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On the entropy for unstable fermionic and bosonic states

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HIGHLIGHTS

- We address the problem of the entropy for complex-energy states.
- The formalism does not require the use of a complex temperature.
- The entropy is calculated starting from a proper definition of the density operator.
- We use complex coherent states and path integrals to compute the density operator.
- The entropy, subject to an interpretation, is bigger for unstable quantum states than for real energy states.

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ABSTRACT

We focus on the calculation of the entropy for decaying states in non-relativistic quantum mechanics. The starting point is the Friedrichs model in second quantization language. In this model, the Hamiltonian admits a spectral representation which includes resonances and Gamow states explicitly. In order to avoid the limitations posed by the definition of canonical probabilities in the presence of a complex spectrum, and/or the use of complex temperatures, we construct the partition function performing a path integration over coherent states. It is shown that the path integration yields results which are correct, at leading order, within the framework of the thermal perturbation theory. Finally, we obtain an expression for the canonical entropy of a quantum decaying system composed of fermion- and boson-states.

1. Introduction

The work of Gamow [1] opened the avenue leading to the quantum mechanical treatment of decaying states, both the mathematics and the physical aspects of resonances have called the attention of a generation of physicists and mathematicians. A concise introduction to the problem may be found in Refs. [2–5]. More recently [6], the subject has been presented in a comprehensive way, to accommodate for applications of the concept to nuclear structure and nuclear reaction problems. So far, the quantum mechanical principles underlying the physics and mathematics of resonances have been consistently formulated in the context of analytic extensions [3] of the standard hermitian eigenvalue problem. A more compact formulation, which is based on Gamow densities, can be found in Ref. [7]. Considerably less effort was devoted to the understanding of the aspects related to the formulation of the problem in the context of the quantum statistical mechanics. A decaying system should also obey thermodynamical laws, in a broad sense not limited by the notion of probabilities, exactly as it happens in the just referred quantum mechanical frame for resonances [3,6]. From the physical point of view

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the problem has strong links with the measurement of decay properties of resonances and with the role of interactions between resonances and the environment. In this paper we address a subject that has been scarcely treated in the literature, that is the calculation of the canonical entropy for non relativistic quantum decaying systems. Here, we extend the scope of our previous work [8] and treat both bosons and fermions.

Kobayashi and Shimbori [9], and Kobayashi [10], have elaborated on the notion of entropy for complex-energy systems. In these papers the real and imaginary parts of the energy of a resonance are treated, as a matter of fact, as independent systems, and the canonical partition function for the resonance is given as a product of canonical partition functions for the real and imaginary parts of the energy, so that the total entropy is the sum of both contributions. Then, in this picture, decaying processes transfer entropy from the imaginary part to the real part and the rate of this transference depends on time. Each part has its own temperature, which suggests a notion of a complex temperature [9,10].

Another point of view has been advocated by the Brussels group [11]. The authors have introduced the notion of entropy operator for Gamow states, in the spirit of the entropy operator defined by Misra, Prigogine and Courbage [12]. This operator belongs to a family of operators defined on an extended space.

In our approach, we have obtained an expression for the entropy of decaying states without the introduction of the entropy operator of Ref. [12], and avoiding the use of complex temperatures. As in Ref. [9], we assume thermodynamical equilibrium for decaying systems, which is an acceptable approximation if the half life of the system is sufficiently large, or equivalently if its width is sufficiently small. As in Ref. [11], we use the Friedrichs model [13] as a laboratory, which has the advantage of being a solvable model for resonance phenomena. However, the Friedrichs model does not solve the problems concerning the definition of mean values and scalar products of Gamow states (vector states for resonances) [14]. Instead of using dyadic products of Gamow vectors [11], we have chosen a quite different strategy. In fact, we have formulated the Friedrichs model in second quantization language to construct coherent states. This tool and the use of path integral methods will allow us to obtain an approximate version of the canonical partition function of a system with resonances, which naturally leads to an approximate expression for the canonical entropy of decaying states. The resulting entropy is complex, a fact that requires an interpretation [8].

This paper is organized as follows: in Section 2, we give a second quantized version of the Friedrichs model. Section 3 is devoted to the path integral formalism over coherent states. In Section 4, we use the same technique to calculate the entropy of a resonant state, and advance an interpretation of it, together with the comparison of results for a simple case. The explicit calculations, corresponding to this section, are presented in Appendix A. In Sections 5 and 6 we extend our method to obtain the entropy of unstable canonical states belonging to a more general situation, that is by treating fermions and bosons, common in realistic nuclear systems. Conclusions are drawn in Section 7. Further details of the calculations are given in Appendix B.

2. The Friedrichs model in second quantification language

The Friedrichs model [13], is an exactly solvable model for decaying phenomena in quantum mechanics, which shows all the features of resonant scattering [15]. For this reason, the Friedrichs model represents a quite general description of resonances in non-relativistic quantum physics, with applications to atomic, nuclear and particle physics as well. The unperturbed sector of the model is H_0. Its spectrum has a simple absolutely continuous sector, given by the set of all non-negative numbers $\mathbb{R}^+ \equiv [0, \infty)$, and a positive eigenvalue embedded in the continuous spectrum. Typically, this would be the case of an isolated atomic state and an external field without interaction. To this Hamiltonian it is added an interaction, represented by the potential V, that intertwines the continuous and discrete sectors of the spectrum of H_0. The interaction is written as a function of an energy dependent form factor, $f(\omega)$, and weighted by a real parameter λ. The total Hamiltonian is $H = H_0 + \lambda V$.

Here, we are going to introduce the Friedrichs model in an operator language like the one used in second quantization. This presentation was discussed in Refs. [16,17]. It assumes the existence of a vacuum state $|0\rangle$ from where the bound state as well as pure states with energy ω in the continuum are created by applying the corresponding creation operators. In this context, the explicit form of H_0 is written as

$$H_0 = \omega_0\, a^\dagger a + \int_0^\infty \mathrm{d}\omega\, \omega\, b_\omega^\dagger b_\omega, \tag{1}$$

where $a^\dagger$ (a) is the creation (annihilation) operator of a bound state of energy ω_0, and $b_\omega^\dagger$ (b_ω) is the creation (annihilation) operator for the state of energy ω. The potential V is given by

$$V = \int_0^\infty \mathrm{d}\omega\, f(\omega)\; (a^\dagger b_\omega + a b_\omega^\dagger). \tag{2}$$

Creation and annihilation operators fulfill the following commutation relations:

$$[a, a^\dagger] = 1; \qquad [b_\omega, b_{\omega'}^\dagger] = \delta(\omega - \omega') \tag{3}$$

and all other commutators vanish. For the states we use the notation

$$|1\rangle = a^\dagger|0\rangle; \qquad |\omega\rangle = b_\omega^\dagger|0\rangle. \tag{4}$$

Note that

$$a|0\rangle = b_\omega|0\rangle = 0. \tag{5}$$

As a consequence of the interaction, the bound state of H_0 vanishes, and it is replaced by a resonance that, under some conditions on the form factor $f(\omega)$, depends analytically on the coupling constant λ. This resonance is a pole of the analytic continuation through the real axis of the function

$$g(z) = \langle 1|\frac{1}{z-H}|1\rangle. \tag{6}$$

The function $g(z)$ is often called the reduced resolvent of H at the complex number z, not in the spectrum of H. This function is analytic with no poles on (an open set of) the complex plane with a branch cut on the positive semiaxis $\mathbb{R}^+ = [0,\infty)$, admitting an analytic continuation through this cut. It is useful to use the explicit form of its inverse $\eta(z)$ ($g(z) = 1/\eta(z)$) given by Refs. [15,16,18]

$$\eta(z) = \omega_0 - z - \int_0^\infty d\omega\, \frac{\lambda^2 f^2(\omega)}{\omega - z}. \tag{7}$$

The function representing the boundary values of $\eta(z)$ on the positive semi-axis from above to below $\eta(\omega + i0)$, and from below to above $\eta(\omega - i0)$, are here denoted as $\eta^+(\omega)$ and $\eta^-(\omega)$ respectively. They are complex conjugate of each other. The analytic continuation of $\eta(\omega + i0)$ has a zero located at $z_R = E_R - i\Gamma/2$, the resonance pole of $g(z)$. The real part E_R of z_R is identified with the resonance energy and the imaginary part $\Gamma/2$ with the half width. Its inverse $\tau = 2\hbar/\Gamma$ is the half life [2]. The explicit form of z_R is not interesting for our purposes and it can be found in Refs. [15,18].

In Ref. [16], the following creation and annihilation operators are defined:

$$A_{\rm IN}^\dagger := \int_\gamma d\omega\, \frac{\lambda f(\omega)}{\omega - z_R}\, b_\omega^\dagger - a^\dagger, \tag{8}$$

$$A_{\rm OUT} := \int_\gamma d\omega\, \frac{\lambda f(\omega)}{\omega - z_R}\, b_\omega - a, \tag{9}$$

$$B_{\omega,\rm IN}^\dagger := b_\omega^\dagger + \frac{\lambda f(\omega)}{\tilde\eta^+(\omega)} \left\{ \int_0^\infty d\omega'\, \frac{\lambda f(\omega')}{\omega' - \omega - i0}\, b_{\omega'}^\dagger - a^\dagger \right\}, \tag{10}$$

$$B_{\omega,\rm OUT} := b_\omega + \frac{\lambda f(\omega)}{\tilde\eta^+(\omega)} \left\{ \int_0^\infty d\omega'\, \frac{\lambda f(\omega')}{\omega' - \omega - i0}\, b_{\omega'} - a \right\}, \tag{11}$$

where,

$$\frac{1}{\tilde\eta^+(\omega)} := \frac{1}{\eta^+(\omega)} + 2\pi i K \delta(\omega - z_R). \tag{12}$$

In these equations γ is the circular contour around the pole and K is the residue of $1/\eta^+(\omega)$ on the pole at z_R. The meaning of the denominators in (10) and (11) is standard in distribution theory [19]. Operators (8)–(11) satisfy the following commutation relations (which are valid at leading order in λ):

$$[A_{\rm OUT}, A_{\rm IN}^\dagger] = 1; \qquad \frac{\eta^+(\omega)}{\eta^-(\omega)}\, [B_{\omega,\rm OUT}, B_{\omega',\rm IN}^\dagger] = \delta(\omega - \omega'), \tag{13}$$

and all other commutators vanish.

The Hamiltonian can be written in terms of these operators as:

$$H = z_R A_{\rm IN}^\dagger A_{\rm OUT} + \int_0^\infty d\omega\, \omega\, \frac{\eta^+(\omega)}{\eta^-(\omega)}\, B_{\omega,\rm IN}^\dagger B_{\omega,\rm OUT}. \tag{14}$$

In this notation, $A_{\rm IN}^\dagger$ and $A_{\rm OUT}$ are, respectively, the creation and annihilation operators [16] of the decaying Gamow vector $|\psi^D\rangle$:

$$|\psi^D\rangle = A_{\rm IN}^\dagger|0\rangle, \qquad A_{\rm OUT}|\psi^D\rangle = |0\rangle. \tag{15}$$

Note that, although from the explicit expressions (8) and (9), $A_{\rm IN}^\dagger$ and $A_{\rm OUT}$ are not formal adjoint of each other, for all practical purpose they can be taken as such. Both operators are defined in terms of a distributional kernel with a pole in the lower half of the complex plane, which means that they can be multiplied by each other. Naturally, it would not be the case if the distributional kernels would have been defined in different half planes. From the point of view of the time asymmetric quantum theory [20–23], both operators are defined for $t > 0$, that is the decaying part of a resonance process [6].

3. The entropy for the harmonic oscillator from the path integrals approach

The procedure to obtain the canonical entropy corresponding to a system with a given Hamiltonian H is well known [24–26]. Consider the partition function Z for a system with a time independent Hamiltonian H then, the canonical entropy is given by

$$S = k\left(1-\beta\,\frac{\partial}{\partial\beta}\right)\log Z, \tag{16}$$

where $Z = \mathrm{tr}\{\mathrm{e}^{-\beta H}\}$, $\beta = 1/(kT)$, k is the Boltzmann constant, and T is the absolute temperature.

If H is the Hamiltonian of a harmonic oscillator (for simplicity we shall take only one dimension), and $|n\rangle$ are the eigenvectors, the partition function is given by the expression

$$Z = \mathrm{tr}\,\{\mathrm{e}^{-\beta H}\} = \sum_{n=0}^{\infty}\langle n|\mathrm{e}^{-\beta H}|n\rangle. \tag{17}$$

Then, the value of the canonical entropy for the one dimensional harmonic oscillator can be directly obtained from (16). A straightforward calculation gives:

$$S = -k\log[2\sinh(\beta\hbar\omega/2)] + k\,\frac{\beta\hbar\omega}{2}\,\coth\left(\frac{\beta\hbar\omega}{2}\right). \tag{18}$$

The use of path integrals to calculate partition functions was introduced by Feynman and Hibbs [25] in their celebrated book. We have adopted the path integrals to write the partition functions in a basis of coherent states, as we shall discuss now. To start with, we express the canonical ensemble in terms of coherent states, and then use path integrals to compute the density operator. As is well known, coherent states are defined from a vacuum state $|0\rangle$ as

$$|\alpha\rangle := \mathrm{e}^{\alpha a^\dagger-\alpha^* a}|0\rangle, \tag{19}$$

where $a^\dagger$ and a are the creation and annihilation operators for the harmonic oscillator, respectively, acting on the vacuum state $|0\rangle$, which is the ground state of the harmonic oscillator. The states (19) are eigenstates of the annihilation operator a with eigenvalues (α). Take now the density operator $\rho = \mathrm{e}^{-\beta H}$ and use the strategy of path integrals to estimate its matrix elements with respect to the coherent states (19). This is for any pair of complex numbers α_i and α_f,

$$\langle\alpha_i|\rho|\alpha_f\rangle = \lim_{N\mapsto\infty}\rho_N(\alpha_i,\alpha_f), \tag{20}$$

where [26]

$$\begin{aligned}\rho_N(\alpha_i,\alpha_f) = \int\prod_{k=1}^{N}\left(\frac{\mathrm{d}^2\alpha_k}{\pi}\right)\,\exp\Bigg\{-\tau\Bigg[\sum_{n=1}^{N}H_+(\alpha_{n-1},\alpha_n)\\ +\sum_{n=1}^{N+1}\left\{\left(\frac{\alpha_n^*-\alpha_{n-1}^*}{2\tau}\right)\,\alpha_n-\alpha_{n-1}^*\,\left(\frac{\alpha_n-\alpha_{n-1}}{2\tau}\right)\right\}\Bigg]\Bigg\},\end{aligned} \tag{21}$$

with $\alpha_0 = \alpha_i$, $\alpha_{N+1} = \alpha_f$ and $\tau = \beta/N$. We write $\alpha_i = x_i + \mathrm{i}y_i$ and $\mathrm{d}\alpha_i = \mathrm{d}x_i\mathrm{d}y_i$, so that we have $2N$ integrals in the variables $x_1,\ldots,x_N,y_1,\ldots,y_N$. The integration limits are $-\infty$ and ∞ in all cases, since there must be one coherent state for any complex number. The factor $H_+(\alpha,\alpha')$ is defined as

$$H_+(\alpha,\alpha') = \frac{\langle\alpha|H|\alpha'\rangle}{\langle\alpha|\alpha'\rangle},\qquad \langle\alpha|\alpha'\rangle = \exp\left\{-\frac{|\alpha|^2}{2}-\frac{|\alpha'|^2}{2}+\alpha^*\alpha'\right\}. \tag{22}$$

The explicit calculation of (22) is given in the Appendix A, and the final result is

$$\rho(\alpha_i,\alpha_f) := \frac{1}{\langle\alpha_i|\alpha_f\rangle}\,\rho_N(\alpha_i,\alpha_f) = \exp\left\{-\frac{1}{2}\,\beta\hbar\omega\right\}\times\exp\{-\beta\hbar\omega\alpha_i^*\alpha_f\}, \tag{23}$$

which does not depend on N. Thus, (23) represents approximate matrix elements, in terms of the coherent states, of the canonical partition function of the harmonic oscillator.

Let us use (16) to calculate the entropy. The partition function is the trace of ρ, and from (23) one gets

$$\begin{aligned}Z &= \int\frac{\mathrm{d}^2\alpha}{\pi}\,\rho(\alpha,\alpha)\\ &= \frac{1}{\pi}\,\mathrm{e}^{-(\beta\hbar\omega)/2}\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\;\mathrm{e}^{-\beta\hbar\omega(x^2+y^2)}\\ &= \mathrm{e}^{-(\beta\hbar\omega)/2}\,\frac{1}{\beta\hbar\omega},\end{aligned} \tag{24}$$

where in performing the integration we have taken $\alpha = x + iy$. This expression for the partition function leads to the entropy

$$S \approx k(1 - \log(\beta\hbar\omega)), \tag{25}$$

which certainly gives an approximation for (18) which is valid in the limit $\beta\hbar\omega \ll 1$, as it is seen by inserting the expansions $\coth x = 1/x + \cdots$ and $\sinh x = x + \cdots$ in (18). In this limit ($\beta\hbar\omega \ll 1$) Eq. (25) is always positive, since $\log(\beta\hbar\omega) < 0$.

The moral of the present Section is that the use of coherent states to calculate the matrix representing the density operator is a technique amenable for extensions to systems where the standard notion of probabilities cannot be applied. This is the case of the calculation of the entropy for decaying states, which will be discussed in the next section.

4. About the notion of entropy for quantum decaying states

Thus, we face the problem of giving an approximate expression for the canonical entropy of a quantum system with resonances. Here, we shall use again (16) and an approximate value of the partition function. This approximate partition function will be found again with the use of the path integral method and coherent states, as we did for the harmonic oscillator in the previous section.

First of all, it is important to remark that no other alternative seems possible in this case, as an equation equivalent to (17) is, in principle, unapplicable, for the reasons which are explained below. In fact, the most natural basis to compute now the trace as in (17) is $\{|\psi^D\rangle, |\omega^+\rangle\}$, where $|\psi^D\rangle$ is the decaying Gamow vector and $|\omega^+\rangle$ is a complete set of generalized eigenvectors of the total Hamiltonian H. The expression of these vectors in the basis $\{|1\rangle, |\omega\rangle\}$ of eigenvectors of the unperturbed Hamiltonian H_0 ($H_0|1\rangle = \omega_0|1\rangle, H_0|\omega\rangle = \omega|\omega\rangle$), $\omega_0 > 0,\ \omega \in [0, \infty)$ is [18]

$$|\psi^D\rangle = |1\rangle + \int_0^\infty \frac{\lambda f(\omega)}{z_R - \omega + i0}\,|\omega\rangle\,d\omega \tag{26}$$

$$|\omega^+\rangle = |\omega\rangle + \frac{\lambda f(\omega)}{\eta^+(\omega)}\left(|1\rangle + \int_0^\infty d\omega' \frac{\lambda f(\omega')}{\omega - \omega' + i0}\,|\omega'\rangle\right), \tag{27}$$

where λ and $f(\omega)$ are the coupling constant and the form factor, respectively, as explained in Section 2. The appearance of the term $+i0$ in the denominator of (26) and (27) is customary in the theory of distributions [19]. From (26) and (27), one sees that expressions like $\langle\psi^D|\psi^D\rangle$ or $\langle\omega^+|\omega^+\rangle$ are ill defined. In fact, since discrete and continuous subspaces of a self adjoint operator are mutually orthogonal, we have that

$$\langle 1|1\rangle = 1; \qquad \langle 1|\omega\rangle = \langle\omega|1\rangle = 0; \qquad \langle\omega|\omega'\rangle = \delta(\omega - \omega'). \tag{28}$$

Take now the expression for $|\psi^D\rangle$ in (26) and use the products in (28) to obtain

$$\langle\psi^D|\psi^D\rangle = 1 + \int_0^\infty d\omega\,\frac{\lambda^2 f(\omega)}{(z_R^* - \omega - i0)(z_R - \omega + i0)}. \tag{29}$$

The integral (29) represents the action of a distribution on the function $f(\omega)$. This distribution is the product of $(z_R^* - \omega - i0)^{-1}$ times $(z_R - \omega + i0)^{-1}$, which is not well defined. A similar problem would arise if we consider $\langle\omega'^+|\omega^+\rangle$.

Since in the Friedrichs model, $H|\psi^D\rangle = z_R|\psi^D\rangle$ and $H|\omega^+\rangle = \omega|\omega^+\rangle$ [18], we conclude that a trace of the form

$$\mathrm{tr}\,e^{-\beta H} = \langle\psi^D|e^{-\beta H}|\psi^D\rangle + \int_0^\infty \langle\omega^+|e^{-\beta H}|\omega^+\rangle\,d\omega \tag{30}$$

is not well defined.

Then, we shall follow a similar approximation that the one discussed in Section 3, not only to obtain an approximate value of the entropy for a system with a decaying state, but also to grasp on the concept itself.

As in the case of the harmonic oscillator, we define the coherent state $|\alpha\rangle$ and its bra $\langle\alpha|$, for all complex number α, as:

$$|\alpha\rangle := \exp\{\alpha A_{\mathrm{IN}}^\dagger - \alpha^* A_{\mathrm{OUT}}\}\,|0\rangle,$$
$$\langle\alpha| := \langle 0|\exp\{\alpha^* A_{\mathrm{OUT}} - \alpha A_{\mathrm{IN}}^\dagger\}, \tag{31}$$

where $|0\rangle$ is the vacuum state. Making use of the commutation relations (31) it becomes evident that these coherent states satisfy the same properties than the coherent states (19). In particular,

$$A_{\mathrm{OUT}}|\alpha\rangle = \alpha|\alpha\rangle; \qquad \langle\alpha|A_{\mathrm{IN}}^\dagger = \alpha^*\langle\alpha|;$$
$$\int_{\mathbb{C}} \frac{d^2\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = 1; \qquad d^2\alpha = (d\mathrm{Real}\,\alpha)(d\mathrm{Im}\,\alpha), \tag{32}$$

where $\mathbb{C}$ denotes the field of complex numbers. The normal expansion (23) is now

$$H_+(\alpha, \alpha') = z_R\,\alpha^*\alpha'. \tag{33}$$

Next, we follow, step by step, the procedure described in the Appendix A, for the case of the harmonic oscillator. Then, instead of (23), we have the following equation

$$\rho(\alpha_i, \alpha_f) = \exp\{-\beta\, z_R\, \alpha_i^* \alpha_f\}, \tag{34}$$

which gives

$$Z = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\beta\, z_R\, x^2} \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\beta\, z_R\, y^2} = \frac{1}{\beta\, z_R} \tag{35}$$

where $z_R = E_R - \mathrm{i}\frac{\Gamma}{2}$. Then, using (16), we arrive at the result:

$$S = k\left[1 - \ln\left(\beta\sqrt{E_R^2 + \frac{\Gamma^2}{4}}\right) - \mathrm{i}\arctan\left(\frac{\Gamma}{2E_R}\right)\right]. \tag{36}$$

In (36) we have taken the principal branch of $\log z$. The result (36) reminds the case of the harmonic oscillator, (25), except for the presence of an imaginary term. If $\Gamma \to 0$ both results do indeed coincide, after replacing $\hbar\omega$ by E_R. The presence of a complex entropy, for the case of Gamow vectors, requires some interpretation on the meaning of its imaginary part. The situation is quite similar to the existence of complex energy for decaying states, where the imaginary part is interpreted as the inverse of the half life. The quasi stability assumption used in the present discussion also implies small values for Γ.

Note that the resonance in the Friedrichs model is caused by the interaction of the system with the background, which plays the role of the thermodynamical bath. Then, we suggest that the real part of the entropy (36) is the entropy of the system and that the imaginary part of it is the entropy transferred from the system to the background. Should the thermodynamical entropy be identified with the modulus of (36), one concludes that the total entropy for a decaying state is bigger than the entropy of a stable system.

5. An exact (albeit formal) expression for the canonical entropy

As we have stated earlier, the expression for the canonical density associated to Gamow states, could be defined as:

$$\rho = \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}\,[\mathrm{e}^{-\beta H}]}, \tag{37}$$

where H is given by (14). For the moment, we shall not specify the trace, except for its invariance with respect to cyclic permutations. The denominator of (37) is the canonical partition function for Gamow states:

$$Z = \mathrm{Tr}\,[\mathrm{e}^{-\beta H}] \tag{38}$$

and the entropy for the canonical Gamow state is given by (16). The derivative of Z (38) with respect to β

$$-\frac{\partial Z}{\partial \beta} = \mathrm{Tr}\,[\mathrm{e}^{-\beta H}\, H] \tag{39}$$

leads to

$$-\frac{\partial Z}{\partial \beta} = z_R\, \mathrm{Tr}\,[\mathrm{e}^{-\beta H}\, A_{\mathrm{IN}}^\dagger\, A_{\mathrm{OUT}}] + \int_0^\infty \mathrm{d}\omega\, \omega\, \frac{\eta^+(\omega)}{\eta^-(\omega)}\, \mathrm{Tr}\,[\mathrm{e}^{-\beta H}\, B_{\omega\,\mathrm{IN}}^\dagger\, B_{\omega\,\mathrm{OUT}}]. \tag{40}$$

To calculate the traces in (40) we must take into account the commutation relations (13) between the operators involved. Then, using ρ as in (37), one has for the first term in the right hand side of (40):

$$\mathrm{Tr}\,[\rho\, A_{\mathrm{IN}}^\dagger\, A_{\mathrm{OUT}}] = \mathrm{Tr}\,[\rho(-1 + A_{\mathrm{OUT}}\, A_{\mathrm{IN}}^\dagger)]. \tag{41}$$

By the definition of ρ in (37), its trace must be equal to one, so that (41) must be equal to

$$-1 + \mathrm{Tr}\,[\rho\, A_{\mathrm{OUT}}\, A_{\mathrm{IN}}^\dagger] = -1 + \mathrm{Tr}\,[A_{\mathrm{IN}}^\dagger\, \rho\, A_{\mathrm{OUT}}], \tag{42}$$

where the last identity results in the cyclic property of the trace. To proceed, let τ be a real number. For any operator O, we define its transform $O(\tau)$

$$O(\tau) = \mathrm{e}^{\tau H}\, O\, \mathrm{e}^{-\tau H}. \tag{43}$$

In the present context, O may be either $A_{\mathrm{IN}}^\dagger$ or A_{OUT}. In the first case,

$$[H, A_{\mathrm{IN}}^\dagger(\tau)] = \mathrm{e}^{\tau H}\, [H, A_{\mathrm{IN}}^\dagger]\, \mathrm{e}^{-\tau H} = \frac{\partial}{\partial \tau}\, A_{\mathrm{IN}}^\dagger(\tau). \tag{44}$$

Since

$$[H, A_{\rm IN}^\dagger] = z_R A_{\rm IN}^\dagger. \tag{45}$$

Eqs. (44) and (45) yield

$$\begin{aligned}\frac{\partial}{\partial\tau} A_{\rm IN}^\dagger(\tau) &= e^{\tau H}(z_R A_{\rm IN}^\dagger) e^{-\tau H} \\ &= z_R A_{\rm IN}^\dagger(\tau).\end{aligned} \tag{46}$$

A similar result is obtained for $A_{\rm OUT}$, namely:

$$\frac{\partial}{\partial\tau} A_{\rm OUT}(\tau) = -z_R A_{\rm OUT}(\tau). \tag{47}$$

From (46) and (47), one gets

$$A_{\rm IN}^\dagger(\tau) = e^{\tau z_R} A_{\rm IN}^\dagger; \qquad A_{\rm OUT}(\tau) = e^{-\tau z_R} A_{\rm OUT}. \tag{48}$$

The first equation in (48) reads

$$e^{\tau H} A_{\rm IN}^\dagger e^{-\tau H} = e^{\tau z_R} A_{\rm IN}^\dagger \Longrightarrow A_{\rm IN}^\dagger e^{-\tau H} = e^{\tau z_R} e^{-\tau H} A_{\rm IN}^\dagger. \tag{49}$$

We take the trace in (42), use (37) and (49), and for $\tau = \beta$ we find:

$$\mathrm{Tr}\,[A_{\rm IN}^\dagger \rho A_{\rm OUT}] = e^{z_R\beta}\,\mathrm{Tr}\,[\rho A_{\rm IN}^\dagger A_{\rm OUT}] \tag{50}$$

then, combining (41), (42) and (50), one gets

$$\mathrm{Tr}\,[\rho A_{\rm IN}^\dagger A_{\rm OUT}] = \frac{1}{e^{z_R\beta} - 1}. \tag{51}$$

In order to obtain the second trace in (40), we use a similar procedure taking into account the commutation relations (13) between the B-operators ($\omega \neq \omega'$):

$$\mathrm{Tr}\,[e^{-\beta H} B_{\omega' IN} B_{\omega OUT}] = \frac{\eta^-(\omega)}{\eta^+(\omega)}\,\frac{1}{e^{\beta\omega}}\,\delta(\omega - \omega'). \tag{52}$$

We observe that (52) depends on the continuous part (background) only and not on the value of z_R of the Gamow sector of the spectrum. This term is not defined (or it is infinite) for $\omega = \omega'$, but we can always drop it from the trace (40) since the background term is separated from the Gamow sector. Then, in the calculation of the partition function for the canonical Gamow state, we keep only the finite contributions to the trace (40).

With these in mind, Eq. (40) yields:

$$-\frac{1}{Z}\frac{\partial Z}{\partial\beta} = \frac{z_R}{e^{\beta z_R} - 1}. \tag{53}$$

Thus,

$$\begin{aligned}\log Z &= -\int \frac{d\beta\, z_R}{e^{\beta z_R} - 1} \\ &= -\log(1 - e^{-\beta z_R}) \\ &= z_R\beta - \log(e^{\beta z_R} - 1).\end{aligned} \tag{54}$$

Using this expression in (16), the entropy for the Gamow sector of the spectrum reads

$$S = k\left(\frac{\beta z_R\, e^{\beta z_R}}{e^{\beta z_R} - 1} - \log(e^{\beta z_R} - 1)\right). \tag{55}$$

The conclusion is that (36) could be considered as a reasonable approximation for (55). We have obtained this result by means of two different methods, a fact which enforces the validity of the adopted procedure.

Note that, we have not made a specific definition of the trace in order to obtain (55). In fact, for the derivation of (55), we have just assumed that the trace of a product of operators is invariant under cyclic permutation of the operators. The fact that our result is independent of the notion of trace, allows us to circumvent the problems associated to the definition of norms in the presence of Gamow states.

6. Interaction between a Gamow plus a fermion: an expression of the entropy for the resulting unstable system

In recent publications [17], we have studied a Friedrichs type model in which we have added a fermion to the standard Friedrichs model. Here, we are assuming that the total Hamiltonian is given by

$$H = H_F + \varepsilon|i\rangle\langle i| + H_f + \int_0^\infty V(\omega)[|i\rangle\langle i,\omega| + |i,\omega\rangle\langle i|]\,\mathrm{d}\omega, \tag{56}$$

where

- H_F is the Hamiltonian for the standard Friedrichs model written in the language of state vectors:

  $$H_F = \omega_0|\omega_0\rangle\langle\omega_0| + \lambda\int_0^\infty f(\omega)[|\omega\rangle\langle\omega_0| + |\omega_0\rangle\langle\omega|]\,\mathrm{d}\omega + \int_0^\infty \omega|\omega\rangle\langle\omega|\mathrm{d}\omega,$$

  where (i) $H_F|\omega_0\rangle = \omega_0|\omega_0\rangle$, i.e., $|\omega_0\rangle$ is a bound state of H_F with eigenvalue $\omega_0 > 0$; (ii) $H_F|\omega\rangle = \omega|\omega\rangle,\ \forall\,\omega \in [0,\infty)$, which means that the states $|\omega\rangle$ are the non-normalizable eigenvectors of H_F for the continuum; (iii) λ is a real coupling constant; (iv) $f(\omega)$ is a real function usually taken square integrable. See Ref. [27] and references quoted therein.
- $|i\rangle$ is the state of a free fermion;
- $H_f := h|i,\omega_0\rangle\langle i| + h^*|i\rangle\langle i,\omega_0|$ represents the interaction between the fermion $|i\rangle$ and the discrete boson in the Friedrichs model $|\omega_0\rangle$;
- $V(\omega)$ is the form factor for the interaction between the fermion and the external field;
- we use the notation $|i,\omega_0\rangle = |i\rangle\otimes|\omega_0\rangle$ and $|i,\omega\rangle = |i\rangle\otimes|\omega\rangle$. Observe that we are assuming an interaction between the fermion and the boson field corresponding to the original Friedrichs model.

This model has been studied in Ref. [17]. If again, we denote by $|\psi^D\rangle$ and $|\psi^G\rangle$ the Gamow vectors corresponding to the resonance of H with resonance pole $z_R = E_R - \mathrm{i}\Gamma/2$, we have proven that this Hamiltonian admits a Berggren type decomposition [28] which can be written in the following form:

$$H = \varepsilon|i\rangle\langle i| + z_R|\psi^D\rangle\langle\psi^G| + \Lambda(z_R)\,[|i,\psi^D\rangle\langle i| + |i\rangle\langle i,\psi^G|] + B_G. \tag{57}$$

Here, B_G denotes a background integral. As explained in the previous section it will be omitted in the calculations. The value of the strength (or vertex function) $\Lambda(z_R)$ has been determined in Ref. [17] and it is written

$$\Lambda(z_R) = \lambda\mathrm{PV}\int_0^\infty \frac{f(\omega)V(\omega)}{z_R-\omega}\,\mathrm{d}\omega - \mathrm{i}\pi\lambda f(z_R)V(z_R), \tag{58}$$

where PV means the Cauchy Principal Value. The form factors $f(\omega)$ and $V(\omega)$ are admitted to have an analytic continuation on the complex plane [27,17].

If we denote by $b^\dagger$ and b the creation and annihilation operators for the fermion, respectively, the relevant part of the Hamiltonian (without the background) can be written in the second quantization language as:

$$H = \varepsilon b^\dagger b + z_R A^\dagger_{\mathrm{IN}}A_{\mathrm{OUT}} + \Lambda(z_R)\,b^\dagger b\,[A^\dagger_{\mathrm{IN}} + A_{\mathrm{OUT}}]. \tag{59}$$

Note that:

$$\{b,b^\dagger\} = 1, \qquad [A_{\mathrm{OUT}},A^\dagger_{\mathrm{IN}}] = 1, \qquad [b^\dagger,A^\dagger_{\mathrm{IN}}] = \cdots = 0. \tag{60}$$

The last equation in (60) means that the fermion creation and annihilation operators commute with the operators which create (or annihilate) the Gamow state $|\psi^D\rangle$. We have written explicitly just one of these commutators, the dots represent the remainder. Eq. (59) comes up trivially from the expansion of (57) in a Berggren basis. In Ref. [17], we have shown that the interaction included in the total Hamiltonian (58) produces two distinct resonances. In order to obtain them, we have used a standard procedure, which is solving the eigenvalue equation $H|\phi\rangle = z|\phi\rangle$ for z complex, where $|\phi\rangle$ as expressed in the Berggren basis formed by the vectors $\{|i,0\rangle, |i,\psi^D\rangle\}$ should have the form [17]:

$$|\phi\rangle = \alpha|i,0\rangle + \delta|i,\psi^D\rangle. \tag{61}$$

(The zero in $|i,0\rangle$ denotes the absence of the Gamow state $|\psi^D\rangle$.) In the two dimensional space spanned by the vectors $|i,0\rangle$ and $|i,\psi^D\rangle$, we may assume that these basis vectors are orthonormal. The restriction of the Hamiltonian to the two dimensional space spanned by this basis is given by

$$\begin{pmatrix} \varepsilon & \Lambda(z_R) \\ \Lambda(z_R) & \varepsilon + z_R \end{pmatrix}. \tag{62}$$

The eigenvalues of this matrix are the complex energies

$$E_\pm = \varepsilon + \frac{z_R}{2} \pm \frac{1}{2}\sqrt{z_R^2 + 4\Lambda^2(z_R)}. \tag{63}$$

In the sequel, we shall separate the unperturbed part, H_0, of H from the interaction V in the following form:

$$H = H_0 + V, \qquad H_0 = \varepsilon b^\dagger b + z_R A_{\rm IN}^\dagger A_{\rm OUT}, \tag{64}$$

$$V = \Lambda(z_R)\, b^\dagger b\, [A_{\rm IN}^\dagger + A_{\rm OUT}]. \tag{65}$$

Note that, V is linear in the coupling constant of the original Friedrichs λ because of (58).

The Gamow states for the interacting system can be trivially calculated and give, for E_+:

$$\begin{aligned}\phi_+ &= \alpha_+|i,0\rangle + \delta_+|i,\psi^D\rangle \\ &= \frac{1}{\Delta}\left\{2\Lambda(z_R)\,|i,0\rangle + \left(z_R + \sqrt{z_R^2 + 4\Lambda^2(z_R)}\right)|i,\psi^D\rangle\right\},\end{aligned} \tag{66}$$

and, for E_-:

$$\begin{aligned}\phi_- &= \alpha_-|i,0\rangle + \delta_-|i,\psi^D\rangle \\ &= \frac{1}{\Delta}\left\{-\left(z_R + \sqrt{z_R^2 + 4\Lambda^2(z_R)}\right)|i,0\rangle + 2\Lambda(z_R)\,|i,\psi^D\rangle\right\}.\end{aligned} \tag{67}$$

Here Δ is the trivial normalization constant that results assuming the orthonormality of $|i,0\rangle$ and $|i,\psi^D\rangle$. Note that the Gamow vectors (66) and (67) are not orthogonal, since $\Lambda(z_R)$ is complex and therefore matrix (62) is not hermitian.

To calculate the entropy for these Gamow states we work again under the hypothesis of thermodynamical equilibrium as we did for the precedent analysis. The expression for the partition function is

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H(\lambda)} = \mathrm{e}^{-\beta\Omega(\lambda)}. \tag{68}$$

Eq. (68) defines $\Omega(\lambda)$ and H is our Hamiltonian (64)–(65) written in the form $H = H_0 + \lambda W$ ($V = \lambda W$). From (68) we obviously have:

$$\frac{\partial Z}{\partial\lambda} = -\beta\, \mathrm{Tr}\,\{\mathrm{e}^{-\beta H(\lambda)}\, W\}. \tag{69}$$

Noting that the average $\langle W\rangle$ then

$$\langle W\rangle = \frac{\mathrm{Tr}\,\{\mathrm{e}^{-\beta H(\lambda)}\, W\}}{Z}, \tag{70}$$

and (69) is equal to

$$\frac{\partial Z}{\partial\lambda} = -\frac{\beta}{\lambda}\, \mathrm{e}^{-\beta\Omega(\lambda)}\, \langle\lambda W\rangle. \tag{71}$$

Similarly, Eq. (68) gives

$$\frac{\partial Z}{\partial\lambda} = -\mathrm{e}^{-\beta\Omega(\lambda)}\, \beta\, \frac{\partial\Omega(\lambda)}{\partial\lambda}. \tag{72}$$

Then, (71) and (72) together give

$$\frac{\partial\Omega(\lambda)}{\partial\lambda} = \frac{1}{\lambda}\, \langle\lambda W\rangle, \tag{73}$$

which implies that:

$$\Omega(\lambda) = \Omega_0 + \int_0^1 \frac{1}{\lambda}\langle\lambda W\rangle\, \mathrm{d}\lambda, \tag{74}$$

where Ω_0 is defined by the relation:

$$\mathrm{e}^{-\beta\Omega_0} = \mathrm{Tr}\{\mathrm{e}^{-\beta H_0}\}. \tag{75}$$

Eq. (74) may be regarded as a perturbative expansion of the statistical potential Ω, in powers of λ in the interval $0 < \lambda < 1$.

The next point is to calculate $\langle W\rangle$, using the thermal propagator method [26], then.

$$\langle\lambda W\rangle = \frac{1}{2}\sum_k \lim_{k\mapsto k'} \lim_{\tau\mapsto\tau'} \left(\frac{\partial}{\partial\tau} + \varepsilon_k\right)\, G(k,k',\tau-\tau'). \tag{76}$$

Here,

$$G(k,k',\tau-\tau') = -\langle T_\tau[a_k(\tau)a_{k'}^\dagger(\tau)]\rangle, \tag{77}$$

is the thermal propagator, where T_τ is the ordering operator with respect to τ. The operators a_k appearing in (77) stand for the annihilation operators b ($k = 1$) and A_{OUT} ($k = 2$), while the creation operators $a_k^\dagger$ read for $b^\dagger$ ($k = 1$) and $A_{\mathrm{IN}}^\dagger$ ($k = 2$). The brackets $\langle - \rangle$ represent traces with respect to the Berggren basis (without background), or equivalently with respect to the basis given by the Gamow vectors $\phi_\pm$, which are eigenvectors of H. In general, (77) can be written as

$$G(k, k', \tau) = -\sum_n \langle n|\mathrm{e}^{-\tau H}\mathrm{e}^{\tau H} a_k \mathrm{e}^{-\beta H} a_{k'}^\dagger |n\rangle$$
$$= -\sum_{nn'} \left(\mathrm{e}^{-\beta E_n}\mathrm{e}^{\tau(E_n - E_{n'})}\right) \langle n|a_k|n'\rangle\langle n'|a_{k'}^\dagger|n\rangle \tag{78}$$

where $|n\rangle = |\phi_\pm\rangle$. From the form of these vectors, given in (66)–(67), and taking the limits indicated in (76), it is seen that, since

$$\langle\phi_\pm|b|\phi_\pm\rangle = \langle\phi_\pm|b^\dagger|\phi_\pm\rangle = 0, \tag{79}$$

one gets

$$G(2, 2, 0) = -\mathrm{e}^{-\beta E_+}\langle\phi_+|A_{\mathrm{OUT}}|\phi_+\rangle\langle\phi_+|A_{\mathrm{IN}}^\dagger|\phi_+\rangle - \mathrm{e}^{-\beta E_+}\langle\phi_+|A_{\mathrm{OUT}}|\phi_-\rangle\langle\phi_-|A_{\mathrm{IN}}^\dagger|\phi_+\rangle$$
$$- \mathrm{e}^{-\beta E_-}\langle\phi_-|A_{\mathrm{OUT}}|\phi_+\rangle\langle\phi_+|A_{\mathrm{IN}}^\dagger|\phi_-\rangle - \mathrm{e}^{-\beta E_-}\langle\phi_-|A_{\mathrm{OUT}}|\phi_-\rangle\langle\phi_-|A_{\mathrm{IN}}^\dagger|\phi_-\rangle. \tag{80}$$

Then, since

$$\langle\phi_+|A_{\mathrm{OUT}}|\phi_+\rangle = \langle(\alpha_+^*\langle i, 0| + \delta_+^*\langle i, \psi^D|)\, |A_{\mathrm{OUT}}|\alpha_+|i, 0\rangle + \delta_+|i, \psi^D\rangle\rangle$$
$$= \langle(\alpha_+^*\langle i, 0| + \delta_+^*\langle i, \psi^D|)\, |A_{\mathrm{OUT}}|\delta_+|i, 0\rangle\rangle = \alpha_+^*\delta_+, \tag{81}$$

and

$$\langle\phi_+|A_{\mathrm{IN}}^\dagger|\phi_+\rangle = \langle(\alpha_+^*\langle i, 0| + \delta_+^*\langle i, \psi^D|)\, |A_{\mathrm{IN}}^\dagger|\alpha_+|i, 0\rangle + \delta_+|i, \psi^D\rangle\rangle$$
$$= \langle(\alpha_+^*\langle i, 0| + \delta_+^*\langle i, \psi^D|)\, |A_{\mathrm{IN}}^\dagger|\alpha_+|i, \psi^D\rangle\rangle = \delta_+^*\alpha_+. \tag{82}$$

As a matter of fact, $A_{\mathrm{IN}}^\dagger|i, \psi^D\rangle$ is not zero, but a vector with two Gamows and its scalar product with the vectors with one or zero Gamows vanishes. Proceeding in this way, Eq. (80) yields

$$G(2, 2, 0) = -\mathrm{e}^{-\beta E_+}(|\alpha_+|^2|\delta_+|^2 + \alpha_+^*\delta_- \delta_+^*\alpha_-) - \mathrm{e}^{-\beta E_-}(\alpha_-^*\beta_+\beta_-^*\alpha_+ + |\alpha_-|^2|\delta_-|^2)$$
$$= -\mathrm{e}^{-\beta E_+}A(\lambda) - \mathrm{e}^{-\beta E_-}B(\lambda). \tag{83}$$

Note that the coefficients A and B are functions of λ due to the dependence of $\Lambda(z_R)$ on λ. Recall that $E_\pm$ depends on λ as well, both through $\Lambda(z_R)$ and z_R. In fact, z_R can be written in terms of powers of λ as [15]

$$z_R = \omega_0 + \lambda^2\int_0^\infty \frac{|f(\omega)|^2}{\omega_0 - \omega + \mathrm{i}0}\,\mathrm{d}\omega + o(\lambda^4). \tag{84}$$

The final expression for $\Omega(\lambda)$ is the following:

$$\Omega(\lambda) = \Omega_0 + \int_0^1 \frac{1}{\lambda}(E_+\mathrm{e}^{-\beta E_+}A(\lambda) + E_-\mathrm{e}^{-\beta E_-}B(\lambda))\,\mathrm{d}\lambda = \Omega_0 + \Omega_I. \tag{85}$$

The integral in (85) can be calculated by writing the function under the integral as a Taylor series in λ:

$$\frac{1}{\lambda}(E_+\mathrm{e}^{-\beta E_+}A(\lambda) + E_-\mathrm{e}^{-\beta E_-}B(\lambda)) = a_1\lambda + a_2\lambda^2 + \cdots. \tag{86}$$

Then the integration is trivial, which is cumbersome here is the determination of the coefficients $\{a_i\}$ in (86). In order to show that the Taylor series in (86) exists, we just need to show that both $A(\lambda)$ and $B(\lambda)$ are quadratic at the first order. We show this in Appendix B.

To determine Ω_0 we use the method outlined in the previous section. Noting that the creation and annihilation operators for the Gamow $|\psi^D\rangle$ and the fermion $|i\rangle$ states commute, we have that (60) also hold here. On the other hand, using the Wick theorem, we have that

$$[\varepsilon b^\dagger b, b^\dagger] = \varepsilon b^\dagger, \qquad [\varepsilon b^\dagger b, b] = -\varepsilon b, \tag{87}$$

then,

$$[H_0, b^\dagger] = \varepsilon b^\dagger, \qquad [H_0, b] = -\varepsilon b \tag{88}$$

so that the expression for the trace of $e^{-\beta H_0}\, b^\dagger b$ must be analogous to (51). Thus, for $\rho_0 = e^{-\beta H_0}/Z$ with $Z = \mathrm{Tr}\,[e^{-\beta H_0}]$ we get

$$-\frac{1}{Z}\,\frac{\partial Z}{\partial \beta} = \mathrm{Tr}[\rho_0\, H_0] = z_R\,\mathrm{Tr}[\rho_0\, A_{\mathrm{IN}}^\dagger A_{\mathrm{OUT}}] + \varepsilon\,\mathrm{Tr}[\rho_0\, b^\dagger b]$$
$$= \frac{z_R}{e^{z_R\beta}-1} + \frac{\varepsilon}{e^{\varepsilon\beta}+1}. \tag{89}$$

and, consequently,

$$-\beta\Omega_0 = \frac{(z_R-\varepsilon)\beta}{2} + \log\cosh\frac{\beta\varepsilon}{2} - \log\sinh\frac{\beta z_R}{2}. \tag{90}$$

In order to obtain the value of the entropy associated to this sector of the statistical potential Ω_0, we use (16) taking $\log Z = -\beta\Omega_0$. It is given by

$$S_0 = k\left(\log\cosh\frac{\beta\varepsilon}{2} - \log\sinh\frac{\beta z_R}{2} - \frac{\beta\varepsilon}{2}\tanh\frac{\beta\varepsilon}{2} - \frac{\beta z_R}{2}\coth\frac{\beta z_R}{2}\right). \tag{91}$$

To this expression, which is the dominant part of the entropy, we should add the polynomial in λ (second term in the r.h.s of Eq. (74)) resulting from the integral of the propagator (83).

7. Conclusions

In this paper, we have pointed out to some of the difficulties concerning the application of concepts of Statistical Mechanics to complex-energy vectors. We have presented a suitable alternative to the probabilistic description, by implementing a representation of the decaying vectors, obtained in the framework of the Friedrichs model, written them in terms of coherent states and by performing a path integration over these states to get the density matrix operator. The results are quite encouraging because, at the level of approximation used to calculate the density operator, we do not have to introduce some ad-hoc notions like complex temperatures or treat independently real and imaginary entropies. We think that this is a promising first step towards a novel formulation of the statistical mechanics for decaying systems. We aim at completing this program in the near future.

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Appendix A. Explicit calculation of the kernel (23)

In this Appendix we are giving the explicit calculation of (23) from (21) and (22). The normal expansion of H gives:

$$H_+(\alpha,\alpha') = \hbar\omega(\alpha^*\alpha' + 1/2). \tag{A.1}$$

Then, we use (A.1) in (21). As a result, we obtain a product of five exponentials:

$$\exp\left\{-\frac{1}{2}\,\tau N\hbar\omega\right\} \times \exp\left\{-\frac{1}{2}\,(|\alpha_i|^2 + |\alpha_f|^2)\right\} \times \exp\left\{-\sum_{k=1}^{N}|\alpha_k|^2\right\}$$
$$\times\ \exp\left\{(1-\tau\hbar\omega)\sum_{k=1}^{N}\alpha_{k-1}^*\alpha_k\right\} \times \exp\{\alpha_N^*\alpha_{N+1}\}. \tag{A.2}$$

Since $\tau = \beta/N$, the former of these products gives $\exp\{-\beta\hbar\omega/2\}$. Then, we integrate (21) starting from the integral in α_1, then α_2 etc., up to α_N. Take $\sigma := 1-\tau\hbar\omega$. After (A.2), the terms in (18) including α_1 give:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dy_1\,\exp\{-x_1^2 - y_1^2 + \sigma(\alpha_i^*(x_1+iy_1)) + \sigma(x_1-iy_1)\alpha_2\}$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}dy_1\ \exp\left\{-\left[x_1 - \frac{\sigma}{2}(\alpha_i^*+\alpha_2)\right]^2 + \frac{\sigma^2}{4}(\alpha_i^*+\alpha_2)^2\right.$$
$$\left. -\left[y_1 - i\frac{\sigma}{2}(\alpha_i^*-\alpha_2)\right]^2 - \frac{\sigma^2}{4}(\alpha_i^*-\alpha_2)^2\right\}. \tag{A.3}$$

Now, we discard the terms $\frac{\sigma}{2}(\alpha_i+\alpha_2)$ and $\mathrm{i}\frac{\sigma}{2}(\alpha_i^*-\alpha_2)$ in the above exponents, since we aim to obtain a finite value for the integral. This approximation is standard in path integrations and it amounts to neglecting infinite vacuum contributions [25]. After these approximations, the above integrals take the form

$$\frac{1}{\pi}\ \exp\{\sigma^2\alpha_i^*\alpha_2\}\int_{-\infty}^{\infty}\mathrm{d}x_1\,\mathrm{e}^{-x_1^2}\int_{-\infty}^{\infty}\mathrm{d}y_1\,\mathrm{e}^{-y_1^2}=\exp\{\sigma^2\alpha_i^*\alpha_2\}. \tag{A.4}$$

Once we have performed the integral over α_1, we proceed with the integral over α_2, namely:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}x_2\int_{-\infty}^{\infty}\mathrm{d}y_2\ \exp\left\{-\left[x_2-\frac{\sigma}{2}\ (\sigma\alpha_i^*+\alpha_3)\right]^2+\frac{\sigma^2}{4}\ (\sigma\alpha_i^*+\alpha_3)^2\right.$$
$$\left.-\left[y_2-\mathrm{i}\frac{\sigma}{2}\ (\sigma\alpha_i^*-\alpha_3)\right]^2-\frac{\sigma^2}{4}\ (\sigma\alpha_i^*-\alpha_3)^2\right\}. \tag{A.5}$$

If we again discard the terms $\frac{\sigma}{2}\ (\sigma\alpha_i^*+\alpha_3)$ and $\mathrm{i}\frac{\sigma}{2}\ (\sigma\alpha_i^*-\alpha_3)$ in the exponentials we get

$$\frac{1}{\pi}\ \exp\{\sigma^3\alpha_i^*\alpha_3\}\int_{-\infty}^{\infty}\mathrm{d}x_2\,\mathrm{e}^{-x_2^2}\int_{-\infty}^{\infty}\mathrm{d}y_2\,\mathrm{e}^{-y_2^2}=\exp\{\sigma^3\alpha_i^*\alpha_3\}. \tag{A.6}$$

Continuing with this procedure, after integrating up to the variable α_{N-1}, we arrive at the exponent

$$\exp\{\sigma^{N-1}\alpha_i^*\alpha_N\} \tag{A.7}$$

which should be included in the last integration appearing in Eq. (21), that is:

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}x_N\int_{-\infty}^{\infty}\mathrm{d}y_N\ \exp\left\{-\left[x_N-\frac{1}{2}\ (\sigma^N\alpha_i^*+\alpha_f)\right]^2+\frac{1}{4}\ (\sigma^N\alpha_i^*+\alpha_f)^2\right.$$
$$\left.-\left[y_N-\frac{i}{2}(\sigma^N\alpha_i^*-\alpha_f)\right]^2-\frac{1}{4}\ (\sigma^N\alpha_i^*-\alpha_f)^2\right\}. \tag{A.8}$$

Then, dropping the terms $\frac{1}{2}\ (\sigma^N\alpha_i^*+\alpha_f)$ and $\frac{i}{2}(\sigma^N\alpha_i^*-\alpha_f)$, we obtain

$$\exp\{\sigma^N\alpha_i^*\alpha_f\}, \tag{A.9}$$

so that (21) reads:

$$\exp\left\{-\frac{1}{2}\ \beta\hbar\omega\right\}\times\exp\left\{-\frac{1}{2}\ (|\alpha_i|^2+|\alpha_f|^2)\right\}\times\exp\{\sigma^N\alpha_i^*\alpha_f\}. \tag{A.10}$$

Note that $\sigma^N\approx 1-N\tau\hbar\omega=1-\beta\hbar\omega$. Then, (A.10) can be written as:

$$\exp\left\{-\frac{1}{2}\ \beta\hbar\omega\right\}\times\exp\left\{-\frac{1}{2}\ (|\alpha_i|^2+|\alpha_f|^2)\right\}\times\exp\{(1-\beta\hbar\omega)\alpha_i^*\alpha_f\}$$
$$=\exp\left\{-\frac{1}{2}\ \beta\hbar\omega\right\}\times\exp\left\{-\frac{1}{2}\ (|\alpha_i|^2+|\alpha_f|^2-2\alpha_i^*\alpha_f)\right\}\times\exp\{-\beta\hbar\omega\alpha_i^*\alpha_f\}$$
$$=\langle\alpha_i|\alpha_f\rangle\ \exp\left\{-\frac{1}{2}\ \beta\hbar\omega\right\}\times\exp\{-\beta\hbar\omega\alpha_i^*\alpha_f\}. \tag{A.11}$$

Then, we impose a condition of normalization in the matrix elements of ρ, which consists in dividing them by $\langle\alpha_i|\alpha_f\rangle$ given in (22). Finally, this leads to (23) and the proof of this formula is complete.

Appendix B. Lowest order for $A(\lambda)$ and $B(\lambda)$

In this appendix we prove that $A(\lambda)$ is quadratic at the first order, i.e., $A(\lambda)=A_0\lambda^2+o(\lambda^3)$. The same result is valid for $B(\lambda)$ and the proof is similar. Due to (66), (67) and (83), we have:

$$A(\lambda)=\frac{4|\Lambda(z_R)|^2\,|z_R+\sqrt{z_R+4\Lambda(z_R)}|^2}{[4|\Lambda(z_R)|^2+|z_R+\sqrt{z_R+4\Lambda(z_R)}|^2]^2}=\frac{N}{D}, \tag{B.1}$$

where N and D represent the numerator and the denominator of (B.1), respectively.

Let us write the numerator N at the lowest order in λ. After (58) and (84), we get:

$$N \approx 4\lambda^2 \left| \mathrm{PV} \int_0^\infty \frac{f(\omega)V(\omega)}{\omega_0 - \omega}\, \mathrm{d}\omega - \mathrm{i}\pi f(\omega_0)V(\omega_0) \right| \, 4\omega_0^2 = 4c_0\lambda^2, \tag{B.2}$$

where c_0 is defined by (B.2).

Note that the form factor for a Friedrichs model (here $f(\omega)$ and $V(\omega)$) cannot vanish at the point ω_0, otherwise there is no resonance (see Ref. [15, p. 107]), so that $f(\omega_0)V(\omega_0) \neq 0$. Hence, N is of quadratic order in λ. The denominator in the lowest order in λ reads:

$$D = [4A_0\lambda^2 + 4\omega^2]^2. \tag{B.3}$$

From here,

$$\frac{1}{D} = \frac{1}{16\omega_0^2} + o(\lambda^4), \tag{B.4}$$

so that,

$$\frac{N}{D} = A_0\lambda^2 + o(\lambda^3), \qquad A_0 = \frac{c_0}{4\omega_0^2}. \tag{B.5}$$

The proof for $B(\lambda)$ is similar.

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