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SOME EXTENSIONS OF CLASSES INVOLVING PAIR OF WEIGHTS RELATED TO THE BOUNDEDNESS OF MULTILINEAR COMMUTATORS ASSOCIATED TO GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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Abstract. We deal with the boundedness properties of higher order commutators related to some generalizations of the multilinear fractional integral operator of order m, I_α^m, from a product of weighted Lebesgue spaces into adequate weighted Lipschitz spaces, extending some previous estimates for the linear case. Our study includes two different types of commutators and sufficient conditions on the weights in order to guarantee the continuity properties described above. We also exhibit the optimal range of the parameters involved. The optimality is understood in the sense that the parameters defining the corresponding spaces belong to a certain region, being the weights trivial outside of it. We further show examples of weights for the class which cover the mentioned area.

1. Introduction

Many classical operators in Harmonic Analysis whose continuity properties were extensively studied, have shown to behave in a suitable way when their multilinear versions act in the corresponding multilinear spaces. For example, in [6] the authors proved that both, the multilinear Calderón-Zygmund operators and their commutators with BMO symbols, are bounded from a product of weighted Lebesgue spaces to an associated weighted Lebesgue space, with weights belonging to the $A_{\vec{p}}$ multilinear class. This article established the starting point of the weighted theory in this general context. Motivated by the result in [6], similar estimates for multilinear fractional integral operators were obtained in [7], and in [4] for their commutators with BMO symbols. The weights involved in both articles belong to a class that generalizes those given in [6] and in [8] for the linear case.

Regarding the two-weighted theory, also in [7] K. Moen gave a complete discussion showing sufficient bump conditions in order to guarantee the continuity properties of the multilinear fractional integral operator acting between a wider class of Lebesgue spaces than those obtained in the one-weight theory. The obtained results generalize the linear version given, for example, in [17] and [10].

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In [2] the authors studied the two-weighted boundedness properties of the multilinear fractional integral operator between a product of weighted Lebesgue spaces into appropiate weighted Lipschitz spaces associated to a parameter δ. They characterized the classes of weights related to this problem, showing also the optimal range of the parameters involved. The optimality is understood in the sense that the parameters defining the corresponding spaces belong to a certain region, being the weights trivial outside of it. These results extend the corresponding proved in [14] for the linear case.

Our main interest in this article is the study of the boundedness properties for higher order commutators related to some generalizations of the multilinear fractional integral operator of order m, I_α^m, defined by

$$I_\alpha^m \vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{\prod_{i=1}^m f_i(y_i)}{(\sum_{i=1}^m |x-y_i|)^{mn-\alpha}}\, d\vec{y},$$

where $0<\alpha<mn$, $\vec{f}=(f_1,f_2,\dots,f_m)$ and $\vec{y}=(y_1,y_2,\dots,y_m)$. These operators are given by

$$T_\alpha^m \vec{f}(x) = \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y}) \prod_{i=1}^m f_i(y_i)\, d\vec{y},$$

where K_α satisfies certain size and regularity conditions (see Section 2). Particularly, the size condition allows us to conclude that $|T_\alpha^m \vec{f}| \le C I_\alpha^m \vec{g}$, where $\vec{g}=(|f_1|,\dots,|f_m|)$. This estimation guarantees the boundedness of T_α^m with different weights, between a product of Lebesgue spaces into a related Lebesgue space when the class of multilinear weights involved is an extension of that given in [7] for the continuity properties of I_α^m. Nevertheless, this argument cannot be applied for other spaces such that BMO, Lipschitz or Morrey because of the fact that they have not a growth property such as Lebesgue spaces. So, we shall focus our attention in studying the boundedness properties of the commutators of T_α^m acting between a product of weighted Lebesgue spaces into certain weighted versions of the aforementioned spaces. In the linear case this problem was studied in [15] for higher order commutators including the case of the operator I_α, which had been previously given in [14] for the two-weighted setting in the linear context. For similar problems involving other weighted type of Lipschitz spaces see [5], [13] and [16]. On the other hand, in the multilinear case and for $T_\alpha^m = I_\alpha^m$ (that is $K_\alpha(x,\vec{y}) = (\sum_{i=1}^m |x-y_i|)^{\alpha-mn}$), generalizations of two-weighted problems can be found in [2] and [3] (see also [1] for the unweighted problem).

In this paper we study the boundedness of commutators of fractional operators, including I_α^m, between a product of weighted Lebesgue spaces and weighted generalizations of those introduced in [9]. For a weight w, the latter are denoted by $\mathbb{L}_w(\delta)$ and defined as the collection of locally integrable functions f for which the inequality

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\delta/n}} \int_B |f(x)-f_B|\, dx \le C$$

holds for every ball $B\subset\mathbb{R}^n$, where $f_B = |B|^{-1}\int_B f$.

We shall consider two different types of higher order commutators of T_α^m. Given an m-tuple of functions $\mathbf{b}=(b_1,\dots,b_m)$, where each component belongs to L^1_{loc}, the first multilinear commutator we shall be dealing with, $T^m_{\alpha,\mathbf{b}}$, is given by the expression

$$T^m_{\alpha,\mathbf{b}}\vec{f}(x) = \sum_{j=1}^m T^m_{\alpha,b_j}\vec{f}(x),$$

where T^m_{α,b_j} is formally defined by

$$T^m_{\alpha,b_j}\vec{f}(x) = \int_{(\mathbb{R}^n)^m}(b_j(x)-b_j(y_j))K_\alpha(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,d\vec{y}.$$

On the other hand, the second type of commutator that we shall consider can be expressed as

$$\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}(x) = \int_{(\mathbb{R}^n)^m}K_\alpha(x,\vec{y})\prod_{i=1}^m(b_i(x)-b_i(y_i))f_i(y_i)\,d\vec{y}.$$

The last two equalities above are a consequence of their definitions given in Section 2, and both commutators were introduced in [12] and [11], respectively.

We shall also be dealing with multilinear symbols with components belonging to the classical Lipschitz spaces $\Lambda(\delta)$ (for more information see Section 2).

Let $\vec{p}=(p_1,p_2,\dots,p_m)$ be a vector of exponents such that $1\le p_i\le\infty$ for every i. Let β, δ and $\tilde{\delta}$ be real constants. Given w, $\vec{v}=(v_1,v_2,\dots,v_m)$ and $\vec{p}$, we say that $(w,\vec{v})\in\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$ if there exists a positive constant C such that the inequality

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n}}\prod_{i=1}^m\left(\int_{\mathbb{R}^n}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-y|)^{(n-\beta_i+\delta/m)p_i'}}\,dy\right)^{1/p_i'}\le C$$

holds for every ball $B=B(x_B,R)$, with the obvious changes when $p_i=1$. The numbers β_i satisfy $\sum_{i=1}^m\beta_i=\beta$ and also $0<\beta_i<n$, for every i (see Section 2 for further details related to these classes of weights).

We shall now state our main results. From now on, $1/p=\sum_{i=1}^m 1/p_i$. See Section 2 for details.

THEOREM 1.1. *Let $0<\alpha<mn$ and T^m_α be a multilinear fractional operator with kernel K_α satisfying* (2.2) *and* (2.3). *Let $0<\delta<\min\{\gamma,mn-\alpha\}$, $\tilde{\alpha}=\alpha+\delta$ and $\vec{p}$ a vector of exponents that satisfies $p>n/\tilde{\alpha}$. Let $\mathbf{b}=(b_1,\dots,b_m)$ be a vector of symbols such that $b_i\in\Lambda(\delta)$, for $1\le i\le m$. Let $\tilde{\delta}\le\delta$ and $(w,\vec{v})$ be a pair of weights belonging to the class $\mathbb{H}_m(\vec{p},\tilde{\alpha},\tilde{\delta})$ such that $v_i^{-p_i'}\in\mathrm{RH}_m$, for every i such that $1<p_i\le\infty$. Then the multilinear commutator $T^m_{\alpha,\mathbf{b}}$ is bounded from $\prod_{i=1}^m L^{p_i}(v_i^{p_i})$ to $\mathbb{L}_w(\tilde{\delta})$, that is, there exists a positive constant C such that the inequality*

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B|T^m_{\alpha,\mathbf{b}}\vec{f}(x)-(T^m_{\alpha,\mathbf{b}}\vec{f})_B|\,dx\le C\prod_{i=1}^m\|f_iv_i\|_{p_i}$$

holds for every ball B and every $\vec{f}$ such that $f_iv_i\in L^{p_i}$, for $1\le i\le m$.

Concerning the commutator $\mathscr{T}^m_{\alpha,\mathbf{b}}$ we have the following result.

THEOREM 1.2. *Let $0 < \alpha < mn$ and T_α^m be a multilinear fractional operator with kernel K_α satisfying* (2.2) *and* (2.3). *Let $0 < \delta < \min\{\gamma, (mn-\alpha)/m\}$, $\tilde{\alpha} = \alpha + m\delta$ and $\vec{p}$ a vector of exponents that satisfies $p > n/\tilde{\alpha}$. Let $\mathbf{b} = (b_1, \dots, b_m)$ be a vector of symbols such that $b_i \in \Lambda(\delta)$, for $1 \le i \le m$. Let $\tilde{\delta} \le \delta$ and $(w, \vec{v})$ be a pair of weights belonging to the class $\mathbb{H}_m(\vec{p}, \tilde{\alpha}, \tilde{\delta})$ such that $v_i^{-p_i'} \in \mathrm{RH}_m$, for every $1 < p_i \le \infty$. Then the multilinear commutator $\mathscr{T}_{\alpha,\mathbf{b}}^m$ is bounded from $\prod_{i=1}^m L^{p_i}(v_i^{p_i})$ to $\mathbb{L}_w(\tilde{\delta})$, that is, there exists a positive constant C such that the inequality*

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}} \int_B |\mathscr{T}_{\alpha,\mathbf{b}}^m \vec{f}(x) - (\mathscr{T}_{\alpha,\mathbf{b}}^m \vec{f})_B|\, dx \le C \prod_{i=1}^m \|f_i v_i\|_{p_i}$$

holds for every ball B and every $\vec{f}$ such that $f_i v_i \in L^{p_i}$, for $1 \le i \le m$.

Note that the restriction on the parameter δ implicated is different in each theorem due to the nature of the considered commutators.

When we deal with $w = \prod_{i=1}^m v_i$, which is the natural substitute of the one weight theory in the linear case, we shall say that $\vec{v} \in \mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$. Then we have the following lemma.

LEMMA 1.3. *Let $0 < \beta < mn$, $\tilde{\delta} \in \mathbb{R}$, and $\vec{p}$ a vector of exponents. If $\vec{v} \in \mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$, then $\tilde{\delta} = \beta - n/p$.*

The proof of this result follows similar arguments as those of Theorem 1.3 in [2] and we shall omit it. As a consequence of this lemma we can prove that if $\tilde{\delta} < \tau = (\beta - mn)(1 - 1/m) + \delta/m$, then condition (2.5) can be reduced to the class $A_{\vec{p},\infty}$, defined as the collection of multilinear weights $\vec{v} = (v_1, \dots, v_m)$ for which the inequality

$$[\vec{v}]_{A_{\vec{p},\infty}} = \sup_{B \subset \mathbb{R}^n} \left\| \mathscr{X}_B \prod_{i=1}^m v_i \right\|_\infty \prod_{i=1}^m \left(\frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} < \infty \tag{1.1}$$

holds (see Corollary 6.4).

The paper is organized as follows. In Section 2 we give the main definitions required in the sequel. In Section 3 we state and prove some auxiliary results that will be useful for the proof of the main theorems, which are contained in Section 4 and 5. Finally in Section 6 we prove some properties of the class $\mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$ and show the optimality of the associated parameters.

2. Preliminaries and definitions

Throughout the paper C will denote an absolute constant that may change in every occurrence. By $A \lesssim B$ we mean that there exists a positive constant c such that $A \le cB$. We say that $A \approx B$ when $A \lesssim B$ and $B \lesssim A$.

By $m \in \mathbb{N}$ we denote the multilinear parameter involved in our estimates. Given a set E, with E^m we shall denote the cartesian product of E m times. We shall be dealing with operators given by the expression

$$T_\alpha^m \vec{f}(x) = \int_{(\mathbb{R}^n)^m} K_\alpha(x, \vec{y}) \prod_{i=1}^m f_i(y_i)\, d\vec{y}, \tag{2.1}$$

for $0<\alpha<mn$, where $\vec{f}=(f_1,\dots,f_m)$, $\vec{y}=(y_1,\dots,y_m)$ and K_α is a kernel satisfying the size condition

$$|K_\alpha(x,\vec{y})| \lesssim \frac{1}{(\sum_{i=1}^m |x-y_i|)^{mn-\alpha}} \tag{2.2}$$

and an additional smoothness condition given by

$$|K_\alpha(x,\vec{y})-K_\alpha(x',\vec{y})| \lesssim \frac{|x-x'|^\gamma}{(\sum_{i=1}^m |x-y_i|)^{mn-\alpha+\gamma}}, \tag{2.3}$$

for some $0<\gamma\leq 1$, whenever $\sum_{i=1}^m |x-y_i|>2|x-x'|$. It is easy to check that $T_\alpha^m=I_\alpha^m$ defined above, when we consider $K_\alpha(x,\vec{y})=(\sum_{i=1}^m |x-y_i|)^{\alpha-mn}$.

We shall introduce two versions of commutators of the operators above. For a specified linear operator T and a function $b\in L^1_{\mathrm{loc}}$ we recall that the classical commutator T_b or $[b,T]$ is given by the expression

$$[b,T]f=bTf-T(bf).$$

When we deal with multilinear functions and symbols, it will be necessary to emphasize how we proceed to perform the commutation. If $b\in L^1_{\mathrm{loc}}$, T is a multilinear operator and $\vec{f}=(f_1,f_2\dots,f_m)$ we write

$$[b,T]_j(\vec{f})=bT(\vec{f})-T((f_1,\dots,bf_j,\dots,f_m)),$$

that is, $[b,T]_j$ is obtained by commuting b with the j-th entry of $\vec{f}$.

The first version of the commutator is defined as follows. Given an m-tuple $\mathbf{b}=(b_1,\dots,b_m)$, with $b_i\in L^1_{\mathrm{loc}}$ for every i, we define the multilinear commutator of T_α^m by the expression

$$T^m_{\alpha,\mathbf{b}}\vec{f}(x)=\sum_{j=1}^m T^m_{\alpha,b_j}\vec{f}(x),$$

where

$$T^m_{\alpha,b_j}\vec{f}(x)=[b_j,T_\alpha^m]_j\vec{f}(x).$$

As a consequence of (2.1) it is not difficult to see that

$$T^m_{\alpha,b_j}\vec{f}(x)=\int_{(\mathbb{R}^n)^m}(b_j(x)-b_j(y_j))K_\alpha(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,d\vec{y}.$$

We now introduce the second type of commutator of T_α^m. The multilinear product commutator $\mathscr{T}^m_{\alpha,\mathbf{b}}$ is defined iteratively as follows

$$\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}=[b_m,\dots[b_2,[b_1,T_\alpha^m]_1]_2\dots]_m\vec{f}.$$

The expression above does not involve a simple notation, so we shall provide an alternative way to denote this commutator (see Section 3).

By means of (2.1) we can also obtain an integral representation for this operator (see Proposition 3.1 below), namely

$$\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}(x) = \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y})\prod_{i=1}^m (b_i(x)-b_i(y_i))f_i(y_i)\,d\vec{y}.$$

By a weight we understand any positive and locally integrable function.

Given $\delta \in \mathbb{R}$ and a weight w we say that a locally integrable function $f \in \mathbb{L}_w(\delta)$ if there exists a positive constant C such that the inequality

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\delta/n}}\int_B |f(x)-f_B|\,dx \le C \tag{2.4}$$

holds for every ball B, where $f_B = |B|^{-1}\int_B f$. The smallest constant C in (2.4) will be denoted by $\|f\|_{\mathbb{L}_w(\delta)}$. If $\delta = 0$ the space $\mathbb{L}_w(\delta)$ coincides with a weighted version of the BMO spaces introduced in [8], the classical Lipschitz functions when $0<\delta<1$ and the Morrey spaces when $-n<\delta<0$. These classes of functions were also studied in [14].

Regarding the symbols, we shall be dealing with the $\Lambda(\delta)$ Lipschitz spaces given, for $0<\delta<1$, by the collection of functions b verifying

$$|b(x)-b(y)| \le C|x-y|^\delta.$$

The smallest constant for this inequality to hold will be denoted by $\|b\|_{\Lambda(\delta)}$. For a given $\mathbf{b} = (b_1,\dots,b_m)$, with $b_i \in \Lambda(\delta)$ for every $1\le i\le m$, we define

$$\|\mathbf{b}\|_{(\Lambda(\delta))^m} = \max_{1\le i\le m}\|b_i\|_{\Lambda(\delta)}.$$

Let $S_m = \{0,1\}^m$. Given a set B and $\sigma = (\sigma_1,\sigma_2,\dots,\sigma_m) \in S_m$, we define

$$B^{\sigma_i} = \begin{cases} B, & \text{if } \sigma_i = 1 \\ \mathbb{R}^n\backslash B, & \text{if } \sigma_i = 0. \end{cases}$$

With the notation $\mathbf{B}^\sigma$ we will understand the cartesian product $B^{\sigma_1}\times B^{\sigma_2}\times\ldots\times B^{\sigma_m}$.

For $\sigma = (\sigma_1,\sigma_2,\dots,\sigma_m)$ we also define

$$\bar{\sigma}_i = \begin{cases} 1 \text{ if } \sigma_i = 0; \\ 0 \text{ if } \sigma_i = 1, \end{cases}$$

for every $1\le i\le m$ and $|\sigma| = \sum_{i=1}^m \sigma_i$.

We now describe the classes of weights involved in our estimates. Let δ be a fixed real constant. If $1\le p_i\le\infty$ for every i, the m-tuple $\vec{p} = (p_1,p_2,\dots,p_m)$ will be called a vector of exponents. We shall also denote $1/p = \sum_{i=1}^m 1/p_i$. Let β and $\tilde{\delta}$ be real constants. Given w, $\vec{v} = (v_1,v_2,\dots,v_m)$ and a vector of exponents $\vec{p}$, we say that $(w,\vec{v}) \in \mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$ if there exists a positive constant C such that the inequality

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n}}\prod_{i=1}^m\left(\int_{\mathbb{R}^n}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-y|)^{(n-\beta_i+\delta/m)p_i'}}\,dy\right)^{1/p_i'} \le C \tag{2.5}$$

holds for every ball $B=B(x_B,R)$. The numbers β_i satisfy $\sum_{i=1}^m \beta_i=\beta$ and also $0<\beta_i<n$, for every i, which leads to $0<\beta<mn$. We shall also see that the parameters $\tilde{\delta}$ and β are related to δ. When $p_i=1$ for some i the integral above is understood as

$$\left\|\frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\beta_i+\delta/m}}\right\|_\infty .$$

Let

$$\mathscr{I}_1=\{1\le i\le m : p_i=1\} \tag{2.6}$$

and

$$\mathscr{I}_2=\{1\le i\le m : 1<p_i\le\infty\}. \tag{2.7}$$

Condition (2.5) implies that

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{\tilde{\delta}/n+1/p-\beta/n}}\prod_{i\in\mathscr{I}_1}\|v_i^{-1}\mathscr{X}_B\|_\infty\prod_{i\in\mathscr{I}_2}\left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'}\le C \tag{2.8}$$

and

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n}}\prod_{i\in\mathscr{I}_1}\left\|\frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B-\cdot|^{n-\beta_i+\frac{\delta}{m}}}\right\|_\infty\prod_{i\in\mathscr{I}_2}\left(\int_{\mathbb{R}^n\setminus B}\frac{v_i^{-p_i'}}{|x_B-\cdot|^{(n-\beta_i+\frac{\delta}{m})p_i'}}\right)^{1/p_i'}\le C. \tag{2.9}$$

We shall refer to the inequalities above as the local and global conditions, respectively.

Furthermore, given $\sigma\in S_m$ we can estimate the i-th factor in (2.5) depending whether $\sigma_i=0$ or $\sigma_i=1$. Thus we have that condition (2.5) implies

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n+\theta(\sigma)}}\prod_{i:\sigma_i=1}\|v_i^{-1}\mathscr{X}_B\|_{p_i'}\prod_{i:\sigma_i=0}\left\|\frac{v_i^{-p_i'}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B-\cdot|^{n-\beta_i+\delta/m}}\right\|_{p_i'}\le C, \tag{2.10}$$

where $\theta(\sigma)=\sum_{i:\sigma_i=1}1-\beta_i/n+\delta/(mn)$.

We recall that a weight w belongs to the *reverse Hölder* class RH_s, $1<s<\infty$, if there exists a positive constant C such that the inequality

$$\left(\frac{1}{|B|}\int_B w^s\right)^{1/s}\le\frac{C}{|B|}\int_B w$$

holds for every ball B in $\mathbb{R}^n$. The smallest constant for which the inequality above holds is denoted by $[w]_{\mathrm{RH}_s}$. It is not difficult to see that $\mathrm{RH}_t\subset\mathrm{RH}_s$ whenever $1<s<t$. We say that $w\in\mathrm{RH}_\infty$ if

$$\sup_B w\le\frac{C}{|B|}\int_B w,$$

for some positive constant C.

3. Auxiliary results

We devote this section to state and prove some facts that will be useful in our estimates.

The following proposition establishes an alternative way to denote the product commutator $\mathscr{T}^m_{\alpha,\mathbf{b}}$.

PROPOSITION 3.1. *Let T^m_α be a multilinear operator as in* (2.1) *and* $\mathbf{b} = (b_1,\dots,b_m)$ *where $b_i \in L^1_{\text{loc}}$ for $1 \le i \le m$. Then we have that*

$$\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}(x) = \sum_{\sigma\in S_m}(-1)^{m-|\sigma|}\left(\prod_{i=1}^m b_i^{\sigma_i}(x)\right)T^m_\alpha(f_1b_1^{\bar{\sigma}_1},\dots,f_mb_m^{\bar{\sigma}_m})(x).$$

Furthermore, we have the integral representation

$$\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}(x) = \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y})\prod_{i=1}^m(b_i(x)-b_i(y_i))f_i(y_i)\,d\vec{y}.$$

Proof. Let us introduce some notation in order to make our calculations simpler. Fix a symbol $\mathbf{b} = (b_1,\dots,b_m)$ and let F_k be the operator resulting after perform k iterative commutings, $1 \le k \le m$, that is

$$F_k\vec{f} = [b_k,\dots[b_2,[b_1,T^m_\alpha]_1]_2\dots]_k\vec{f}.$$

Given $\sigma \in S_m$, let us also denote with

$$\vec{g}^{\,k}_{\mathbf{b},\sigma} = (g_1,g_2,\dots,g_m)$$

where

$$g_i = \begin{cases} f_ib_i^{\bar{\sigma}_i} & \text{if } 1 \le i \le k,\\ f_i & \text{if } k < i \le m.\end{cases}$$

We shall proceed by induction in order to show that

$$F_k\vec{f} = \sum_{\sigma\in S_k}(-1)^{k-|\sigma|}\left(\prod_{i=1}^k b_i^{\sigma_i}\right)T^m_\alpha\left(\vec{g}^{\,k}_{\mathbf{b},\sigma}\right). \tag{3.1}$$

Notice that the case $k = 1$ is immediate after performing one commutation only. Let us assume that the expression above holds for k and we shall prove it for $k+1$. By the definition and the inductive hypothesis we have

$$\begin{aligned} F_{k+1}\vec{f} &= [b_{k+1},F_k]_{k+1}\vec{f}\\ &= b_{k+1}\sum_{\sigma\in S_k}(-1)^{k-|\sigma|}\left(\prod_{i=1}^k b_i^{\sigma_i}\right)T^m_\alpha\left(\vec{g}^{\,k}_{\mathbf{b},\sigma}\right)\\ &\quad - \sum_{\sigma\in S_k}(-1)^{k-|\sigma|}\left(\prod_{i=1}^k b_i^{\sigma_i}\right)T^m_\alpha(f_1b_1^{\bar{\sigma}_1},\dots,f_kb_k^{\bar{\sigma}_k},f_{k+1}b_{k+1},f_{k+2},\dots,f_m). \end{aligned}$$

Observe that $\{\theta \in S_{k+1}\} = \{\theta \in S_{k+1} : \theta_{k+1} = 1\} \cup \{\theta \in S_{k+1} : \theta_{k+1} = 0\}$. By rewriting the sums above we get

$$\begin{aligned} F_{k+1}\vec{f} &= \sum_{\theta \in S_{k+1}, \theta_{k+1}=1} (-1)^{k-(|\theta|-1)} \left(\prod_{i=1}^{k+1} b_i^{\theta_i} \right) T_\alpha^m \left(\vec{g}_{\mathbf{b},\theta}^{k+1} \right) \\ &\quad + \sum_{\theta \in S_{k+1}, \theta_{k+1}=0} (-1)^{k+1-|\theta|} \left(\prod_{i=1}^{k+1} b_i^{\theta_i} \right) T_\alpha^m \left(\vec{g}_{\mathbf{b},\theta}^{k+1} \right) \\ &= \sum_{\theta \in S_{k+1}} (-1)^{k+1-|\theta|} \left(\prod_{i=1}^{k+1} b_i^{\theta_i} \right) T_\alpha^m \left(\vec{g}_{\mathbf{b},\theta}^{k+1} \right). \end{aligned}$$

Therefore (3.1) holds for every $1 \le k \le m$, and the case $k = m$ allows us to conclude the desired estimate.

In order to show the integral representation, we shall prove that

$$F_k\vec{f} = \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y}) \prod_{i=1}^{k} (b_i(x) - b_i(y_i)) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y}. \tag{3.2}$$

We proceed again by induction on k. If $k = 1$ we have that

$$\begin{aligned} F_1\vec{f}(x) &= b_1(x) T_\alpha^m \vec{f}(x) - T_\alpha^m((b_1 f_1, f_2, \dots, f_m))(x) \\ &= \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y}) (b_1(x) - b_1(y_1)) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y}. \end{aligned}$$

Now assume the representation holds for k and let us prove it for $k+1$. Indeed, using the definition of F_k and the inductive hypothesis we get

$$\begin{aligned} F_{k+1}\vec{f}(x) &= [b_{k+1}, F_k]_{k+1} \vec{f}(x) \\ &= b_{k+1}(x) F_k \vec{f}(x) - F_k(f_1, \dots, f_k, f_{k+1} b_{k+1}, f_{k+2}, \dots, f_m)(x) \\ &= b_{k+1}(x) \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y}) \prod_{i=1}^{k} (b_i(x) - b_i(y_i)) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y} \\ &\quad - \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y}) b_{k+1}(y_{k+1}) \prod_{i=1}^{k} (b_i(x) - b_i(y_i)) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y} \\ &= \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y}) \prod_{i=1}^{k+1} (b_i(x) - b_i(y_i)) \prod_{i=1}^{m} f_i(y_i)\, d\vec{y}. \end{aligned}$$

This shows that the representation for $k+1$ also holds. Putting $k = m$ we get the integral representation for $\mathscr{T}_{\alpha,\mathbf{b}}^m$. □

LEMMA 3.2. *Let $m \in \mathbb{N}$ and a_i, b_i and c_i be real numbers for $1 \le i \le m$. Then*

$$\prod_{i=1}^{m} (a_i - b_i) - \prod_{i=1}^{m} (c_i - b_i) = \sum_{j=1}^{m} (a_j - c_j) \prod_{i<j} (a_i - b_i) \prod_{i>j} (c_i - b_i).$$

Proof. We proceed by induction on m. If $m=1$ it is immediate, both sides are equal to a_1-c_1 since the products on the right-hand side are equal to 1.

Assume that the equality holds for $m=k$. Let us prove it for $m=k+1$. We have that

$$\begin{aligned}\prod_{i=1}^{k+1}(a_i-b_i)-\prod_{i=1}^{k+1}(c_i-b_i) &= (a_{k+1}-b_{k+1})\prod_{i=1}^{k}(a_i-b_i)-\prod_{i=1}^{k+1}(c_i-b_i)\\ &= (a_{k+1}-c_{k+1})\prod_{i=1}^{k}(a_i-b_i)\\ &\quad +(c_{k+1}-b_{k+1})\left(\prod_{i=1}^{k}(a_i-b_i)-\prod_{i=1}^{k}(c_i-b_i)\right).\end{aligned}$$

Using the inductive hypothesis we arrive to

$$\begin{aligned}\prod_{i=1}^{k+1}(a_i-b_i)-\prod_{i=1}^{k+1}(c_i-b_i) &= (a_{k+1}-c_{k+1})\prod_{i=1}^{k}(a_i-b_i)\\ &\quad +(c_{k+1}-b_{k+1})\sum_{j=1}^{k}(a_j-c_j)\prod_{i<j}(a_i-b_i)\prod_{i>j}(c_i-b_i)\\ &= \sum_{j=1}^{k+1}(a_j-c_j)\prod_{i<j}(a_i-b_i)\prod_{i>j}(c_i-b_i),\end{aligned}$$

so the result also holds for $m=k+1$. This completes the proof. □

The next lemma establishes a useful relation between $A_{\vec{p}}$ and $A_{\vec{p},q}$ classes that we shall need in the sequel. Given $\vec{p}=(p_1,\dots,p_m)$ with $1/p=\sum_{i=1}^m 1/p_i$ and $1\le p_i\le\infty$ for every i, we say that $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{p}}$ if

$$\sup_B\left(\frac{1}{|B|}\int_B\prod_{i=1}^m w_i^{p/p_i}\right)^{1/p}\prod_{i\in\mathscr{I}_1}\left\|w_i^{-1}\mathscr{X}_B\right\|_\infty\prod_{i\in\mathscr{I}_2}\left(\frac{1}{|B|}\int_B w_i^{1-p_i'}\right)^{1/p_i'}<\infty,$$

and this supremum is denoted by $[\vec{w}]_{A_{\vec{p}}}$. Since $p_i\ge 1$ for every i, we get that $p\ge 1/m$.

On the other hand, given $0<q<\infty$ and $\vec{p}$ as above, we say that $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{p},q}$ if

$$\sup_B\left(\frac{1}{|B|}\int_B\prod_{i=1}^m w_i^{q}\right)^{1/q}\prod_{i\in\mathscr{I}_1}\left\|w_i^{-1}\mathscr{X}_B\right\|_\infty\prod_{i\in\mathscr{I}_2}\left(\frac{1}{|B|}\int_B w_i^{-p_i'}\right)^{1/p_i'}<\infty,$$

and this supremum is denoted by $[\vec{w}]_{A_{\vec{p},q}}$. When $q=\infty$ this class corresponds to $A_{\vec{p},\infty}$ given in (1.1).

LEMMA 3.3. *Let $\vec{p}=(p_1,\dots,p_m)$ be a vector of exponents, $0<q<\infty$ and $\vec{w}=(w_1,\dots,w_m)$. Assume that*

$$\frac{1}{p_i}+\frac{1}{mq}-\frac{1}{mp}>0 \tag{3.3}$$

for every $1 \le i \le m$*. We define* $\lambda_m = 1/(mp)' + 1/(mq)$,

$$\ell_i = \begin{cases} 1 & \text{if } i \in \mathscr{I}_1 \\ (\lambda_m p_i')' & \text{if } i \in \mathscr{I}_2, \end{cases}$$

ℓ *such that* $1/\ell = \sum_{i=1}^m 1/\ell_i$ *and* $\vec{z} = (z_1, \ldots, z_m)$ *where* $z_i = w_i^{q\ell_i/\ell}$*, for each* i*. Then* $\vec{w} \in A_{\vec{p},q}$ *if and only if* $\vec{z} \in A_{\vec{\ell}}$*.*

Proof. We first notice that condition (3.3) guarantees that $\ell_i > 1$ for $i \in \mathscr{I}_2$. It is immediate from the definition that

$$\prod_{i=1}^m z_i^{\ell/\ell_i} = \prod_{i=1}^m w_i^q.$$

Observe that

$$\frac{1}{\ell} = \sum_{i=1}^m \frac{1}{\ell_i} = \sum_{i \in \mathscr{I}_1} 1 + \sum_{i \in \mathscr{I}_2} \frac{1}{(\lambda_m p_i')'} = m - \frac{1}{\lambda_m}\left(m - \frac{1}{p}\right) = \frac{1}{q\lambda_m}.$$

Also notice that for $i \in \mathscr{I}_2$

$$z_i^{1-\ell_i'} = w_i^{q\ell_i(1-\ell_i')/\ell} = w_i^{-\ell_i'/\lambda_m} = w_i^{-p_i'}$$

and

$$\frac{1}{\ell_i'} = \frac{1}{\lambda_m p_i'} = \frac{q}{\ell p_i'}.$$

These identities imply that

$$\left(\frac{1}{|B|}\int_B \prod_{i=1}^m z_i^{\ell/\ell_i}\right)^{1/\ell} \prod_{i \in \mathscr{I}_1} \|z_i^{-1}\mathscr{X}_B\|_\infty \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|B|}\int_B z_i^{1-\ell_i'}\right)^{1/\ell_i'}$$

can be rewritten as

$$\left[\left(\frac{1}{|B|}\int_B \prod_{i=1}^m w_i^q\right)^{1/q} \prod_{i \in \mathscr{I}_1} \|w_i^{-1}\mathscr{X}_B\|_\infty \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|B|}\int_B w_i^{-p_i'}\right)^{1/p_i'}\right]^{q/\ell},$$

from where the equivalence follows. □

REMARK 1. When $m = 1$ condition (3.3) trivially holds, and we get that $w \in A_{p,q}$ if and only if $w^q \in A_{1+q/p'}$, a well-known relation between A_p and $A_{p,q}$ classes.

4. Proof of Theorem 1.1

We devote this section to prove Theorem 1.1. We shall begin with an auxiliary lemma that will be useful for this purpose.

LEMMA 4.1. *Let $0<\alpha<mn$, $0<\delta<mn-\alpha$, $\tilde{\alpha}=\alpha+\delta$ and $\vec{p}$ a vector of exponents that satisfies $p>n/\tilde{\alpha}$. Let $\tilde{\delta}\leq\delta$ and $(w,\vec{v})$ be a pair of weights belonging to the class $\mathbb{H}_m(\vec{p},\tilde{\alpha},\tilde{\delta})$ such that $v_i^{-p_i'}\in \mathrm{RH}_m$ for every $i\in\mathscr{I}_2$. Then there exists a positive constant C such that for every ball B and every $\vec{f}$ such that $f_iv_i\in L^{p_i}$, $1\leq i\leq m$, we have that*

$$\int_B |I_{\tilde{\alpha},m}\vec{g}(x)|\,dx \leq C\frac{|B|^{1+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}\prod_{i=1}^m \|f_iv_i\|_{p_i},$$

where $\vec{g}=(f_1\mathscr{X}_{2B},f_2\mathscr{X}_{2B},\dots,f_m\mathscr{X}_{2B})$.

Proof. We shall follow similar lines as in the proof of Lemma 3.1 in [2]. We include a sketch for the sake of completeness.

Using (2.6) and (2.7), we shall split the set $\mathscr{I}_2$ into $\mathscr{I}_2^1$ and $\mathscr{I}_2^2$ where

$$\mathscr{I}_2^1=\{i\in\mathscr{I}_2: 1<p_i<\infty\}\quad\text{and}\quad \mathscr{I}_2^2=\{i\in\mathscr{I}_2: p_i=\infty\}.$$

Let $m_i=\#\mathscr{I}_i$, for $i=1,2$ and $m_2^j=\#\mathscr{I}_2^j$, also for $j=1,2$. Then $m=m_1+m_2=m_1+m_2^1+m_2^2$. Then, by denoting $\tilde{B}=2B$, for $x\in B$ we have that

$$\begin{aligned}
|I_{\tilde{\alpha},m}\vec{g}(x)| &\leq \int_{\tilde{B}^m}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\tilde{\alpha}}}\,d\vec{y}\\
&\leq \left(\prod_{i\in\mathscr{I}_2^2}\|f_iv_i\|_\infty\right)\int_{\tilde{B}^m}\frac{\prod_{i\in\mathscr{I}_1\cup\mathscr{I}_2^1}|f_i(y_i)|\prod_{i\in\mathscr{I}_2^2}v_i^{-1}(y_i)}{(\sum_{i=1}^m|x-y_i|)^{mn-\tilde{\alpha}}}\,d\vec{y}\\
&\leq \left(\prod_{i\in\mathscr{I}_2^2}\|f_iv_i\|_\infty\right)\left(\prod_{i\in\mathscr{I}_1}\|f_i\mathscr{X}_{\tilde{B}}\|_1\right)\int_{\tilde{B}^{m_2}}\frac{\prod_{i\in\mathscr{I}_2^1}|f_i(y_i)|\prod_{i\in\mathscr{I}_2^2}v_i^{-1}(y_i)}{(\sum_{i\in\mathscr{I}_2}|x-y_i|)^{mn-\tilde{\alpha}}}\,d\vec{y}\\
&= \left(\prod_{i\in\mathscr{I}_2^2}\|f_iv_i\|_\infty\right)\left(\prod_{i\in\mathscr{I}_1}\|f_i\mathscr{X}_{\tilde{B}}\|_1\right)I(x,B).
\end{aligned}$$

Since $p>n/\tilde{\alpha}$ we have that

$$\tilde{\alpha}>n/p=n\sum_{i=1}^m\frac{1}{p_i}=m_1n+\frac{n}{p^*},$$

where $1/p^*=\sum_{i\in\mathscr{I}_2}1/p_i$. Then we can split $\tilde{\alpha}=\tilde{\alpha}^1+\tilde{\alpha}^2$, where $\tilde{\alpha}^1>m_1n$ and $\tilde{\alpha}^2>n/p^*$. Therefore

$$mn-\tilde{\alpha}=m_2n-\tilde{\alpha}^2+m_1n-\tilde{\alpha}^1.$$

Let us sort the sets $\mathscr{I}_2^1$ and $\mathscr{I}_2^2$ increasingly, so

$$\mathscr{I}_2^1=\left\{i_1,i_2,\dots,i_{m_2^1}\right\}\quad\text{and}\quad \mathscr{I}_2^2=\left\{i_{m_2^1+1},i_{m_2^1+2},\dots,i_{m_2}\right\}.$$

We now define $\vec{g} = (g_1, \ldots, g_{m_2})$, where

$$g_j = \begin{cases} |f_{i_j}| & \text{if } \ 1 \le j \le m_2^1; \\ v_{i_j}^{-1} & \text{if } \ m_2^1 + 1 \le j \le m_2. \end{cases}$$

Then we can proceed in the following way

$$\begin{aligned} I(x,B) &\lesssim \int_{\tilde{B}^{m_2}} \frac{\prod_{i \in \mathscr{I}_2^1} |f_i(y_i)| \prod_{i \in \mathscr{I}_2^2} v_i^{-1}(y_i) (\sum_{i \in \mathscr{I}_2} |x - y_i|)^{\tilde{\alpha}^1 - nm_1}}{(\sum_{i \in \mathscr{I}_2} |x - y_i|)^{m_2 n - \tilde{\alpha}^2}} \, d\vec{y} \\ &\lesssim |\tilde{B}|^{\tilde{\alpha}^1/n - m_1} \int_{\tilde{B}^{m_2}} \frac{\prod_{i \in \mathscr{I}_2^1} |f_i(y_i)| \prod_{i \in \mathscr{I}_2^2} v_i^{-1}(y_i)}{(\sum_{i \in \mathscr{I}_2} |x - y_i|)^{m_2 n - \tilde{\alpha}^2}} \, d\vec{y} \\ &= |\tilde{B}|^{\tilde{\alpha}^1/n - m_1} \int_{\tilde{B}^{m_2}} \frac{\prod_{j=1}^{m_2} g_j(y_{i_j})}{(\sum_{j=1}^{m_2} |x - y_{i_j}|)^{m_2 n - \tilde{\alpha}^2}} \, d\vec{y} \\ &\lesssim |\tilde{B}|^{\tilde{\alpha}^1/n - m_1} I_{\tilde{\alpha}^2, m_2}(\vec{g} \mathscr{X}_{\tilde{B}^{m_2}})(x). \end{aligned}$$

Next we define the vector of exponents $\vec{r} = (r_1, \ldots, r_{m_2})$ in the following way

$$r_j = \begin{cases} m_2 p_{i_j} / (m_2 - 1 + p_{i_j}) & \text{if } \quad 1 \le j \le m_2^1; \\ m_2 & \text{if } \ m_2^1 + 1 \le j \le m_2. \end{cases}$$

This definition yields

$$\frac{1}{r} = \sum_{j=1}^{m_2} \frac{1}{r_j} = \sum_{j=1}^{m_2^1} \left(\frac{1}{m_2} + \frac{m_2 - 1}{m_2 p_{i_j}} \right) + \sum_{j = m_2^1 + 1}^{m_2} \frac{1}{m_2} = \frac{m_2^1}{m_2} + \frac{m_2 - 1}{m_2 p^*} + \frac{m_2^2}{m_2} = 1 + \frac{m_2 - 1}{m_2 p^*}.$$

Notice that $1/r > 1/p^*$ and also $n/p^* < \tilde{\alpha}^2$ by construction. Then there exists an auxiliary number $\tilde{\alpha}_0$ such that $n/p^* < \tilde{\alpha}_0 < n/r$. Indeed, if $\tilde{\alpha}^2 < n/r$ we can directly pick $\tilde{\alpha}_0 = \tilde{\alpha}^2$. Otherwise $\tilde{\alpha}_0 < \tilde{\alpha}^2$. We shall first assume that $m_2 \ge 2$. We set

$$\frac{1}{q} = \frac{1}{r} - \frac{\tilde{\alpha}_0}{n}.$$

Observe that $0 < 1/q < 1$ since

$$\frac{1}{r} - 1 = \left(1 - \frac{1}{m_2}\right) \frac{1}{p^*} < \frac{1}{p^*} < \frac{\tilde{\alpha}_0}{n}.$$

Using the well-known continuity property $I_{\tilde{\alpha}_0, m_2} \colon \prod_{j=1}^{m_2} L^{r_j} \to L^q$ with respect to the Lebesgue measure (see, for example, [7]) we obtain

$$\begin{aligned} \int_B I(x,B) \, dx &\lesssim |\tilde{B}|^{\tilde{\alpha}^1/n - m_1 + (\tilde{\alpha}^2 - \tilde{\alpha}_0)/n} \left(\int_B |I_{\tilde{\alpha}_0, m_2}(\vec{g} \mathscr{X}_{\tilde{B}^{m_2}})(x)|^q \, dx \right)^{1/q} |B|^{1/q'} \\ &\lesssim |\tilde{B}|^{(\tilde{\alpha} - \tilde{\alpha}_0)/n - m_1 + 1/q'} \left(\int_{\mathbb{R}^n} |I_{\tilde{\alpha}_0, m_2}(\vec{g} \mathscr{X}_{\tilde{B}^{m_2}})(x)|^q \, dx \right)^{1/q} \end{aligned}$$

$$\lesssim |\tilde{B}|^{(\tilde{\alpha}-\tilde{\alpha}_0)/n-m_1+1/q'} \prod_{j=1}^{m_2} \|g_j \mathscr{X}_{\tilde{B}}\|_{r_j}.$$

Observe that $r_j < p_{i_j}$ for every $1 \leq j \leq m_2^1$. Since $v_i^{-p_i'} \in \mathrm{RH}_m \subseteq \mathrm{RH}_{m_2}$ for every $i \in \mathscr{I}_2$, applying Hölder's inequality and then the reverse Hölder condition on these weights we get

$$\begin{aligned}
\prod_{j=1}^{m_2} \|g_j \mathscr{X}_{\tilde{B}}\|_{r_j} &= \prod_{i \in \mathscr{I}_2^1} \left(\int_{\tilde{B}} |f_i|^{r_i} v_i^{r_i} v_i^{-r_i} \right)^{1/r_i} \prod_{i \in \mathscr{I}_2^2} \left(\int_{\tilde{B}} v_i^{-m_2} \right)^{1/m_2} \\
&\leq \prod_{i \in \mathscr{I}_2^1} \|f_i v_i\|_{p_i} \left(\int_{\tilde{B}} v_i^{-m_2 p_i'} \right)^{1/(m_2 p_i')} \prod_{i \in \mathscr{I}_2^2} \left(\int_{\tilde{B}} v_i^{-m_2} \right)^{1/m_2} \\
&\leq |\tilde{B}|^{m_2^1/m_2 - 1/(m_2 p^*) + m_2^2/m_2} \prod_{i \in \mathscr{I}_2^1} \left[v_i^{-p_i'} \right]_{\mathrm{RH}_{m_2}} \|f_i v_i\|_{p_i} \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-p_i'} \right)^{1/p_i'} \\
&\quad \times \prod_{i \in \mathscr{I}_2^2} \left[v_i^{-1} \right]_{\mathrm{RH}_{m_2}} \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right) \\
&\lesssim |\tilde{B}|^{1-1/(m_2 p^*)} \prod_{i \in \mathscr{I}_2^1} \|f_i v_i\|_{p_i} \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-p_i'} \right)^{1/p_i'} \prod_{i \in \mathscr{I}_2^2} \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right).
\end{aligned}$$

Combining the estimates above with condition (2.8), we finally arrive to

$$\begin{aligned}
\int_B |I_{\tilde{\alpha},m} \vec{g}(x)|\, dx &\leq \left(\prod_{i \in \mathscr{I}_2^2} \|f_i v_i\|_\infty \right) \left(\prod_{i \in \mathscr{I}_1} \|f_i \mathscr{X}_{\tilde{B}}\|_1 \right) \int_B I(x,B)\, dx \\
&\lesssim \left(\prod_{i \in \mathscr{I}_2^2} \|f_i v_i\|_\infty \right) \left(\prod_{i \in \mathscr{I}_1} \|f_i \mathscr{X}_{\tilde{B}}\|_1 \right) |\tilde{B}|^{(\tilde{\alpha}-\tilde{\alpha}_0)/n - m_1 + 1/q' + 1 - 1/(m_2 p^*)} \\
&\quad \times \prod_{i \in \mathscr{I}_2^1} \|f_i v_i\|_{p_i} \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-p_i'} \right)^{1/p_i'} \prod_{i \in \mathscr{I}_2^2} \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-1} \right) \\
&\lesssim \left(\prod_{i=1}^{m} \|f_i v_i\|_{p_i} \right) \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_i^{-p_i'} \right)^{1/p_i'} \prod_{i \in \mathscr{I}_1} \left\| v_i^{-1} \mathscr{X}_{\tilde{B}} \right\|_\infty \\
&\quad \times |\tilde{B}|^{(\tilde{\alpha}-\tilde{\alpha}_0)/n - m_1 + 1/q' + 1 - 1/(m_2 p^*)} \\
&\lesssim \|w \mathscr{X}_{\tilde{B}}\|_\infty^{-1} |\tilde{B}|^{\tilde{\delta}/n - \tilde{\alpha}/n + 1/p + (\tilde{\alpha}-\tilde{\alpha}_0)/n - m_1 + 1/q' + 1 - 1/(m_2 p^*)} \left(\prod_{i=1}^{m} \|f_i v_i\|_{p_i} \right) \\
&\lesssim \|w \mathscr{X}_B\|_\infty^{-1} |B|^{1+\tilde{\delta}/n} \left(\prod_{i=1}^{m} \|f_i v_i\|_{p_i} \right).
\end{aligned}$$

Thus we can achieve the desired estimate provided $m_2 \geq 2$. We shall now consider $0 \leq m_2 < 2$. There are only three possible cases:

1. $m_2 = 0$. In this case we have $m_2^1 = m_2^2 = 0$ and this implies $\vec{p} = (1,1,\dots,1)$. This situation is not possible, because $p > n/\tilde{\alpha}$.

2. $m_2^1 = 0$ and $m_2^2 = 1$. In this case $1/p = m-1$. Condition $p > n/\tilde{\alpha}$ implies $\tilde{\alpha} > (m-1)n$. Let i_0 be the index such that $p_{i_0} = \infty$. Using Fubini's theorem, we can proceed in the following way

$$\int_B \int_{\tilde{B}^m} \frac{\prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |x-y_i|)^{mn-\tilde{\alpha}}}\, d\vec{y}\, dx = \int_{\tilde{B}^m} \prod_{i=1}^m |f_i(y_i)| \left(\int_B \left(\sum_{i=1}^m |x-y_i| \right)^{\tilde{\alpha}-mn} dx \right) d\vec{y}.$$

Since

$$\begin{aligned} \int_B \left(\sum_{i=1}^m |x-y_i| \right)^{\tilde{\alpha}-mn} dx &\lesssim \int_0^{4R} \rho^{\tilde{\alpha}-mn} \rho^{n-1}\, d\rho \\ &\lesssim |B|^{\tilde{\alpha}/n-m+1}, \end{aligned}$$

by (2.8), we get

$$\begin{aligned} \int_B |I_{\tilde{\alpha},m}\vec{g}(x)|\, dx &\lesssim |B|^{\tilde{\alpha}/n-m+2} \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \left(\prod_{i\in\mathscr{I}_1} \left\| v_i^{-1} \mathscr{X}_{\tilde{B}} \right\|_\infty \right) \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_{i_0}^{-1} \right) \\ &\lesssim \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|\tilde{B}|^{\tilde{\alpha}/n-m+2+\tilde{\delta}/n-\tilde{\alpha}/n+1/p}}{\|w\mathscr{X}_{\tilde{B}}\|_\infty} \\ &\lesssim \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|B|^{1+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}. \end{aligned}$$

3. $m_2^1 = 1$ and $m_2^2 = 0$. If i_0 denotes the index for which $1 < p_{i_0} < \infty$, the condition $p > n/\tilde{\alpha}$ implies that

$$\frac{\tilde{\alpha}}{n} > \frac{1}{p} = m-1+\frac{1}{p_{i_0}},$$

and thus $\tilde{\alpha} > (m-1)n$. We repeat the estimate given in the previous case. This yields to

$$\begin{aligned} \int_B |I_{\tilde{\alpha},m}\vec{g}(x)|\, dx &\lesssim |B|^{\tilde{\alpha}/n-m+1+1/p'_{i_0}} \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \left(\prod_{i\in\mathscr{I}_1} \left\| v_i^{-1} \mathscr{X}_{\tilde{B}} \right\|_\infty \right) \left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} v_{i_0}^{-p'_{i_0}} \right)^{1/p'_{i_0}} \\ &\lesssim \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|\tilde{B}|^{\tilde{\alpha}/n-m+1+1/p'_{i_0}+\tilde{\delta}/n-\tilde{\alpha}/n+1/p}}{\|w\mathscr{X}_{\tilde{B}}\|_\infty} \\ &\lesssim \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|B|^{1+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}. \end{aligned}$$

We covered all the possible cases for m_2 and the proof is complete. □

Proof of Theorem 1.1 . It will be enough to prove that

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B |T^m_{\alpha,b_j}\vec{f}(x)-c_j|\,dx \le C\prod_{i=1}^m \|f_iv_i\|_{p_i}, \tag{4.1}$$

for some positive constant c_j and every ball B, for each j and with C independent of B and j. Indeed, if (4.1) holds we take $c=\sum_{j=1}^m c_j$ and therefore

$$\begin{aligned}\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B |T^m_{\alpha,\mathbf{b}}\vec{f}(x)-c|\,dx &\le \sum_{j=1}^m \frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B |T^m_{\alpha,b_j}\vec{f}(x)-c_j|\,dx\\ &\le Cm\prod_{i=1}^m \|f_iv_i\|_{p_i}\end{aligned}$$

and the proof would be complete. Then we shall proceed to prove (4.1).

Fix $1\le j\le m$ and a ball $B=B(x_B,R)$. We decompose $\vec{f}=(f_1,f_2,\dots,f_m)$ as $\vec{f}=\vec{f}_1+\vec{f}_2$, where $\vec{f}_1=(f_1\mathscr{X}_{2B},f_2\mathscr{X}_{2B},\dots,f_m\mathscr{X}_{2B})$. We take

$$c_j=\left(T^m_{\alpha,b_j}\vec{f}_2\right)_B=\frac{1}{|B|}\int_B T^m_{\alpha,b_j}\vec{f}_2(z)\,dz.$$

We also notice that

$$\frac{1}{|B|}\int_B T^m_{\alpha,b_j}\vec{f}_2(x)\,dx=\sum_{\sigma\in S_m,\sigma\neq\mathbf{1}}\frac{1}{|B|}\int_B\int_{(\mathbf{2B})^\sigma}(b_j(x)-b_j(y_j))K_\alpha(x,\vec{y})\prod_{i=1}^m f_i(y_i)\,d\vec{y}\,dx. \tag{4.2}$$

In order to prove (4.1) we write

$$\begin{aligned}\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B |T^m_{\alpha,b_j}\vec{f}(x)-c_j|\,dx &\le \frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\left(\int_B |T^m_{\alpha,b_j}\vec{f}_1(x)|\,dx+\frac{1}{|B|}\int_B|T^m_{\alpha,b_j}\vec{f}_2(x)-c_j|\,dx\right)\\ &=\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\left(I+\frac{1}{|B|}II\right).\end{aligned}$$

Let us first estimate I. Applying Lemma 4.1 we get

$$\begin{aligned}I=\int_B |T^m_{\alpha,b_j}\vec{f}_1(x)\,dx| &\le \int_B\int_{(2B)^m}|b_j(x)-b_j(y_j)|\,|K_\alpha(x,\vec{y})|\prod_{i=1}^m|f_i(y_i)|\,d\vec{y}\,dx\\ &\lesssim \|b_j\|_{\Lambda(\delta)}\int_B\int_{(2B)^m}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\tilde{\alpha}}}\,d\vec{y}\,dx\\ &\lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m}\int_B |I_{\tilde{\alpha},m}\vec{f}_1(x)|\,dx\\ &=\|\mathbf{b}\|_{(\Lambda(\delta))^m}\frac{|B|^{1+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}\prod_{i=1}^m\|f_iv_i\|_{p_i}.\end{aligned}$$

Consequently,

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\,I\lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m}\prod_{i=1}^m\|f_iv_i\|_{p_i}.$$

We now turn our attention to II. By (4.2) we can write

$$\begin{aligned}
II &\le \sum_{\sigma\in S_m,\sigma\neq\mathbf{1}} \int_B\int_B\int_{(\mathbf{2B})^\sigma} \big|(b_j(x)-b_j(y_j))K_\alpha(x,\vec{y})-(b_j(z)-b_j(y_j))K_\alpha(z,\vec{y})\big| \\
&\qquad \times \prod_{i=1}^m |f_i(y_i)|\,d\vec{y}\,dx\,dz \\
&\le \sum_{\sigma\in S_m,\sigma\neq\mathbf{1}} \int_B\int_B\int_{(\mathbf{2B})^\sigma} \big|(b_j(x)-b_j(y_j))(K_\alpha(x,\vec{y})-K_\alpha(z,\vec{y}))\big| \prod_{i=1}^m |f_i(y_i)|\,d\vec{y}\,dx\,dz \\
&\quad + \sum_{\sigma\in S_m,\sigma\neq\mathbf{1}} \int_B\int_B\int_{(\mathbf{2B})^\sigma} \big|(b_j(x)-b_j(z))K_\alpha(z,\vec{y})\big| \prod_{i=1}^m |f_i(y_i)|\,d\vec{y}\,dx\,dz \\
&= \sum_{\sigma\in S_m,\sigma\neq\mathbf{1}} (I_1^\sigma+I_2^\sigma).
\end{aligned}$$

We shall estimate each sum separately. Fix $\sigma\in S_m,\sigma\neq\mathbf{1}$. We start with I_1^σ. Since we are assuming $\sigma\neq\mathbf{1}$, condition (2.3) implies that

$$\begin{aligned}
|K_\alpha(x,\vec{y})-K_\alpha(z,\vec{y})| &\lesssim \frac{|x-z|^\gamma}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha+\gamma}} \\
&\lesssim \frac{|B|^{\gamma/n}}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha+\gamma}}.
\end{aligned}$$

Therefore we have that

$$\begin{aligned}
I_1^\sigma &\lesssim \|b_j\|_{\Lambda(\delta)}|B|^{\gamma/n}\int_B\int_B\int_{(\mathbf{2B})^\sigma} \frac{|x-y_j|^\delta\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha+\gamma}}\,d\vec{y}\,dx\,dz \\
&\lesssim \|b_j\|_{\Lambda(\delta)}|B|^{1+\gamma/n}\int_B\int_{(\mathbf{2B})^\sigma} \frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\tilde{\alpha}+\delta+\gamma-\delta}}\,d\vec{y}\,dx \\
&\lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m}|B|^{1+\delta/n}\int_B\int_{(\mathbf{2B})^\sigma} \frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x-y_i|)^{mn-\tilde{\alpha}+\delta}}\,d\vec{y}\,dx \\
&= \|\mathbf{b}\|_{(\Lambda(\delta))^m}|B|^{1+\delta/n}\int_B J_1(x,\sigma)\,dx.
\end{aligned}$$

By separating the factors in J_1 and applying Hölder's inequality we arrive to

$$\begin{aligned}
J_1(x,\sigma) &\lesssim \left(\prod_{i:\sigma_i=1}\int_{2B}\frac{|f_i(y_i)|}{|2B|^{1-\tilde{\alpha}_i/n+\delta/(mn)}}\,dy_i\right)\left(\prod_{i:\sigma_i=0}\int_{\mathbb{R}^n\setminus 2B}\frac{|f_i(y_i)|}{|x-y_i|^{n-\tilde{\alpha}_i+\delta/m}}\,dy_i\right) \\
&\lesssim \prod_{i=1}^m\|f_iv_i\|_{p_i}\left(\prod_{i:\sigma_i=1}\left\|\frac{v_i^{-1}\mathscr{X}_{2B}}{|2B|^{1-\tilde{\alpha}_i/n+\delta/(mn)}}\right\|_{p_i'}\right)\left(\prod_{i:\sigma_i=0}\left\|\frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus 2B}}{|x-\cdot|^{n-\tilde{\alpha}_i+\delta/m}}\right\|_{p_i'}\right) \\
&= \left(\prod_{i=1}^m\|f_iv_i\|_{p_i}\right)|2B|^{-\sum_{i:\sigma_i=1}(1-\tilde{\alpha}_i/n+\delta/(mn))}\prod_{i:\sigma_i=1}\|v_i^{-1}\mathscr{X}_{2B}\|_{p_i'}
\end{aligned}$$

$$\times \prod_{i:\sigma_i=0} \left\| \frac{v_i^{-1} \mathscr{X}_{\mathbb{R}^n \setminus 2B}}{|x-\cdot|^{n-\tilde{\alpha}_i+\delta/m}} \right\|_{p_i'}$$
$$\lesssim \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|2B|^{(\tilde{\delta}-\delta)/n}}{\|w\mathscr{X}_{2B}\|_\infty},$$

by virtue of condition (2.10). Thus the estimate

$$J_1(x,\sigma) \lesssim \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|2B|^{(\tilde{\delta}-\delta)/n}}{\|w\mathscr{X}_{2B}\|_\infty} \tag{4.3}$$

yields to

$$I_1^\sigma \lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m} \left(\prod_{i=1}^m \|f_i v_i\|_{p_i} \right) \frac{|B|^{2+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}. \tag{4.4}$$

We now proceed to estimate I_2^σ. We have that

$$I_2^\sigma \lesssim \|b_j\|_{\Lambda(\delta)} |B|^{\delta/n} \int_B \int_B \int_{(\mathbf{2B})^\sigma} \frac{\prod_{i=1}^m |f_i(y_i)|}{(\sum_{i=1}^m |z-y_i|)^{mn-\alpha}} \, d\vec{y}\, dx\, dz$$
$$\lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m} |B|^{2+\delta/n} \left(\prod_{i:\sigma_i=1} \int_{2B} \frac{|f_i(y_i)|}{|2B|^{1-\alpha_i/n}} \, dy_i \right) \left(\prod_{i:\sigma_i=0} \int_{\mathbb{R}^n \setminus 2B} \frac{|f_i(y_i)|}{|x_B-y_i|^{n-\alpha_i}} \, dy_i \right).$$

Since $\tilde{\alpha}_i = \alpha_i + \delta_i/m$ for each i, applying Hölder's inequality we can write

$$\prod_{i:\sigma_i=1} \int_{2B} \frac{|f_i(y_i)|}{|2B|^{1-\alpha_i/n}} \, dy_i \lesssim \prod_{i:\sigma_i=1} \frac{\|f_i v_i\|_{p_i}}{|2B|^{\sum_{i:\sigma_i=1}(1/p_i-\alpha_i/n)}} \left\| v_i^{-1} \mathscr{X}_{2B} \right\|_{p_i'}$$
$$= \prod_{i:\sigma_i=1} \frac{\|f_i v_i\|_{p_i}}{|2B|^{\sum_{i:\sigma_i=1}(1/p_i-\tilde{\alpha}_i/n+\delta/(mn))}} \left\| v_i^{-1} \mathscr{X}_{2B} \right\|_{p_i'}$$

and

$$\prod_{i:\sigma_i=0} \int_{\mathbb{R}^n \setminus 2B} \frac{|f_i(y_i)|}{|x_B-y_i|^{n-\alpha_i}} \, dy_i \lesssim \|f_i v_i\|_{p_i} \left\| \frac{v_i^{-1} \mathscr{X}_{\mathbb{R}^n \setminus 2B}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\tilde{\alpha}_i+\delta/m}} \right\|_{p_i'}.$$

Combining these estimates and using condition (2.10) we arrive to

$$I_2^\sigma \lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m} |B|^{2+\delta/n} \prod_{i=1}^m \|f_i v_i\|_{p_i} \frac{|2B|^{(\tilde{\delta}-\delta)/n}}{\|w\mathscr{X}_B\|_\infty} = \|\mathbf{b}\|_{(\Lambda(\delta))^m} \frac{|B|^{2+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty} \prod_{i=1}^m \|f_i v_i\|_{p_i}. \tag{4.5}$$

Therefore, applying the estimates obtained in (4.4) and (4.5) we conclude that

$$\frac{1}{|B|} II \lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m} \frac{|B|^{1+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty} \prod_{i=1}^m \|f_i v_i\|_{p_i}.$$

This completes the proof of (4.1) and we are done. □

5. Proof of Theorem 1.2

We devote this section to prove Theorem 1.2. We shall first establish an auxiliary lemma, which is essentially the boundedness given in Lemma 4.1 with different parameters. The proof can be achieved by following the same steps and we shall omit it.

LEMMA 5.1. *Let $0<\alpha<mn$, $0<\delta<(n-\alpha)/m$, $\tilde{\alpha}=\alpha+m\delta$ and $\vec{p}$ a vector of exponents that satisfies $p>n/\tilde{\alpha}$. Let $\tilde{\delta}\le\delta$ and $(w,\vec{v})$ be a pair of weights belonging to the class $\mathbb{H}_m(\vec{p},\tilde{\alpha},\tilde{\delta})$ such that $v_i^{-p_i'}\in \mathrm{RH}_m$ for every $i\in\mathscr{I}_2$. Then there exists a positive constant C such that for every ball B and every $\vec{f}$ such that $f_iv_i\in L^{p_i}$, $1\le i\le m$, we have that*

$$\int_B |I_{\tilde{\alpha},m}\vec{g}(x)|\,dx\le C\frac{|B|^{1+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}\prod_{i=1}^m\|f_iv_i\|_{p_i},$$

where $\vec{g}=(f_1\mathscr{X}_{2B},f_2\mathscr{X}_{2B},\dots,f_m\mathscr{X}_{2B})$.

Proof of Theorem 1.2. It will be enough to prove that

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B|\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}(x)-c|\,dx\le C\prod_{i=1}^m\|f_iv_i\|_{p_i}, \tag{5.1}$$

for some constant c and every ball B, with C independent of B and $\vec{f}$.

Fix a ball $B=B(x_B,R)$. By proceeding as in the proof of Theorem 1.1, we split $\vec{f}=\vec{f}_1+\vec{f}_2$, where $\vec{f}_1=(f_1\mathscr{X}_{2B},f_2\mathscr{X}_{2B},\dots,f_m\mathscr{X}_{2B})$. We take

$$c=\left(\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2\right)_B=\frac{1}{|B|}\int_B\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(z)\,dz.$$

By Proposition 3.1, for $z\in B$ we have that

$$\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(z)=\sum_{\sigma\in S_m,\sigma\neq\mathbf{1}}\int_{(\mathbf{2B})^\sigma}K_\alpha(z,\vec{y})\prod_{i=1}^m(b_i(z)-b_i(y_i))f_i(y_i)\,d\vec{y}. \tag{5.2}$$

Thus

$$\begin{aligned}\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B|\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}(x)-c|\,dx&\le\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\int_B|\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_1(x)|\,dx\\&\quad+\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{2+\tilde{\delta}/n}}\int_B\int_B|\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(x)-\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(z)|\,dz\,dx\\&=\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}(I+II).\end{aligned}$$

Let us first estimate I. Applying Proposition 3.1, (2.2) and Lemma 5.1 we get

$$I=\int_B|\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_1(x)|\,dx\le\int_B\int_{(2B)^m}|K_\alpha(x,\vec{y})|\prod_{i=1}^m|b_i(x)-b_i(y_i)|\,|f_i(y_i)|\,d\vec{y}\,dx$$

$$\leq C\prod_{i=1}^{m}\|b_i\|_{\Lambda(\delta)}\int_B\int_{(2B)^m}|K_{\tilde{\alpha}}(x,\vec{y})|\prod_{i=1}^{m}|f_i(y_i)|\,d\vec{y}\,dx$$

$$\leq C\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}\int_B|I_{\tilde{\alpha},m}\vec{f}_1(x)|\,dx$$

$$\leq C\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}\frac{|B|^{1+\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}\prod_{i=1}^{m}\|f_iv_i\|_{p_i}.$$

Consequently,

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{1+\tilde{\delta}/n}}\,I\leq C\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}\prod_{i=1}^{m}\|f_iv_i\|_{p_i}.$$

We continue with the estimate of II. We shall see that

$$|\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(x)-\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(z)|\leq C\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}\frac{|B|^{\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}\prod_{i=1}^{m}\|f_iv_i\|_{p_i},\tag{5.3}$$

for every $x,z\in B$. This would imply that $II\leq C\prod_{i=1}^m\|f_iv_i\|_{p_i}$.

For $x\in B$, using (5.2), we can write

$$\begin{aligned}
&|\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(x)-\mathscr{T}^m_{\alpha,\mathbf{b}}\vec{f}_2(z)|\\
&\quad\leq\sum_{\sigma\in S_m,\sigma\neq\mathbf{1}}\int_{(\mathbf{2B})^\sigma}\left|K_\alpha(x,\vec{y})\prod_{i=1}^{m}(b_i(x)-b_i(y_i))-K_\alpha(z,\vec{y})\prod_{i=1}^{m}(b_i(z)-b_i(y_i))\right|\\
&\qquad\times\prod_{i=1}^{m}|f_i(y_i)|\,d\vec{y}\\
&\quad\leq\sum_{\sigma\in S_m,\sigma\neq\mathbf{1}}\int_{(\mathbf{2B})^\sigma}|K_\alpha(x,\vec{y})-K_\alpha(z,\vec{y})|\prod_{i=1}^{m}|b_i(x)-b_i(y_i)|\,|f_i(y_i)|\,d\vec{y}\\
&\qquad+\sum_{\sigma\in S_m,\sigma\neq\mathbf{1}}\int_{(\mathbf{2B})^\sigma}|K_\alpha(z,\vec{y})|\left|\prod_{i=1}^{m}(b_i(x)-b_i(y_i))-\prod_{i=1}^{m}(b_i(z)-b_i(y_i))\right|\\
&\qquad\times\prod_{i=1}^{m}|f_i(y_i)|\,d\vec{y}\\
&\quad=\sum_{\sigma\in S_m,\sigma\neq\mathbf{1}}(I_1^\sigma+I_2^\sigma).
\end{aligned}$$

Fix $\sigma\in S_m,\sigma\neq\mathbf{1}$. Let us first estimate I_1^σ. Applying condition (2.3) we have that

$$\begin{aligned}
|K_\alpha(x,\vec{y})-K_\alpha(z,\vec{y})|&\lesssim\frac{|x-z|^\gamma}{(\sum_{i=1}^m|x-y_i|)^{mn-\alpha+\gamma}}\\
&\lesssim\frac{|B|^{\gamma/n}}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\alpha+\gamma}}.
\end{aligned}$$

Therefore we have that

$$I_1^\sigma\lesssim|B|^{\gamma/n}\prod_{i=1}^{m}\|b_i\|_{\Lambda(\delta)}\int_{(\mathbf{2B})^\sigma}\frac{(\sum_{i=1}^m|x_B-y_i|)^{m\delta}\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\alpha+\gamma}}\,d\vec{y}$$

$$\lesssim \|\mathbf{b}\|^m_{(\Lambda(\delta))^m}|B|^{\gamma/n}\int_{(\mathbf{2B})^\sigma}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\tilde{\alpha}+\delta+\gamma-\delta}}\,d\vec{y}$$
$$\lesssim \|\mathbf{b}\|^m_{(\Lambda(\delta))^m}|B|^{\delta/n}\int_{(\mathbf{2B})^\sigma}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\tilde{\alpha}+\delta}}\,d\vec{y},$$

since $\gamma>\delta$ and $\vec{y}\in(\mathbf{2B})^\sigma$ implies that $|x_B-y_j|\geq C|B|^{1/n}$ for at least one index $1\leq j\leq m$. From this expression we can use the estimate (4.3) performed in page 17 in order to obtain

$$I_1^\sigma\lesssim \|\mathbf{b}\|^m_{(\Lambda(\delta))^m}\frac{|B|^{\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty}\prod_{i=1}^m\|f_iv_i\|_{p_i}.$$

Next we proceed to estimate I_2^σ. Fix $\sigma\in S_m,\sigma\neq\mathbf{1}$. Applying Lemma 3.2 we have that

$$\left|\prod_{i=1}^m(b_i(x)-b_i(y_i))-\prod_{i=1}^m(b_i(z)-b_i(y_i))\right|\leq\sum_{j=1}^m|b_j(x)-b_j(z)|\prod_{i>j}|b_i(z)-b_i(y_i)|\prod_{i<j}|b_i(x)-b_i(y_i)|$$
$$\lesssim\left(\prod_{i=1}^m\|b_i\|_{\Lambda(\delta)}\right)|B|^{\delta/n}\sum_{j=1}^m\prod_{i>j}|z-y_i|^\delta\prod_{i<j}|x-y_i|^\delta.$$

Since x and z belong to B and $\vec{y}\in(\mathbf{2B})^\sigma$, we have that

$$|x-y_i|\lesssim\sum_{j=1}^m|x_B-y_j|$$

and also

$$|z-y_i|\lesssim\sum_{j=1}^m|x_B-y_j|,$$

for each i, regardless y_i belongs to $2B$ or $\mathbb{R}^n\backslash 2B$. Therefore we arrive to

$$\left|\prod_{i=1}^m(b_i(x)-b_i(y_i))-\prod_{i=1}^m(b_i(z)-b_i(y_i))\right|\lesssim\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}|B|^{\delta/n}\left(\sum_{j=1}^m|x_B-y_j|\right)^{(m-1)\delta}.$$

Using this estimate we can proceed with I_2^σ as follows

$$I_2^\sigma\lesssim\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}|B|^{\delta/n}\int_{(\mathbf{2B})^\sigma}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\alpha+(1-m)\delta}}\,d\vec{y}$$
$$\lesssim\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}|B|^{\delta/n}\int_{(\mathbf{2B})^\sigma}\frac{\prod_{i=1}^m|f_i(y_i)|}{(\sum_{i=1}^m|x_B-y_i|)^{mn-\tilde{\alpha}+\delta}}\,d\vec{y}$$
$$\lesssim\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}|B|^{\delta/n}\prod_{i:\sigma_i=1}\int_{2B}\frac{|f_i(y_i)|}{|2B|^{1-\tilde{\alpha}_i/n+\delta/(mn)}}\,dy_i\prod_{i:\sigma_i=0}\int_{\mathbb{R}^n\backslash 2B}\frac{|f_i(y_i)|}{|x_B-y_i|^{n-\tilde{\alpha}_i+\delta/m}}\,dy_i.$$

Applying Hölder's inequality and condition (2.10) we get

$$I_2^\sigma\lesssim\|\mathbf{b}\|^m_{(\Lambda(\delta))^m}|B|^{\delta/n-\theta(\sigma)}\prod_{i=1}^m\|f_iv_i\|_{p_i}\prod_{i:\sigma_i=1}\|v_i^{-1}\mathscr{X}_{2B}\|_{p_i'}\prod_{i:\sigma_i=0}\left\|\frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\backslash 2B}}{|x_B-\cdot|^{n-\tilde{\alpha}_i+\delta/m}}\right\|_{p_i'}$$

$$\lesssim \|\mathbf{b}\|_{(\Lambda(\delta))^m}^m \frac{|B|^{\tilde{\delta}/n}}{\|w\mathscr{X}_B\|_\infty} \prod_{i=1}^m \|f_i v_i\|_{p_i},$$

where $\theta(\sigma) = \sum_{i:\sigma_i=1} (1-\tilde{\alpha}_i/n + \delta/(mn))$. So (5.3) holds and the proof is complete. □

6. The class $\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$

In this section we give a complete study of the class $\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$ related with the boundedness properties stated in our main results. Recall that $(w,\vec{v})$ belongs to the class $\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$ if there exists a positive constant C for which the inequality

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n}} \prod_{i=1}^m \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}}{(|B|^{1/n} + |x_B - y|)^{(n-\beta_i+\delta/m)p_i'}}\, dy \right)^{1/p_i'} \leq C$$

holds for every ball $B = B(x_B, R)$.

We begin with a characterizaction of this class of weights in terms of the global condition (2.9). The proof follows similar lines as Lemma 2.1 in [2] and we shall omit it. We recall the notation $\mathscr{I}_1 = \{i : p_i = 1\}$ and $\mathscr{I}_2 = \{i : 1 < p_i \leq \infty\}$.

LEMMA 6.1. *Let $0 < \beta < mn$, $\tilde{\delta} \in \mathbb{R}$, $\vec{p}$ a vector of exponents and $(w,\vec{v})$ a pair of weights such that $v_i^{-1} \in \mathrm{RH}_\infty$ for $i \in \mathscr{I}_1$ and $v_i^{-p_i'}$ is doubling for $i \in \mathscr{I}_2$. Then, condition $\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$ is equivalent to* (2.9).

As an immediate consequence of this lemma we have the following.

COROLLARY 6.2. *Under the hypotheses of Lemma* 6.1 *we have that conditions* (2.9) *implies* (2.8).

The next lemma establishes a useful property in order to give examples of weights in the considered class. We shall assume that $\beta_i = \beta/m$ for every i.

LEMMA 6.3. *Let $0 < \beta < mn$, $\tilde{\delta} < \tau = (\beta - mn)(1-1/m) + \delta/m$, $\vec{p}$ a vector of exponents and $(w,\vec{v})$ a pair of weights satisfying condition* (2.8)*. Then $(w,\vec{v})$ satisfies* (2.9).

Proof. The proof follows similar lines as in Lemma 2.3 in [2]. We shall give a scheme for the sake of completeness. Let $\theta = n - \beta/m + \delta/m$. Let B be a ball and $B_k = 2^k B$, for $k \in \mathbb{N}$. If $i \in \mathscr{I}_1$ we have that

$$\begin{aligned} \left\| \frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B - \cdot|^\theta} \right\|_\infty &\leq \sum_{k=1}^\infty \left\| \frac{v_i^{-1}\mathscr{X}_{B_{k+1}\setminus B_k}}{|x_B - \cdot|^\theta} \right\|_\infty \\ &\lesssim \sum_{k=1}^\infty |B_k|^{-\theta/n} \left\| v_i^{-1}\mathscr{X}_{B_{k+1}} \right\|_\infty . \end{aligned}$$

On the other hand, for $i \in \mathscr{I}_2$

$$\left(\int_{\mathbb{R}^n\setminus B} \frac{v_i^{-p_i'}(y)}{|x_B-y|^{\theta p_i'}}\,dy\right)^{1/p_i'} \le \left(\sum_{k=1}^{\infty}\int_{B_{k+1}\setminus B_k} \frac{v_i^{-p_i'}(y)}{|x_B-y|^{\theta p_i'}}\,dy\right)^{1/p_i'}$$
$$\lesssim \sum_{k=1}^{\infty}|B_k|^{-\theta/n}\left(\int_{B_{k+1}} v_i^{-p_i'}\right)^{1/p_i'}.$$

By taking $\vec{k}=(k_1,k_2\ldots,k_m)$, the left-hand side of (2.9) can be bounded by a multiple constant of

$$\sum_{\vec{k}\in\mathbb{N}^m}\prod_{i\in\mathscr{I}_1}|B_{k_i}|^{-\theta/n}\left\|v_i^{-1}\mathscr{X}_{B_{k_i+1}}\right\|_\infty \prod_{i\in\mathscr{I}_2}|B_{k_i}|^{-\theta/n}\left(\int_{B_{k_i+1}} v_i^{-p_i'}\right)^{1/p_i'} = \sum_{\vec{k}\in\mathbb{N}^m} I\left(B,\vec{k}\right).$$

Observe that $\mathbb{N}^m \subset \bigcup_{i=1}^m K_i$, where $K_i=\{\vec{k}=(k_1,k_2,\ldots,k_m): k_i\ge k_j$ for every $j\}$. Let us estimate the sum over K_1, being similar for the other sets. Therefore

$$\sum_{\vec{k}\in K_1} I\left(B,\vec{k}\right) \le \sum_{k_1=1}^{\infty}|B_{k_1}|^{-\frac{\theta}{n}}\prod_{i\in\mathscr{I}_1}\left\|v_i^{-1}\mathscr{X}_{B_{k_1+1}}\right\|_\infty \prod_{i\in\mathscr{I}_2}\left(\int_{B_{k_1+1}} v_i^{-p_i'}\right)^{1/p_i'}\prod_{i\ne 1}\sum_{k_i=1}^{k_1}|B_{k_i}|^{-\frac{\theta}{n}}.$$

Notice that

$$\sum_{k_i=1}^{k_1}|B_{k_i}|^{-\theta/n} = |B|^{-\theta/n}\sum_{k_i=1}^{k_1}2^{-k_i\theta} \lesssim |B_{k_1}|^{-\theta/n}\sum_{k_i=1}^{k_1}2^{(k_1-k_i)\theta} \lesssim |B_{k_1}|^{-\theta/n}2^{k_1\theta}.$$

Thus, from the estimation above and (2.8) we obtain that

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n}}\sum_{\vec{k}\in K_1} I\left(B,\vec{k}\right) \lesssim |B|^{(\delta-\tilde{\delta})/n}\sum_{k_1=1}^{\infty}2^{(m-1)k_1\theta}\,|B_{k_1}|^{-m\theta/n}\,\|w\mathscr{X}_{B_{k_1+1}}\|_\infty$$
$$\times\prod_{i\in\mathscr{I}_1}\left\|v_i^{-1}\mathscr{X}_{B_{k_1+1}}\right\|_\infty \prod_{i\in\mathscr{I}_2}\left(\int_{B_{k_1+1}} v_i^{-p_i'}\right)^{1/p_i'}$$
$$\lesssim \sum_{k_1=1}^{\infty}2^{-k_1(\theta-mn-\tilde{\delta}+\beta)},$$

being the last sum finite since $\tilde{\delta}<\tau$. □

Recall that when we restrict the pair of weights $(w,\vec{v})$ to satisfy the relation $w=\prod_{i=1}^m v_i$, condition $\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$ is denoted by $\vec{v}\in\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$. The following result establishes that, for a suitable range of the parameter $\tilde{\delta}$, this class is equivalent to $A_{\vec{p},\infty}$.

COROLLARY 6.4. *Let $\delta\in\mathbb{R}$, $0<\beta<mn$, $\tilde{\delta}<\tau=(\beta-mn)(1-1/m)+\delta/m$. Then $\vec{v}\in\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$ if and only if $\vec{v}\in A_{\vec{p},\infty}$.*

Proof. Let $\vec{v} \in \mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$. Then condition (2.8) holds. On the other hand, by Lemma 1.3, we have that $\tilde{\delta} = \beta - n/p$. This implies that $\vec{v} \in A_{\vec{p},\infty}$.

Conversely, let $\vec{v} \in A_{\vec{p},\infty}$. Since $\tilde{\delta} < \tau$, by Lemma 6.3 we have that $\vec{v}$ satisfies (2.9). By Lemma 6.1 it will be enough to check that $v_i^{-1} \in \mathrm{RH}_\infty$ for $i \in \mathscr{I}_1$ and $v_i^{-p_i'}$ is doubling for $i \in \mathscr{I}_2$. Let us first check that $v_i^{-1} \in \mathrm{RH}_\infty$ for $i \in \mathscr{I}_1$. Let $r = p/(mp-1)$, fix $i_0 \in \mathscr{I}_1$ and observe that

$$\begin{aligned}
\frac{1}{|B|}\int_B v_{i_0}^r &= \frac{1}{|B|}\int_B \prod_{i=1}^m v_i^r \prod_{i \neq i_0} v_i^{-r} \\
&\leq \left\| \mathscr{X}_B \prod_{i=1}^m v_i \right\|_\infty^r \prod_{i \in \mathscr{I}_1, i \neq i_0} \|v_i^{-1}\mathscr{X}_B\|_\infty^r \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{r/p_i'} \\
&\leq \frac{[\vec{v}]_{A_{\vec{p},\infty}}^r}{\|v_{i_0}^{-1}\mathscr{X}_B\|_\infty^r} \\
&= [\vec{v}]_{A_{\vec{p},\infty}}^r \inf_B v_{i_0}^r.
\end{aligned}$$

Therefore, $v_{i_0}^r$ is an A_1 weight. Then we can conclude $v_{i_0}^{-1}$ is an RH_∞ weight.

On the other hand, observe that $A_{\vec{p},\infty} \subseteq A_{\vec{p},q}$ for every $q > 0$. If we pick $q = p$ we can apply Lemma 3.3 to conclude that $\vec{z} = (z_1, \ldots, z_m)$ belongs to $A_{\vec{\ell}}$, where $\ell = p$, $z_i = v_i^{\ell_i}$ and $\ell_i = p_i$ for every i. This implies (see, for example, Theorem 3.6 in [6]) that $z_i^{1-\ell_i'} \in A_{m\ell_i'}$, that is, $v_i^{-p_i'} \in A_{mp_i'} \subseteq A_\infty$, so it is a doubling weight for every $i \in \mathscr{I}_2$. This completes the proof. □

The following two theorems allows us to describe the region where we can find nontrivial weights in $\mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$ in terms of the parameters p, β and $\tilde{\delta}$.

THEOREM 6.5. *Let $\delta \in \mathbb{R}$ be fixed. Let $0 < \beta < mn$, $\tilde{\delta} \in \mathbb{R}$ and $\vec{p}$ be a vector of exponents. The following statements hold:*

(a) *If $\tilde{\delta} > \delta$ or $\tilde{\delta} > \beta - n/p$ then condition $\mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$ is satisfied if and only if $v_i = \infty$ a.e. for some $1 \leq i \leq m$.*

(b) *The same conclusion holds if $\tilde{\delta} = \beta - n/p = \delta$.*

(c) *If $\tilde{\delta} < \beta - mn$, then condition $\mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$ is satisfied if and only if $v_i = \infty$ a.e. for some $1 \leq i \leq m$ or $w = 0$ a.e.*

Proof. Let $(w, \vec{v}) \in \mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$. We start with the proof of item (a). We shall first assume that $\tilde{\delta} > \delta$. Picking a ball $B = B(x_B, R)$ such that x_B is a Lebesgue point of w^{-1}, from (2.5) we obtain

$$\prod_{i \in \mathscr{I}_1} \left\| \frac{v_i^{-1}}{(|B|^{1/n} + |x_B - \cdot|)^{n-\beta_i+\delta/m}} \right\|_\infty \prod_{i \in \mathscr{I}_2} \left(\int_{\mathbb{R}^n} \frac{v_i^{-p_i'}}{(|B|^{1/n} + |x_B - \cdot|)^{(n-\beta_i+\delta/m)p_i'}} \right)^{\frac{1}{p_i'}} \lesssim \frac{|B|^{(\tilde{\delta}-\delta)/n}}{\|w\mathscr{X}_B\|_\infty}$$

$$\lesssim \frac{w^{-1}(B)}{|B|R^{\delta-\tilde{\delta}}},$$

for every $R>0$. By letting R approach to zero we can deduce that there exists $1\leq i\leq m$ such that $v_i=\infty$ almost everywhere.

Let us now consider the case $\tilde{\delta}>\beta-n/p$. Pick a ball $B=B(x_B,R)$, being x_B a Lebesgue point of w^{-1} and of every v_i^{-1}. Then condition (2.8) implies that

$$\prod_{i=1}^m \frac{1}{|B|}\int_B v_i^{-1} \leq \prod_{i\in\mathscr{I}_1} \|v_i^{-1}\mathscr{X}_B\|_\infty \prod_{i\in\mathscr{I}_2}\left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'} \lesssim \frac{|B|^{\frac{\tilde{\delta}}{n}-\frac{\beta}{n}+\frac{1}{p}}}{\|w\mathscr{X}_B\|_\infty} \lesssim \frac{w^{-1}(B)}{|B|}R^{\tilde{\delta}-\beta+n/p}$$

for every $R>0$. If we let again R approach to zero, we obtain

$$\prod_{i=1}^m v_i^{-1}(x_B)=0,$$

and then $\prod_{i=1}^m v_i^{-1}$ is zero a.e. This allows us to conclude that the set $\bigcap_{i=1}^m\{v_i^{-1}>0\}$ has null measure. Since $v_i(y)>0$ for almost every y and every i, there exists j such that $v_j=\infty$ a.e.

We now proceed with item (b). Suppose $\tilde{\delta}=\beta-n/p=\delta$. We define

$$\xi=\sum_{i=1}^m \frac{1}{p_i'}=m-\frac{1}{p}.$$

Applying Hölder's inequality we obtain

$$\left(\int_{\mathbb{R}^n}\frac{\left(\prod_{i\in\mathscr{I}_2}v_i^{-1}\right)^{1/\xi}}{(|B|^{1/n}+|x_B-\cdot|)^{\sum_{i\in\mathscr{I}_2}(n-\beta_i+\delta/m)/\xi}}\right)^{\xi} \lesssim \prod_{i\in\mathscr{I}_2}\left(\int_{\mathbb{R}^n}\frac{v_i^{-p_i'}}{(|B|^{1/n}+|x_B-\cdot|)^{(n-\beta_i+\delta/m)p_i'}}\right)^{\frac{1}{p_i'}}$$

and since $(w,\vec{v})\in\mathbb{H}_m(\vec{p},\beta,\tilde{\delta})$,

$$\prod_{i\in\mathscr{I}_1}\left\|\frac{v_i^{-1}}{(|B|^{1/n}+|x_B-\cdot|)^{n-\beta_i+\delta/m}}\right\|_\infty \left(\int_{\mathbb{R}^n}\frac{(\prod_{i\in\mathscr{I}_2}v_i^{-1})^{1/\xi}}{(|B|^{1/n}+|x_B-\cdot|)^{\sum_{i\in\mathscr{I}_2}(n-\beta_i+\delta/m)/\xi}}\right)^{\xi} \lesssim \frac{w^{-1}(B)}{|B|},$$

and we can deduce that for every ball B

$$\left(\int_{\mathbb{R}^n}\frac{(\prod_{i=1}^m v_i^{-1})^{1/\xi}}{(|B|^{1/n}+|x_B-y|)^{(mn-\beta+\delta)/\xi}}\,dy\right)^{\xi} \lesssim \frac{w^{-1}(B)}{|B|}.$$

From this inequality, we can continue adapting an argument presented in [14], to conclude that there exists i such that $v_i=\infty$ a.e.

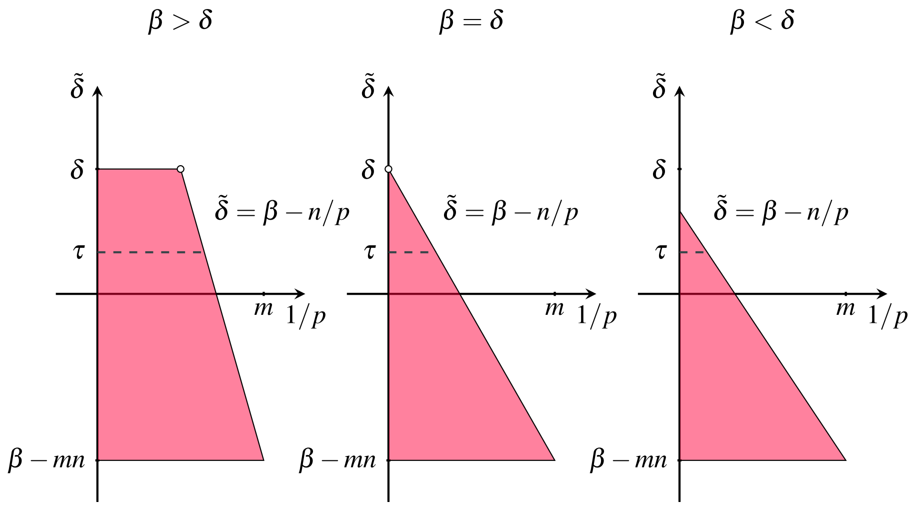
We end with the proof of item (c). Let $\tilde{\delta}<\beta-mn$. Given a ball $B=B(x_B,R)$ and $B_0\subset B$, condition (2.8) implies that

$$\|w\mathscr{X}_{B_0}\|_\infty\prod_{i=1}^m\|v_i^{-1}\mathscr{X}_{B_0}\|_{p_i'} \leq \|w\mathscr{X}_B\|_\infty\prod_{i=1}^m\|v_i^{-1}\mathscr{X}_B\|_{p_i'} \lesssim R^{\tilde{\delta}-\beta+mn}.$$

The right-hand side of the inequality above tends to zero when R approaches to ∞, which implies that either $\|w\mathscr{X}_{B_0}\|_\infty = 0$ or $\|v_i^{-1}\mathscr{X}_{B_0}\|_{p_i'} = 0$, for some i. As B_0 is arbitrary we obtain either $w = 0$ or $v_i = \infty$ for some i, respectively. □

THEOREM 6.6. *Let $\delta \in \mathbb{R}$ be fixed. Given $0 < \beta < mn$, there exist pairs of weights $(w, \vec{v})$ satisfying* (2.5) *for every $\vec{p}$ and $\tilde{\delta}$ such that $\beta - mn \le \tilde{\delta} \le \min\{\delta, \beta - n/p\}$, excluding the case $\tilde{\delta} = \delta$ when $\beta - n/p = \delta$.*

The next figure depicts the area in which we can find nontrivial pair of weights belonging to $\mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$, for a fixed value δ and depending on β.



Since the classes $\mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$ have a similar structure as those defined in [2], the proof of the theorem above will follow similar lines as that in Theorem 5.1 of [2], with adequate changes. We include a sketch for the sake of completeness. We shall need the following auxiliary lemma.

LEMMA 6.7. *For a ball $B = B(x_B, R)$ in $\mathbb{R}^n$ and $\alpha > -n$, we have that*

$$\int_B |x|^\alpha \, dx \approx R^n \left(\max\{R, |x_B|\}\right)^\alpha.$$

Proof of Theorem 6.6. Recall that $\tau = (\beta - mn)(1 - 1/m) + \delta/m$ is the number appearing in Lemma 6.3, we shall split the proof into the following cases:

(a) $\beta - mn < \tilde{\delta} < \tau \le \beta - n/p$;

(b) $\beta - mn < \tilde{\delta} \le \beta - n/p < \tau$;

(c) $\beta - mn < \tilde{\delta} = \tau < \delta < \beta - n/p$;

(d) $\beta - mn < \tilde{\delta} = \tau < \beta - n/p < \delta$;

(e) $\tau < \tilde{\delta} \leq \min\{\delta, \beta - n/p\}$;

(f) $\tilde{\delta} = \beta - mn$.

Let us prove (a). Recall that $\mathscr{I}_1 = \{i : p_i = 1\}$, $\mathscr{I}_2 = \{i : p_i > 1\}$ and let $m_j = \#\mathscr{I}_j$, for $j = 1, 2$. Since $m_1 < m$ by the restrictions on the parameters, we can take

$$0 < \varepsilon < \frac{mn - \beta + \tilde{\delta}}{m - m_1}.$$

For $1 \leq i \leq m$ we define

$$\xi_i = \begin{cases} 0 & \text{if } i \in \mathscr{I}_1, \\ \frac{n}{p_i'} - \varepsilon & \text{if } i \in \mathscr{I}_2. \end{cases}$$

Let $\rho = \sum_{i=1}^m \xi_i + \tilde{\delta} - \beta + n/p > 0$. Then we take

$$w(x) = |x|^\rho \quad \text{and} \quad v_i(x) = |x|^{\xi_i}.$$

By virtue of Lemma 6.3 it will be enough to show that $(w, \vec{v})$ verifies condition (2.8). Let $B = B(x_B, R)$ and assume that $|x_B| \leq R$. If $i \in \mathscr{I}_2$, by Lemma 6.7 we get

$$\left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'} = \left(\frac{1}{|B|}\int_B |x|^{-\xi_i p_i'}\,dx\right)^{1/p_i'} \approx R^{-\xi_i},$$

and $\left\|v_i^{-1}\mathscr{X}_B\right\|_\infty = 1$ for $i \in \mathscr{I}_1$. On the other hand, $\|w\mathscr{X}_B\|_\infty \lesssim R^\rho$ since $\rho > 0$. Therefore,

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{\tilde{\delta}/n - \beta/n + 1/p}} \prod_{i \in \mathscr{I}_1} \left\|v_i^{-1}\mathscr{X}_B\right\|_\infty \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'} \lesssim R^{\rho - \sum_{i=1}^m \xi_i - \tilde{\delta} + \beta - n/p} \leq C.$$

We now consider the case $|x_B| > R$. We have that

$$\|w\mathscr{X}_B\|_\infty \lesssim |x_B|^\rho,$$

whilst for $i \in \mathscr{I}_2$

$$\left(\frac{1}{|B|}\int_B |x|^{-\xi_i p_i'}\,dx\right)^{1/p_i'} \approx |x_B|^{-\xi_i}.$$

Consequently, since $\tilde{\delta} < \beta - n/p$ we get

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{\tilde{\delta}/n - \beta/n + 1/p}} \prod_{i \in \mathscr{I}_1} \left\|v_i^{-1}\mathscr{X}_B\right\|_\infty \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'} \lesssim |x_B|^{\rho - \sum_{i=1}^m \xi_i - \tilde{\delta} + \beta - n/p} \leq C,$$

which completes the proof of (a).

We now prove (b). In this case we take $w = 1$ and $v_i = |x|^{\xi_i}$, $\xi_i = (\beta - \tilde{\delta})/m - n/p_i$ for every $1 \leq i \leq m$. By Lemma 6.3 it will be enough to prove that $(w, \vec{v})$ satisfies condition (2.8). Pick a ball $B = B(x_B, R)$ and assume that $|x_B| \leq R$. If $i \in \mathscr{I}_1$ we get $\xi_i < 0$, since we are assuming $\tilde{\delta} > \beta - mn$. In this case we get

$$\left\| v_i^{-1} \mathscr{X}_B \right\|_\infty \approx R^{-\xi_i}.$$

On the other hand, for $i \in \mathscr{I}_2$ we have $\xi_i < n/p_i'$, so Lemma 6.7 yields

$$\left(\frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \approx R^{-\xi_i}.$$

These two estimates imply that

$$\frac{\|w \mathscr{X}_B\|_\infty}{|B|^{\tilde{\delta}/n - \beta/n + 1/p}} \prod_{i \in \mathscr{I}_1} \left\| v_i^{-1} \mathscr{X}_B \right\|_\infty \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \lesssim \frac{R^{-\sum_{i=1}^m \xi_i}}{R^{\tilde{\delta} - \beta + n/p}} = 1.$$

If $|x_B| > R$, we have that $\left\| v_i^{-1} \mathscr{X}_B \right\|_\infty \lesssim |x_B|^{-\xi_i}$ and also

$$\left(\frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \approx |x_B|^{-\xi_i}$$

by Lemma 6.7. Thus

$$\frac{\|w \mathscr{X}_B\|_\infty}{|B|^{\tilde{\delta}/n - \beta/n + 1/p}} \prod_{i \in \mathscr{I}_1} \left\| v_i^{-1} \mathscr{X}_B \right\|_\infty \prod_{i \in \mathscr{I}_2} \left(\frac{1}{|B|} \int_B v_i^{-p_i'} \right)^{1/p_i'} \lesssim \frac{|x_B|^{-\sum_{i=1}^m \xi_i}}{R^{\tilde{\delta} - \beta + n/p}} \leq 1,$$

since $\tilde{\delta} \leq \beta - n/p$. This concludes the proof of item (b).

In order to prove (c) we pick $(\beta - \tau)/m - n/p_i < \xi_i < n/p_i'$ for $i \in \mathscr{I}_2$ and $\xi_i = 0$ for $i \in \mathscr{I}_1$. We also take $\rho = \sum_{i=1}^m \xi_i + \tau - \beta + n/p > 0$ and define $w(x) = |x|^\rho$ and $v_i(x) = |x|^{\xi_i}$, for $1 \leq i \leq m$. We first notice that

$$\prod_{i \in \mathscr{I}_1} \left\| \frac{v_i^{-1} \mathscr{X}_{\mathbb{R}^n \setminus B}}{|x_B - \cdot|^{n - \beta/m + \delta/m}} \right\|_\infty \leq R^{-\sum_{i \in \mathscr{I}_1} (n - \beta/m + \delta/m)}.$$

By virtue of Lemma 6.1 we have to prove that condition (2.9) holds. Using the estimate above, it will be enough to show that

$$R^{\delta - \tilde{\delta} - \sum_{i \in \mathscr{I}_1} (n - \beta/m + \delta/m)} \|w \mathscr{X}_B\|_\infty \prod_{i \in \mathscr{I}_2} \left(\int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n - \beta/m + \delta/m) p_i'}} \, dy \right)^{1/p_i'} \leq C \tag{6.1}$$

for every ball B. We shall first assume that $|x_B| \leq R$. Let $B_k = B\left(x_B, 2^k R\right)$ for $k \in \mathbb{N}_0$ and $i \in \mathscr{I}_2$. By Lemma 6.7, we get

$$\left(\int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n - \beta/m + \delta/m) p_i'}} \, dy \right)^{1/p_i'} \lesssim \sum_{k=0}^\infty (2^k R)^{-n + \beta/m - \delta/m} \left(\int_{B_{k+1} \setminus B_k} |y|^{-\xi_i p_i'} \, dy \right)^{1/p_i'}$$

$$\lesssim \sum_{k=0}^{\infty}(2^kR)^{-n+\beta/m-\delta/m-\xi_i+n/p_i'}$$
$$\lesssim R^{-n/p_i+\beta/m-\delta/m-\xi_i},$$

since $-n/p_i+\beta/m-\delta/m-\xi_i<0$ by the election of ξ_i. Since $\tilde{\delta}=\tau$, the left-hand side of (6.1) is bounded by a multiple constant of

$$R^{\delta-\tilde{\delta}-\sum_{i\in\mathscr{I}_1}(n-\beta/m+\delta/m)+\rho-\sum_{i\in\mathscr{I}_2}(n/p_i-\beta/m+\delta/m+\xi_i)}=R^{-\tilde{\delta}-n/p+\beta+\rho-\sum_{i=1}^m\xi_i}=1.$$

We now assume $|x_B|>R$. There exists a number N such that $2^NR<|x_B|\le 2^{N+1}R$. If $i\in\mathscr{I}_2$ we have that

$$\left(\int_{\mathbb{R}^n\setminus B}\frac{v_i^{-p_i'}(y)}{|x_B-y|^{(n-\beta/m+\delta/m)p_i'}}\,dy\right)^{1/p_i'}\lesssim\sum_{k=0}^{\infty}(2^kR)^{-n+\beta/m-\delta/m}\left(\int_{B_k}|y|^{-\xi_ip_i'}\,dy\right)^{1/p_i'}$$
$$=\sum_{k=0}^{N}+\sum_{k=N+1}^{\infty}=S_1^i+S_2^i.$$

Let $\theta_i=n/p_i+(\delta-\beta)/m$, for $1\le i\le m$. We shall first prove that if $\theta_i<0$, then

$$S_j^i\lesssim|x_B|^{-\xi_i-\theta_i},\tag{6.2}$$

for $j=1,2$. Indeed, by Lemma 6.7 we obtain

$$S_1^i\lesssim\sum_{k=0}^{N}(2^kR)^{-n+\beta/m-\delta/m+n/p_i'}|x_B|^{-\xi_i}$$
$$\lesssim|x_B|^{-\xi_i}R^{-\theta_i}\sum_{k=0}^{N}2^{-k\theta_i}$$
$$\lesssim|x_B|^{-\xi_i}(2^NR)^{-\theta_i}$$
$$\lesssim|x_B|^{-\xi_i-\theta_i}.$$

For S_2^i we apply again Lemma 6.7 in order to get

$$S_2^i\lesssim\sum_{k=N+1}^{\infty}\left(2^kR\right)^{-n+\beta/m-\delta/m+n/p_i'-\xi_i}\lesssim\sum_{k=N+1}^{\infty}\left(2^kR\right)^{-\xi_i-\theta_i}$$
$$=\left(2^{N+1}R\right)^{-\xi_i-\theta_i}\sum_{k=0}^{\infty}2^{-k(\xi_i+\theta_i)}\lesssim|x_B|^{-\xi_i-\theta_i},$$

since $\theta_i+\xi_i=n/p_i+(\delta-\beta)/m+\xi_i>0$.

We now assume that $\theta_i=0$. Proceeding similarly as in the previous case, we have

$$S_1^i\lesssim|x_B|^{-\xi_i}N\lesssim|x_B|^{-\xi_i}\log_2\left(\frac{|x_B|}{R}\right),$$

and

$$S_2^i \lesssim |x_B|^{-\xi_i}$$

since $\xi_i > 0$ when $\theta_i = 0$. Consequently,

$$S_1^i + S_2^i \lesssim |x_B|^{-\xi_i}\left(1+\log_2\left(\frac{|x_B|}{R}\right)\right) \lesssim |x_B|^{-\xi_i}\log_2\left(\frac{|x_B|}{R}\right). \tag{6.3}$$

We finally consider the case $\theta_i > 0$. For S_2^i we can proceed exactly as in the case $\theta_i < 0$ and get the same bound. On the other hand, for S_1^i we have that

$$S_1^i \lesssim \sum_{k=0}^{N}(2^kR)^{-n+\beta/m-\delta/m+n/p_i'}|x_B|^{-\xi_i} \lesssim |x_B|^{-\xi_i}R^{-\theta_i}\sum_{k=0}^{N}2^{-k\theta_i} \lesssim |x_B|^{-\xi_i-\theta_i}2^{N\theta_i}.$$

Therefore, if $i \in \mathscr{I}_2$ and $\theta_i > 0$ we get

$$S_1^i + S_2^i \lesssim |x_B|^{-\xi_i-\theta_i}\left(1+2^{N\theta_i}\right) \lesssim 2^{N\theta_i}|x_B|^{-\xi_i-\theta_i}. \tag{6.4}$$

Combining (6.2), (6.3) and (6.4) we obtain

$$\begin{aligned}\prod_{i\in\mathscr{I}_2}\left(\int_{\mathbb{R}^n\setminus B}\frac{v_i^{-p_i'}(y)}{|x_B-y|^{(n-\beta/m+\delta/m)p_i'}}\,dy\right)^{1/p_i'} &\lesssim \prod_{i\in\mathscr{I}_2,\theta_i<0}|x_B|^{-\xi_i-\theta_i}\prod_{i\in\mathscr{I}_2,\theta_i=0}|x_B|^{-\xi_i}\log_2\left(\frac{|x_B|}{R}\right)\\ &\quad\times\prod_{i\in\mathscr{I}_2,\theta_i>0}|x_B|^{-\xi_i-\theta_i}\,2^{N\theta_i}\\ &\lesssim |x_B|^{-\sum_{i\in\mathscr{I}_2}(\xi_i+\theta_i)}\,2^{N\sum_{i\in\mathscr{I}_2,\theta_i>0}\theta_i}\\ &\quad\times\left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathscr{I}_2,\theta_i=0\}},\end{aligned}$$

that is,

$$\prod_{i\in\mathscr{I}_2}\left\|\frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B-\cdot|^{n-\beta/m+\delta/m}}\right\|_{p_i'} \lesssim |x_B|^{-\sum_{i\in\mathscr{I}_2}(\xi_i+\theta_i)}\,2^{N\sum_{i\in\mathscr{I}_2,\theta_i>0}\theta_i}\left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathscr{I}_2,\theta_i=0\}}, \tag{6.5}$$

so the left-hand side of (6.1) can be bounded by a multiple constant of

$$R^{\delta-\tilde{\delta}-(n-\beta/m+\delta/m)m_1}|x_B|^{\rho-\sum_{i\in\mathscr{I}_2}(\xi_i+\theta_i)}\,2^{N\sum_{i\in\mathscr{I}_2,\theta_i>0}\theta_i}\left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathscr{I}_2,\theta_i=0\}}$$

which is equivalent to

$$\left(\frac{|x_B|}{R}\right)^{\tau-\delta+(n-\beta/m+\delta/m)m_1+\sum_{i\in\mathscr{I}_2,\theta_i>0}\theta_i}\left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathscr{I}_2,\theta_i=0\}}. \tag{6.6}$$

Since $\theta_i < n+(\delta-\beta)/m$ for $i \in \mathscr{I}_2$, there exists $\varepsilon > 0$ that verifies

$$\sum_{i\in\mathscr{I}_2,\theta_i>0}\theta_i + \varepsilon\#\{i\in\mathscr{I}_2,\theta_i=0\} \leq \left(n+\frac{\delta-\beta}{m}\right)\#\{i\in\mathscr{I}_2,\theta_i>0\}. \tag{6.7}$$

Using the fact that $\log_2 t \lesssim \varepsilon^{-1}t^{\varepsilon}$ for every $t \geq 1$, we can majorize (6.6) by a constant factor provided that

$$\tau-\delta+\left(n+\frac{\delta-\beta}{m}\right)(m_1+\#\{i\in\mathscr{I}_2:\theta_i>0\})\leq \tau-\delta+\left(n+\frac{\delta-\beta}{m}\right)(m-1)=0.$$

Indeed, if this last inequality did not hold, then we would have that $\theta_i>0$ for every $i\in\mathscr{I}_2$. We also observe that $\theta_i>0$ for $i\in\mathscr{I}_1$. This would lead to $n/p>\beta-\delta$, a contradiction.

In order to prove (d) we only consider two cases. If there exists some $i\in\mathscr{I}_2$ such that $\theta_i\leq 0$, the proof follows exactly as in (c). If not, that is $\theta_i>0$ for every $i\in\mathscr{I}_2$, observe that

$$\begin{aligned}\tau-\delta+\left(n+\frac{\delta-\beta}{m}\right)m_1+\sum_{i\in\mathscr{I}_2,\theta_i>0}\theta_i&=\tau-\delta+\sum_{i=1}^{m}\left(\frac{n}{p_i}+\frac{\delta-\beta}{m}\right)\\&=\tau+\frac{n}{p}-\beta\\&<0,\end{aligned}$$

then we can choose $\varepsilon>0$ small enough so that the resulting exponent for $|x_B|/R$ in (6.6) becomes negative.

We now proceed with the proof of (e). Let us first suppose that $\tilde{\delta}<\min\{\delta,\beta-n/p\}$. We take $\rho=\tilde{\delta}-\tau>0$ and $\xi_i=(\delta-\tau)/m-\theta_i$, for every i. Then we define $w(x)=|x|^{\rho}$ and $v_i=|x|^{\xi_i}$, $1\leq i\leq m$. These functions are locally integrable since $\rho>0$ and $\xi_i<n/p_i'$. Furthermore, $\xi_i<0$ for $i\in\mathscr{I}_1$, so $v_i^{-1}\in \mathrm{RH}_\infty$ for these index. Then, by Lemma 6.1, it will be enough to show that condition (2.9) holds. Fix a ball $B=B(x_B,R)$ and assume that $|x_B|<R$. Then we get

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n}}\lesssim R^{\delta-\tilde{\delta}+\rho}=R^{\delta-\tau}. \tag{6.8}$$

On the other hand, if $i\in\mathscr{I}_1$ we have

$$\begin{aligned}\left\|\frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B-\cdot|^{n-\beta/m+\delta/m}}\right\|_\infty&\lesssim\sum_{k=0}^{\infty}\left\|\frac{v_i^{-1}\mathscr{X}_{B_{k+1}\setminus B_k}}{|x_B-\cdot|^{n-\beta/m+\delta/m}}\right\|_\infty\\&\lesssim\sum_{k=0}^{\infty}\left(2^kR\right)^{-\xi_i-n+\beta/m-\delta/m}\\&\lesssim R^{(\tau-\delta)/m},\end{aligned}$$

since $\tau<\tilde{\delta}<\delta$. This yields

$$\prod_{i\in\mathscr{I}_1}\left\|\frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B-\cdot|^{n-\beta/m+\delta/m}}\right\|_\infty\lesssim R^{m_1(\tau-\delta)/m}. \tag{6.9}$$

Finally, since $\xi_i + \theta_i = (\delta - \tau)/m > 0$ for $i \in \mathscr{I}_2$, we can proceed as in page 28 to obtain

$$\prod_{i \in \mathscr{I}_2} \left(\int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\beta/m+\delta/m)p_i'}}\, dy \right)^{1/p_i'} \lesssim R^{m_2(\tau-\delta)/m}. \tag{6.10}$$

By combining (6.8), (6.9) and (6.10), the left-hand side of (2.9) is bounded by a constant C.

We now consider the case $|x_B| > R$. We have that

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{(\tilde{\delta}-\delta)/n}} \lesssim R^{\delta-\tilde{\delta}}|x_B|^\rho. \tag{6.11}$$

Since $|x_B| > R$, there exists a number $N \in \mathbb{N}$ such that $2^N R < |x_B| \le 2^{N+1}R$. For $i \in \mathscr{I}_1$ we write

$$\begin{aligned}\left\| \frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B - \cdot|^{n-\beta/m+\delta/m}} \right\|_\infty &\lesssim \sum_{k=0}^{N} \left\| \frac{v_i^{-1}\mathscr{X}_{B_{k+1}\setminus B_k}}{|x_B - \cdot|^{n-\beta/m+\delta/m}} \right\|_\infty + \sum_{k=N+1}^{\infty} \left\| \frac{v_i^{-1}\mathscr{X}_{B_{k+1}\setminus B_k}}{|x_B - \cdot|^{n-\beta/m+\delta/m}} \right\|_\infty \\ &= S_1^i + S_2^i.\end{aligned}$$

Proceeding as we did in (6.5) with $p_i = 1$, we have that

$$\prod_{i \in \mathscr{I}_1} \left\| \frac{v_i^{-1}\mathscr{X}_{\mathbb{R}^n\setminus B}}{|x_B - \cdot|^{n-\beta/m+\delta/m}} \right\|_\infty \lesssim |x_B|^{-\sum_{i\in\mathscr{I}_1}(\xi_i+\theta_i)} 2^{N\sum_{i\in\mathscr{I}_1}\theta_i}. \tag{6.12}$$

Finally, if $i \in \mathscr{I}_2$ our choice of ξ_i allows us to follow the argument given in page 29 to conclude that

$$\begin{aligned}\prod_{i \in \mathscr{I}_2} \left(\int_{\mathbb{R}^n \setminus B} \frac{v_i^{-p_i'}(y)}{|x_B - y|^{(n-\beta/m+\delta/m)p_i'}}\, dy \right)^{1/p_i'} &\lesssim |x_B|^{-\sum_{i\in\mathscr{I}_2}(\xi_i+\theta_i)} 2^{N\sum_{i\in\mathscr{I}_2,\theta_i>0}\theta_i} \\ &\quad \times \left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathscr{I}_2,\theta_i=0\}}.\end{aligned}$$

Combining the inequality above with (6.11) and (6.12), the left-hand side of (2.9) can be bounded by a multiple constant of

$$R^{\delta-\tilde{\delta}}|x_B|^{\rho-\sum_{i=1}^m(\theta_i+\xi_i)} 2^{N\sum_{i:\theta_i>0}\theta_i} \left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathscr{I}_2,\theta_i=0\}}$$

which is equal to

$$\left(\frac{R}{|x_B|}\right)^{\delta-\tilde{\delta}-\sum_{i:\theta_i>0}\theta_i} \left(\log_2\left(\frac{|x_B|}{R}\right)\right)^{\#\{i\in\mathscr{I}_2,\theta_i=0\}}.$$

If $\theta_i < 0$ for every i then the exponent of $R/|x_B|$ is positive. On the other hand, if $\theta_i \ge 0$ for every i, then

$$\delta - \tilde{\delta} - \sum_{i:\theta_i>0}\theta_i = \delta - \tilde{\delta} - \sum_{i=1}^m \theta_i = \delta - \tilde{\delta} - \frac{n}{p} + \beta - \delta > 0,$$

since $\tilde{\delta} < \beta - n/p$. In both cases we can repeat a similar argument as in (6.7) to conclude that $(w, \vec{v})$ belongs to $\mathbb{H}_m(\vec{p}, \beta, \tilde{\delta})$. Let us observe that, for example, if $\beta \le \delta$ then every θ_i is nonnegative.

If $\tilde{\delta} = \delta < \beta - n/p$ or $\tilde{\delta} = \beta - n/p < \delta$ the same estimation as above works when we take $\theta_i < 0$ for every i. The second case also works when $\theta_i > 0$ for every i.

We finish with the proof of item (f). In this case we fix $\rho > 0$ and take $w(x) = (1+|x|^\rho)^{-m_1}$. If g_i are nonnegative fixed functions in $L^{p_i'}(\mathbb{R}^n)$ for $i \in \mathscr{I}_2$, we define

$$v_i(x) = \begin{cases} e^{|x|} & \text{if } i \in \mathscr{I}_1, \\ g_i^{-1} & \text{if } i \in \mathscr{I}_2. \end{cases}$$

Fix a ball $B = B(x_B, R)$. It is enough to check condition (2.8), since $\tilde{\delta} = \beta - mn < \tau$. Notice that

$$\|w\mathscr{X}_B\|_\infty \prod_{i\in\mathscr{I}_1} \left\|v_i^{-1}\mathscr{X}_B\right\|_\infty \le \prod_{i\in\mathscr{I}_1} \left\|(1+|\cdot|^\rho)^{-1}\mathscr{X}_B\right\|_\infty \left\|e^{-|\cdot|}\mathscr{X}_B\right\|_\infty \le 1.$$

Therefore,

$$\frac{\|w\mathscr{X}_B\|_\infty}{|B|^{\tilde{\delta}/n-\beta/n+1/p}} \prod_{i\in\mathscr{I}_1} \left\|v_i^{-1}\mathscr{X}_B\right\|_\infty \prod_{i\in\mathscr{I}_2} \left(\frac{1}{|B|}\int_B v_i^{-p_i'}\right)^{1/p_i'} \lesssim \prod_{i\in\mathscr{I}_2} \|g_i\|_{p_i'},$$

for every ball B. This concludes the proof of (f). □

Declarations

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