International Journal of Wavelets, Multiresolution
and Information Processing
Vol. 11, No. 2 (2013) 1350015 (19 pages)

DOI: 10.1142/S021969131350015X

A NEW REFINEMENT WAVELET–GALERKIN METHOD IN A SPLINE LOCAL MULTIRESOLUTION ANALYSIS SCHEME FOR BOUNDARY VALUE PROBLEMS

VICTORIA VAMPA* and MARÍA T. MARTÍN†

Facultad de Ciencias Exactas
Universidad Nacional de La Plata, Argentina
**victoriavampa@gmail.com*
†teremartin.map@gmail.com

EDUARDO SERRANO

Universidad de General San Martín
Buenos Aires, Argentina
eduser@gmail.com

Received 5 June 2012
Revised 20 December 2012
Published 27 March 2013

In this work, a new Wavelet–Galerkin method for boundary value problems is presented. It improves the approximation in terms of scaling functions obtained through a collocation scheme combined with variational equations. A B-spline multiresolution structure on the interval is designed in order to refine the solution recursively and efficiently using wavelets. Numerical examples are given to verify good convergence properties of the proposed method.

Keywords: B-spline functions; second order boundary problems; wavelets; multiresolution analysis; Wavelet–Galerkin.

AMS Subject Classification: 22E46, 53C35, 57S20

1. Introduction

In solving differential equations, wavelets provide a robust and accurate alternative to traditional methods and their advantage is really appreciated when they are applied to problems having localized singular behavior. The solution is approximated by an expansion of scaling functions and wavelets, with the convenience that multiscale and localization properties can be exploited. The choice of wavelet basis is governed by several factors including the desired order of numerical accuracy and computational effort.

A good feature of wavelet methods is the possibility to apply adaptive techniques. In some cases multiscale bases are combined with finite element methods and adaptive refinement strategies are designed. Examples are the multiscale lifting method and the dynamically adaptive algorithm developed by Chen *et al.*[3] and Bindal *et al.*[4] respectively.

On the other hand, some authors applied adaptive procedures in wavelet collocation methods, as the method introduced by Cai *et al.*[13] and Kumar *et al.*[14] which uses cubic splines and allows the optimization of the number of basis functions used for the solution of the problem.

The Galerkin method using variational equations and appropriate elemental functions, is a good alternative, producing an efficient regularization action: in weak formulations for a given equation, the approximating functions can be relatively less regular and easier to construct, see Ref. 16.

Vampa *et al.* in Ref. 5, used Daubechies scaling functions to solve differential equations -in a finite element context- for structural mechanics problems. Later, in a recent article, Ref. 17, Vampa *et al.* presented a modified Wavelet–Galerkin method using B-spline scaling functions to solve boundary value problems. This proposal combines variational equations with a collocation scheme and gives an approximation at an initial scale. In this work, a refinement process using wavelets is developed. It improves the approximation recursively with minimal computational effort.

Numerical examples are used to demonstrate the applicability of the proposed method, whose approximate solutions are computed in scaling-spline form and improved with wavelets. The approximations were validated and convergence of the proposed method was found to compare favorably to other numerical solutions.

The outline of the paper is as follows: a Wavelet–Galerkin method using scaling functions as basis functions for a second order linear differential operator is introduced in Sec. 2. How to design a multiresolution structure on the interval to solve the boundary value problem presented is described in Sec. 3. In 3.1 a Multiresolution Analysis (MRA) structure on the interval is defined. In 3.2 we give a summary of basic B-spline properties and the construction of subspaces of scaling functions and wavelets corresponding to cubic B-splines on the interval is shown, in an MRA framework. In 3.3 a Modified Wavelet–Galerkin Method to approximate the solution of the boundary value problem, combining variational and collocation equations, is developed. In 3.4 we describe how wavelets can be designed and can be used to improve the approximate solution. In Sec. 3.5 the error is analyzed and convergence is demonstrated. A bound of the approximation error is presented. Numerical examples are described and analyzed in Sec. 4. A comparison of the proposed method in contrast with other numerical methods is also shown. In Sec. 5, conclusions are presented.

2. Wavelet–Galerkin Method

We consider the following one dimensional linear boundary value problem on the interval $I = [0, 1]$:

$$Lu = -u''(x) + p(x)u'(x) + q(x)u(x) = f(x) \quad u(0) = u(1) = 0 \tag{2.1}$$

where $p(x)$, $q(x)$ and $f(x)$ are continuous functions on I and u is a function in certain Hilbert space V. If Eq. (2.1) cannot be solved exactly, one has to rely on approximation methods. We seek an approximation $\tilde{u}$ of u which lies in a certain finite dimensional subspace $V_h = \text{span}\{\Phi_1, \Phi_2, \dots, \Phi_N\} \subset V$.

Let $\langle \cdot, \cdot \rangle$ be the inner product of the space V. Note that $a(u,v) = \langle Lu, v \rangle$ defines a bilinear form on $V \times V$, so that the variational or weak formulation corresponding to the problem Eq. (2.1), is to seek $u \in V$, such that

$$a(u,v) = \langle f, v \rangle, \quad \forall\, v \in V. \tag{2.2}$$

The analogous finite dimensional problem is to find $\tilde{u} \in V_h$ such that:

$$a(\tilde{u}, v_h) = \langle f, v_h \rangle, \quad \forall\, v_h \in V_h. \tag{2.3}$$

It is well known that if $a(\cdot,\cdot)$ is continuous, V-elliptic and $\langle f, v \rangle$ is a continuous linear form in V, both problems Eqs. (2.2) and (2.3) have a unique solution, (Lax-Milgram theorem[1]). From Céa's lemma[1] the following error bounds are valid:

$$\|u - \tilde{u}\|_V^2 \le \frac{C}{\gamma}\, inf_{\,v \in \widehat{V}_h} \|u - v\|_V^2 \tag{2.4}$$

where C and γ are constants corresponding to continuity and coercivity of the bilinear form $a(\cdot,\cdot)$, and h is a measure of the partition of I considered, and

$$\|u - \tilde{u}\|_V^2 \le C h^r |u|_{H^{r+1}}^2 \tag{2.5}$$

where r depends on the regularity of the solution.

Going back to Eq. (2.1), the associated bilinear form is:

$$a(u,v) = \int_0^1 (u'(x)v'(x) + p(x)u'(x)v(x) + q(x)u(x)v(x))dx \tag{2.6}$$

for u and $v \in V^0 \subset L^2(I)$, the subspace of functions with homogeneous boundary conditions.

As we are looking for the approximate solution $\tilde{u} \in V_h$, $\tilde{u} = \sum_{k=1}^N \alpha_k \Phi_k$, from Eq. (2.3), we have

$$\sum_{k=1}^{N} \alpha_k a(\Phi_k, \Phi_n) = \langle f, \Phi_n \rangle, \quad n = 1, 2, \dots, N \tag{2.7}$$

and we arrive at the problem of solving a matrix equation

$$A\alpha = b \tag{2.8}$$

where $A(n,k) = a(\Phi_k, \Phi_n)$ and $b(n) = \langle f, \Phi_n \rangle$.

For computational aspects, it is convenient to have a sparse matrix A with a low condition number and basis functions with a small support, regularity and orthogonality. It is also desirable that the basis functions should be simple to evaluate, differentiate and integrate. Finally, one wants the scheme to be refinable in order to allow that the approximation $\tilde{u}$ can be improved, modifying recursively the subspace V_h. If the basis functions Φ_k are generated from dilations and translations of a mother generating function, calculations become simpler. This suggests considering an MRA structure. Furthermore, if self-similarity given by scale relations is satisfied, a hierarchical approximation to the exact solution is obtained and it is possible to refine and improve the precision of the approximate solution.

In conclusion, MRA schemes, see Ref. 12, would provide a powerful mathematical tool for function approximation and multiscale representation of the solution of differential equations corresponding to the problem in Eq. (2.1). It is important to point out that, as these structures are generally defined on the whole real line, they must be restricted adequately to the interval I where the differential problem is formulated.

3. A New Method in the Context of an MRA on the Interval

As described by Chui,[12] a MRA on $L^2(\mathbb{R})$ consists of a sequence of embedded closed subspaces $V_j \subset L^2(\mathbb{R})$, $j \in \mathbb{Z}$,

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$$

that satisfies several properties and typically is constructed by first identifying the subspace V_0 and the scaling function ϕ. Denoting

$$\phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k), \tag{3.1}$$

for each $j \in \mathbb{Z}$, the family $\{\phi_{j,k} : k \in \mathbb{Z}\}$ is a basis of V_j.

Associated with the scaling function ϕ there exists a function ψ called the *mother wavelet* such that the collection $\{\psi(x-k), k \in \mathbb{Z}\}$ is a Riesz basis[12] of W_0, the orthogonal complement of V_0 in V_1. If we consider,

$$\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k), \tag{3.2}$$

for each $j \in \mathbb{Z}$, the family $\{\psi_{j,k} : k \in \mathbb{Z}\}$ is a basis of W_j, the orthogonal complement of V_j in V_{j+1}. It is noteworthy that wavelets allow the refinement of the representation space taking into account that

$$V_{j+1} = V_j \oplus W_j. \tag{3.3}$$

As was mentioned at the end of Section 2, multiresolution structures in $L^2(\mathbb{R})$, have to be restricted to $L^2(I)$, to solve boundary value problems on I, see Ref. 9 and Ref. 11. If Haar bases are considered for $L^2(\mathbb{R})$, it suffices to take the restrictions of these functions to I. Things are not so trivial when one starts from smoother wavelets on the line. It is not clear *a priori* how to adapt the functions in such a way that the result is an orthonormal basis of $L^2(I)$. Several solutions have been proposed for this problem. A first solution is to extend the functions supported on I to the whole line by making them vanish for $x \notin I$. This approach may introduce a discontinuity at the edges and consequently, large wavelets coefficients are obtained near the edges, and too many wavelets are used. Another alternative consists in periodizing, but, unless the function itself is already periodic, it again introduces a discontinuity.

In the following section an MRA on the interval with B-splines as scaling functions is described, and is constructed using orthogonality conditions in a way similar to when it is designed in $\mathrm{L}^2(\mathbb{R})$.

3.1. *Definition of an MRA structure on an interval*

An MRA in $L^2(I)$ is defined as a sequence of finite dimensional subspaces,

$$\widehat{V}^I_{J\min} \subset \widehat{V}^I_{J\min+1} \cdots \subset \widehat{V}^I_{-1} \subset \widehat{V}^I_0 \subset \widehat{V}^I_1 \subset \widehat{V}^I_2 \subset \cdots \tag{3.4}$$

which starts at a scale $Jmin \leq 0$ (which depends on the interval I) that verifies the following properties:

- $\dim \widehat{V}^I_{j+1} \sim 2 \dim \widehat{V}^I_j$,
- $V^I_{J\min}$ contains all polynomials up to a certain degree m
- $\overline{\bigcup \widehat{V}^I_j} = \mathrm{L}^2(I)$,
- $\widehat{W}^I_j \subset \widehat{V}^I_{j+1}$ is the orthogonal complement of $\widehat{V}^I_j$ in $\widehat{V}^I_{j+1}$ of dimension 2^j.

Let us assume that the support of the scaling function $\phi(x) \in V_0$ is $[0, S]$, $S \in N$, and the support of the wavelet $\psi \in W_0$ is $[-S+1, S]$. Then, at scale j, $2^j + S - 1$ basis functions intersect the interval $[0, 1]$, $2^j - S + 1$ are interior and $2S - 2$ are boundary functions ($(S-1)$ such that $0 \in int(sop(\phi_j))$ and $(S-1)$ such that $1 \in int(sop(\phi_j))$).

Considering j_0 such that $2^{j_0} \geq S$, we define for $j \geq j_0$, $\phi^I_{j,k}(x) = \phi_{j,k}(x)\chi_{[0,1]}(x)$ and

$$\widehat{V}^I_j = \mathrm{gen}\{\phi^I_{j,k}, 1 - S \leq k \leq 2^j - 1\}. \tag{3.5}$$

It is easy to see that $\widehat{V}^I_j \subset \widehat{V}^I_{j+1}$ and also that the above properties can be demonstrated (see Refs. 7 and 11).

The way to construct wavelets is different. As the $sop(\psi_{j,k}) = [(1+k-S)/2^j, (S+k)/2^j]$, $k \in N$, then wavelets $\psi_{j,k}$ intersect $[0, 1]$ if $1-S \leq k \leq 2^j + S - 2$ and they are interiors if $S - 1 \leq k \leq 2^j - S$. This imposes that $2^{j_0} \geq 2S - 1$.

On the other hand, the orthogonal complement of $\widehat{V}^I_j$ in $\widehat{V}^I_{j+1}$, $\widehat{W}^I_j$, has dimension 2^j. As $2^j + (2S - 2)$ wavelets intersect the interval $[0, 1]$, this implies that the restrictions $\psi_{j,k}\chi_{[0,1]}$ do not constitute a basis for $\widehat{W}^I_j$, these wavelets are overcompleted. As shown by Meyer in Ref. 10, designing boundary wavelets to generate a basis for $\widehat{W}^I_j$ must be done carefully.

3.2. *Cubic-B-spline subspaces*

Spline wavelets are extremely regular and usually symmetric or anti-symmetric. They can be designed to have compact support and they have explicit expressions which facilitate not only theoretical formulation, but also numerical implementations with a computer, see Refs. 9 and 19.

Let us consider *B-spline functions of order* $m+1$, that is, connected piecewise polynomials of degree m having $m-1$ continuous derivatives. The joining points are called *knots* and they are typically equally-spaced and positioned at the integers.

These functions can be defined recursively by convolutions:[12]

$$\varphi_1(x) = \chi_{[0,1]}(x)$$
$$\varphi_{m+1}(x) = \varphi_m * \varphi_1(x) \tag{3.6}$$

and constitute the scaling functions of the MRA structure.

Among many properties that B-splines have, the most important ones for our method are the following:

- Two-scale relation
$$\varphi_{m+1}(x) = 2^{-m} \sum_{k=0}^{m+1} \binom{m+1}{k} \varphi_{m+1}(2x-k). \tag{3.7}$$
- Differentiation
$$\frac{d^k}{dx^k}\varphi_{m+1}(x) = \Delta^k \varphi_{m+1-k}(x) \tag{3.8}$$
where Δ^k is the k-order difference operator and $1 \le k \le m-1$ i.e. corresponds to a reduction of the spline degree by k.
- Inner products
$$\int_{\mathbb{R}} \varphi_{m+1}(x-k)\varphi_{n+1}(x-l)dx = \varphi_{m+n+2}(n+1+l-k) \tag{3.9}$$
i.e., correspond to simple evaluations of higher order splines at integer points. This property is obtained from the convolution product and is useful in weak formulations of differential problems.

In the B-spline MRA,[9,11] V_0 is the subspace generated by the translates of the *scaling function* φ_{m+1} and for each $j \in \mathbb{Z}$, the family $\{\varphi_{m+1,j,k}{:}k \in \mathbb{Z}\}$ where

$$\varphi_{m+1,j,k}(x) := 2^{j/2}\varphi_{m+1}(2^j x - k), \tag{3.10}$$

is a basis of V_j. These subspaces, V_j, constitute an MRA in $\mathrm{L}^2(\mathbb{R})$.

B-splines of order $m = 3$ are used in this work. As they are functions in C^2, a hierarchical approximation of the solution for the second order problem Eq. (2.1) can be obtained and accurate results can most likely be expected (see Ref. 18).

In the cubic B-spline MRA framework, the scaling function φ_4 has support in $[0,4]$ and $\varphi_{4,j,k}(x) := 2^{j/2}\varphi_4(2^j x - k)$ is a basis of V_j. For simplicity, $\varphi_{4,j,k}(x)$ will be denoted by $\varphi_{j,k}(x)$.

In order to define the MRA restricted to the interval $I = [0,1]$ let us denote by $\varphi^I_{j,k}(x) = \varphi_{j,k}\chi_{[0,1]}(x)$ for $j \ge 2$. These scaling functions satisfy interesting properties (see Ref. 6), they are supported on $[2^{-j}k, 2^{-j}(k+4)]$ and are splines in $\mathbb{Z}/2^j$. They are *interior splines* if $0 \le k \le 2^j - 4$ and *boundary splines* if $-3 \le k \le -1$ or $2^j - 3 \le k \le 2^j - 1$.

According to Eq. (3.5), subspaces $\widehat{V}_j^I$ constitute an MRA in $[0,1]$, and dim $\widehat{V}_j^I = 2^j + 3$. Each subspace consists of piecewise polynomials of degree $m = 3$ with knots in $0 \leq k/2^j \leq 1$ and can reproduce polynomials of degree $r \leq 3$ (see Ref. 18).

The scaling functions $\varphi_{j,k}^I(x)$ form a Riesz basis for $\widehat{V}_j^I$, (see Ref. 12), and satisfy the two-scale relation. If we denote

$$[\widehat{\varphi}_j^I] = (\varphi_{j,-3}^I, \varphi_{j,-2}^I, \dots\dots, \varphi_{j,2^j-1}^I) \tag{3.11}$$

a $1 \times (2^j+3)$ dimensional array, and $\widehat{H}_j$, the two scale matrix of dimension $(2^{j+1}+3) \times (2^j+3)$, the following relation is satisfied,[6]

$$[\widehat{\varphi}_j^I] = [\widehat{\varphi}_{j+1}^I] \cdot \widehat{H}_j. \tag{3.12}$$

The grammian matrix $\widehat{P}_j^I \in R^{(2^j+3)\times(2^j+3)}$, associated with the bases $[\widehat{\varphi}_j^I]$, is

$$\widehat{P}_j^I = [\widehat{\varphi}_j^I]^t \cdot [\widehat{\varphi}_j^I] = (\langle \varphi_{j,k}^I, \varphi_{j,n}^I \rangle)_{-3\leq n,k\leq 2^j-1}. \tag{3.13}$$

The following step is the definition of a suitable basis for the wavelet space $\widehat{W}_j^I$, the orthogonal complement of $\widehat{V}_j^I$ in $\widehat{V}_{j+1}^I$ in such a way that (similar to Eq. (3.3)),

$$\widehat{V}_{j+1}^I = \widehat{V}_j^I \oplus \widehat{W}_j^I \tag{3.14}$$

is verified.

As was mentioned before there exist different proposals such as those designed by Mallat[9] and Meyer.[10] In this work, the construction of suitable wavelets is motivated by the construction for the whole line, and it is described in the following proposition:

Proposition 3.1. *Suppose matrix $\widehat{G}_j$ of dimension $(2^{j+1}-3)\times$ and 2^j, $j \geq 2$ is such that its columns are in the null space of $\widehat{H}_j^t.\widehat{P}_{j+1}^I$ of dimension 2^j, then*

$$[\widehat{\psi}_j^I] = [\widehat{\varphi}_{j+1}^I] \cdot \widehat{G}_j \tag{3.15}$$

is a basis for the orthogonal complement $\widehat{W}_j^I$.

Proof. Requiring the orthogonality condition,

$$[\widehat{\varphi}_j^I]^t \cdot [\widehat{\psi}_j^I] = 0 \tag{3.16}$$

and taking into account Eqs. (3.12) and (3.15) gives the desired result. □

The simple and recursive structure of both matrices $\widehat{H}_j$ and $\widehat{P}_{j+1}^I$ can be exploited in the construction of the matrix $\widehat{G}_j$ in such a way that a band matrix is obtained.

Consequently, with the construction described in the proposition above, interior wavelets are not modified. The same occurred with scaling functions: only wavelets basis corresponding to the edges are different and are adequately designed (see Refs. 7 and 10). It is important to point out that the edge functions -as they are finite linear combinations of some shifts of φ_4- have the same regularity.

Let us now consider the space of *interior* scaling functions

$$V_j^I = \text{gen}\{\varphi_{j,k}^I, 0 \le k \le 2^j - 4\}, \quad j \ge 2 \tag{3.17}$$

of dimension $2^j - 3$ and denote by W_j^I the orthogonal complement of V_j^I in V_{j+1}^I of dimension 2^j.

It can be demonstrated that $\overline{\bigcup V_j^I} = \mathrm{L}^2[0,1]$ and that the sequence of subspaces V_j^I defines an *interior* MRA in $[0,1]$.

The construction of a basis for W_j^I is analogous to the method described in Proposition 1. In this case, matrix G_j is of dimension $(2^{j+1} - 3) \times 2^j$, $j \ge 2$ with columns in the null space of $\widehat{H}_j^t \cdot P_{j+1}^I$ of dimension 2^j, then $[\psi_j^I] = [\varphi_{j+1}^I] \cdot G_j$ is a basis for the orthogonal complement W_j^I.

In Fig. 1 splines corresponding to scale $j = 3$ (six boundary splines and five interior splines) are shown. For the same scale, wavelets constructed following Proposition 1 (three boundary wavelets designed at the left edge and two interior wavelets) are presented in Fig. 2.

3.3. *Modified Wavelet–Galerkin method*

If we consider the variational problem Eq. (2.7) in V_j^I, the following $(2^j - 3)$-dimensional matrix system has to be solved:

$$A_{4,j}\ \alpha_j = b_{4,j} \tag{3.18}$$

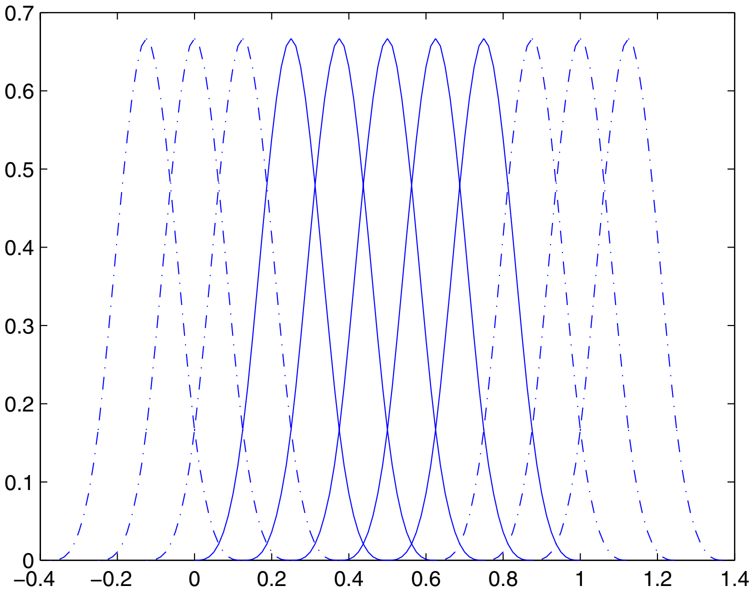

Fig. 1. Boundary and interior scaling functions, corresponding to cubic B-splines.

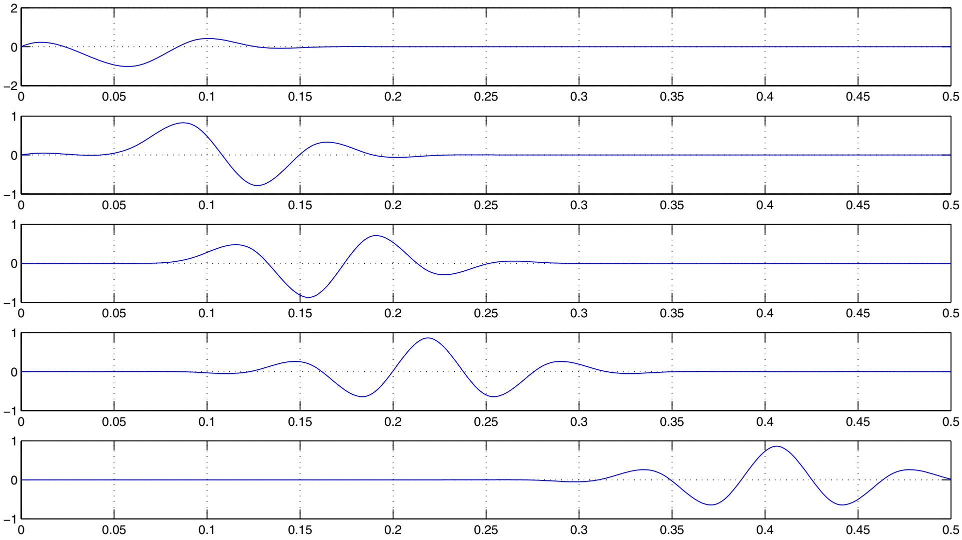

Fig. 2. Boundary and interior wavelet functions corresponding to cubic B-splines.

where (see Ref. 6 for details) for $0 \leq n,\ k \leq 2^j - 4$,

$$A_{4,j}(n,k) = -2^{2j}\varphi_8''(4+n-k) + 2^j p_j(n,k)\varphi_8'(4+n-k) \\ +\ q_j(n,k)\varphi_8(4+n-k) \tag{3.19}$$

and, according to the notation introduced in Sec. 3.2, φ_8 is the B-spline of order eight.

$$b_{4,j}(n) = \langle f, \varphi_{j,n} \rangle \tag{3.20}$$

Taking into account the drawbacks described in Ref. 17 concerning convergence, our proposal is to combine variational equations with a collocation scheme, using both spaces V_j^I and $\widehat{V}_j^I$ to construct an algebraic system to obtain $\widehat{u}_j$. We called this method the *Modified Wavelet–Galerkin Method* which is described below and yields a better approximation $\widehat{u}_j$ in scale j (with a higher rate of convergence), of the form $\widehat{u}_j = \sum_{k=-3}^{2^j-1} \widehat{\alpha}_{j,k}\varphi_{j,k}^I$:

(1) *Algebraic system*:

(a) *Variational equations*: they are obtained from the variational formulation, considering that the unknown function u is in $\widehat{V}_j^I$ and the test function v is in V_j^I. This leads to a rectangular system of dimension $(2^j - 3) \times (2^j + 3)$:

$$\widehat{A}_{4,j}\widehat{\alpha}_j = \widehat{b}_{4,j}. \tag{3.21}$$

The matrix and vector elements are similar to the ones described in Eqs. (3.19) and (3.20), considering $\varphi_{j,k}^I$ in $\widehat{V}_j^I$.

(b) *Collocation equations*: they are obtained from the requirement that the residual should vanish at the ends of the interval and at collocations points, 2^{-j} and $1-2^{-j}$,

$$
\begin{aligned}
&u''(0)+p(0)u'(0)+q(0)u(0)=f(0)\\
&u''(2^{-j})+p(2^{-j})u'(2^{-j})+q(2^{-j})u(2^{-j})=f(2^{-j})\\
&u''(1-2^{-j})+p(1-2^{-j})u'(1-2^{-j})+q(1-2^{-j})u(1-2^{-j})=f(1-2^{-j})\\
&u''(1)+p(1)u'(1)+q(1)u(1)=f(1)
\end{aligned}
\tag{3.22}
$$

(c) *Boundary conditions*: are obtained from the requirement that the solution satisfies the boundary conditions of the problem,

$$
\begin{aligned}
\widehat{\alpha}_{-3}\ \varphi_{j,-3}(0)+\widehat{\alpha}_{-2}\ \varphi_{j,-2}(0)+\widehat{\alpha}_{1}\ \varphi_{j,-1}(0)=0\\
\widehat{\alpha}_{2^j-2}\ \varphi_{j,2^j-2}(1)+\widehat{\alpha}_{2^j-1}\ \varphi_{j,2^j-1}(1)+\widehat{\alpha}_{2^j}\ \varphi_{j,2^j}(1)=0
\end{aligned}
\tag{3.23}
$$

(2) *Approximate solution in* $\widehat{V}_j^I$: After assembling all of the equations listed above, the 2^j+3 coefficients $\widehat{\alpha}_{jk}$ corresponding to the approximate solution are obtained by solving the square algebraic system.

It is important to notice that the matrix corresponding to the algebraic system Eqs. (3.21)–(3.23) is a Toeplitz and a band matrix. When rows corresponding to additional equations (from collocation and boundary equations) are added, the matrix maintains structure properties and an efficient resolution of the algebraic system is possible. This is also valid for the nonconstant coefficients case.

3.4. *Wavelet formulation for refinement*

An *a-posteriori* error estimation of the approximation in V_j^I may indicate the convenience of increasing the scale.

In this section we describe how wavelets can be used to increase the scale i.e. to obtain an approximation in scale $j+1$, once the approximation $\widehat{u}_j$ in scale j,

$$
\widehat{u}_j=\sum_{k=-3}^{2^j-1}\widehat{\alpha}_{j,k}\,\varphi_{j,k}^I \tag{3.24}
$$

has been obtained. One possibility is to repeat the process described previously. Another attractive strategy consists of improving the approximation recursively using wavelets to express the details at higher scales. In this way, large computational savings could be achieved.

In the first alternative, the algebraic system Eqs. (3.21)–(3.23) should be solved, in $j+1$ scale, thereby obtaining an expression similar to that of Eq. (3.24).

It is important to remark that in this case, as only scaling functions are used, the MRA structure is not exploited.

In the second option another basis of $\widehat{V}^I_{j+1}$ is considered, giving the following expression for $\widehat{u}_{j+1}$

$$\widehat{u}_{j+1} = \sum_{k=-3}^{-1} \widehat{\alpha}_{j+1,k}\, \varphi^I_{j+1,k} + \sum_{k=0}^{2^{j+1}-4} \widehat{\alpha}_{j+1,k}\, \varphi^I_{j+1,k} + \sum_{2^{j+1}-3}^{2^{j+1}-1} \widehat{\alpha}_{j+1,k}\, \varphi^I_{j+1,k} \tag{3.25}$$

$$\widehat{u}_{j+1} = \sum_{k=-3}^{-1} \widehat{\alpha}_{j+1,k}\ \varphi^I_{j+1,k} + \sum_{k=0}^{2^{j}-4} \beta^I_{j,k}\varphi^I_{j,k} + \sum_{k=2^{j+1}-3}^{2^{j+1}-1} \widehat{\alpha}_{j+1,k}\varphi^I_{j+1,k} + \sum_{k=1}^{2^j} \nu_{j,k}\psi_{j,k} \tag{3.26}$$

taking into account the relation of Eq. (3.14).

In this proposal, the $2^{j+1}-3$ variational equations

$$\langle L\widehat{u}_{j+1}, \varphi^I_{j+1,n}\rangle = \langle f, \varphi^I_{j+1,n}\rangle, \quad 0 \le n \le 2^{(j+1)} - 4 \tag{3.27}$$

are replaced with:

$$\langle L\widehat{u}_{j+1}, \varphi^I_{j,n}\rangle = \langle f, \varphi^I_{j,n}\rangle, \quad 0 \le n \le 2^{j} - 4 \tag{3.28}$$

$$\langle L\widehat{u}_{j+1}, \psi^I_{j,n}\rangle = \langle f, \psi^I_{j,n}\rangle, \quad 1 \le n \le 2^{j} \tag{3.29}$$

while the equations corresponding to both edges in this new base are similar to the ones obtained before. Concerning the new algebraic system to solve, we are in position to establish the following result:

Proposition 3.2. *Let us consider the* $2^{j+1}-3$ *variational equations* (3.27) *obtained with the Modified Wavelet–Galerkin Method in scale* $j+1$. *If variational equations* (3.27) *are replaced by* (3.28) *and* (3.29), *both algebraic systems are equivalent.*

The practical implication of the proposition above is that, using this new structure of the variational equations, the number of unknowns is reduced and the refinement with wavelets leads to a very efficient algorithm.

Our goal is to use a representation of the approximation in the space $\widehat{V}_{j+1}$ in the following form,

$$\widehat{u}_{j+1} = \widehat{u}_j + [\widehat{u}_{j+1} - \widehat{u}_j] = \widehat{u}_j + \widehat{v}_j \tag{3.30}$$

and consider for the increment $\widehat{v}_j \in \widehat{V}_{j+1}$ the expansion:

$$\widehat{v}_j = \sum_{k=-3}^{2^{j+1}-1} \gamma_{j+1,k}\widehat{\varphi}^I_{j+1,k} = [\widehat{\varphi}^I_{j+1}]\cdot[\gamma_{j+1}] \tag{3.31}$$

with the scaling functions of the space $\widehat{V}_{j+1}$.

Remark 3.1. It is important to note that $\widehat{v}_j$ and $\widehat{u}_j$ are not orthogonal.

The way in which the number of unknowns is reduced is made explicit by the following theorem:

Theorem 3.1. (*Reduction of the number of unknowns*) *Let D_j be the matrix corresponding to the interior products $\langle L\varphi^I_{j,k}, \varphi^I_{j,n}\rangle_{-3\le n,k\le 2^j-1}$ and let $\widehat{v}_j = [\widehat{\varphi}^I_{j+1}]\cdot[\gamma_{j+1}]$ be the increment in Eq.* (3.30). *If $\widehat{v}_j$ satisfies the orthogonality condition*

$$\langle L\widehat{v}_j, \widehat{\varphi}^I_{j,n}\rangle = 0 \quad 0 \le n \le 2^j - 4 \tag{3.32}$$

then, there exists a matrix N_j of dimension $(2^{j+1}+3)\times(2^j+6)$, recursive and of simple structure such that the increment coefficients in the basis of scaling functions of $\widehat{V}_{j+1}$ are

$$[\gamma_{j+1}] = N_j[\widehat{\alpha}_{j+1}]. \tag{3.33}$$

In this way the number of unknowns is reduced from $(2^{j+1}+3)$ to (2^j+6).

Proof. The condition Eq. (3.32), which is associated to an homogeneous linear system of $2^j - 3$ equations and $2^{j+1}+3$ unknowns, follows directly if we substitute (3.30) in (3.28),

$$\langle L\widehat{u}_j, \widehat{\varphi}^I_{j,n}\rangle + \langle L\widehat{v}_j, \widehat{\varphi}^I_{j,n}\rangle = \langle f, \widehat{\varphi}^I_{j,n}\rangle, \tag{3.34}$$

and taking into account variational equations in scale j, $\langle L\widehat{u}_j,\, \widehat{\varphi}^I_{j,n}\rangle = \langle f,\, \widehat{\varphi}^I_{j,n}\rangle$.

Replacing (3.31) in (3.32) and considering matrices derived from the differential operator L (as the grammian matrix, $\widehat{P}^I_j$ Eq. (3.13)), a homogeneous system is obtained. The reduction is a consequence of the null space dimension, which is $(2^{j+1}+3)-(2^j-3) = (2^j+6)$. □

Remark 3.2. It is very important to point out that the number of equations to be solved is minimal. In addition, in the theorem above, it is supposed that matrix N_j is of full rank. This will depend on the matrix D_j, and so depends on L. If this is not the case and the matrix N_j is not of full rank, the reduction of unknowns will not be altered.

The previous discussions about the linear system that has to be solved in this new proposal are summarized in the following theorem.

Theorem 3.2. *Let $\widehat{u}_j$ be the approximation obtained from Modified Wavelet–Galerkin Method at scale j, from Eqs.* (3.21)–(3.23) *and $\widehat{v}_j$ the increment ($\widehat{v}_j = [\widehat{\varphi}^I_{j+1}]\cdot[\gamma_{j+1}]$). Then, coefficients $[\widehat{\alpha}_{j+1}]$ are obtained solving the system which consists in 2^j variational equations,*

$$\langle L\widehat{v}_j, \psi_{j,n}\rangle = \langle f - L\widehat{u}_j, \psi_{j,n}\rangle \tag{3.35}$$

and six equations corresponding to the edges.

Proof. If we substitute $\widehat{u}_{j+1}$ with the decomposition Eq. (3.30) in Eq. (3.29), 2^j variational equations are obtained. As the term $\langle L\widehat{u}_j, \psi_{j,n}\rangle$ is known at scale j, it is now in the independent term. Adding six equations, corresponding to both edges written in terms of $[\gamma_{j+1}]$ coefficients, yields the linear system to be solved. Finally, with coefficients $[\widehat{\alpha}_{j+1}]$ corresponding to the approximation at scale $j+1$ and using the relation Eq. (3.33), $\widehat{v}_j$, is found. □

It can be concluded that the MRA provides the structure to obtain $\widehat{u}_{j+1}$ in an efficient manner: solving a linear system -at scale $j+1$- of dimension 2^j+6 in the space $\widehat{V}^I_{j+1}$ (Eq. (3.34)).

The iterative refinement algorithm proposed can be summarized as follows:

- Step 1 Choose an initial scale $j = j_0$ and find scaling function coefficients solving the linear system Eqs. (3.21)–(3.23). This yields the approximation $\widehat{u}_j = \sum_{k=-3}^{2^j-1} \widehat{\alpha}_{j,k}\varphi^I_{j,k}$
- Step 2 Find the coefficients $[\widehat{\alpha}_{j+1}]$ of $\widehat{v}_j$ solving the system Eq. (3.35)
- Step 3 Given an adequate threshold ϵ, the following cutting criterion is applied: if $\|\widehat{v}_j\|_2^2 < \epsilon$, STOP a good approximation is obtained, IF NOT, go to the following step
- Step 4 $\widehat{u}_{j+1} = \widehat{u}_j + \widehat{v}_j$,
  $j = j+1$ go back to Step 2

3.5. *Approximation error analysis*

This section is devoted to the analysis of approximation error in scale j using the Modified Wavelet–Galerkin Method developed in Sec. 3.3. It must be considered that the approximation $\widehat{u}_j$ in $\widehat{V}^I_j$ is obtained by solving the system Eq. (3.21) and six additional equations, once the (2^j+3) coefficients $\widehat{\alpha}_{jk}$ are calculated.

As the approximation in scale j is not the solution of a variational problem in $\widehat{V}^I_j$, we will use a strategy to apply Céa's lemma presented in Sec. 2. We will demonstrate that it is possible to design a subspace $\widehat{U}^I_j$ of $\widehat{V}^I_j$ in such a way that the approximation is the solution of a classic variational problem in $\widehat{U}^I_j$.

Recalling that the difference in the proposed method lies in the design of additional equations corresponding to both edges, it is natural to suppose that the subspace $\widehat{U}^I_j$ required will be generated by

$$\left\{\tilde{\varphi}^I_{j,0}, \varphi^I_{j,1}, \varphi^I_{j,2}, \ldots\ldots, \varphi^I_{j,N-1}, \tilde{\varphi}^I_{j,N}\right\} \tag{3.36}$$

where $N = 2^j - 4$, $\varphi^I_{j,k}$, $1 \le k \le N-1$ are the interior scaling functions, excluding both $\varphi^I_{j,0}$ and $\varphi^I_{j,N}$ but adding two new functions $\tilde{\varphi}^I_{j,0}$ and $\tilde{\varphi}^I_{j,N}$ at left and right edges, respectively.

Scaling functions are used to define $\tilde{\varphi}^I_{j,0}$ and $\tilde{\varphi}^I_{j,N}$, i.e. $\varphi^I_{j,-3}$, $\varphi^I_{j,-2}$, and $\varphi^I_{j,-1}$ corresponding to left end and $\varphi^I_{j,N+1}$, $\varphi^I_{j,N+2}$ and $\varphi^I_{j,N+3}$ to the right edge of the interval.

For simplicity we only recall error analysis at the edge $x = 0$, proposing that

$$\tilde{\varphi}^I_{j,0} = c_{-3}\varphi^I_{j,-3} + c_{-2}\varphi^I_{j,-2} + c_{-1}\varphi^I_{j,-1} + c_0\varphi^I_{j,0}$$

verifies the following conditions:

$$\begin{aligned} \tilde{\varphi}^I_{j,0}(0) &= 0 \\ L\tilde{\varphi}^I_{j,0}(0) &= f(0) \\ L\tilde{\varphi}^I_{j,0}(2^{-j}) &= f(2^{-j}) \\ \langle L\widehat{u} - f, \tilde{\varphi}^I_{j,0}\rangle &= 0. \end{aligned} \tag{3.37}$$

It can be shown that if the subspace $\widehat{U}^I_j$ is generated by (3.36), the inclusion $\widehat{U}^I_j \subset \widehat{V}^I_j$ is satisfied. Moreover, the approximate solution $\widehat{u}_j$ is in $\widehat{U}^I_j$ (see Ref. 6 for details).

Then, the following theorem holds:

Theorem 3.3. *The solution $\widehat{u}_j$ in $\widehat{V}^I_j$ of the Modified Wavelet–Galerkin Method is also the solution of a pure variational problem in the subspace $\widehat{U}^I_j$.*

Consequently, it is now possible to use Céa's lemma (see Eq. (2.4)), with the following error estimation:

Corollary 3.1. *Let $\widehat{u}_j = \sum_{k=-3}^{2^j-1} \widehat{\alpha}_{j,k}\varphi^I_{j,k}$ be solution of the algebraic system Eqs.* (3.21)–(3.23), *then,*

$$\|u - \widehat{u}_j\|^2_{\widehat{V}^I_j} \leq \frac{C}{\gamma} inf_{v\in\widehat{U}^I_j} \|u - v\|^2_{\widehat{V}^I_j} \tag{3.38}$$

where C and γ are constants corresponding to continuity and coercivity of the bilinear form a.

From (3.38) it is derived that the solution obtained with the Modified Wavelet–Galerkin Method minimizes norm error (with a constant factor) and converges to the exact solution as the scale j increases.

It is demonstrated in Ref. 18 that the interpolatory cubic spline function S_h, which coincides with a smooth function $u \in C^4$ with uniform spacing h, satisfies:

$$\|u - S_h\|^2_{H^1} \leq \frac{35}{24} h^4 \|u\|_\infty. \tag{3.39}$$

As a consequence of the above results, the following bound is valid for the approximation error:

$$\|u - \widehat{u}_j\|^2_{L^2} \leq C\left(\frac{1}{2^j}\right)^4. \tag{3.40}$$

Another error estimation can be done taking into account (3.30) and (3.31). By normalizing, one has

$$\|\widehat{v}_j\|^2_2 \leq C_j \sum_{k=-3}^{2^{j+1}-1} |\gamma_{j+1,k}|^2 \tag{3.41}$$

The functions $\varphi^I_{j,k}$ constitute a Riesz basis of $\widehat{V}^I_{j+1}$, and so, (3.41) is verified for certain constants C_j, with $C_j \leq C$, for all j, see Ref. 12. These coefficients constitute a natural expression for the error in the L^2 norm.

Furthermore, as the functions $\varphi^I_{j,k}$ are well localized, a local error estimation can be obtained and can be used in local improvement strategies, in adapting refinement schemes.

Remark 3.3. It is important to note that estimations are available not only for the increment $\widehat{v}_j$, but for the residual $f - L\widehat{u}_j$ as well.

4. Numerical Experiments

In this section we present some numerical experiments concerning boundary value problems.

B-spline scaling functions were employed as described in Secs. 3.2 and 3.3 and wavelets were designed and used in solution refinement to increase the scale (Sec. 3.4). Approximation errors are also shown and compared with theoretical estimations which were described in Sec. 3.5.

Taking into account estimation errors presented in Sec. 3.5, the following semi-norm was used to measure errors:

$$\|v\|_{j,\infty} = \max_{k=0,1,\ldots,2^j-1} \left| v\left(\frac{k}{2^j}\right)\right|, \tag{4.1}$$

As the expression

$$E(j) = \|u - u_j\|_{j,\infty} = C2^{-jR} \tag{4.2}$$

is valid, one has $\frac{E(j)}{E(j+1)} = 2^R$ and convergence order R is obtained:

$$R = \log_2(E(j)) - \log_2(E(j+1)). \tag{4.3}$$

Example 4.1. The problem

$$Lu = -u''(x) + u(x) = (4\pi^2 + 1)\sin(2\pi x) \quad u(0) = 0 \quad u(1) = 0$$

has the exact solution

$$u(x) = \sin(2\pi x).$$

This is presented in order to make a comparison with the results published in Ref. 15, where Daubechies scaling functions were used.

Values in the first column of Table 1 were obtained by the proposed method. They are between the values of the second and third column which correspond to the errors using Daubechies scaling functions[8] of order $N = 4$ and $N = 6$ respectively. These results were presented in Ref. 15 and convergence is $O(2^{-Nj})$. Theoretical convergence orders are then verified, and this is related to the $N/2$ null moments that scaling functions of Daubechies have (see Ref. 6 for more details).

Example 4.2. The following family of boundary value problems with nonconstant coefficients is analyzed,

$$Lu = -u''(x) + \frac{2}{x+\lambda}u'(x) + u(x) = f(x) \quad u(0) = 0 \quad u(1) = 0$$

Table 1. Relative errors for different scales, Example 1.

j	$\|u-\widehat{u}_j\|_{j,\infty} B-$ splines	$\|u-\tilde{u}_j\|_{j,\infty} D4$	$\|u-\widehat{u}_j\|_{j,\infty} D6$
3	4.6×10^{-3}	1.4×10^{-3}	2.9×10^{-5}
4	9.4×10^{-6}	6.4×10^{-5}	5.3×10^{-7}
5	7.1×10^{-8}	1.2×10^{-6}	1.3×10^{-9}
6	5.0×10^{-10}	2.2×10^{-8}	3.4×10^{-12}
7	3.5×10^{-12}	4.0×10^{-10}	8.3×10^{-15}
8	2.5×10^{-14}	7.4×10^{-12}	1.7×10^{-17}
9	2.46×10^{-15}	1.4×10^{-13}	8.48×10^{-18}

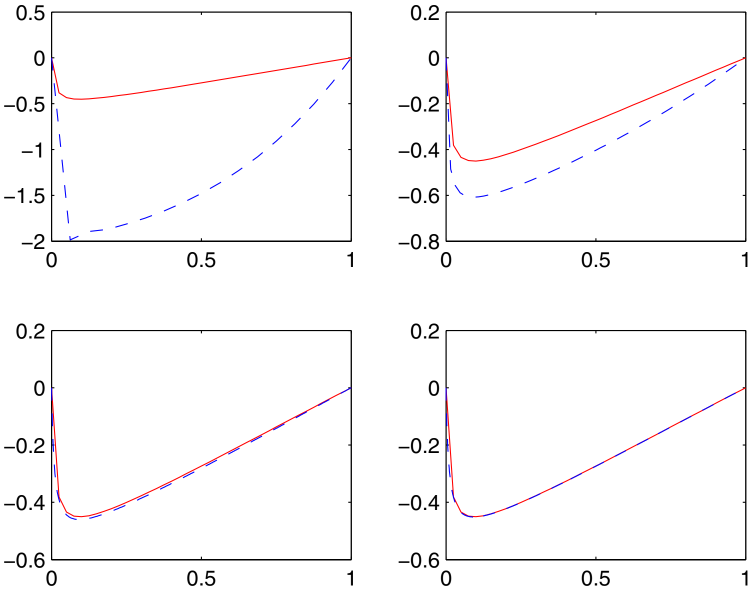

Fig. 3. Approximations corresponding to scales $j = 4, 6, 8, 10$, $\lambda = 0.01$, Example 4.2.

with $f(x)$ being the function that corresponds to the solution,

$$u(x) = c_1 \frac{\sin(x+\lambda)}{(x+\lambda)} + c_2 \frac{\cos(x+\lambda)}{(x+\lambda)} + \frac{1}{(x+\lambda)}$$

where,

$$c_1 = \frac{-\cos(1+\lambda)+\cos(\lambda)}{\sin(\lambda)\cos(1+\lambda)-\cos(\lambda)\sin(1+\lambda)}$$

$$c_2 = \frac{-\sin(\lambda)+\sin(1+\lambda)}{\sin(\lambda)\cos(1+\lambda)-\cos(\lambda)\sin(1+\lambda)}.$$

It can be observed that for large values of λ the exact solution is regular. However, it is almost singular for small values (it has a high gradient at $x = 0$ and is regular in

Table 2. Error for different values of λ, at scale $j = 10$, Example 4.2.

λ	$\|u - \widehat{u}_j\|_{j,\infty}$
5	3.1×10^{-10}
1	4.5×10^{-8}
0.5	2.0×10^{-7}
0.01	6.7×10^{-4}

Table 3. Relative errors for different scales, $x_0 = 0.75$, $\lambda = 50$, Example 4.3.

j	$\|u - \widehat{u}_j\|_{j,\infty}$
5	3.8×10^{-2}
6	2.0×10^{-2}
7	4.9×10^{-3}
8	1.1×10^{-3}
9	2.9×10^{-4}

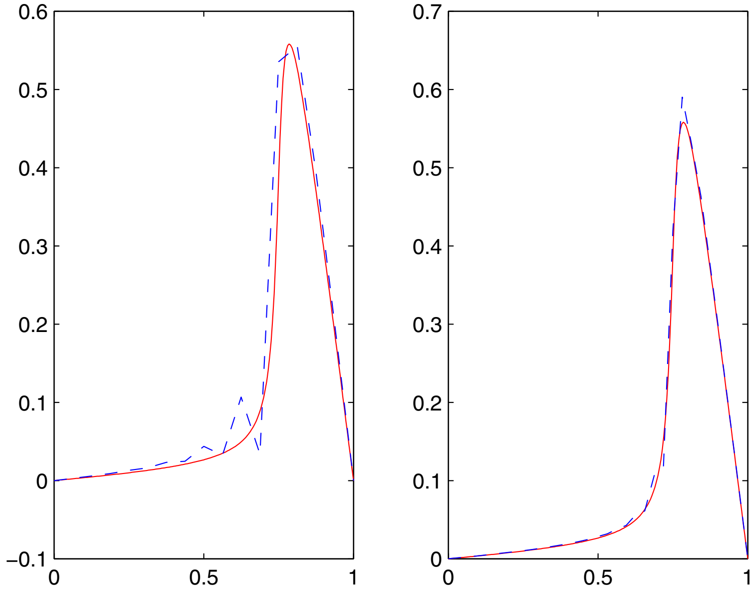

Fig. 4. Approximate solutions at scales $j = 4$ and $j = 5$, $x_0 = 0.75$, $\lambda = 50$, Example 4.3.

the rest of the domain). For the analysis, values of the parameter $0.01 \leq \lambda \leq 5$ were considered. In Fig. 3, convergence is shown when scaling is increased for $\lambda = 0.01$ and in Table 2 errors for different values of λ at scale $j = 10$ are presented.

Example 4.3. The function u defined by

$$u(x) = (1 - x) \cdot (\tan^{-1}(\lambda(x - x_0)) + \tan^{-1}(\lambda x_0))$$

is the solution of the differential problem,[2]

$$Lu = -(k(x)u'(x))' = f(x) \quad u(0) = 0 \quad u(1) = 0$$

where,

$$k(x) = \frac{1}{\lambda} + \lambda(x - x_0)^2$$

and

$$f(x) = 2(1 + \lambda(x - x_0)(tan^{-1}(\lambda(x - x_0)) + tan^{-1}(\lambda x_0)).$$

In Table 3, errors for different scales are presented. A value of the parameter λ that corresponds to a solution with a high gradient at x_0 was chosen. In Fig. 4, the approximations for scales $j = 4$ and $j = 5$ are shown.

5. Conclusions

In this paper a new technique for numerical solution refinement is developed. The scheme proposed reduces computational complexity and is open to future extensions.

The main proposal, described in Sec. 3, combines numerical and computational advantages of B-spline functions with wavelets capacities, in a Galerkin-variational context. The method is developed to approximate the solution of boundary value problems for singular second order ordinary differential equations.

A multiresolution structure on the interval is defined and provides a more efficient and cost-effective way to improve the approximate solution obtained at an initial scale. Based on the norm of increment coefficients, local error estimators are presented. In addition, an adaptive algorithm can be implemented which refines the solution in the partial domain of interest, offering the most promising avenue for future developments.

References

1. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, (North Holland, 1978).
2. E. Becker, G. Carey and T. Oden, *Finite Elements. An Introduction*, Vol. I (Prentice-Hall, 1981).
3. X. Chen, Z. He, J. Xiang and B. Li, A dynamic multiscale lifting computation method using Daubechies wavelet, *J. Comput. Appl. Math.* **188** (2006) 228–245.
4. A. Bindal, J. G. Khinast and M. G. Ierapetritou, Adaptive multiscale solution of dynamical systems in chemical processes using wavelets, *Comput. Chemical Eng.* **27** (2003) 131–142.
5. L. Alvarez Díaz, M. T. Martín and V. Vampa, Daubechies wavelet beam and plate finite elements, *Finite Elements Anal. Design* **45** (2008) 200–209.
6. V. Vampa, *Desarrollo de herramientas basadas en la transformada wavelet para su aplicacin en la resolución numérica de ecuaciones diferenciales*, Tesis de Doctorado en Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, (2011).

7. A. Cammilleri and E. P. Serrano, Spline multiresolution analysis on the interval, *Latin Amer. Appl. Research* **31** (2001) 65–71.
8. I. Daubechies, *Ten Lectures on Wavelets*. CBMS-NSF Regional Conference. Series in Applied Mathematic. Departament of Mathematics. University of Lowell. MA SIAM: Philadelphia. PA (1992).
9. S. G. Mallat, *A Wavelet Tour of Signal Processing — The Sparse Way*, (Academic Press — Elsevier-MA EEUU, 2009).
10. Y. Meyer, *Ondelettes et Operateurs, I*: *Ondelettes, II*: *Operateurs de Calderón-Zygmund, III*: (*with R. Coifman*), *Operateurs Multilineaires* (Hermann, Paris, 1990). English translation of Vol. 1, Wavelets and Operators (Cambridge University Press Cambridge, 1993).
11. D. Walnut, *An Introduction to Wavelet Analysis*, in Applied and Numerical Harmonic Analysis Series (Birkhauser Eds., Boston, 2002).
12. C. Chui, *An Introduction to Wavelets* (Academic Press, New York, 1992).
13. W. Cai and J. Wang, Adaptive multiresolution collocation methods for initial boundary value problems of nonlinear pdes., *SIAM J. Numer. Anal.* **33** (1996) 937–970.
14. V. Kumar and M. Mehra, Cubic spline adaptive wavelet scheme to solve singularly perturbed reaction diffusion problems, *Int. J. Wavelets, Multiresolut. Inf. Process.* **5** (2007) 317–331.
15. O. Muller Nielsen, *Wavelets in Scientific Computing*, Ph.D. Thesis. Technical University of Denmark, Department of Mathematical Modeling, (1998).
16. J. N. Reddy, On the numerical solution of differential equations by the finite element method, *Indian J. Pure and Appl. Math.* **16**(12) (1985) 1512–1528.
17. V. Vampa, M. T. Martín and E. Serrano, A hybrid method using wavelets for the numerical solution of boundary value problems on the interval, *Appl. Math. Comput.* **217** (2010) 3355–3367.
18. I. J. Schoenberg, Cardinal interpolation and spline functions, *J. Appr. Theory* **2** (1969) 167–206.
19. M. Unser, Ten good reasons for using spline wavelets, *Proc. SPIE Wavelets Appli. Signal Image Process.* **3169** (1997) 422–431.