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An inverse bilevel equilibrium problem for public transport demand estimation

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Abstract: We consider the problem of demand estimation for public transport networks. Given an origin-destination matrix representing the public transport demand, the distribution of flow among different lines can be obtained assuming that it corresponds to a certain equilibrium characterized by an optimization problem.
The knowledge of the origin-destination matrix is expensive and sometimes unaffordable in practice. Traditionally, it is estimated using statistical or econometrical considerations. In this work, we explore the estimation through the numerical solution of a bilevel optimization problem. One disadvantage of this formulation is the difficulty of obtaining descent directions, therefore we proposed a derivative-free method for the resolution of the optimization problem. The method is firstly tested on small networks using a derivative-free optimization method and then, using an approach based on simulation. This simulation-optimization methodology showed as good results as analytical modeling, opening the door to handle bigger networks where analytical computation is hard to accomplish.

Keywords: Transit assignment; Public-transport demand; Bi-level optimization; Simulation; Equilibrium model

1. INTRODUCTION

Transit assignment models have become an interesting research area because knowing the passenger behavior allows comparing different planning scenarios in terms of network performance. Such models typically assume that the transport demand is known.

Many models for passenger behavior have been proposed. Most of them consider that when a passenger decides to travel between certain O-D pairs and is waiting for a vehicle at a stop, he must decide which transit line he should take to minimize his total expected travel time (including access, wait and in-vehicle time). Other models consider that passengers seek to minimize their generalized cost, which includes not only the total travel time but also in-vehicle crowding and fares, among others. Among the first models that considered congestion effects, we can cite Spiess and Florian (1989) that work with the concept of hyperpath composed by "strategies of attractive lines", but failed to be realistic in cases of high demand because waiting times are considered flow independent, assuming that passengers always can take the first bus that arrives at the stop. This is not necessarily true in a congested network.

De Cea and Fernández (1993) began to consider the congestion effects at bus stops and inside the bus. This model was improved in Cominetti and Correa (2001) formulating a transit equilibrium problem that uses effective frequency functions that vanish if the in-vehicle flow exceeds its capacity (see 4). The main limitation of these methods is that the technical assumptions are very limiting in the first case and there no efficient algorithms to compute the solution in both cases.

Cepeda et al. (2006) decided to continue this idea and reformulated the equilibrium problem as the minimization of a nonconvex and nondifferentiable gap function. To solve this problem a heuristic method was proposed, using an adaptation of the Method of Successive Averages (MSA) and obtaining the lines flow vector. This method can be applied on high-scale networks without computational drawbacks but can generate line flows that exceed the capacity when the demands are high. To improve this method, Codina and Rosell (2017) presented an algorithm with strict capacities that finds the solution of the

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fixed point inclusion formulation derived from the problem of variational inequality proposed by Codina (2013). At each iteration an assignment problem is solved, using Lagrangian duality and a cutting-planes method.

The use of the previous models of transit assignment in any planning study requires the knowledge of the transport demand, commonly known as the origin-destination matrix. Obtaining that matrix could be very expensive and sometimes unaffordable in practice. As has been made for the case of traffic assignment (see Walpen et al. (2015)), in this work we explore its estimation through some directly measurable quantities like the real frequencies of the buses. As we know how to compute, given the demand, the flows, and hence the frequencies, we pose a kind of inverse problem whose solution estimates the actual demand. We implemented this idea in Bhouri et al. (2020) and here we reproduce those results and compare them within a simulation approach. As far as we know, previous works about public transport demand estimation do not use this approach. Most of them are based on statistical or econometrical considerations, see Ortuzar and Willumsen (2001); Cascetta (2009); Dike et al. (2018); García-Ferrer et al. (2006).

In the next section, we present a detailed description of the assignment model following the one presented in Cepeda et al. (2006). In section 3 we pose the inverse problem used for demand estimation and in section 4 we present the numerical experiments made with the example given in Cepeda et al. (2006).

2. TRANSIT ASSIGNMENT MODEL

Following the notation of previous works as Spiess and Florian (1989); Cominetti and Correa (2001); Cepeda et al. (2006); Codina (2013) we consider a directed graph $G = (N, A)$ where N is the node set and A the set of arcs, each one with cardinality N_N and N_A. The set of nodes is composed of the bus-stop nodes N_s and the line nodes N_l. The arcs are divided into the alighting and boarding arcs connecting the bus-stop nodes with the line nodes, the on-board arcs (or line segments) connecting line-nodes, and the walk arcs connecting bus-stop nodes, see Figure 1 for a sketch.

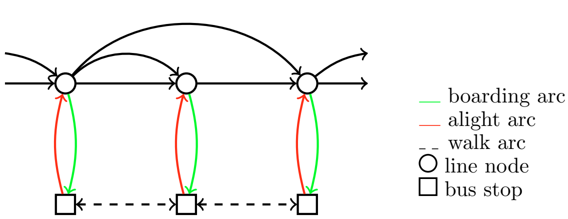


Fig. 1. Public transport network.

For some origin-destination (od) pairs $(i, d) \in W \subset N \times N$, there is a transport demand called g_i^d, and we call D the set of all nodes d that are destinations of some od pair. For a node i we call A_i^+ the set of outgoing arcs and A_i^- the incoming arcs set. We also define the node-arc incidence matrix $A \in \mathbb{R}^{N_N \times N_A}$ where $A_{ia} = 1$ iff $a \in A_i^+$, $A_{ia} = -1$ iff $a \in A_i^-$ and otherwise zero.

We call v_a^d the flow through arc a with destination $d \in D$. For each destination d we define the set of feasible flows with destination d and the set of total feasible flows as

$$V^d = \left\{ v^d \in \mathbb{R}_+^{N_A} : A v^d = g^d \right\}, \tag{1}$$

$$V = \left\{ v \in \mathbb{R}_+^{N_A} : v = \sum_d v^d, v^d \in V^d, \forall d \right\}. \tag{2}$$

We call $V(g)$ the set of feasible flows for the demand g, that is the set of all $v_a^d \geq 0$ such that $v_a^d = 0$ for all $a \in A_d^+$ and satisfying the flow conservation constraints:

$$g_i^d + \sum_{a \in A_i^-} v_a^d = \sum_{a \in A_i^+} v_a^d, \quad \forall i \neq d. \tag{3}$$

Two functions of the full flow vector v are associated to each arc, the travel time function $t_a(v)$ and the effective frequency $f_a(v)$. Both have non negative values and the frequencies can have the constant value $+\infty$. As mentioned in Cepeda et al. (2006) the case when t_a and f_a are constants is called the *uncongested* case and the case where only the frequencies f_a are fixed is called the *semicongested* case. Here we will consider a third case where the travel time function is constant but the frequencies are not. To model the impact of the bus load on the frequency the function 4 is used:

$$f_a(v) = \begin{cases} \mu \left[1 - \left(\frac{v_a}{\mu c - v_{a'} + v_a} \right)^{\beta} \right], & \text{if } v_{a'} < \mu c, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $v_a = \sum_{d \in D} v_a^d$ is the total flow boarding at stop and using arc a and $v_{a'}$ is the total flow after the stop ($v_{a'} \geq v_a$). The parameter μ is the nominal frequency of the lines and c is the physical capacity of the buses, thus, $\mu c - v_{a'}$ is the residual capacity waiting at the stop.

The rationale behind the model is that each passenger at each node chooses an arc to continue its trip. The decision is based on minimizing the total travel time. Thus, at each node a Common Line Problem should be solved: passengers select a nonempty subset of common lines $s \subseteq A$ and board the first vehicle that arrives at the stop and belongs to this set. The chosen strategy minimizes their total expected travel time. In addition, now the frequencies depend on the flows. In the paper, Cepeda et al. (2006) it is shown that the corresponding (equilibrium) flow $v \in V^*(g)$ is the global minimizer of the so-called gap function G of the flow v, that we write here also as a function of the demand g,

$$G(v, g) = \sum_{d \in D} \left[\sum_{a \in A} t_a(v) v_{a'}^d + \sum_{i \neq d} \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} - g_i^d \tau_i^d(v) \right], \tag{5}$$

where t_a is the travel time, τ_j^d is the total expected travel time from j to d, A_i^+ is the set of arcs emerging from i, f_a models the impact of the congestion on the frequency, μ is the nominal frequency of the line and c its capacity, β is a calibrated parameter and $v_{a'}$ is the on-board flow right after the stop.

Then the transit assignment for a given demand g is obtained minimizing $G(v, g)$ over the flows in $V(g)$. It is known, also by the work Cepeda et al. (2006), that the optimal value is 0. This is because function G is the difference between the total time experienced by passengers (travel time + maximum waiting time at stops) and the total expected travel time of the system. A detailed explanation about the construction and interpretation of gap function and its optimal value can be found in Cepeda et al. (2006).

To solve the assignment problem in Cepeda et al. (2006); Codina and Rosell (2017) the authors propose the MSA (Mean Successive Average) method. It means that starting with an all-or-nothing assignment, at each iteration travel times are updated and a new assignment (for fixed travel times and frequencies) is averaged with the previous one. Interestingly enough, in contrast to the traffic assignment problem, here we have a computable stopping criterium as we know that $G(v, g) = 0$ for an equilibrium. The assignment with fixed travel times and frequencies is made using the Hyperpath Dijkstra method as it was proposed in Cepeda et al. (2006); Spiess and Florian (1989).

For the sake of completeness we reproduce the MSA algorithm below:

Result: Flow at equilibrium
Let $\alpha_k \in (0, 1)$ such that $\alpha_k \to 0$ and $\sum_{k=0}^{\infty} \alpha_k = \infty$;
Find $v^0 \in V(g)$ and let $k = 0$;
while $G(v^k) > \epsilon G(v^0)$ **do**
  Compute $t_a = t_a(v^k)$ and $f_a = f_a(v^k)$;
  Compute the shortest hyperpath for each $d \in D$;
  Compute the induced flows $\hat{v}_a^d$;
  Update $v^{k+1} = (1 - \alpha_k)v^k + \alpha_k \hat{v}$;
  Set $k = k + 1$;
end

In order to obtain the first flow $v(0)$, an all-or-nothing assignment is made computing the shortest hyperpath for $t_a = t_a(0)$ and $f_a = f_a(0)$. If $f_a(v^0) = 0$ for some arc a, then the next iteration will be unfeasible. To avoid this situation, the effective frequency can be augmented to $\tilde{f}_a(v) = max\{f_a(v), \varepsilon\}$, for a small enough $\varepsilon > 0$. In this way, even for a large flow, there will always be a feasible arc.

3. DEMAND ESTIMATION PROBLEM

Assuming that the model carefully represents the real dynamic of the passengers, it is possible to use it to detect anomalies or changes in the demand data when the observed flow or frequencies are different from the computed ones.

Here we focus on correcting the given demand to comply with the observed frequencies. That is, given a nominal demand $\bar{g}$ and a subset of arcs $A_{obs} \subset A$ over which the frequency $\bar{f}$ is measured (observed), we look for the demand g that minimizes

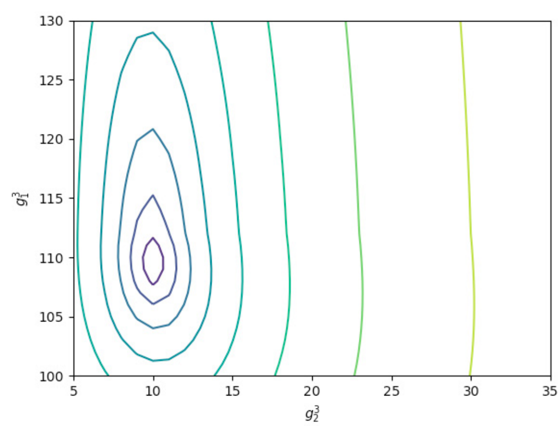
$$\min_{g,v} \sum_{a \in A_{obs}} \left(\frac{\bar{f}_a - f_a}{\bar{f}_a}\right)^2 + \gamma \sum_{a \in A} \left(\frac{\bar{g}_a - g_a}{\bar{g}_a}\right)^2 \quad (6)$$

$$s.t.$$

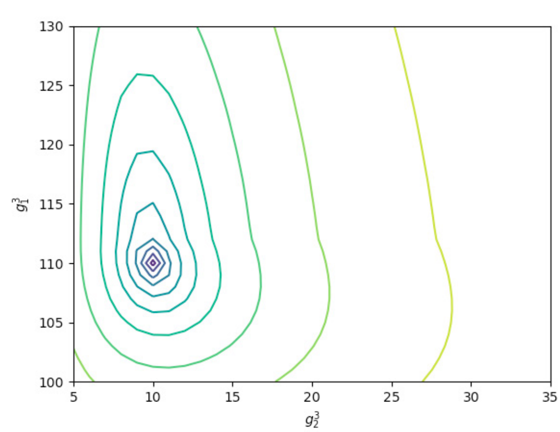
$$v \in V(g), \quad (7)$$

$$G(v, g) = 0. \quad (8)$$

More general quadratic criteria can be considered, for example including a coefficient for each arc that represents the confidence of the measures on that arc. The regularization parameter γ represents the trade-off between adjusting the observed frequencies and conserving the nominal demand; in Figure 2 we show the level curves computed for different values of γ in the case of the first example in section 4. The regularization term has a beneficial effect on the convexity of the problem and also on the uniqueness of its solution (see again Figure 2, where sublevel sets are "more convex" for γ higher), but large values of γ make the problem to ignore the observations. This behavior is also observed in the second example in section 4; in Figure 3 we show the distances between nominal and estimated demand (blue line) and measured and obtained frequencies (orange line) for different values of γ. It can be seen how



(a) $\gamma = 1/5$



(b) $\gamma = 1/100$

Fig. 2. Level curves (log scale)

as γ grows, the distance $d(\bar{g}_a, g_a) = \sqrt{\sum_{a \in A} \left(\frac{\bar{g}_a - g_a}{\bar{g}_a} \right)^2}$ decreases, but $d(\bar{f}_a, f_a) = \sqrt{\sum_{a \in A_{obs}} \left(\frac{\bar{f}_a - f_a}{\bar{f}_a} \right)^2}$ increases.

Nevertheless, even for large values of γ, i.e, for a more convex problem, the numerical solution of this bilevel problem is rather involved because the flow $v(g)$ is given implicitly by $G(v, g) = 0$ and there is not an easy way to compute variations of v with respect to g.

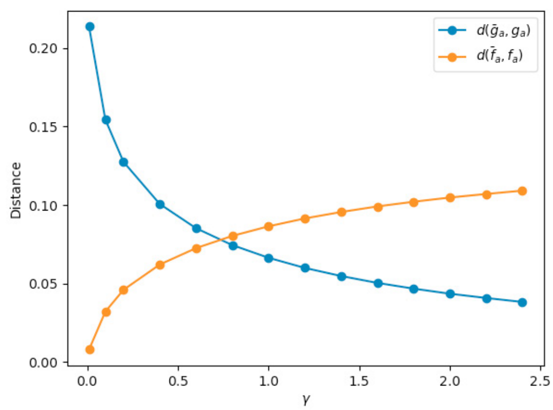


Fig. 3. Distances between nominal and estimated demand, and observed and obtained frequencies for Example 2

3.1 Simulation-optimization approach

For complex traffic networks, it is very difficult to deal with individual passenger dynamics using an analytical method. There are too many parameters to set up, whose values can affect the result of the estimation. Moreover, passenger arrival could not follow a stable probabilistic distribution (like the exponential inter-arrival times supposed by Cepeda et al. (2006)). In those cases, the usage of an agent-based simulation could help with the modeling of different scenarios in a more realistic way. In a simulation, many parameters are estimated by the simulation itself, preventing the fall into prejudicial over-simplification. Then, the simulation can be used as a black-box function in conjunction with any optimization method which does not require the analytical formulation of the function to optimize.

We use the tool SUMO (Lopez et al. (2018)), an urban traffic simulator, with support for handling buses and pedestrians. SUMO was used in conjunction with RSUMO (Baquela (2013)), a small library developed in R (R Core Team (2020)) to perform traffic analysis. Given the traffic network parameters (bus demands and frequencies) SUMO simulates the dynamic of passengers and buses and calculates traveling and waiting times. Using these estimated waiting times, the effective frequency can be estimated too. Also, due to the passenger arrivals are stochastic, running multiple simulations allows estimating the expected behavior of these indicators. Using this data, it is possible to estimate the effective frequencies instead of computing them with the analytic formula in 4, leading also to a different way of computing the objective function defined in 6.

4. NUMERICAL EXPERIMENTS

For a first numerical experiment and for the sake of comparison with already published results we consider the small example that is proposed in Cepeda et al. (2006) (Section 4.1.1). We reproduce their computations and compare them with our methodology obtaining the same assignment results.

For the numerical examples, we will assume that we have an observed frequency data and the objective is to estimate the O-D matrix that induces those frequencies.

To find the minimizers in 6 we use the Nelder-Mead method (see Lagarias et al. (1998)). It is a derivative free method included in Python through the command *optimize.fmin* (Python v. 3.8.5), and we considered a precision value of 0.01.

4.1 Cepeda et al. network

Consider the network in Figure 4 with three nodes and two transit lines connecting them: L_1 (local line, connecting nodes 1, 2, and 3) and L_2 (express line, connecting node 1 with node 3). Suppose that we have demands of 10 trips from node 1 to node 2, 100 trips from node 1 to node 3, and 10 trips from node 2 to node 3. Considering that the capacity of each bus is 20 passengers by bus, the dwell time at stops is 0.01 minutes and the effective frequencies are defined by 4 with $\beta = 0.2$.

Finally suppose that the frequencies of lines L_1 and L_2 are 6 and 16 vehicles per hour, respectively, and travel times over each arc are $t_{12} = 20.01$, $t_{23} = 20.01$ and $t_{13} = 24.01$ minutes.

In order to obtain the equilibrium assignment, we applied the MSA Algorithm. It is important to note that demands g_1^2 and g_2^3 can only use the line L_1 while demand g_1^3 can choose L_1 or L_2. Taking this into account we obtained the following arc volumes:

$$v_{12} = 25.7, \quad v_{23} = 25.7, \quad v_{13} = 84.3$$

where it can be seen that passengers who want to travel from node 1 to node 3 choose a strategy that considers both lines, local and express.

For this assignment the total time (travel + wait) of each strategy for each demand g_i^d satisfies the equilibrium condition $T_s^d = \tau_i^d$. In the particular case of g_1^3 the total travel time is equal to 40.02 minutes.

The effective frequencies based on these assignment are $f_{12} = 0.0265$, $f_{23} = 0.0374$ y $f_{13} = 0.0625$.

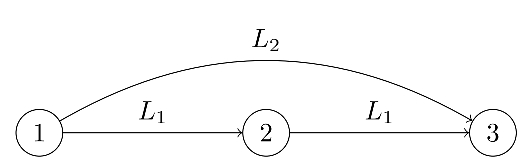


Fig. 4. Small network proposed by Cepeda et al. (2006)

Suppose we can measure the current effective frequencies and based on them and a nominal demand we want to estimate the current O-D matrix. Consider, for example, the following observed frequencies:

$$\bar{f}_{12} = 0.0215, \quad \bar{f}_{23} = 0.0362, \quad \bar{f}_{13} = 0.0624$$

These frequencies are obtained when we perform the flow assignment with $g_1^2 = 10$, $g_1^3 = 110$ and $g_2^3 = 10$. Taking into account these frequencies and considering the nominal O-D matrix $\bar{g}_1^2 = 10$, $\bar{g}_1^3 = 100$ and $\bar{g}_2^3 = 10$ we solve the problem (6-8) with $\gamma = 1/5$ and obtain the estimated O-D matrix $g_1^2 = 10.05$, $g_1^3 = 109.5$ and $g_2^3 = 9.98$, which can be considered a good approach to the assumed real O-D matrix $\bar{g}_1^2 = 10$, $\bar{g}_1^3 = 110$ and $\bar{g}_2^3 = 10$. The progress of the objective function of problem (6-8) during the O-D matrix estimation can be seen in Figure 6.

Simulation approach results In order to compare both approaches, the network was also modeled in SUMO. The bus speed was tuned so as to obtain the same travel times for lines L1 an L2. Efficient frequencies were estimated measuring the passenger travel and waiting times at the stops. Then, the optimization algorithm was again the Nelder-Mead algorithm but the version included in the R function *optim*. For each point generated by the Nelder-Mead algorithm, 30 simulation processes were ran and their results were averaged. The obtained estimation for the OD matrix was $g_1^2 = 10.03$, $g_1^3 = 110.5$ and $g_2^3 = 9.04$, which is also a good estimation for the real OD matrix.

4.2 Example 2

In order to reproduce the previous methodology in another network with a small increase in difficulty, we consider a new example with four nodes and four lines serving it as shown in Figure 5. The data of each line are summarized in Table 1. Considering demands $g_1^3 = g_1^4 = g_4^3 = 100$ the MSA Algorithm was applied and the results are exposed in Table 2. Table 3 summarizes the arc flows obtained summing over all destinations and considering all demands. The effective frequencies obtained for this assignment are also shown there.

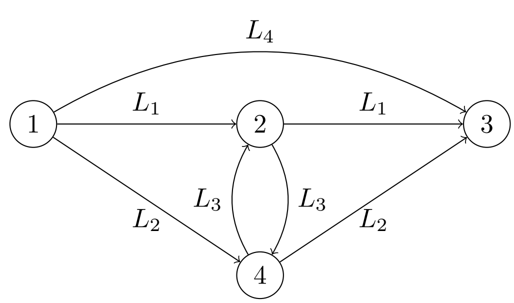


Fig. 5. Network with 4 nodes and 4 lines

In order to estimate the O-D matrix we have the measured frequencies:

$$f_{12} = 0.0243, \; f_{23} = 0.0591, \; f_{14} = 0.0489, \; f_{43} = 0.0515$$
$$f_{24} = 0.0977, \; f_{42} = 0.1419, \; f_{13} = 0.0428,$$

that are obtained when an assignment is made with $g_1^3 = 120$ and $g_1^4 = g_4^3 = 100$.

Using the nominal demands $g_1^3 = g_1^4 = g_4^3 = 100$ we solved the problem (6-8) with $\gamma = 1/100$ and observed frequencies obtained for a demand of $g_1^3 = 120$ and $g_1^4 = g_4^3 = 100$, obtaining the following demand estimation $g_1^3 = 118.86$, $g_1^4 = 100.75$ and $g_4^3 = 100.18$. The progress of the objective function of problem (6-8) during the O-D matrix estimation can be seen in Figure 6.

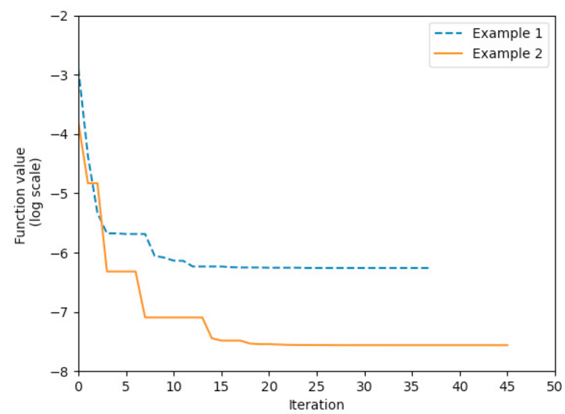


Fig. 6. Progress of objective function in (6).

Simulation approach results Following the same scheme as in 4.1.1, we estimated the OD matrix through simulation and optimization. The estimation we obtained is $g_1^3 = 118.43$, $g_1^4 = 100.84$ and $g_4^3 = 100.07$, which is similar to the result we obtained using the purely mathematical approach.

5. CONCLUSIONS

In this work, we have proposed an approach to public transport demand estimation. This approach can be used not only for demand estimation but also to detect demand changes after detecting changes in waiting and traveling times.

Given a model of flow distribution for public transport according to its demand, we propose the solution of an inverse problem to update the demand for observed flow variations. Preliminary results show that it can be done with derivative-free optimization algorithms over small-sized networks. We present two examples in which the estimated O-D matrix is good and very close to the O-D matrix that generates the observed frequencies after making the assignment. A disadvantage of the proposed numerical solution is the local convergence of the method. Indeed, even if using a nominal target demand could improve these properties, the nominal demand is also unknown in practice. When the target demand is too far from the demand that effectively uses public transport, the results will not be very accurate. The numerical analysis for larger networks and the search for an analytical derivation of descent directions are currently under work.

The use of a derivative-free method allowed us to model traffic flows using agent-based simulations. The results obtained were similar to the analytical model. Due to modeling bigger scenarios with simulation seems to be easier than modeling them analytically, and results are similar, we hope that more complex and larger traffic nets and scenarios could be analyzed with this tool. Also, we

Table 1. Service data

Line	Route	Travel times (min)	Frequencies (veh/h)
L_1	$1 \to 2 \to 3$	$t_{12} = t_{23} = 20.01$	8
L_2	$1 \to 4 \to 3$	$t_{14} = t_{43} = 22.01$	16
L_3	$2 \to 4 \to 2$	$t_{24} = t_{42} = 5.01$	16
L_4	$1 \to 3$	$t_{13} = 28.01$	10

Table 2. Disaggregated flows resulting for assignment in example 2.

Demand	Arc flows	Lines used	Total cost
g_1^4	$v_{12}^4 = 32.54,\ v_{14}^4 = 67.46,\ v_{24}^4 = 32.54$	L_1, L_2, L_3	$T_s^4 = 39.1260$
g_4^3	$v_{23}^3 = 6.37,\ v_{43}^3 = 93.63,\ v_{42}^3 = 6.37$	L_1, L_2, L_3	$T_s^3 = 41.0354$
g_1^3	$v_{12}^3 = 21.69,\ v_{23}^3 = 21.69,\ v_{14}^3 = 39.38\ v_{43}^3 = 39.38,$ $v_{13}^3 = 38.93,\ v_{43}^3 = 39.38,\ v_{13}^3 = 38.93$	L_1, L_2, L_4	$T_s^3 = 45.1520$

Table 3. Total flows and effective frequencies resulting for assignment in example 2.

Results \ Arc (i,j)	(1,2)	(2,3)	(1,4)	(4,3)	(2,4)	(4,2)	(1,3)
Arc flows	54.24	28.06	106.83	133.01	32.54	6.37	38.93
Effective frequencies	0.0259	0.0613	0.0525	0.0526	0.0978	0.1448	0.0465

could take into account dynamic effects, like jams, high variance in arrival time, and pedestrian walking from stop to stop. These additional effects are currently under work too.

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