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Convergence of solutions of a one-phase Stefan problem with Neumann boundary data to a self-similar profile

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Abstract. We study a one-dimensional one-phase Stefan problem with a Neumann boundary condition on the fixed part of the boundary. We construct the unique self-similar solution, and show that starting from arbitrary initial data, solution orbits converge to the self-similar solution.

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1. Introduction

1.1. On the content of the paper

The Stefan problem has been extensively studied in the past decades. Despite the number of articles and books published on this topic, [8–11], there are still open problems left, see for instance [1,3]. One of the questions which requires further attention is the long time behavior of the one-phase Stefan problem, where the heat flux is specified at the fixed boundary, namely the Neumann problem:

$$\begin{array}{lll} (i) & u_t(x,t) - u_{xx}(x,t) = 0, & t > 0,\ 0 < x < s(t), \\ (ii) & -u_x(0,t) = \dfrac{h}{\sqrt{t+1}}, \quad u(s(t),t) = 0, & t > 0, \\ (iii) & \dot{s}(t) = -u_x(s(t),t), & t > 0, \\ (iv) & u(x,0) = u_0(x), & 0 < x < s(0) = b_0, \end{array} \tag{1}$$

where we assume that $h > 0$.

We stress that this type of boundary condition is reasonable from the modeling view point. Namely, the choice of h means that the water in the container is heated at $x = 0$ that leads to ice melting at $x = s(t)$. We choose (1−ii), because it simple, yet it leads to non-trivial behavior of solutions.

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The decay of data presented in (1$-ii$) is consistent with the parabolic scaling, i.e. when we change variables by introducing $\eta = \frac{x}{\sqrt{t+1}}$, then the first condition in (1$-ii$) will be transformed to a constant in time, see the first condition in (3$-ii$). Note that we have shifted the initial time by 1, in order to avoid an artificial singularity at the initial time $t = 0$. Let us note that the existence of a unique smooth solution (u, s) to Problem (1) has already been established. We recall the assumptions of this result in Subsection 1.2. Here u may be called the temperature and s is the position of the interface.

Our approach to study the long time behavior of Problem (1) follows a general heuristics saying that the time asymptotics is determined by the steady states (there is none for (1)) or special solutions such as self-similar solutions or travelling waves. In fact, we show in Corollary 1 that there is exactly one self-similar solution (v, σ), which has the form, $v(x,t) = U\left(\frac{x}{\sqrt{t+1}}\right)$ and $\sigma(t) = \omega\sqrt{t+1}$ for some constant ω and a profile function U. Our main result states that the self-similar solution is attracting.

Theorem 1. *Suppose that* (u_0, b_0) *satisfies the conditions*

$$0 \le u_0 \in W^{1,\infty}(0,+\infty), \quad 0 < u_0(0) \quad \textit{and } u_0(x) = 0 \textit{ in } [b_0, \infty). \tag{2}$$

Let (u, s) *be the corresponding solution of Problem* (1). *Then,*

1) $\lim_{t\to\infty} s(t)/\sqrt{t+1} = \omega$;
2) $\lim_{t\to\infty} \sup_{x/\sqrt{t+1}\in[0,\omega]} \left| u(x,t) - U\left(\frac{x}{\sqrt{t+1}}\right)\right| = 0.$

Our method of proof is based upon recent results obtained by [3,4], who use the comparison principle in an essential way. The argument dwells on the possibility of trapping a given solution to (1) between two solutions with known time asymptotic behavior. In order to make this method work we transform (1) to a problem on a bounded domain with the help of similarity variables. The self-similar solution of (1) corresponds to the steady state solution of the transformed system. Its uniqueness is of crucial importance for the proof.

Let us stress the main difference between [3,4] and the present article. The authors of [3,4] present a quite technical proof to show that the space derivative of the solution uniformly converges to its limit as $t \to \infty$. Here, we completely avoid such a claim, so that our proof is simpler and more direct, which would make our method easier to adapt to a different setting.

We should point out that there are a number of results dealing with the asymptotic behavior of solutions of Stefan problems, mainly in the case of Dirichlet data on the fixed boundary. However, even for Dirichlet data, there are not so many articles besides [3,4] simultaneously addressing the behavior of the temperature profile u and the shape of the interface s.

1.2. Existence and uniqueness of the solution

Let us stress that (2) is our standing set of assumptions on the initial conditions. Moreover, the condition $u_0(0) > 0$ is necessary to construct proper lower solutions, but is not needed in the Proposition below:

Proposition 1. *Assume that the initial condition u_0 satisfies (2) and that $h > 0$. Then, there exists a unique classical solution (u, s) of Problem (1) for all $t > 0$, in the following sense:*

$$s \in C^1((0,\infty)) \cap C([0,\infty)), \quad u \in C^{2,1}(\{(x,t) : t > 0, 0 < x < s(t)\}),$$
$$u \in C(\{(x,t) : t \geq 0, 0 \leq x \leq s(t)\}), u_x \in C(\{(x,t) : t > 0, 0 \leq x \leq s(t)\}).$$

When we want to emphasize the dependence of (u, s) on the initial conditions we will write $u = u(\cdot,\cdot,(u_0, s_0))$, $s = s(\cdot,(u_0, s_0))$.

We refer to [7, Chapter 8, Theorem 2] for the proof of this proposition.

Strictly speaking, the original statement in [7] required u_0 to be of class C^1; however, we may relax this assumption in view of [2, Theorem 5.1].

The organization of this paper is as follows. In Sect. 2.1 we discuss the existence and uniqueness of the self-similar solution. Subsection 2.2 is devoted to the study of upper and lower solutions as well as to estimates following from monotonicity. In the last Section, Section 3, we present the proof of the convergence result which is based on the comparison principle.

2. Self-similar, lower and upper solutions

2.1. Self-similar solution

We start by re-expressing Problem (1) in terms of the self-similar variables. In other words, we set

$$W(\eta,\tau) = u(x,t) \quad \text{and} \quad b(\tau) = \frac{s(t)}{\sqrt{t+1}},$$

where $\eta = \dfrac{x}{\sqrt{t+1}}$ and $\tau = \ln(t+1)$, to obtain the problem

$$\begin{array}{lll} (i) & W_\tau(\eta,\tau) - W_{\eta\eta}(\eta,\tau) - \frac{\eta}{2} W_\eta(\eta,\tau) = 0 & \tau > 0,\ 0 < \eta < b(\tau), \\ (ii) & -W_\eta(0,\tau) = h, \quad W(b(\tau),\tau) = 0 & \tau > 0, \\ (iii) & \dot{b}(\tau) + \frac{b(\tau)}{2} = -W_\eta(b(\tau),\tau) & \tau > 0, \\ (iv) & b(0) = b_0 > 0, \quad W(\eta,0) = u_0(\eta) & 0 < \eta < b_0. \end{array} \tag{3}$$

Let us remark that the existence and uniqueness of the stationary solution of problem (3) were given in [12].

Lemma 1. *The associated stationary problem to (3), which is given by*

$$\begin{array}{ll} (i) & W_{\eta\eta}(\eta) + \frac{\eta}{2} W_\eta(\eta) = 0, \quad 0 < \eta < \omega, \\ (ii) & -W_\eta(0) = h, \quad W(\omega) = 0, \\ (iii) & \frac{\omega}{2} = -W_\eta(\omega). \end{array} \tag{4}$$

admits a unique solution given by the pair (U, ω) such that

$$U(\eta) = h \int_\eta^\omega e^{-\frac{s^2}{4}}\, ds, \qquad \eta \in [0,\omega] \tag{5}$$

and ω is the unique positive solution of the equation $h = \frac{x}{2} e^{\frac{x^2}{4}}$.

We immediately conclude from this result that

Corollary 1. *If we set* $u(x,t) = U\left(\frac{x}{\sqrt{t+1}}\right)$ *and* $\sigma = \omega\sqrt{t+1}$, *then* (u,σ) *is a solution to* (1–*i*)–(1–*iii*).

2.2. Lower and upper solutions

Similarly as in [3] we define notions of lower and upper solutions of (3).

Definition 1. We say that a pair of smooth functions $(\underline{W}, \underline{b})$ (resp. $(\overline{W}, \overline{b})$) is a lower (resp. upper) solution of Problem (3) if

$$\begin{array}{lll}
(i) & W_\tau(\eta,\tau) - W_{\eta\eta}(\eta,\tau) - \frac{\eta}{2}W_\eta(\eta,\tau) \le 0 \quad (\text{resp.} \ge 0), & \tau > 0,\ 0 < \eta < b(\tau),\\
(ii) & -W_\eta(0,\tau) \le h \quad (\text{resp.} \ge h), & \tau > 0,\\
(iii) & W(b(\tau),\tau) = 0 & \tau > 0,\\
(iv) & \dot{b}(\tau) + \frac{b(\tau)}{2} \le -W_\eta(b(\tau,\tau)) \quad (\text{resp.} \ge -W_\eta(b(\tau,\tau))) & \tau > 0,\\
(v) & b(0) \le b_0 \quad (\text{resp. } b(0) \ge b_0) & \\
(vi) & W(\eta,0) \le u_0(\eta) \quad (\text{resp. } W(\eta,0) \ge u_0(\eta)) & 0 < \eta < b(0).
\end{array} \tag{6}$$

The following comparison principle is a fundamental tool in our article.

Theorem 2. *Let* $(\underline{W}_1(\eta,\tau), \underline{b}_1(\tau))$ *(respectively,* $(\overline{W}_2(\eta,\tau), \overline{b}_2(\tau))$*) be the extensions by zero of the lower (respectively, upper solutions) of* (3) *corresponding to the data* (h_1, u_{01}, b_{01}) *(respectively,* (h_2, u_{02}, b_{02})*). If* $h_1 \le h_2$, $u_{01} \le u_{02}$ *and* $b_{01} \le b_{02}$, *then* $\underline{b}_1(\tau) \le \overline{b}_2(\tau)$ *for every* $\tau > 0$ *and* $\underline{W}_1(\eta,\tau) \le \overline{W}_2(\eta,\tau)$ *for every* $\eta \ge 0$ *and* $\tau \ge 0$.

Proof. The proof is rather similar to those presented by [6, Lemma 2.2 and Remark 2.3] and [5, Lemma 3.5]. We omit it here. □

In fact, we will construct lower and upper solutions, which are independent of time. For this purpose, we present the perturbed stationary problem

$$\begin{array}{ll}
(i) & W_{\eta\eta}(\eta) + \frac{1}{2}\lambda\eta W_\eta(\eta) = 0 \quad 0 < \eta < b_\lambda,\\
(ii) & -W_\eta(0) = \tilde{h}, \qquad W(b_\lambda) = 0,\\
(iii) & \frac{b_\lambda}{2} = -W_\eta(b_\lambda),
\end{array} \tag{7}$$

whose solution is given by the pair (U_λ, b_λ), where $U_\lambda(\eta) = \tilde{h}\int_\eta^{b_\lambda} e^{-\lambda s^2/4}\mathrm{d}s$, for given λ, and b_λ is the unique solution to

$$\tilde{h} = \frac{b_\lambda}{2}\, e^{\lambda b_\lambda^2/4}. \tag{8}$$

Remark 1. It is easy to see that for every $\lambda > 0$ and η in $(0, b_\lambda)$, $U_\lambda \ge 0$, $(U_\lambda)_\eta < 0$ and $(U_\lambda)_{\eta\eta} > 0$. In particular, U_λ is a linear function for $\lambda = 0$, and it is a strictly convex function if $\lambda > 0$.

Lemma 2. *Let* (U_λ, b_λ) *be a solution of* (7). *If* $\tilde{h} \le h$, *then for all* $\lambda > 1$ (U_λ, b_λ) *is an independent of time lower solution to* (3).

Proof. First we show that if $\lambda > 1$ is sufficiently large, then solutions of (7) are lower solutions. Indeed

$$-\left\{(U_\lambda)_{\eta\eta} + \frac{\eta}{2}(U_\lambda)_\eta\right\} = -\left\{(U_\lambda)_{\eta\eta} + \lambda\frac{\eta}{2}(U_\lambda)_\eta\right\} - \frac{\eta - \lambda\eta}{2}(U_\lambda)_\eta$$
$$= \frac{\eta(\lambda - 1)}{2}(U_\lambda)_\eta < 0, \tag{9}$$

for all $\eta \in (0, b_\lambda)$. Now, from (8) and the inequality $e^x > x$ for all $x > 0$, it holds that $0 < b_\lambda = 2\tilde{h}e^{-\lambda b_\lambda^2/4} \le \dfrac{8\tilde{h}}{\lambda b_\lambda^2}$, which implies that $b_\lambda \to 0$ as $\lambda \to \infty$; thus we can choose λ large enough so that $b_\lambda < b_0$.

Next we show that we can choose λ such that $U_\lambda \le u_0$. On the one hand we have that

$$u_0(\eta) = u_0(0) + \int_0^\eta u_0'(s)\,ds \ge u_0(0) - M\eta, \quad \text{for all } 0 \le \eta \le b_0.$$

On the other hand, for every $0 < \eta < b_\lambda$, we deduce from the strict convexity of U_λ discussed in Remark 1 that

$$U_\lambda(\eta) < U_\lambda(0) + \frac{U_\lambda(b_\lambda) - U_\lambda(0)}{b_\lambda}\eta = U_\lambda(0) - \frac{U_\lambda(0)}{b_\lambda}\eta. \tag{10}$$

Also we remark that

$$U_\lambda(0) = \tilde{h}\int_0^{b_\lambda} e^{-\lambda s^2/4}\mathrm{d}s \le \tilde{h}b_\lambda \to 0, \quad \lambda \to \infty, \tag{11}$$

and recall that by the hypothesis (2) $u_0(0) > 0$. Thus, if we choose λ large enough so that $b_\lambda < \dfrac{u_0(0)}{M}$ and $U_\lambda(0) < u_0(0)$, it follows that

$$U_\lambda(\eta) \le u_0(\eta) \quad \text{for all } \eta \in [0, b_\lambda]. \tag{12}$$

We conclude that (U_λ, b_λ) is a lower solution according to Definition 1. □

Now, we define the pair $(\underline{W}_\lambda, \underline{b}_\lambda)$ by

$$\underline{b}_\lambda = b_\lambda \quad \text{and} \quad \underline{W}_\lambda(\eta) := \begin{cases} U_\lambda(\eta) & \text{if } 0 \le \eta \le \underline{b}_\lambda, \\ 0 & \text{if } \eta > \underline{b}_\lambda. \end{cases} \tag{13}$$

Next we propose an upper solution which is a straight line on its support. It is easy to verify that the pair $(\overline{W}, \bar{b})$ defined by

$$\bar{b} \ge b_0 \text{ with } \bar{b} \ge 2h \quad \text{and} \quad \overline{W}(\eta) := \begin{cases} \dfrac{\bar{b}}{2}(\bar{b} - \eta), & \text{if } 0 \le \eta \le \bar{b}, \\ 0 & \text{if } \eta > \bar{b} \end{cases} \tag{14}$$

is an upper solution. Next, we give an additional condition in order to ensure that $\overline{W} \ge u_0$ on the interval $[0, b_0]$. Since

$$u_0(\eta) \le M(b_0 - \eta) \text{ for all } \eta \in (0, b_0),$$

we deduce that if $\bar{b} \ge \sqrt{2Mb_0}$, then

$$\overline{W}(\eta) \ge u_0(\eta) \text{ for all } \eta \in (0, b_0).$$

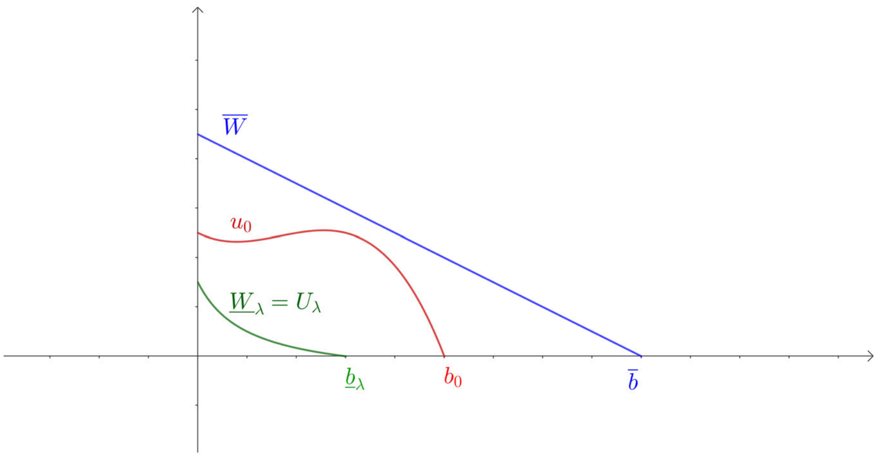

Figure 1. Lower and upper solutions

3. Convergence

Problems (1) and (3) are equivalent, because they are related by a non-singular change of variables. Hence, we automatically have solutions to (3). In this section the dependence of solutions on their data, (W_0, b_0), plays an important role. Thus, we will write $(W, b) = (W(\cdot, \cdot, (W_0, b_0)), b(\cdot, (W_0, b_0)))$ to emphasize this dependence.

In this section, the pair $(\underline{W}_\lambda, \underline{b}_\lambda)$ is the lower solution for a fixed λ given by (13) and $(\overline{W}, \overline{b})$ is the upper solution given in (14). We shall write

$$\underline{W}(\eta, \tau) := W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda)), \quad \underline{b}(\tau) = b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \tag{15}$$

and

$$\overline{W}(\eta, \tau) := W(\eta, \tau, (\overline{W}, \overline{b})), \quad \overline{b}(\tau) = b(\tau, (\overline{W}, \overline{b})) \tag{16}$$

to denote solutions of (3) with initial conditions $(\underline{W}_\lambda, \underline{b}_\lambda)$ and $(\overline{W}, \overline{b})$.

Lemma 3. Positivity and boundedness
There holds:

$$0 \leq \underline{W}_\lambda(\eta) \leq \underline{W}(\eta, \tau) \leq W(\eta, \tau, (u_0, b_0)) \leq \overline{W}(\eta, \tau) \leq \overline{W}(\eta) \leq \frac{\overline{b}^2}{2},$$

and

$$0 \leq b_\lambda \leq b(\tau, (u_0, b_0)) \leq \overline{b}.$$

Proof. Repeatedly apply the comparison principle Theorem 2. □

Lemma 4. [3] Monotonicity in time

(a) *The functions $\underline{W}(\eta, \tau)$ and $\underline{b}(\tau)$, are non-decreasing in time.*
(b) *The functions $\overline{W}(\eta, \tau)$ and $\overline{b}(\tau)$, are non-increasing in time.*

Proof. We only prove part a). From Theorem 2 we deduce that

$$W(\eta, s, (\underline{W}_\lambda, \underline{b}_\lambda)) \geq \underline{W}_\lambda(\eta) \quad \text{and} \quad b(s, (\underline{W}_\lambda, \underline{b}_\lambda)) \geq \underline{b}_\lambda, \quad \text{for all } s \geq 0. \tag{17}$$

Now, for a fixed $s = \sigma$, we consider the pair (W^σ, b^σ) where

$$W^\sigma(\eta) := W(\eta, \sigma, (\underline{W}_\lambda, \underline{b}_\lambda)) \quad \text{and} \quad b^\sigma := b(\sigma, (\underline{W}_\lambda, \underline{b}_\lambda)). \tag{18}$$

In particular we have $W^\sigma(\eta) \geq \underline{W}_\lambda(\eta)$ and $b^\sigma \geq \underline{b}_\lambda$. Then we apply again Theorem 2 to deduce that for every $\tau \geq 0$

$$W(\eta, \tau, (W^\sigma, b^\sigma) \geq W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \quad \text{and} \quad b(\tau, (W^\sigma, b^\sigma)) \geq b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda)). \tag{19}$$

Returning to (17), now consider $s = \tau + \sigma$ for $\tau \geq 0$. It holds that the pair $(W(\eta, \tau + \sigma, (\underline{W}_\lambda, \underline{b}_\lambda)), b(\tau + \sigma, (\underline{W}_\lambda, \underline{b}_\lambda)))$ is a solution to problem (3) for the initial conditions (18) for every $\tau > 0$. From the uniqueness of the solution we deduce that for all $\tau \geq 0$ we have

$$\begin{aligned} W(\eta, \tau, (W^\sigma, b^\sigma) =& W(\eta, \tau + \sigma, (\underline{W}_\lambda, \underline{b}_\lambda)) \text{ and} \\ b(\tau, (W^\sigma, b^\sigma)) =& b(\tau + \sigma, (\underline{W}_\lambda, \underline{b}_\lambda)). \end{aligned} \tag{20}$$

Substituting (20) in (19) we deduce that

$$\begin{aligned} W(\eta, \tau + \sigma, (\underline{W}_\lambda, \underline{b}_\lambda)) \geq& W(\eta, \tau, (\underline{W}_\lambda, \underline{b}_\lambda)) \text{ and} \\ b(\tau + \sigma, (\underline{W}_\lambda, \underline{b}_\lambda)) \geq& b(\tau, (\underline{W}_\lambda, \underline{b}_\lambda)), \end{aligned} \tag{21}$$

which completes the proof of part a). □

We remark that if $(\underline{W}, \underline{b})$ (resp. $(\overline{W}, \bar{b})$) is defined in (15) (resp. (16)), then the Lemmas 3 and 4 imply that for every $\lambda > 0$

$$0 < b_\lambda \leq \lim_{\tau \to \infty} \underline{b}(\tau) = \underline{b}^\infty \leq \lim_{\tau \to \infty} \bar{b}(\tau) = \bar{b}^\infty \leq \bar{b}.$$

In addition, the Lemmas 3 and 4 imply the convergence of $\underline{W}$ and $\overline{W}$, namely

$$0 \leq \lim_{\tau \to \infty} \underline{W}(\eta, \tau) = \underline{W}^\infty(\eta) \leq \frac{\bar{b}^2}{2}, \text{ for all } \eta \in [0, \underline{b}^\infty]; \tag{22}$$

and

$$0 \leq \lim_{\tau \to \infty} \overline{W}(\eta, \tau) = \overline{W}^\infty(\eta) \leq \frac{\bar{b}^2}{2} \text{ for all } \eta \in [0, \bar{b}^\infty). \tag{23}$$

Finally we state and prove the main result of this paper, which in turn implies the result of Theorem 1.

Theorem 3. *Let $(W(\eta, \tau, (u_0, b_0)), b(\tau, (u_0, b_0)))$ be the solution to the free boundary problem (3) associated to the initial data (u_0, b_0). If (U, ω) is the unique steady state of (3) given by Lemma 1, then*

(a) $\lim_{\tau \to \infty} b(\tau) = \omega$,

(b) *$W(\cdot, \tau)$ converges to U uniformly on $[0, \beta]$ for any $\beta \in (0, \omega)$, when $\tau \to \infty$.*

(c) *At each time τ let us extend W to $(0, \infty)$ by the formula*

$$\tilde{W}(\eta, \tau) = \begin{cases} W(\eta, \tau) & \eta \in [0, b(\tau)], \\ 0 & \eta \in (b(\tau), \infty). \end{cases}$$

Then, $\tilde{W}(\cdot,\tau)$ *converges uniformly to* $\tilde{W}^\infty$ *on* $[0,\infty)$*, where*

$$\tilde{W}^\infty(\eta) = \begin{cases} U, \ \eta \in [0,\omega], \\ 0, \ \ \eta \in (\omega,\infty). \end{cases}$$

Proof. Step 1. We have to identify the limits $\underline{W}_\infty$ and $\overline{W}^\infty$, and improve the convergence. For this purpose, we shall show the estimate,

$$\int_T^{T+1}\int_0^{b(\tau)} W_\eta^2(\eta,\tau)\, d\eta d\tau \le M_1, \tag{24}$$

where $W=\underline{W}$ or $W=\overline{W}$ are the functions defined in (15) and (16), respectively.

Proposition 1 ensures existence and uniqueness of classical solutions. When we want to restrict our attention to $\Omega_{T,1} := \{(\eta,\tau) : \eta \in (0,b(\tau)), \tau \in (T,T+1)\}$, then these solutions are understood as continous up the boundary of $\Omega_{T,1}$ with the additional following properties $W_\eta \in C(\overline{\Omega}_{T,1})$ and $W_\tau, W_{\eta\eta} \in C(\Omega_{T,1})$, but this does not guarentee that $W_{\eta\eta}$ and W_τ are square integrable over $\Omega_{T,1}$. This is why we set $\psi^\delta \in W^{1,\infty}(\{(\eta,\tau) : \eta \in (0,b(\tau)), \tau \in (0,\infty)\})$

$$\psi^\delta(\eta,\tau) = \begin{cases} 0 & \eta \in [0,\delta), \\ \frac{1}{\delta}(\eta-\delta) & \eta \in [\delta,2\delta), \\ 1 & \eta \in [2\delta, b(\tau)-2\delta), \\ 1-\frac{1}{\delta}(\eta-b(\tau)+2\delta) & \eta \in [b(\tau)-2\delta, b(\tau)-\delta), \\ 0 & \eta \in [b(\tau)-\delta,\infty) \end{cases}$$

and $\phi^\delta \in W^{1,\infty}((0,\infty))$, where

$$\phi^\delta(\tau) = \begin{cases} 0 & \tau \in [0,T+\delta), \\ \frac{1}{\delta}(\tau-T-\delta) & \tau \in [T+\delta,T+2\delta), \\ 1 & \tau \in [T+2\delta, T+1-2\delta)), \\ 1-\frac{1}{\delta}(\tau-(T+1-2\delta)) & \tau \in [T+1-2\delta, T+1-\delta), \\ 0 & \tau \in [T+1-\delta,\infty). \end{cases}$$

Finally, we set $\varphi^\delta(\eta,\tau) = \psi^\delta(\eta,\tau)\phi^\delta(\tau)$. Now, let us multiply the equation (3)$-(i)$ by $\varphi^\delta W$ and integrate over $\Omega_{T,1}$. We arrive at

$$L_1^\delta(T) := \int_{\Omega_{T,1}} W W_\tau \varphi^\delta \, d\eta d\tau = \int_{\Omega_{T,1}} \left(W W_{\eta\eta} + \frac{\eta}{2} W_\eta W \right) \varphi^\delta \, d\eta d\tau =: R_1^\delta(T).$$

We first analyze the left-hand-side, we see that integration by parts yields,

$$2L_1^\delta(T) = \int_{\Omega_{T,1}} \varphi^\delta (W^2)_\tau \, d\tau d\eta = -\int_{\Omega_{T,1}} \varphi_\tau^\delta \ W^2 \, d\tau d\eta.$$

The boundary terms vanish, because the support of φ^δ does not intersect $\partial\Omega_{T,1}$. Now, we want to compute the limit $L_1(T) = \lim_{\delta\to 0^+} L_1^\delta(T)$. We remark that

$$L_1(T) = \lim_{\delta\to 0^+}\left(-\frac{1}{2\delta}\int_T^{T+1}\int_{b(\tau)-2\delta}^{b(\tau)-\delta}\dot b(\tau)\phi^\delta(\tau)W^2(\eta,\tau)\,d\eta d\tau + \frac{1}{2\delta}\int_{T+1-2\delta}^{T+1-\delta}\int_0^{b(\tau)}\psi^\delta W^2\,d\eta d\tau - \frac{1}{2\delta}\int_{T+\delta}^{T+2\delta}\int_0^{b(\tau)}\psi^\delta W^2\,d\eta d\tau\right).$$

For the limit in the first integral, note that from the continuity of the integrand, we can apply the mean value property for integrals to obtain that

$$\lim_{\delta\to 0^+}\frac{1}{2\delta}\int_T^{T+1}\int_{b(\tau)-2\delta}^{b(\tau)-\delta}\dot b(\tau)\phi^\delta(\tau)W^2(\eta,\tau)\,d\eta d\tau = \frac{1}{2}\int_T^{T+1}\dot b(\tau)W^2(b(\tau),\tau)\,d\tau.$$

Applying the condition at the moving boundary, we deduce that

$$\lim_{\delta\to 0^+}\frac{1}{2\delta}\int_T^{T+1}\int_{b(\tau)-2\delta}^{b(\tau)-\delta}\dot b(\tau)\phi^\delta(\tau)W^2(\eta,\tau)\,d\eta d\tau = 0.$$

As for the other terms, we proceed in a similar way to deduce that

$$\lim_{\delta\to 0^+}\frac{1}{2\delta}\int_{T+1-2\delta}^{T+1-\delta}\int_0^{b(\tau)}\psi^\delta W^2\,d\eta d\tau = \frac{1}{2}\int_0^{b(T+1)}W^2(\eta,T+1)\,d\eta,$$

and that

$$\lim_{\delta\to 0^+}\frac{1}{2\delta}\int_{T+\delta}^{T+2\delta}\int_0^{b(\tau)}\psi^\delta W^2\,d\eta d\tau = \frac{1}{2}\int_0^{b(T)}W^2(\eta,T)\,d\eta.$$

Hence, we conclude that

$$L_1(T) = \frac{1}{2}\int_0^{b(T+1)}W^2(\eta,T+1)\,d\eta - \frac{1}{2}\int_0^{b(T)}W^2(\eta,T)\,d\eta.$$

Next we consider the term $R_1^\delta(T)$. We integrate by parts to obtain

$$R_1^\delta(T) = -\int_{\Omega_{T,1}}\left(W_\eta^2\varphi^\delta + WW_\eta\varphi_\eta^\delta + \frac{1}{4}W^2(\eta\varphi_\eta^\delta + \varphi^\delta)\right)d\eta d\tau.$$

Again here, the boundary terms vanish, because the support of φ^δ is contained in $\Omega_{T,1}$. We remark that the same argument as above leads us to

$$\begin{aligned}
&\lim_{\delta\to 0^+}\int_{\Omega_{T,1}}WW_\eta\varphi_\eta^\delta\,d\eta d\tau \\
&\quad= \lim_{\delta\to 0^+}\frac{1}{\delta}\int_T^{T+1}\left(\int_\delta^{2\delta}WW_\eta\phi^\delta\,d\eta d\tau - \int_{b(\tau)-2\delta}^{b(\tau)-\delta}WW_\eta\phi^\delta\,d\eta d\tau\right) \\
&\quad= \int_T^{T+1}(-hW(0,\tau) - W(b(\tau),\tau)W_\eta(b(\tau),\tau))\,d\eta d\tau \\
&\quad= -\int_T^{T+1}hW(0,\tau)\,d\eta d\tau,
\end{aligned}$$

where we have also used the boundary condition (3−ii) as well as the condition at the interface (3−iii). Similarly one can show that

$$\lim_{\delta\to 0^+}\int_{\Omega_{T,1}} \frac{1}{4}W^2(\eta\varphi^\delta_\eta+\varphi^\delta))\,d\eta d\tau=\frac{1}{4}\int_{\Omega_{T,1}} W^2(\eta,\tau)\,d\eta d\tau.$$

Hence, we conclude

$$\begin{aligned}R_1(T)&=\lim_{\delta\to 0^+}R_1^\delta(T)\\&=-\int_{\Omega_{T,1}} W_\eta^2\,d\tau d\eta+h\int_T^{T+1}W(0,\tau)\,d\tau-\frac{1}{4}\int_{\Omega_{T,1}} W^2\,d\tau d\eta.\end{aligned}$$

In view of Lemma 3 a possible choice of the constant in (24) M_1 is given by

$$M_1=h\frac{\overline{b}^2}{2}+\frac{\overline{b}^5}{8}.$$

Step 2. Applying the mean value theorem for integrals in (24) we deduce from Step 1 that there exist two sequences of points $\tau_n',\tau_n''\in[n,n+1)$ such that

$$\int_0^{\underline{b}(\tau_n')}\underline{W}_\eta^2(\eta,\tau_n')\,d\eta,\quad \int_0^{\overline{b}(\tau_n'')}\overline{W}_\eta^2(\eta,\tau_n'')\,d\eta\le M_1. \tag{25}$$

Since $\overline{b}^\infty\le\overline{b}(\tau_n'')$, it follows that (25) implies the bound

$$\|\overline{W}_\eta(\cdot,\tau_n'')\|_{L^2(0,\overline{b}^\infty)}\le\sqrt{M_1}.$$

Moreover, if we fix any $\beta\in(0,\underline{b}^\infty)$, then for all sufficiently large n we have $\underline{b}(\tau_n')\in(\beta,\underline{b}^\infty)$. Hence, (25) implies the bound

$$\|\underline{W}_\eta(\cdot,\tau_n')\|_{L^2(0,\beta)}\le\sqrt{M_1}$$

for all $\beta\in(0,\underline{b}^\infty)$. Hence, we can select subsequences (not relabelled) such that

$$\underline{W}_\eta(\cdot,\tau_n')\rightharpoonup\psi^\beta \text{ in } L^2(0,\beta)\quad\text{and}\quad \overline{W}_\eta(\cdot,\tau_n'')\rightharpoonup\Psi \text{ in } L^2(0,\overline{b}^\infty).$$

Due to (22) and (23) the limits of $\underline{W}(\cdot,\tau_n')$ and $\overline{W}(\cdot,\tau_n'')$ are uniquely defined, so is the case for $\underline{W}_\eta(\cdot,\tau_n')$ and $\overline{W}_\eta(\cdot,\tau_n'')$. Hence, we deduce that ψ^β and Ψ do not depend on the choice of the sequence τ_n and using (22) and (23) we conclude that ψ^β is the weak derivative $\underline{W}_\eta^\infty$ in every interval $(0,\beta)$ and that $\Psi=\overline{W}_\eta^\infty$. So that $\underline{W}^\infty\chi_{(0,\beta)}\in H^1(0,\beta)$ for all $\beta\in(0,\underline{b}^\infty)$ and thus $\overline{W}^\infty\in H^1(0,\overline{b}^\infty)$. In fact, $\underline{W}^\infty\in H^1(0,\underline{b}^\infty)$. Indeed, in view of Lebesgue monotone convergence theorem we have

$$\lim_{\beta\to\underline{b}^\infty}\int_0^{\underline{b}^\infty}(\underline{W}_\eta^\infty)^2(\eta)\chi_{(0,\beta)}(\eta)\,d\eta=\int_0^{\underline{b}^\infty}(\underline{W}_\eta^\infty)^2(\eta)\,d\eta\le M_1.$$

Step 3. We claim that $(\underline{W}^\infty,\underline{b}^\infty)$ and $(\overline{W}^\infty,\overline{b}^\infty)$ are both stationary solutions, namely solutions of (4). Hence they are smooth and equal. Indeed, we multiply the equation (3−i) by $\varphi \in C_c^\infty(\mathbb{R})$ such that $\varphi_\eta(0) = 0$ and we integrate on $\Omega_{T,1}$. Proceeding as in step 2 we set

$$L_2(T) = \int_{\Omega_{T,1}} W_\tau \varphi \, d\eta d\tau = \int_{\Omega_{T,1}} \left(W_{\eta\eta}\varphi + \frac{\eta}{2} W_\eta \varphi \right) d\eta d\tau = R_2(T).$$

$$\begin{aligned} L_2(T) &= \int_{\Omega_{T,1}} W_\tau(\eta,\tau)\varphi(\eta) \, d\eta d\tau \\ &= \int_0^{b(T+1)} W(\eta, T+1)\varphi(\eta) \, d\eta - \int_0^{b(T)} W(\eta, T)\varphi(\eta) \, d\eta, \end{aligned}$$

where $W = \underline{W}$ or $W = \overline{W}$. We then deduce from Lebesgue's dominated convergence theorem that $\lim_{T\to\infty} L_2(T) = 0$.

Next, we investigate the right-hand-side $R_2(T)$. Integration by parts yields

$$\begin{aligned} R_2(T) &= \int_{\Omega_{T,1}} W_{\eta\eta}\varphi \, d\eta d\tau + \int_{\Omega_{T,1}} \frac{\eta}{2} W_\eta \varphi \, d\eta d\tau \\ &= \int_{\Omega_{T,1}} W \left(\varphi_{\eta\eta} - \frac{1}{2}(\eta\varphi)_\eta \right) d\eta d\tau - \int_T^{T+1} \left(\dot{b} + \frac{b}{2} \right) \varphi(b(\tau)) + h\varphi(0). \end{aligned}$$

Now, we pass to the limit as $T \to \infty$. It follows from Lebesgue's dominated convergence theorem that

$$\lim_{T\to\infty} \int_{\Omega_{T,1}} W(\eta,\tau) \left(\varphi_{\eta\eta} - \frac{1}{2}(\eta\varphi)_\eta \right) d\eta d\tau = \int_0^{b^\infty} W^\infty(\eta) \left(\varphi_{\eta\eta} - \frac{1}{2}(\eta\varphi)_\eta \right) d\eta,$$

where (W^∞, b^∞) is either $(\underline{W}^\infty, \underline{b}^\infty)$ or $(\overline{W}^\infty, \overline{b}^\infty)$. Let us denote by Φ an antiderivative of φ. Then,

$$\lim_{T\to\infty} \int_T^{T+1} \dot{b}\varphi(b(\tau)) \, d\tau = \lim_{T\to\infty} (\Phi(b(T+1)) - \Phi(b(T))) = 0.$$

In addition,

$$\lim_{T\to\infty} \int_T^{T+1} \frac{b}{2}\varphi(b(\tau)) \, d\tau = \frac{1}{2} b^\infty \varphi(b^\infty).$$

Finally, we collect all the results concerning $R_2(T)$, while keeping in mind that $\lim_{T\to\infty} L_2(T) = 0$. This yields

$$0 = \lim_{T\to\infty} R_2(T) = \int_0^{b^\infty} W^\infty(\eta) \left(\varphi_{\eta\eta} - \frac{1}{2}(\eta\varphi)_\eta \right) d\eta - \frac{b^\infty}{2}\varphi(b^\infty) + h\varphi(0), \tag{26}$$

for all smooth functions φ in $\mathbb{R}$ such that $\varphi_\eta(0) = 0$. In particular W^∞ satisfies the differential equation (4−i) in the sense of distributions.

Step 4. We recall that $W^\infty \in H^1(0, b^\infty)$. It is easy to infer from (26) that $W^\infty_{\eta\eta} = -\frac{\eta}{2} W^\infty_\eta \in L^2(0, b^\infty)$, which in turn implies that $W^\infty \in H^2(0, b^\infty)$.

Next we search for the boundary condition and the conditions on the moving boundary satisfied by W^∞. After integrating by parts twice in (26) we obtain,

$$0 = \int_0^{b^\infty} \left(W^\infty_{\eta\eta} + \frac{\eta}{2} W^\infty_\eta \right) \varphi \, d\eta + W^\infty \varphi_\eta |_{\eta=0}^{\eta=b^\infty} - W^\infty_\eta \varphi |_{\eta=0}^{\eta=b^\infty} - \frac{b^\infty}{2} \varphi(b^\infty)(W^\infty(b^\infty) + 1) + h\varphi(0),$$

so that

$$0 = W^\infty(b^\infty)\varphi_\eta(b^\infty) - W^\infty_\eta(b^\infty)\varphi(b^\infty) + W^\infty_\eta(0)\varphi(0) - \frac{b^\infty}{2}\varphi(b^\infty)(W^\infty(b^\infty) + 1) + h\varphi(0), \quad (27)$$

for all smooth functions φ on $\mathbb{R}$ such that $\varphi_\eta(0) = 0$. Now, if we additionally choose φ such that $\varphi(b^\infty) = \varphi_\eta(b^\infty) = 0$, then (27) reduces to

$$\varphi(0)(W^\infty_\eta(0) + h) = 0,$$

and since $\varphi(0)$ is arbitrary, we deduce that

$$W^\infty_\eta(0) + h = 0.$$

Thus (27) becomes

$$0 = W^\infty(b^\infty)\varphi_\eta(b^\infty) - W^\infty_\eta(b^\infty)\varphi(b^\infty) - \frac{b^\infty}{2}\varphi(b^\infty)(W^\infty(b^\infty) + 1). \quad (28)$$

Next we suppose that $\varphi(b^\infty) = 0$, but $\varphi_\eta(b^\infty) \neq 0$. Then

$$W^\infty(b^\infty)\varphi_\eta(b^\infty) = 0,$$

and hence

$$W^\infty(b^\infty) = 0.$$

Then, (28) becomes

$$0 = -W^\infty_\eta(b^\infty)\varphi(b^\infty) - \frac{b^\infty}{2}\varphi(b^\infty). \quad (29)$$

Suppose that $\varphi(b^\infty) \neq 0$. Then (29) implies that

$$W^\infty_\eta(b^\infty) = -\frac{b^\infty}{2}.$$

We deduce that the solution pair (W^∞, b^∞) coincides with the unique solution of Problem (4), i.e. $(W^\infty, b^\infty) = (U, \omega)$ or in other words with the unique steady state solution of the time evolution problem, Problem (3).

Step 5. We recall that, in view of step 4, $W^\infty \in H^2(0, b^\infty) \subset C^{1,\frac{1}{2}}([0, b^\infty])$. Moreover, the convergence of $\overline{W}$ to W^∞ (resp. $\underline{\tilde{W}}$ to W^∞) is monotone on $[0, b^\infty]$. Hence, we deduce with the help of Dini's Theorem that this convergence is uniform.

Finally, since $b_\lambda \leq b_0 \leq \bar{b}$ and since $\underline{W}_\lambda \leq u_0 \leq \overline{W}$, the comparison principle implies that $\underline{b}(\tau) \leq b(\tau) \leq \bar{b}(\tau)$ and $\underline{\tilde{W}}(\eta, \tau) \leq \tilde{W}(\eta, \tau) \leq W(\eta, \tau) \leq \overline{W}(\eta, \tau)$ for all (η, τ). We conclude that $b(\tau) \to b^\infty$ and that $\tilde{W}(\tau)$ converges to W^∞ uniformly on compact sets of $[0, \infty)$ as $\tau \to \infty$. $\square$

Now, Theorem 1 easily follows from Theorem 3.

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