The Continuous Gabor Transform and Smoothness Analysis of Besicovitch Almost Periodic Signals

1st R. Florentin
FI-Universidad de Buenos Aires
LABIS-UCA
Buenos Aires, Argentina
electronicaraulflorentin@gmail.com

2nd J.M. Medina
FI-UBA and CONICET
LABIS-UCA
Buenos Aires, Argentina
ORCID 0000-0003-0370-6837

3rd M. Miralles
Universidad de Buenos Aires
LABIS-UCA
Bunos Aires, Argentina
mmiralles@gmail.com

***Abstract*—In this work smoothness analysis of almost-periodic signals is studied. Here, analogously to the case of $L^2(\mathbb{R})$ or finite energy signals, the smoothness of the class of almost periodic bounded power signals is characterized in terms of the decay of its Gabor transform. Moreover, some results are given as equivalence of norms between appropriate function spaces.**

***Index Terms*—Almost-periodic Functions, Gabor Transform, Smoothness**

I. Introduction

Almost-periodic functions are a useful model of persistent signals. In real life, the occurrence of almost-periodic oscillations is much more common than *exact* periodic ones. Almost-periodic functions were extensively studied by H. Bohr, V. Stepanov, N. Wiener, A.S. Besicovitch [3], [17] among other renown scientists. Initially, this theory was concerned with the study of the almost-periodicity of the solutions of differential equations. As shown in [7], for example, if we consider the wave equation

$$u_{xx} = k^2 u_{tt}\,,$$

with the non-ideal boundary condition:

$$u(t,0)=0,\ u_x(t,l)+hu(t,l)=0,\ \ h>0,$$

then we get almost-periodic solutions to the wave equation. A possible physic interpretation could be the following: $u(x,t)$ describes the motion of a vibrating elastic string such that it is fixed at $x=0$ and whose end at $x=l$ has its tension $u_x(t,l)$ proportional to the elongation $u(t,l)$. Apart from mathematical physics, almost-periodic waves or oscillations appear in other dynamical systems and Control Theory [13]. On the other hand, they are a subclass of functions to which the Generalized Harmonic Analysis tools, first developed by Wiener, can be applied to them [1]. As it is discussed in [2], these tools are also well adapted for interpreting spectral bio-electric data, where non-periodic and persistent rhythms appear and the usual finite-energy techniques (i.e. $L^2(\mathbb{R})$) of harmonic analysis cannot be applied. Finally, there has been a substantial research in how some usual time-frequency representations, i.e. Wavelets and Gabor transforms, can be adapted to this scenery. Some positive answers about the representation of almost-periodic signals were given in e.g. [4], [11], [12], [14] and more recently in [5]. Gabor and Wavelet Transform not only give, in some sense, optimal representations of signals but also are useful signal analysis tools, at least in the finite-energy context. We note, however, that this fact it is not discussed, for the almost-periodic case, in none of these referenced works. Here, we shall discuss some of these facts for the Gabor (or Short Time Fourier Transform). In the finite-energy context, smoothness or regularity analysis is very well described in terms of decay of Gabor or Wavelet coefficients or as equivalences of norms. Smoothness analysis is of certain importance in the classification of signals. In contrast to the $L^2(\mathbb{R})$ setup, here we will prove some analogue results for the Gabor Transform of almost-periodic signals. There exist several definitions of almost-periodicity with increasing generality. Here will be concerned with the Besicovitch class of almost-periodic signals. In particular, these functions constitute a closed subspace of almost-periodic signals included in the more general (Hilbert) vector space of Bounded Quadratic Mean functions, i.e. *Bounded Power signals*. The paper is organized as follows: first the Besicovitch class of Almost Periodic signals is introduced. In Section II-A time frequency-analysis of almost periodic functions is discussed. Finally, the main results on smoothness analysis are given in Section III. A brief practical and preliminar example on biomedical time series is presented there.

II. The Besicovitch Class of Almost Periodic Signals and Function Spaces.

As usual, for $p \in [1,\infty)$, we will denote the classical Lebesgue function spaces with $L^p(\mathbb{R})$. When $p=2$, we denote $\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g}(x)dx$. With some abuse, we shall use the same notation when this integral is well defined for functions which not necessarily belong to $L^2(\mathbb{R})$. The Fourier Transform of $f \in L^1(\mathbb{R})$ is given by:

$$\mathcal{F}f(\lambda) = \widehat{f}(\lambda) = \int_{\mathbb{R}} f(x)e^{-i2\pi\lambda.x}dx\,.$$

Analogously, if $\widehat{f}$ is integrable, f can be recovered by the inverse Fourier Transform, $(\widehat{f})^{\vee}$. By a density argument the Fourier Transform can be defined for $f \in L^2(\mathbb{R})$. In fact, in this case, one has the *Plancherel identity:*

$$\|\widehat{f}\|^2_{L^2(\mathbb{R})} = 2\pi\,\|f\|^2_{L^2(\mathbb{R})}$$

expressing the fact that the Fourier Transform, over $L^2(\mathbb{R})$, is a unitary map up to a constant. Fourier transforms are also defined for other classes of measures or functions. For more details see for example [10], [15]. We shall need another class of functions [1], [10]:

Definition II.1. *The space* $AP(\mathbb{R})$ *of* almost periodic functions *is the set of continuous functions* $f : \mathbb{R} \longrightarrow \mathbb{C}$ *with the property that for every* $\epsilon > 0$*, there exists* $L > 0$*, such that every interval of the real line of length greater than L contains a value* τ *satisfying* $\sup_{t\in\mathbb{R}} |f(t+\tau) - f(t)| \leq \epsilon$.

Recall that $AP(\mathbb{R})$ coincides with the uniform norm closure of the space of trigonometric polynomials $\sum_j C(j)e^{i\lambda_j t}$ with $\lambda_j \in \mathbb{R}$ and $C(j) \in \mathbb{C}$.

In $AP(\mathbb{R})$ one can introduce the inner product:

$$(f, g)_{AP(\mathbb{R})} = \lim_{T\longrightarrow\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\overline{g(t)}dt\,.$$

The norm $\| f \|_{AP(\mathbb{R})} = \lim_{T\longrightarrow\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt$ induced by this inner product makes $AP(\mathbb{R})$ a non-complete, non-separable space. The completion of $AP(\mathbb{R})$ with respect to this norm is the Hilbert space $AP_2(\mathbb{R})$ of *Besicovitch almost periodic functions*. As an extension, its norm will be denoted $\| f \|_{AP(\mathbb{R})}$. There, the complex exponentials $(e^{i\lambda t})_{\lambda\in\mathbb{R}}$ form a complete orthonormal basis and the following analogue, due to Wiener [10], of Plancherel identity holds:

$$\| f \|_{AP_2(\mathbb{R})} = \| C(f) \|_{L^2(\mathbb{R},dc)}\,, \tag{1}$$

where

$$C(f)(\lambda) = \lim_{T\longrightarrow\infty} \frac{1}{2T} \int_{-T}^{T} f(t)e^{-i\lambda t}dt$$

denotes the *Bohr Transform* of f and c denotes the counting measure. Obviously, $C(f)(\lambda) = 0$ for all λ except for a finite or countable subset of them. In the applied literature these functions are referred as *finite power signals* in contrast to the usual $L^2(\mathbb{R})$ space of *finite energy signals*. On the other hand, for any $T > 0$, $AP_2(\mathbb{R})$ contains the class of T-periodic signals $L^2(T)$.

A. Gabor Transform and Time-Frequency Analysis of Almost Periodic Signals.

First we recall the definition of the usual Gabor Transform of a *finite-energy* $L^2(\mathbb{R})$ function.

1) Gabor Transform of $L^2(\mathbb{R})$ *functions.:* Let $g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $g \neq 0$, be fixed. For $f \in L^2(\mathbb{R})$, the Gabor (or Windowed Fourier Transform) can be defined by [9]:

$$\mathcal{G}f(w, x) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi iw(t-x)}dt\,. \tag{2}$$

Note that alternatively the following relations hold true:

$$\mathcal{G}f(w, x) = \mathcal{F}(f\overline{g}(\,.\, - x))(w) = \langle f, g_{w,x}\rangle\,,$$

where $g_{w,x} = g(t-x)e^{-2\pi iw(t-x)}$.

The key result for the Gabor Transform is the following form of the Parseval formula:

$$\|\mathcal{G}f\|^2_{L^2(\mathbb{R}^2)} = \|g\|^2_{L^2(\mathbb{R})}\|f\|^2_{L^2(\mathbb{R})} \tag{3}$$

It is of interest to find out for which other function spaces equation (2) is still well defined as well as if some kind of isometric relation, like (3), holds. In fact, (3) expresses that $\mathcal{G}$ is a continuous linear operator with *continuous inverse* over its range.

2) Gabor Transform of Persistent Signals: If $f = \sum_j C(j)e^{i\lambda_j t}$ is a trigonometric polynomial is immediate that $\mathcal{G}f$ exists and moreover, from the properties of the Fourier Transform:

$$\mathcal{G}f(w, x) = \sum_j C(j)\widehat{g}(w - 2\pi\lambda_j)e^{i2\pi\lambda_j x}\,.$$

As the set of trigonometric polynomials is dense in $AP_2(\mathbb{R})$ it is reasonable to try to define $\mathcal{G}f$ for an arbitrary $f \in AP_2$. Supposing, in addition, that the function g of (2) verifies that $\int_{\mathbb{R}} |g(t)|^2(1 + |t|^2)dt < \infty$ and recalling that if $f \in AP_2(\mathbb{R})$ then $\int_{\mathbb{R}} |f(t)|^2(1+|t|^2)^{-1}dt < \infty$, we get that (2) is also well defined for all $f \in AP_2(\mathbb{R})$, i.e. the integral defining $\mathcal{G}f(w, x)$ exists and is finite for every x, w. Once established that $\mathcal{G}f$ is well defined for $f \in AP_2(\mathbb{R})$, in [14] it is proved that:

THEOREM **II.2.** *If* $f \in AP_2(\mathbb{R})$ *or* $f \in L^\infty(\mathbb{R})$ *then* $\mathcal{G}f$ *is well defined and moreover:*

1) *If* $f \in L^\infty(\mathbb{R})$ *then:*

$$\|\mathcal{G}f\|_{L^\infty(\mathbb{R}^2)} \leq \|g\|_{L^1(\mathbb{R})}\|f\|_{L^\infty(\mathbb{R})}\,.$$

2) *If* $f \in AP_2(\mathbb{R})$ *then:*

$$\int_{\mathbb{R}} \|\mathcal{G}f(w,\,.\,)\|^2_{AP_2(\mathbb{R})}dw = \|g\|^2_{L^2(\mathbb{R})}\|f\|^2_{AP_2(\mathbb{R})} \tag{4}$$

Note, that $\mathcal{G}f$ is introduced in [14] but it is not enough clear from their definition in which sense the integral defining $\mathcal{G}f$ has to be interpreted (i.e. as a limit in norm or pointwise). Finally, we observe that the additional condition on the window g, $\int_{\mathbb{R}} |g(t)|^2(1 + |t|^2)dt < \infty$ can be dropped. However, in this case $\mathcal{G}f$ must be defined by a density argument. In both cases the following results hold true.

III. Hölder Continuity and Smoothness Measurements.

In this Section we present our original results.

1) Some Definitions and Auxiliary Results: As usual smoothness is described by the magnitude of the increments of a function, or alternatively by the decay of its Fourier transform, if this is defined in some specific sense. First, if $\alpha \in (0, \infty)$ let us recall that a measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is (uniformly) α-Hölder continuous [15], or $f \in C^{\alpha}(\mathbb{R})$ for short, if there exists $C > 0$ such that

$$|f(t) - f(t')| \leq C|t - t'|^{\alpha}\,, \tag{5}$$

for all $t, t' \in \mathbb{R}$. This allows to introduce the semi-norm $\|f\|_{C^{\alpha}(\mathbb{R})} = \sup_{h\in(0,1)} \frac{1}{h^{\alpha}} \|f(\,.\,+h) - f\|_{L^{\infty}}$. Alternatively, if $f \in AP_2(\mathbb{R})$, we can measure its quadratic regularity introducing the semi-norm:

$$\|f\|_{AP_2^{\alpha}(\mathbb{R})} = \sup_{h\in(0,1)} \frac{1}{h^{\alpha}} \|f(\,.\,+h) - f\|_{AP_2(\mathbb{R})}\,.$$

We will say that $f \in AP_2^{\alpha}(\mathbb{R})$ if $\|f\|_{AP_2^{\alpha}(\mathbb{R})} < \infty$. For the case of almost-periodic functions we can give an analogue definition of Sobolev space. Recall the definition of the Fourier-Bohr Transform of f of Section II, then we define:

$$H^{2,\alpha}_{AP}(\mathbb{R}) = \left\{ f \in AP_2(\mathbb{R}) : \;\sum_{\lambda} |\lambda|^{2\alpha} |C(f)(\lambda)|^2 < \infty \right\}.$$

Similarly to the case of the Lebesgue spaces $L^2(\mathbb{R})$ and the ordinary Fourier Transform, one can prove that $f \in H^{2,\alpha}_{AP}$ if and only if $\|f\|^2_{H^{2,\alpha}_{AP}(\mathbb{R})} = \sum_{\lambda}(1 + |\lambda|^2)^{\alpha}|C(f)(\lambda)|^2 < \infty$. We can also prove the following useful Theorem on the inclusion of these spaces:

THEOREM **III.1.** *Let $f \in AP_2(\mathbb{R})$. Then:*

1) *There exists a positive constant $K_1(\alpha)$ such that:*

$$\|f\|^2_{AP_2^{\alpha}(\mathbb{R})} \leq K_1(\alpha) \sum_{\lambda} |\lambda|^{2\alpha} |C(f)(\lambda)|^2\,.$$

2) *There exists positive constant $C_2(\alpha)$ such that:*

$$\|f\|_{AP_2^{\alpha}(\mathbb{R})} \leq K_2(\alpha) \|f\|_{C^{\alpha}(\mathbb{R})}\,.$$

3) *If $0 < \beta < \alpha$ and $\|f\|_{AP_2^{\alpha}(\mathbb{R})} < \infty$ then there exists $K_3(\alpha, \beta)$ such that:*

$$\sum_{\lambda} |\lambda|^{2\beta} |C(f)(\lambda)|^2 \leq K_3(\alpha, \beta) \left(\|f\|^2_{AP_2(\mathbb{R})} \right)^{\frac{\beta}{\alpha}+1}\,.$$

In particular, the following set inclusions hold:

$$C^{\alpha}(\mathbb{R}) \subset AP_2^{\alpha}(\mathbb{R})$$

and

$$H^{2,\alpha}_{AP}(\mathbb{R}) \subset AP_2^{\alpha}(\mathbb{R}) \subset H^{2,\beta}_{AP}(\mathbb{R})$$

for any $0 < \beta < \alpha$.

The Fourier-Bohr Transform offers a spectral representation for the class of persistent signals given by Besicovitch almost-periodic functions. Note that these functions, as defined, have discrete spectrum. This representation, analogously to the ordinary Fourier transform offers a characterization of the regularity of a function in terms of the decay of its coefficients. However, this case of generalized harmonic analysis has not any time-frequency localization. A first way to introduce time-frequency analysis is by means of the Gabor Transform. Let us present our main results on smoothness using the Gabor transform as analysis tool.

2) Main Results: Recall that by Theorem II.2 -2) $\mathcal{G}f$ is well defined for any $f \in AP_2(\mathbb{R})$. First, we characterize the spaces $H^{2,\alpha}_{AP}(\mathbb{R})$ in terms of the Gabor Transform. We aim to prove an analogue result to those of e.g. [8] (Chapters 2 and 9) , [15] (Chapter 6) or [16] for the $L^2(\mathbb{R})$-Wavelet transform. It so far we can prove the following:

THEOREM **III.2.** *Let $f \in AP_2(\mathbb{R})$, $\alpha \geq 0$.*

1) *There exists a constant $C_1 > 0$ such that:*

$$C_1 \|f\|^2_{H^{2,\alpha}_{AP}(\mathbb{R})} \leq \int_{\mathbb{R}} \|\mathcal{G}f(w,\,.\,)\|^2_{AP_2(\mathbb{R})} (1+|w|^2)^{\alpha} dw\,. \tag{6}$$

2) *Let g be such that $\int_{\mathbb{R}} |\hat{g}(\lambda)|^2(1 + |\lambda|^2)^{\alpha} d\lambda < \infty$. Then there exists $C_2 > 0$ such that:*

$$C_2 \|f\|^2_{H^{2,\alpha}_{AP}(\mathbb{R})} \geq \int_{\mathbb{R}} \|\mathcal{G}f(w,\,.\,)\|^2_{AP_2(\mathbb{R})} (1+|w|^2)^{\alpha} dw\,. \tag{7}$$

3) *Let g be as in statement* (2)*. Then $f \in H^{2,\alpha}_{AP}(\mathbb{R})$ if and only if*

$$\int_{\mathbb{R}} \|\mathcal{G}f(w,\,.\,)\|^2_{AP_2(\mathbb{R})} (1 + |w|^2)^{\alpha} dw < \infty\,.$$

Moreover, there exists positive constants $C_1 \leq C_2$ such that:

$$C_1 \|f\|^2_{H^{2,\alpha}_{AP}(\mathbb{R})} \leq \int_{\mathbb{R}} \|\mathcal{G}f(w,\,.\,)\|^2_{AP_2(\mathbb{R})} (1 + |w|^2)^{\alpha} dw \leq C_2 \|f\|^2_{H^{2,\alpha}_{AP}(\mathbb{R})}\,. \tag{8}$$

Observe that, in some sense, Theorem III.2 is a generalization of Theorem II.2-2). In fact, for $\alpha = 0$, it can be proved that the constants can be adjusted so that $C_1 = C_2$. Combining our Theorems III.1 and III.2 we get the following characterization of quadratic regularity (in terms of the norm of increments of f) using the Gabor transform.

LEMMA **III.3.** *Let $f \in AP_2(\mathbb{R})$ and $\alpha > 0$.*

1) *If*

$$\int_{\mathbb{R}} \|\mathcal{G}f(w,\,.\,)\|^2_{AP_2(\mathbb{R})} (1 + |w|^2)^{\alpha} dw < \infty\,,$$

then $f \in AP_2^{\alpha}(\mathbb{R})$.

2) *If $f \in AP_2^{\alpha}(\mathbb{R})$ then*

$$\int_{\mathbb{R}} \|\mathcal{G}f(w,\,.\,)\|^2_{AP_2(\mathbb{R})} (1 + |w|^2)^{\beta} dw < \infty\,,$$

for all $0 < \beta < \alpha$.

A. *An example.*

A kind of persistent signal is given by the time series recording of a human trait (Figure 1). The following graphics (See at the end) illustrates the frequency domain behaviour of such a signal (Figure 2). Figure 3 is the asymptotic decay of its Bohr-Fourier transform. The global smoothness as a rule of thumb is described in this way, however the Gabor transform also retains this property and moreover in an stable way as equation (8) shows. In fact, Figure 4 is an example of this. These are the graphics of the Gabor or windowed Fourier transform for three different time instants.

Figure1: Human trait. Times series(16324 samples)

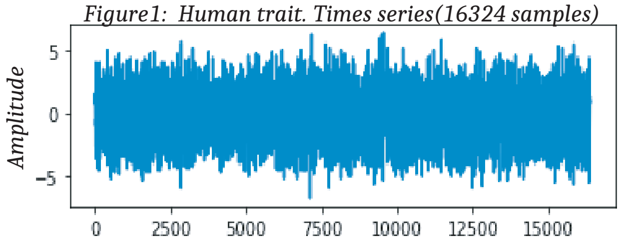


Figure2: Human trait. Times series (FFT)

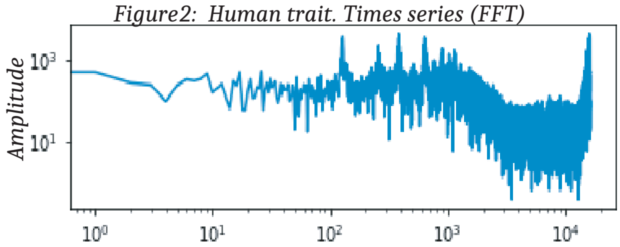
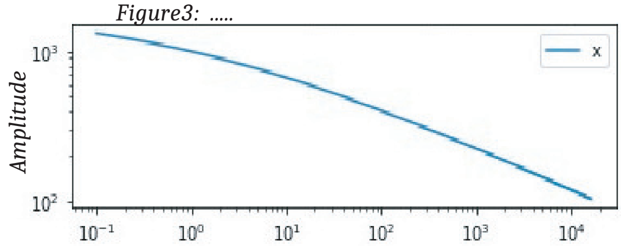


Figure3: .....



IV. Conclusions

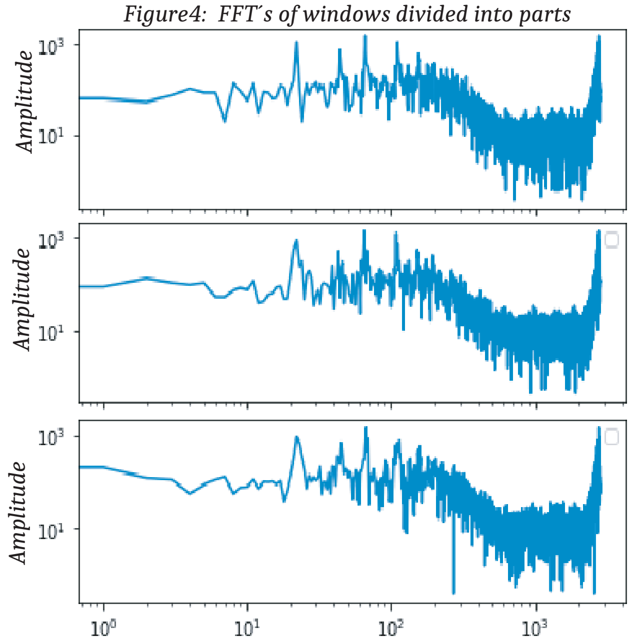
We proved that the Gabor Transform is a good analysis tool for characterizing regularity of almost-periodic signals/functions. First we introduce some smoothness classes of almost-periodic functions and analogously to the usual $L^2(\mathbb{R})$ case, the smoothness is described in terms of the eventual belonging of the signal to a given space of functions with a certain prescribed reguralty. The pertaining to one of these function spaces is characterized by appropriate norm equivalences involving the Gabor Transform of the analyzed signal. Complete proofs and more experimental evidence will be presented in further work elsewhere.

References

[1] Benedetto, J. *Spectral Synthesis*, Springer, 1975.

[2] Benedetto, J. "Generalized Harmonic Analysis and Gabor and Wavelet Systems". Proceedings of Symposia in Applied Mathematics, A.M.S., Vol. 52, pp. 85-113, 1997. Springer, 1975

Figure4: FFT´s of windows divided into parts



[3] Besicovitch A.S. Almost Periodic Functions, Cambridge Univ. Press, 1932.

[4] Boggiatto P., Férnandez C., Galbis A. "Gabor systems and almost periodic functions", Appl. Comput. Harmon. Anal. 42, pp. 65-87, 2017.

[5] Centeno H., Medina J.M. "AP-Frames and Stationary Random Processes" Appl. Comput. Harmon. Anal, Vol. 61, pp.1-24, 2022.

[6] Christensen, O. An Introduction to Frames and Riesz Bases. 2nd. Edition, Birkhäuser, 2016.

[7] Corduneanu C. Almost Periodic Oscilations and Waves. Springer, 2009.

[8] Daubechies I. Ten Lectures on Wavelets, SIAM, 1992.

[9] Gröchenig K., Foundations of Time-Frequency Analysis, Birkhäuser, 2001.

[10] Katznelson Y. An Introduction to Harmonic Analysis. Dover, 1976.

[11] Kim Y.H., Ron A. "Time Frequency Representations of Almost Periodic Functions" Constr. Approx. 29, pp.303-323, 2009.

[12] Kim Y. H. "Representations of Almost-Periodic Functions". J. Fourier Anal Appl. 19, pp. 857-876, 2013.

[13] Mäkila P.M, Partington J.R., Norlander T., "Bounded Power Signal Spaces For Robust Control and Modeling", SIAM J. Control Optim. Vol. 37, 1, pp.97-117.

[14] Partington J. R., "On the Windowed Fourier Transform and Wavelet Transform of Almost Periodic Functions". Appl. Comput. Anal. Appl. 10, pp. 45-60, 2001.

[15] Pinsky M., Introduction to Fourier Analysis and Wavelets (Graduate Studies in Mathematics), American Mathematical Society, 2009.

[16] Rieder A. "The wavelet transform on Sobolev spaces and its approximation properties", Numer. Math., Vol.58, pp. 875-894, 1991.

[17] Wiener N. The Fourier Integral, Cambridge Univ. Press, 1933.