(S\cap P)\times(S\cap Q)$ containing $2.k$ distinct players. It is clear that $k\le\min(|S\cap P|,|S\cap Q|)$. We will denote an assignment game by (P,Q,α), where α is an $m\times n$ matrix whose entries are $\alpha_{i,j}, i\in P, j\in Q$, in the understanding that the characteristic function is constructed according to the rules stated in (3.1), (3.2) and (3.3).

Related to an assignment game (P,Q,α), there is the linear program **P**

$$\max \sum_{i\in P, j\in Q}\alpha_{i,j}.x_{i,j}$$

subject to

$$a)\sum_{i\in P}x_{i,j}\le 1 \text{ for all } j\in Q$$

$$b)\sum_{j\in Q}x_{i,j}\le 1 \text{ for all } i\in P$$

$$c)x_{i,j}\ge 0 \text{ for all } i\in P, j\in Q.$$

In the sequel we will restrict ourselves to the case $|P|=|Q|=n$.

Remark 2 It is well-known that this linear program has extremal solutions involving only 0-1 entries (see Dantzig (1963)). It is well-known too that any extreme solution x of $a),b)$ and $c)$ can be identified with a matching between the players in P and those in Q. Sometimes, some players can remain alone. We will denote by $B(x)$ the family of 1 and 2-person coalitions representing that matching. Conversely, given a partition B of $P\cup Q$ included in K, x_B will stand for the corresponding solution of $a),b)$ and $c)$ given by: $(x_B)_{i,j}=1$ if $\{i,j\}\in B$ and $(x_B)_{i,j}=0$ if $\{i,j\}\notin B$.

A *feasible assignment* for (P,Q,α), is an integer matrix $x=(x_{i,j})_{i\in P,j\in Q}$ satisfying $a),b)$

and $c)$ above, while any solution of $\mathbf{P}$ is an *optimal assignment*.

The dual linear program of $\mathbf{P}$ is $\mathbf{P}^*$ defined as
$$\min \sum_{i\in P} u_i + \sum_{j\in Q} w_j$$
subject, for all $i\in P, j\in Q$, to

$$d) u_i + w_j \geq \alpha_{i,j}$$

$$e) u_i \geq 0, w_j \geq 0.$$

Since the primal program $\mathbf{P}$ has a solution, so does $\mathbf{P}^*$ Moreover, the fundamental theorem of linear programming asserts that, if x is an optimal assignment and (u,w) is a solution of $\mathbf{P}^*$, then $\sum_{i\in P} u_i + \sum_{j\in Q} w_j = \sum_{i\in P, j\in Q} \alpha_{i,j}.x_{i,j} = v(P\cup Q)$.

The pair of vectors $(u,w)\in R^m \times R^n$ is a *feasible payoff* for the game (P,Q,α), if there is a feasible assignment x such that $\sum_{i\in P} u_i + \sum_{j\in Q} w_j = \sum_{i\in P, j\in Q} \alpha_{i,j}.x_{i,j}$. In this case we say that (u,w) and x are *compatible*, and we call $((u,w);x)$ a *feasible outcome*. A feasible payoff is *stable* if it satisfies $d)$ and $e)$ above. It is easy to see that the set of stable payoffs is a subset of the set A of imputations for (P,Q,α), On the other side, the core of the assignment game is

$$C = \{(u,v)\in E : \sum_{i\in S\cap P} u_i + \sum_{j\in S\cap Q} w_j \geq v(S) \text{ for all } S\subseteq P\cup Q\}.$$

The following result relating stable payoffs and points in the core of an assignment game is due to Shapley and Shubik (1972, Theorem 2).

Proposition 2 Let (P,Q,α), be an assignment game. Then

$i)$ the set of stable payoffs and the core of the game are the same.

$ii)$ The core of (P,Q,α), is the (nonempty) set of solutions of $\mathbf{P}^*$.

This result implies that (P,Q,α) is a balanced game.

4. A new computational approach

The core concept of solution has been studied in the framework of restricted games in many directions. One suitable for the purposes of this note is that presented in Boros *et al.* (1997) which we present briefly here. Let (N,v) be a TU-game and K be a nonempty family of coalitions (the set of *essential* coalitions). For further references, any subfamily $B\subseteq K$ will be called, a K -family of coalitions. The -core of (N,v) is defined by

$$C(v,K) = \{x\in R^n : e(N,x)\geq 0 \text{ and } e(S,x)\leq 0 \text{ for all } S\in K\}. \quad (4.1)$$

A characterization theorem similar to Shapley-Bondareva's result also holds for the $C(v,K)$, and it is due to Gurvich and Vasin (1977).

Proposition 3 The core $C(v,K)$, is nonempty if and only if for any (minimal) balanced K-family of coalitions B, the inequality (2.2) holds for any set of balancing weights $(\lambda_S)_{S\in B}$ for B

In the case of an assignment game (P,Q,α), let us consider the family

$$K = \{S\subseteq P\cup Q : |S\cap P|, |S\cap Q| \in \{0,1\} \text{ or } S = (P\cup Q)\}. \quad (4.2)$$

K consists of the unitary coalitions, the 2-player coalitions with one player of the type P and one of the type Q, and the grand coalition $P\cup Q$.

The following result states a strong relationship between the set of stable payoffs and the core $C(v,K)$, whose proof is in fact, included in the proof of Proposition 2.

Theorem 4 Let (P,Q,α) be an assignment game. Then, the set of stable payoffs coincides with $C(v,K)$

Proof Let (u,w) be a stable payoff. As we mentioned at the end of Section 3, (u,w) should be an imputation and thus

$$\sum_{i\in P} u_i + \sum_{j\in Q} w_j = v(P\cup Q), \quad (4.3)$$

and

$$u_i \geq 0, w_j \geq 0 \quad (4.4)$$

for all $i\in P, j\in Q$. Moreover, from the stability conditions we get that

$$u_i + w_j \geq \alpha_{i,j} \text{ and} \quad (4.5)$$

for all $i\in P, j\in Q$. Relationships (4.3), (4.4) and (4.5) together imply that $e(S,(u,w))\leq 0$ for all $S\in K$ with $e(P\cup Q,(u,w)) = 0$ as well. Therefore, $(u,w)\in C(v,K)$.

To see the converse, if $(u,v)\in C(v,K)$, and since $P\cup Q\in K$, then (4.3) holds. Thus, $((u, v); x)$ is a feasible outcome for any optimal solution x of P and consequently, (u, w) is a feasible payoff. On the other side, the inequalitie $e(S,(u,w))\leq 0$ for all $S\in K$ imply the stability conditions (4.4) and (4.5).

We would like to stress on the fact that, whenever K is given by (4.2), $C(v,K) \subseteq A \subseteq E$.

We now define a K-maximal U-transfer scheme inductively by: $x^0 \in E$ and, given $x^0 \in E$,

$$x^{k+1} = x^k + e(S_k, x^k).\beta_{S_k},$$

where S_k is chosen arbitrarily from the set $\psi(x^k) = \{S \in K : e(S,x^k) \geq e(S,x^k) \text{ for all } S \in K\}$. Then, the following convergence result has a proof similar to that of Cesco (1998, Proposition 3.1 and Theorem 3.5) .

Theorem 5 Let (P,Q,α) and K the family of coalitions defined by (4.2). Then, a K-maximal U-transfer scheme converges to a point in $C(v,K)$ if and only if $C(v,K)$ is nonempty.

Remark 3 If $C(v,K)$ is empty, then the algorithm does not converge at all. However, it has been observed numerically, that it always approximates to a limit cycle **c** with $B(\mathbf{c})$ being a minimal balanced K-subfamily of coalitions. The K-algorithm not only provides **c** and $B(\mathbf{c})$ but also the set of balancing weights $\Lambda = (\lambda_S)_{S\in B(\mathbf{c})}$ for $B(\mathbf{c})$.

Given a balanced family of coalitions B and a set of balancing weights $\Lambda = (\lambda_S)_{S\in B}$, the worth of B with respect to Δ is $w(B,\Lambda) = \sum_{S\in B} \lambda_S.v(S)$. If B is minimal balanced, and thus having a unique set of balanced weights, $w(B,\Lambda)$ will be denoted $w(B)$ only.

Theorem 5 is the key to develop a theoretic iterative algorithm to compute a point in $C(v,K)$ and therefore, a dual solution for $\mathbf{P}^*$. However, this approach has the drawback of requiring the value $v(P \cup Q)$, which should be obtained by solving an optimization problem of the form (3.3) providing a dual solution while we need a primal one. To overcome these difficulties, we will modify slightly the approach taking into account the observation stated in Remark 3 and we will present our proposal in the form of a pseudo-code. For simplicity, we will assume that the diagonal of α has at least one non-zero entry. Given an assignment game (P, Q, α) and a real number V, let us call the V-modified assignment a game having the same characteristic function as (P, Q, α), but with V as the worth of the grand coalition $P \cup Q$.

Step 1 Set $V = \sum_{i=1}^{n} \alpha_{i,n+i}$.

Step 2 Run the K-algorithm for the modified V-modified assignment game. If convergence is achieved, $x_{i,j} = 1$ if $j = n+i$ for all $i = 1,\ldots,n$ and $x_{i,j} = 0$ otherwise, is an optimal solution for P. If not, let $\mathbf{c} = (x^k)_{k=1}^m$ be a limit cycle with supp $(\mathbf{c}) = (S_k)_{k=1}^m$, and $\Lambda = (\lambda_S)_{S\in B(\mathbf{c})}$ be the set of balancing weights.

Step 3 Compute $w(B(\mathbf{c})) = \sum_{S\in B(\mathbf{c})} \lambda_S.v(S)$.

Step 4 Set $V = w(B(\mathbf{c}))$.

Step 5 Run the K-algorithm for the V-modified assignment game. If the algorithm reaches a point in the core, $B(\mathbf{c})$ is a matching such that $x_{B(\mathbf{c})}$ is an optimal solution for P and $w(B(\mathbf{c}))$ is the value of the original assignment game (P,Q,α) Otherwise, the algorithm provide a new limit cycle $\mathbf{c} = (x^k)_{k=1}^m$ with a new supporting family of coalitions and a new set of balancing weights $\Lambda = (\lambda_S)_{S\in B(\mathbf{c})}$. Then, go to *Step 3*.

This algorithm ends in a finite number of iterations because the worth of the families generated in *Step 5* (provided they are minimal balanced) form an increasing sequence, and there is only a finite number of these families. The stopping rule in *Steps 5* (and in *Step 2* too) is based on Theorem 7 proved below, and the fact that, if convergence is achieved, the V-modified assignment auxiliary game is balanced with $V = w(B(\mathbf{c}))$ ($V = \sum_{i=1}^{n} \alpha_{i,n+i}$) which implies that the supporting family of **c** $(\{\{1,n+1\},\ldots,\{n,n+n\}\})$ is balanced with maximal worth.

Finding minimal balanced families of coalitions is a key point in the framework of assignment games, mainly because of the following results.

Theorem 6 Let be an assignment game (P,Q,α) Then there is a one-to-one correspondence between the minimal balanced K-subfamilies of coalitions different from $\{P \cup Q\}$ and the extreme assignments of $a), b)$ and $c)$.

Proof In their seminal paper, Shapley and Shubik (1972) showed that any assignment game (P,Q,α) has an optimal assignment and, furthermore, that its core is non-empty. This result along with Theorem 2.7 in Kaneko Wooders (1982) allows us to claim that the only minimal balanced K-families of coalitions are the K-partitions of $P \cup Q$. Now, since the family $B(x)$ associated with an extreme solution of the set of restriction $a), b)$ and $c)$.for P defines an assignment, $B(x)$ is K-partition of K, which is always a minimal balanced K-family of coalitions. On the other hand, given a minimal balanced K-family of coalitions B, different from $P \cup Q$ which is a K-partition of $P \cup Q$ the vector x_B associated to B is clearly an extreme point of the set of restrictions $a), b)$ and $c)$.

Remark 4 The aforementioned result of Shapley and Shubik was proven by using a well-known Birkoff's result about the extremal points of the set of doubly stochastic matrices (see Remark 3). A graph theoretic proof of the claim that the only minimal balanced K-subfamilies of K are the partitions is given in Le Breton *et al.* (1992). A direct proof of that claim is included in Cesco (2007) and will be presented elsewhere.

Theorem 7 Let B be a minimal balanced K -subfamily of coalitions different from $\{P \cup Q\}$ having maximal worth. Then, x_B is a solution of the linear program P. Conversely, if x is an extreme solution of P then $B(x)$ is a minimal balanced subfamily different from $\{P \cup Q\}$ having maximal worth.

Proof Let $B' \neq \{P \cup Q$ be a minimal subfamily of K having maximal worth, and x_B its associated feasible solution of P.We first note that the worth of B,

$$w(B) = \sum_{\{i,j\} \in B} \alpha_{i,j}$$
$$= \sum_{\{i,j\} \in B} \alpha_{i,j}.(x_B)_{i,j}.$$

If B has maximal worth, it follows that $w(B) \geq w(B') = \sum_{\{i,j\} \in B'} \alpha_{i,j}.(x_{B'})_{i,j}$ for any other minimal balanced subfamily $B' \neq \{P \cup Q\}$ of K But this implies that x_B, which is an extreme point of the polyhedron $a), b)$ and $c)$ of restrictions for P,has a maximal objective value among all the other extreme points. So, it is an optimal solution for P. Reversing the former arguments, we prove the converse.

5. Conclusions

We close this note with several comments. First we point out that we have not intended to write a completely formal mathematical paper but, instead, to put forward several facts and some experience gathered from numerical experiments in order to propose a potentially good procedure to compute optimal solutions in the assignment game, from a point of view which seem to be new. However, a lot of work has to be done yet to get a really competitive algorithm. In *Step 5* (as well as in *Step 2*) there is no formal proof, in general, that the support of the limit cycle is a minimal balanced family. Certainly it is a balanced family of coalitions and by solving a reduced linear program it is possible to extract a minimal balanced subfamily. As we stated in Remark 3, we have always found out that the support of the limit cycle is a minimal balanced family and, in the some games having some strong symmetric characteristics, we have been able to prove this fact formally. We conjecture that this is always true, but until this result is gotten, a subroutine to detect if the support of the limit cycle is minimal balanced should be included to find out if the routine to extract a minimal balanced subfamily has to be run or not. The main theoretic weakness of the algorithm described in Section 4 is that there is no general proof about the ending or 'convergence' of the K-algorithm to a limit cycle. As it, this is still an open question. However we mention two facts that encourage our research. It has been proved that the algorithm presented in Cesco (1998) and, in particular, the K-algorithm, always generates bounded sequences of pre-imputations (see for instance, Cesco and Calí (2003)) which support the numerical approximation to the limit cycles observed in all the examples we have tried out. This result allows us to get ε -maximal U-cycles in the sense that for every $\varepsilon > 0$, we can find a sequence of maximal U -transfers $(x^k)_{k=1}^{m+1}, m = m(\varepsilon) > 1,$ such that $\|x^{m+1} - x^1\|_2 \leq \varepsilon$. For many of the practical needs, this ε-cycles seem to be appropriate. Result regarding these cycles will be presented elsewhere. The second fact is that we have proved approximation results of the K-algorithm to maximal limit cycles (Cesco and Calí 2004) although for a different class K of essential coalitions than that related to the assignment games. We expect to extend the proof to different or more general cases, including the one concerning us here.

Another point worth to be mentioned is that, if convergence is not achieved in *Step 2,* the supporting family of the limit cycle obtained instead, not necessarily has to have maximal worth. However, if this were the case, there would be no need to perform the iterations between *Steps 3* and *5*. To get cycles with this desirable property, the initial value $v(P \cup Q)$ (which we arbitrarily set as $\sum_{i=1}^{n} \alpha_{i,n+i}$) should be large enough.

Finally, we would like to point out that, in each iteration of the K-algorithm performed in *Step 5,* the main computational effort is done in making the, at most, $n^2 + n$ comparisons between excesses; this number usually is reduced to, at most, $2.n$ near the end of the iteration. On the other hand, this part of the algorithm is highly parallelizable. These two facts together could be the key facts to obtain very efficient implementations for this procedure.

6. Appendix

In this Appendix we will use the assignment game (P, Q, α) with $P = \{1,2,3\}, Q = \{4,5,6\}$ and $\alpha = \begin{pmatrix} 5 & 8 & 2 \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix}$ from Shapley and Shubik (1972) to illustrate some characteristics of the procedure developed here.

Example 1: This first example shows that if the initial value V in *Step 1* of the algorithm of Section 4 is not set large enough, it may be necessary to run the iterations between *Step 3* and *Step 5*. Let us start with $V = 5 = \alpha_{1,4}$ instead of the suggested initial value $\sum_{i=1}^{n} \alpha_{i,n+i}$ which could be, anyway, not large enough.

The auxiliary K-algorithm used in *Step 2* also requires an initial pre-imputation to start. For some starting points, like

$$x^0 = \begin{pmatrix} 0.4300 \\ 0.6276 \\ 0.6152 \\ 1.8691 \\ 0.8426 \\ 0.6155 \end{pmatrix},$$

for instance, which was a random selection, the limit cycle $\mathbf{c}$ obtained as a result has $B(\mathbf{c}) = \{(1,5),(2,6),(3,4)\}$, which defines an optimal assignment with $w(B(\mathbf{c})) = 16$. But, if

$$x^0 = \begin{pmatrix} 1.3632 \\ 0.8460 \\ 0.3264 \\ 1.0827 \\ 1.3501 \\ 0.0316 \end{pmatrix},$$

which is another random selection for the starting point of the K-algorithm, the limit cycle $\mathbf{c}$ obtained has $B(\mathbf{c}) = \{(1,4),(2,5),(3,6)\}$ with $w(B(\mathbf{c})) = 14$, and thus, it does not define an optimal assignment. Now, modifying the value V to 14 as indicated in *Step 4* and running the K-algorithm again, we always obtain limit cycles $\mathbf{c}$ with $B(\mathbf{c}) = \{(1,5),(2,6),(3,4)\}$ defining the unique optimal assignment for the game.

Example 2: Here we illustrate the algorithm whenever it converges to a core point of the assignment game just in *Step 2*. This happens when the initial value V coincides with the value of the grand coalition $v(P \cup K)$ in the assignment game (P, Q, α). Thus, let us put $V = 16$ in *Step 1*. Starting the K-algorithm in *Step 2* from the initial pre-imputation

$$x^0 = \begin{pmatrix} 16/6 \\ 16/6 \\ 16/6 \\ 16/6 \\ 16/6 \\ 16/6 \end{pmatrix},$$

it converges to the imputation

$$x = \begin{pmatrix} 3.0000 \\ 5.0021 \\ 0.0000 \\ 2.0000 \\ 5.0000 \\ 0.9979 \end{pmatrix}$$

which is a core point. This core point defines an optimal solution for the linear program $\mathbf{P}^*$ and in this case it is easy to find out the optimal assignment that it defines since, given an imputation x in the core of the assignment game (P, Q, α), each coalition S defining the optimal assignment related to x must have $e(S, x) = 0$. Here, the only coalitions with zero excess are $(1,5),(2,6)$ and $(3,4)$. But, in general, given a core point for the assignment game, other coalitions aside from those in an optimal assignment may have zero excess with respect to that imputation as well. This is the case if, in the example we are working here, the starting point for the K-algorithm is, for instance

$$x^0 = \begin{pmatrix} -16/3 \\ 16/3 \\ 16 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then, the core point reached is

$$x = \begin{pmatrix} 5.0000 \\ 6.0000 \\ 1.0000 \\ 1.0000 \\ 3.0000 \\ 0.000 \end{pmatrix},$$

and besides $(1,5),(2,6)$ and $(3,4)$, coalitions $(2,4)$ and $(2,5)$ also have zero excess with respect to x.Thus, an additional procedure has to be designed to construct the assignment related to a core imputation. However, this is avoided if the starting value $V = \sum_{i=1}^{n} \alpha_{i,n+i})$ is chosen as indicated in *Step 1* whenever it is possible.This selection, as well as any other of the form $V = \sum_{i=1}^{n} \alpha_{i,n+\pi(i)}$, π being a permutation of the set $\{1,\ldots,n\}$, guarantees that $V \le v(P \cup Q)$, and in the case that the equality holds, it also guarantees that the K-algorithm converges to a core point in *Step* 2 and that the family of coalitions $\{(1,n+1),\ldots,(n,n+n)\}$ $(\{(1,n+\pi(1)),\ldots,(n,n+\pi(n))\})$ has maximal worth and that it defines an optimal assignment, although it may be not the only one.

We close by showing some statistics regarding the claim made at the end of Section 5. We run a very preliminary version of the K-algorithm on 30 problems with $|P| = |Q| = 3$ with data randomly generated. We recorded the number of iterations for the K-algorithm up to convergence is achieved, with an approximation error of $10-6$, as well as the type of limit found (cycle or core point). We also recorded the iteration number from which a set of coalitions start to appear in a cyclic order up to convergence to a limit cycle or, in the case that the convergence is to a core point, when a reduced group of coalitions repeat itself up to convergence. The next table shows the averages for these quantities. Maximum and minimum values are also showed in parenthesis.

Conv.to a		No. Iter. (Max/ Min)	Cyclic repetition (Max/ Min)
Cycle	24	31.33 (34/18)	3.75 (11/1)
Core point	6	111.83 (337/18)	11 (45/1)

Although this is a very small numerical experience, some conclusions can be gathered from it. Convergence to a limit cycle is much faster than to a core point, and the number of iterations is consistently almost the same. We believe that a rate of convergence could be derived, depending only on the number of players, from a geometric point of view, in a similar way than it was done for the general case of three person games in Cesco and Cali (2008). It is also interesting to note that the cyclic behavior of the coalitions in the limit cycle appears at the very early stages of the K-algorithm. Thus, efficient techniques to detect cycles of coalitions as well as to approximate the worth of the families that they determine could improve substantially the performance of this auxiliary step of the method. In the case that the convergence is to a core point, we note that, since the K-algorithm is an alternanting projection method, techniques to accelerate the convergence could be employed to take advantage of the reduced number of coalitions that appears in the tail of the sequences of coalitions related generated by each run of the K-algorithm.

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