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INFORMATION ENTROPY AND SPIN-SQUEEZING IN ATOMIC THREE-LEVEL SYSTEMS

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We consider the problem of an atomic three-level system in interaction with a radiation field. The time evolution of the system, in atomic ladder and Λ configurations, is solved exactly assuming a coherent-state as the initial atomic state. We calculate the atomic spin-squeezing, the atomic entropy-squeezing, and their variances. We show that the spin-squeezing and the entropy-squeezing exhibit similar time dependence.

Keywords: Atomic squeezing; entropy atomic squeezing; three-level atoms.

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1. Introduction

The concept of entanglement has received a great deal of attention, particularly, in connection with recent developments in the field of quantum information.[1] In this context, entanglement is regarded as a fundamental tool in quantum information processing. Several quantum protocols, such as teleportation,[2] are based on entangled states. The problem of measuring entanglement is an active field of research.[3] A review on this subject can be found in Refs. 4 and 5 and references therein.

One of the relevant quantities which can be used to characterize entanglement is entropy.[6] For a general two-component quantum system, Araki and Lieb have established a well-known relation among the total entropy of the system and the reduced entropies of the components.[7,8] From this relation, Lindblad,[9] Barnett and Phoenix,[10] and Knight and Phoenix,[11] have used the index of correlation to analyze the presence of entanglement. Kraus[12] and Maasen and Uffink[13] have discussed the optimal entropy-uncertainty-relation for a pair of complementary observables in

a finite Hilbert space. This concept was extended by Sánchez–Ruiz[14] for a set of N-complementary observables.

More recently, an alternative way of reformulating the Heisenberg Uncertainty Principle, in terms of entropies, was introduced by Obada *et al.*[15] A related parameter, the so-called entropy-squeezing parameter, was introduced in Ref. 15 to study the entropy-squeezing of a two-level atom interacting with two modes *via* energy-dependent couplings. The problem of the interaction between two quantum systems with SU(1,1) and SU(2) symmetries was considered in Ref. 16. The authors of Ref. 16 have analyzed the occurrence of entanglement in that system by using the entropy-squeezing parameter as an indicator.

The single-atom entropy-squeezing, and the entropy of a system of two-level atoms interacting with a bimodal field in an ideal cavity was studied in Ref. 17. The variance- and entropy-squeezing for two-level atoms in interaction with a non-degenerate parametric amplifier was considered in Ref. 18. The characterization of entropy-squeezing, as an indicator of entanglement in a three-level system interacting with a cavity field, was presented in Ref. 19. It was found that the setting of the initial state and the activation of the atom-field coupling affects the field entropy-squeezing rather dramatically. Also, the link between entanglement and entropy has been used to study coherence and entanglement in the ground state of a bosonic Josephson junction.[20]

In recent works,[21–23] the architecture of potential quantum computers has been reviewed. The design of a quantum computer may require hybrid systems in which atoms serve as memory, and super-conducting circuits are the logical gates. This integration between different quantum systems may imply the use of chip technology. In this context, the authors of Ref. 22 have advanced the concept of atom chips. These devices can be realized by extreme cold atoms in an optical lattice. The authors of Ref. 23 have reviewed the experimental progress towards quantum information processing and quantum simulation using neutral atoms in two-dimensional (2D) arrays of optical micro-traps as 2D registers of q-bits. From a theoretical point of view, the study of entanglement, atomic squeezing and coherence is essential to improve the performance of such systems.

In a different context, collisions in ultra-cold gases have been used to induce quadrature spin-squeezing in two-component Bose condensates.[24,25] The authors of Ref. 26 have proposed a generalization to a higher-dimensional spin-space by measuring squeezing in a spin-1 Bose condensate. This squeezing is associated to negligible occupation of squeezed modes, and is analogous to optical two-mode vacuum-squeezing.

Taking the concepts which have been advanced in Ref. 26 as our main motivation, we study, in this work, the appearance of atomic-squeezing for a system of N three-level atoms interacting among themselves and with a radiation field. The atomic excitations are modeled by the algebra associated to the SU(3) group. We shall consider the time evolution of a coherent state, for the atomic sector of

the calculations. The paper is organized as follows. The formalism is presented in Sec. 2. In Sec. 3, we present and discuss the result of the calculations which we have performed for the proposed models. The conclusions are drawn in Sec. 4.

2. Formalism

We shall consider a system of N identical three-level atoms of ^{87}Rb, in a cavity and interacting with a radiation field.[27–32] The dipole–dipole interaction, of the atomic sector, is modeled after the study of Refs. 27–29 and 33–35, as well. This is a suitable representation for the atomic configurations considered (see the following subsections) and it is a generalization of the two-level case.[36] We shall analyze two schemes, which are related to different three-level configurations: a ladder configuration and a Λ-configuration.

2.1. *Atomic ladder-configuration scheme*

Following the work of Ref. 26, the Hamiltonian of a system of three-level atoms, in the ladder-configuration, is written

$$\begin{aligned} H = \frac{1}{3}\Omega N + \omega_a \left(a^\dagger a + \frac{1}{2}\right) + \omega S_z + \frac{1}{2}\Delta Q_{zz} \\ + \zeta(a^\dagger S_- \ + \ S_+ a) + \lambda \ S_+ S_- \,, \end{aligned} \tag{1}$$

where $a^\dagger(a)$ is the one photon-creation (-annihilation) operator of the photon mode of energy ω_a, and $S_\pm$ and $Q_{\alpha,\beta}$ are the spin and quadrupole operators of the Cartesian dipole-quadrupole decomposition of the SU(3) Lie algebra.[26] The energies Ω, ω and Δ are related to the level energies ω_i $(i = 0, 1, 2)$ by $\Omega = \omega_0 + \omega_1 + \omega_2$, $\omega = (\omega_2 - \omega_0)/2$ and $\Delta = (\omega_2 + \omega_0)/2 - \omega_1$ ($\hbar = 1$ everywhere). In this scheme, the transitions take place between the atomic levels ordered in the sequence $\omega 2 > \omega_1 > \omega_0$. The operators of (1) are defined[26]

$$\begin{aligned} S_\alpha &= -i \sum_{\alpha,\beta,\gamma} \epsilon_{\alpha\beta\gamma} c^\dagger_\gamma c_\beta \,, \\ Q_{\alpha\beta} &= -c^\dagger_\beta c_\alpha - c^\dagger_\alpha c_\beta + \frac{2}{3}\delta_{\alpha\beta} \sum_\gamma c^\dagger_\gamma c_\gamma \,, \end{aligned} \tag{2}$$

where the operators $c^\dagger_\alpha$ are expressed as

$$\begin{aligned} c^\dagger_x &= \frac{1}{\sqrt{2}}(-b^\dagger_2 + b^\dagger_0) \,, \\ c^\dagger_y &= \frac{i}{\sqrt{2}}(b^\dagger_2 + b^\dagger_0) \,, \\ c^\dagger_z &= b^\dagger_1 \,, \end{aligned} \tag{3}$$

in terms of the boson creation (annihilation) operators, $b^\dagger_i(b_i)$, associated to the excitation of the ith atomic level $(i = 0, 1, 2)$. Defining $S^{ij} = b^\dagger_j b_i$, we write the

cartesian components of the operators (2) as

$$\begin{aligned}
S_x &= \frac{1}{\sqrt{2}}(S^{01}+S^{12}+S^{21}+S^{10})\,,\\
S_y &= -\frac{i}{\sqrt{2}}(S^{01}+S^{12}-S^{21}-S^{10})\,,\\
S_z &= (S^{22}-S^{00})\,,\\
Q_{xz} &= \frac{1}{\sqrt{2}}(-S^{01}+S^{12}+S^{21}-S^{10})\,,\\
Q_{xy} &= i(S^{20}-S^{02})\,,\\
Q_{yz} &= -\frac{i}{\sqrt{2}}(-S^{01}+S^{12}-S^{21}+S^{10})\,,\\
Q_{xx} &= \frac{2}{3}N-(S^{00}+S^{22}-S^{02}-S^{20})\,,\\
Q_{yy} &= \frac{2}{3}N-(S^{00}+S^{22}+S^{02}+S^{20})\,,\\
Q_{zz} &= -\frac{4}{3}N+2(S^{00}+S^{22}) = -(Q_{xx}+Q_{yy})\,,
\end{aligned} \tag{4}$$

where $S_\pm = S_x \pm iS_y$. Notice that the term ΔQ_{zz} produces the effect of a quadratic Zeeman operator.[26]

In terms of the operators S^{ij}, the Hamiltonian of Eq. (1) reads

$$\begin{aligned}
H = \omega_a\left(a^\dagger a+\frac{1}{2}\right)+\sum_i \omega_i S^{ii}+2\lambda(S^{01}+S^{12})(S^{21}+S^{10})\\
+\sqrt{2}\,\zeta(a(S^{01}+S^{12})+a^\dagger(S^{21}+S^{10}))\,.
\end{aligned} \tag{5}$$

The collective atomic-states can be identified by the number of atoms in the ground state and in the second excited state, n_0 and n_2, respectively

$$|n_0 n_2\rangle = N(n_0,n_2)\sum_P |n_0^P(1)\cdots n_0^P(N) n_2^P(1)\cdots n_2^P(N)\rangle\,,$$

$$N(n_0,n_2) = \left(\begin{pmatrix} N \\ N-n_0-n_2\end{pmatrix}\begin{pmatrix} n_0+n_2 \\ n_2\end{pmatrix}\right)^{-1/2}, \tag{6}$$

with $n_0 = \sum_{j=1}^N n_0^P(j)$, $n_2 = \sum_{j=1}^N n_2^P(j)$ and $n_i^P(j) = 0,1$. Note that the internal degeneracy of each of the two available atomic states is included in the definition of the basis $|n_0^P(1)\cdots n_0^P(N) n_2^P(1)\cdots n_2^P(N)\rangle$.

Since the Hamiltonian of Eq. (1) contains a bosonic degree of freedom the state which represents (n_a) photons is written as the number state

$$|n_a\rangle = \frac{1}{\sqrt{n_a!}}a^{\dagger n_a}|0\rangle\,. \tag{7}$$

The Hamiltonian of Eq. (1) commutes with the operator

$$P = a^\dagger a + S_z \,. \tag{8}$$

Thus, the vectors of the basis can be written in terms of the eigenvalues of P, $L = n_a + n_2 - n_0$, where n_a is the number of photons in the configuration. Because of the symmetry (8) we can write the basis of product states

$$|n_a, n_0, n_2\rangle = |n_a\rangle \otimes |n_0, n_2\rangle \,. \tag{9}$$

and label it, consequently,

$$|NL\, n_0 n_2\rangle = |L + n_0 - n_2\rangle \otimes |n_0, n_2\rangle \,. \tag{10}$$

In writing Eq. (10), we have used the constrain in the number of atoms, $N = n_0 + n_1 + n_2$, then n_1 does not appear explicitly as a label of the states and only two quantum numbers are needed to specify the atomic sector. In the basis of states with fixed values of N and L, the exact solution is written[31,32]

$$|\Psi_{N,L}\rangle = \sum_{\eta \equiv \{n_a, n_0, n_2\}} c_{N,L}(\eta)|\eta\rangle \,. \tag{11}$$

The configurations in (11) are restricted by the conditions: $L = n_a + n_2 - n_0$ and $N = n_0 + n_1 + n_2$, as said before.

2.2. *Atomic Λ-configuration system*

For the Hamiltonian of the atomic Λ-configuration we write[37]

$$\begin{aligned} H = {} & \omega_a \left(a^\dagger a + \frac{1}{2}\right) + \sum_i \omega_i S^{ii} + \lambda(S^{01} + S^{21})(S^{10} + S^{12}) \\ & + \zeta(a(S^{01} + S^{21}) + a^\dagger(S^{10} + S^{12}))\,. \end{aligned} \tag{12}$$

In this scheme, the transitions take place between the atomic levels ordered in the $\omega_1 > \omega_2 > \omega_0$ (Ref. 37).

In terms of the Cartesian dipole-quadrupole decomposition of the SU(3) Lie algebra, it reads

$$\begin{aligned} H = {} & \frac{1}{3}\Omega N + \omega_a \left(a^\dagger a + \frac{1}{2}\right) + \omega S_z + \frac{\Delta}{2} Q_{zz} \\ & + \frac{\zeta}{2\sqrt{2}}((a^\dagger + a)(S_+ + S_-) + (a^\dagger - a)(Q_+ - Q_-)) \\ & + \frac{\lambda}{8}(S_+ + S_- - Q_+ + Q_-)(S_+ + S_- - Q_- + Q_+)\,, \end{aligned} \tag{13}$$

with $Q_\pm = Q_{xz} \pm iQ_{yz}$. The Hamiltonian of Eq. (13) commutes with the operator

$$R = a^\dagger a - \frac{1}{2} Q_{zz} + N/3 = a^\dagger a + S^{11}\,. \tag{14}$$

In this case, the basis can be written in terms of the eigenvalues of R, $M = n_a + N - n_0 - n_2$:

$$|NM\, n_0 n_2\rangle = |M + n_0 + n_2 - N\rangle \otimes |n_0, n_2\rangle\,. \tag{15}$$

2.3. *Initial condition*

To study the time evolution of the states and observables described by the previous models we follow the formalism presented in Refs. 31, 32, 37. We shall assume that the initial state is the direct product of a coherent photon-state and a coherent spin-state (CSS):

$$|I\rangle = |z_{\rm ph}\rangle \otimes |z_{\rm at}\rangle\,, \tag{16}$$

with

$$|z_{\rm ph}\rangle = N e^{z_{\rm ph} a^\dagger}|0\rangle\,, \tag{17}$$

and

$$|z_{\rm at}\rangle = N e^{z_{\rm at} S_+}|0\rangle\,. \tag{18}$$

The parameter $z_{\rm ph}$ is related to the mean-value of photons in the system through $|z_{\rm ph}|^2 = \langle n_a\rangle$, while $z_{\rm at} = -e^{-i\phi_0}\tan(\theta_0/2)$. The angles (θ_0, ϕ_0) define the direction $\mathbf{n}_0 = (\sin\theta_0\cos\phi_0, \sin\theta_0\sin\phi_0, \cos\theta_0)$, such that $\mathbf{S}\cdot\mathbf{n}_0|z_{\rm at}\rangle = -S|z_{\rm at}\rangle$, with $S = N$. In the standard SU(3) representation (see Ref. 38), the coherent state of the atomic sector requires, for its definition, the use of two complex variables. The state (18) is a vector belonging to the spin subspace of the complete SU(3) representation, and we have chosen it for simplicity, since the atomic sector of the Hamiltonian has been written in terms of spin operators.

2.4. *Spin-squeezing parameter*

Atomic spin-squeezed-states are quantum-correlated systems with reduced fluctuations in one of the collective spin components. Following the work of Ueda and Kitagawa,[39] we shall define a set of orthogonal axes $\{\mathbf{n}_{x'}, \mathbf{n}_{y'}, \mathbf{n}_{z'}\}$, such that $\mathbf{n}_{z'}$ is along the direction of $\langle\mathbf{S}\rangle$. We shall fix the direction x' by looking at the minimum value of $(\Delta S_{x'})^2$, consequently, we define the squeezing factor as

$$\zeta^2_{x'} = \frac{2(\Delta S_{x'})^2}{|\langle\mathbf{S}\rangle|}\,. \tag{19}$$

Then, the system is squeezed if $\zeta^2_{x'} < 1$. So defined, the parameter of Eq. (19) is SU(2) invariant.[40] For completeness, we shall define

$$\zeta^2_{y'} = \frac{2(\Delta S_{y'})^2}{|\langle\mathbf{S}\rangle|}\,. \tag{20}$$

Clearly $\zeta^2_{x'}\zeta^2_{y'} \geq 1$.

Alternatively, we can define the variance-squeezing parameters,

$$V_{x'} = (\Delta S_{x'})^2 - \frac{|\langle \mathbf{S} \rangle|}{\mathbf{2}}\,,$$
$$V_{y'} = (\Delta S_{y'})^2 - \frac{|\langle \mathbf{S} \rangle|}{\mathbf{2}}\,, \tag{21}$$

and, by this definition, the system is squeezed if $V_{x'} < 0$ or $V_{y'} < 0$.

In the previous definition of squeezing we have used the $\{S_x, S_y, S_z\}$ SU(2)-subspace. This choice represents a projection on one of the multiple Bloch spheres of SU(3). Another possible choice of the SU(2) subspace is the one of Ref. 26 which requires an initial state with $\langle S_z \rangle = 0$.

2.5. *Entropy squeezing*

The importance of squeezed states of light for optical devices employed in quantum measurements has been demonstrated during the last decade, particularly in connection with realizations of quantum communications, teleportation and cryptography. Generally speaking, squeezed light is a natural tool in quantum information theory Refs. 15 and 17, (and references therein). In this section, we shall briefly review the concept of entropy-squeezing, in order to relate it with the spin-squeezing mechanism of the previous section. In both cases we shall start from Heisenberg's uncertainty relations, and include fluctuations. In this section, we shall generalize the definitions obtained by other authors, for the case of two-level atoms,,[15,17] to the present case of three-level atoms.

The information entropy $H(S_\sigma)$ (Refs, 15 and 17) of the operators S_σ ($\sigma = x', y', z'$) is given by

$$H(S_\sigma) = -\sum_{j=0}^{2N} P_j(\sigma) \log(P_j(\sigma))\,, \tag{22}$$

where

$$P_j(\sigma) = \langle \sigma, j | \rho_{\rm at}(t) | \sigma, j \rangle\,, \tag{23}$$

is the expectation value of the reduced atomic entropy $\rho_{\rm at}$ ($\rho_{\rm at} = {\rm Tr}_{\rm ph}\rho(t)$) on the jth eigenstate of the operator S_σ. For the present case, three-level atoms, we shall generalize the expressions valid for the SU(2) case (two-level atoms),[15,17] to the SU(3) representation. The details of the derivation of these expressions are given in Appendix A. The quantities $H(S_{x'})$, $H(S_{y'})$, $H(S_{z'})$, when $\mathbf{n}_{z'}$ is along the direction of $\langle \mathbf{S} \rangle$, satisfy the condition

$$H(S_{x'}) + H(S_{y'}) + H(S_{z'}) \geq 2\log(2^{2N}) - \frac{2}{2^{2N}} \sum_k \binom{2N}{k} \log \binom{2N}{k}\,, \tag{24}$$

This condition can also be written as

$$\delta H(S_{x'})\delta H(S_{y'}) \geq \frac{2^{4N}\prod_k \binom{2N}{k}^{-\frac{1}{2^{2N-1}}\binom{2N}{k}}}{\delta H(S_{z'})}\,, \tag{25}$$

with $\delta H(S_\sigma) = e^{H(S_\sigma)}$. The atomic squeezing of the system is determined by using the entropy uncertainty relation [Eq. (25)]. The fluctuation in component S_σ of the spin of the atomic system is said to be squeezed if the information entropy $H(S_\sigma)$ satisfies

$$E(S_\sigma) = \delta H(S_\sigma) - \frac{2^{2N}\prod_k \binom{2N}{k}^{-\frac{1}{2^{2N}}\binom{2N}{k}}}{\sqrt{\delta H(S_{z'})}} < 0\,. \tag{26}$$

In deriving the previous equations we have used the same arguments introduced in the previous one, concerning spin-squeezing. In the following section, we shall show that, by the way of the numerical results, the time dependency of both observables, spin-squeezing and entropy-squeezing, is rather similar and that both of them can be used to characterize the degree of squeezing of the spin of atomic system.

3. Results and Discussion

In this section, we shall present the results of the calculations which we have performed by applying the formalism given by the Hamiltonian of Eqs. (5) and (12). We have calculated entropy,- $E(S'_j)$, and variance-, $V_{j'}$, squeezing parameters for a system consisting of $N = 10$ atoms of ^{87}Rb. The effective level scheme includes, for the ladder configuration, the state $5^2S_{1/2}$ as the lower state ($|0\rangle$), the state $5^2P_{3/2}$ as the intermediate state ($|1\rangle$), and the $5^2D_{5/2}$ as the upper state ($|2\rangle$). The Λ-configuration is realized in rubidium by adopting as active levels the states $5^2S_{1/2}(F = 1)(|0\rangle)$, $5^2P_{3/2}(F = 2)(|1\rangle)$ and $5^2S_{1/2}(F = 2)$ $(|2\rangle)$, respectively. The energy of the photon sector of the Hamiltonian is fixed to the resonant case.[37,41]

3.1. *Results for the ladder configuration*

In the analysis, we have fixed the interaction constants of the Hamiltonian of Eq. (5) to realistic values, extracted from our previous work on ^{87}Rb.[36,37] Since we are using natural units ($\hbar = 1$), the couplings and frequencies are given in units of energy (the scale is arbitrary, for a comparison with the energy scale of the atomic case see Refs. 36 and 37) and the time variable is measured in units of inverse energy. We have studied the behavior of the model for different initial states. Figure 1 shows the results for the variance-squeezing $V_{j'}$ (insets (a), (c) and (e)), and for the entropy-squeezing $E(S_{j'})$ (insets (b), (d) and (f)), in absence of interaction between the atoms and the radiation field ($\lambda = 0.08, \zeta = 0.0$). Different initial atomic states have been considered. The results shown in insets (a) and (b) correspond to an

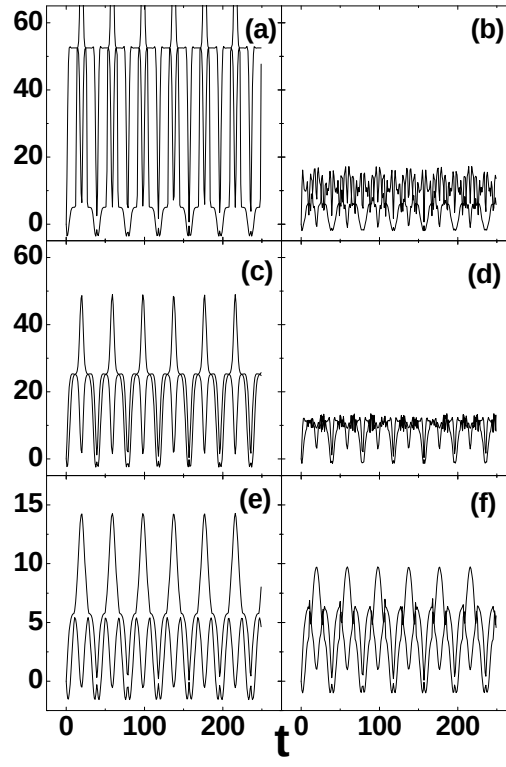

Fig. 1. Time dependence of the entropy-squeezing, $E(S_{j'})$, and variance-squeezing, $V_{j'}$. The time scale is given in arbitrary units. The atoms interact with a coupling constant $\lambda = 0.08$, in absence of interaction with the photon sector, for the Hamiltonian describing the ladder-configuration of subsection 2.1. Insets (a), (c) and (e) show the behavior of the variance-squeezing parameter. In each box, the curve which reaches the largest value is that of $V_{y'}$, and the other represents $V_{x'}$. Insets (b),(d) and (f) show the dependence of the entropy-squeezing parameter $E(S_{j'})$, following the same ordering (that is the larger value in each box represents the y'-component and the smaller one the x'-component, respectively). The same convention has been applied to all of the following figures. Different initial atomic have been considered, namely: Insets (a) and (b) correspond to an initial CSS with $\theta_0 = \pi/2$ and $\phi_0 = 0$, while for insets (c) and (d) we have adopted a initial CSS with $\theta_0 = \pi/4$ and $\phi_0 = 0$. For insets (e) and (f), $\theta_0 = \pi/8$ and $\phi_0 = 0$.

initial CSS with $\theta_0 = \pi/2$ and $\phi_0 = 0$, while for insets (c) and (d) we have adopted an initial CSS with $\theta_0 = \pi/4$ and $\phi_0 = 0$; for insets (e) and (f) the parameters of the initial state are $\theta_0 = \pi/8$ and $\phi_0 = 0$ (see Table 1).[a] As it can be seen from the curves displayed in Fig. 1, the entropy-squeezing parameter provides the same information for the persistence of squeezing than the variance-squeezing parameter. The range of variation of the entropy-squeezing parameter is smaller than the one of the variance squeezing parameter. Figures 2–4 show the time dependence of the

[a]This choice of the phase, for the atomic coherent state results in a definite orientation of the state but it does not affect the pattern of the spin-squeezing, as shown in Refs. 36. Therefore, we shall keep it all-trough the calculations.

Table 1. Orientation angles considered for the calculations of the results shown in Fig. 1.

V_j	$E(S_j)$	θ_0	ϕ_0
(a)	(b)	$\pi/2$	0
(c)	(d)	$\pi/4$	0
(e)	(f)	$\pi/8$	0

Table 2. Average photon numbers, $\langle n_a \rangle$, considered for the calculations of the results shown in Fig. 2.

V_j	$E(S_j)$	$\langle n_a \rangle$
(a)	(b)	40
(c)	(d)	20

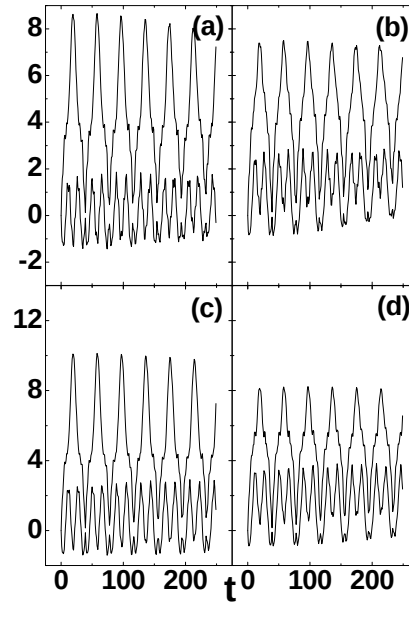

Fig. 2. Entropy squeezing, $E(S_{j'})$, and variance squeezing, $V_{j'}$, as a function of time. The atoms interact in the ladder-scheme with couplings $\lambda = 0.08$ and $\zeta = 0.01$. Insets (a) and (c) show the time dependence of the variance-squeezing parameter, and insets (b) and (d) show the evolution of the entropy-squeezing parameter. The upper values in each box represent the y'-component, and the lower values are associated to the x'-component, of both the variance- and entropy-squeezing. For this case the parameters of the CSS are $\theta_0 = \pi/8$ and $\phi_0 = 0$. The parameter $\langle n_a \rangle$ of the photon sector is fixed at the values $\langle n_a \rangle = 40$ (cases (a) and (b)) and $\langle n_a \rangle = 20$ (cases (c) and (d)).

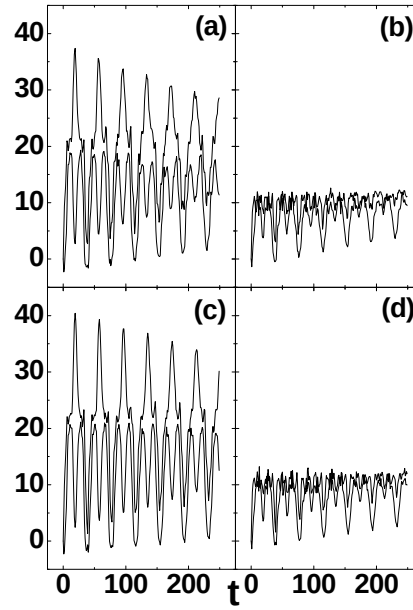

Fig. 3. Entropy squeezing, $E(S_{j'})$, and variance squeezing, $V_{j'}$ (ladder-configuration). The initial atomic state correspond to a CSS with $\theta_0 = \pi/4$ and $\phi_0 = 0$. The rest of the parameters are those given in the captions to Fig. 2.

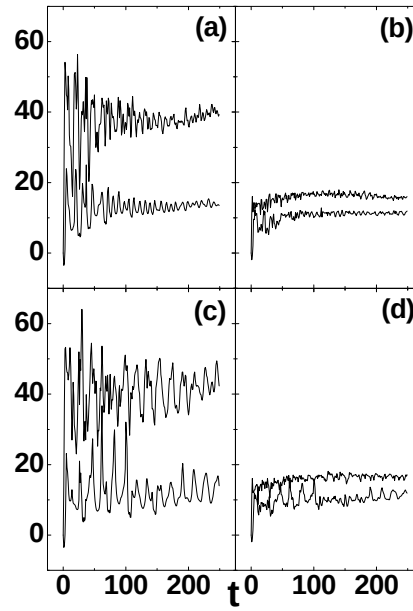

Fig. 4. Entropy squeezing, $E(S_{j'})$, and variance squeezing, $V_{j'}$ (ladder-configuration). The initial atomic state correspond to a CSS with $\theta_0 = \pi/2$ and $\phi_0 = 0$. The rest of the parameters are those of Fig. 2.

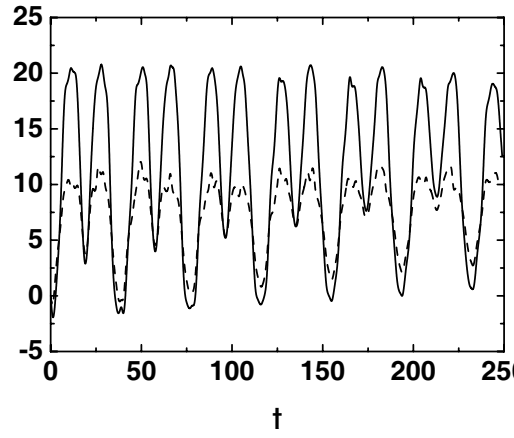

Fig. 5. Entropy squeezing, $E(S_{x'})$ (dashed-line), and variance squeezing, $V_{x'}$ (solid-line), for the ladder-configuration, as a function of time. The parameters used in the calculation are those of Fig. 3 (insets (c) and (d)), and the photon-atom coupling corresponds to the normal phase (see the text for explanations).

entropy-squeezing and variance-squeezing when the interaction with the radiation field is turned on ($\zeta = 0.01$). In these figures, insets (a) and (c) show the variance-squeezing parameter, while insets (b) and (d) display the behavior of the entropy-squeezing parameter. Figure 2 shows results for an initial CSS with $\theta_0 = \pi/8$ and $\phi_0 = 0$. The curves of the insets (a) and (b) show the evolution of a initial photon coherent state with $\langle n_a \rangle = 40$, while the results of insets (c) and (d) correspond to $\langle n_a \rangle = 20$ (see Table 2). In the case of Fig. 3 we have chosen an initial coherent spin state with $\theta_0 = \pi/4$ and $\phi_0 = 0$, and for Fig. 4 the initial atomic coherent state is defined by $\theta_0 = \pi/2$ and $\phi_0 = 0$, for this state $\langle \mathbf{S} \rangle$ is orientated along the x axis. As said before, the results do indeed depend on the choice of the orientation angle ϕ_0, but the pattern of the spin-squeezing is the same for different values of it, as shown in Ref. 36. The results seemingly indicate that both parameters provide the same information about the persistence of the orientation of the spin of the system. The analysis of Figs. 2–4 indicates also that, in presence of atom-photon interaction, and in order to optimize the persistence of atomic squeezing, the orientation of the initial CSS should be close to the axial z axis. The mean number of photons of the initial state do not affect significantly the results. In order to better illustrate the coherence between the time dependence of both observables, in Fig. 5 we show the values of the entropy-squeezing and the variance, as a function of time. From these results it becomes evident that both observables are in phase.

3.2. *Results for the Λ-configuration*

Figures 6-9 show the results for the time evolution of the same system of atoms of Rubidium, this time in the Λ-configuration. The interactions are described by the Hamiltonian of Eq. (13). In the numerical analysis, we have fixed the interaction

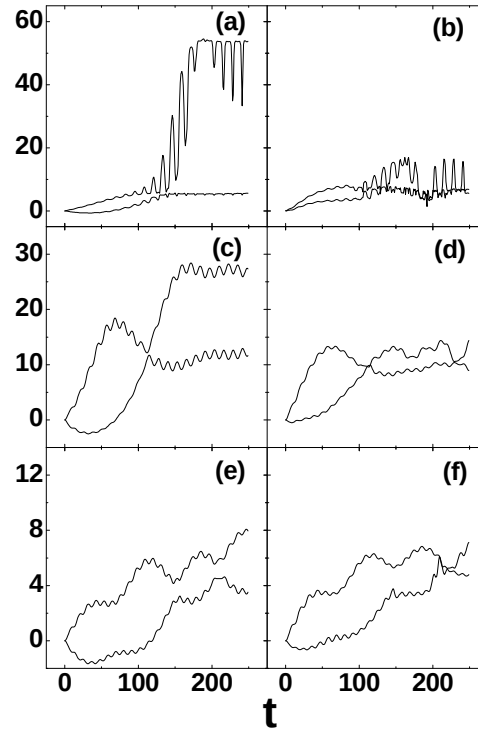

Fig. 6. Time dependence of entropy-squeezing, $E(S_{j'})$ and variance-squeezing, $V_{j'}$. The atomic active levels correspond to the Λ-configuration 2.2 with couplings $\lambda = 0.002$ and $\zeta = 0.0$. The calculated quantities are displayed in the order explained in the captions Fig. 1, and for the initial conditions given in the same Figure.

constants to realistic values[37] ($\lambda = 0.002, \zeta = 0.0$). As before, we have calculated the entropy squeezing and the variance for different initial atomic states. Figure 5 shows the results for the entropy-squeezing and variance-squeezing in absence of interaction between the atoms and the radiation field. Insets (a) and (b) correspond to an initial CSS with $\theta_0 = \pi/2$ and $\phi_0 = 0$, while for insets (c) and (d) we have adopted an initial CSS with $\theta_0 = \pi/4$ and $\phi_0 = 0$. For the curves shown in insets (e) and (f) of Fig. 5 the orientation angles were fixed at $\theta_0 = \pi/8$ and $\phi_0 = 0$. As it can be seen from these results, the entropy-squeezing parameter is more restrictive with respect to the persistence of squeezing than the variance parameter, that is the range of variation of the entropy squeezing parameter is smaller than the one of the variance squeezing parameter. Figures 6–8 show the behavior of the entropy-squeezing, $E(S_{j'})$, and variance-squeezing, $V_{j'}$, for the same system, when the interaction with the radiation field is turned on ($\zeta = 0.008$). Insets (a) and (c) display the time dependence of the variance-squeezing parameter, and insets (b) and (d) show that of the entropy-squeezing parameter. Figure 6 shows results for an initial CSS with $\theta_0 = \pi/8$ and $\phi_0 = 0$. The curves of insets (a) and (b) show

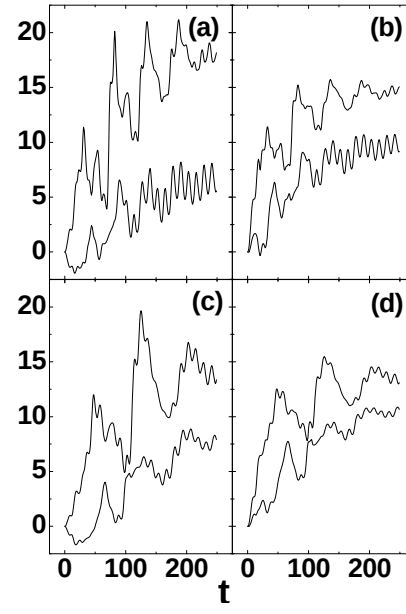

Fig. 7. Entropy squeezing, $E(S_{j'})$, and variance squeezing, $V_{j'}$, for the atoms in the Λ-configuration 2.2, with couplings $\lambda = 0.002$ and $\zeta = 0.008$. The other parameters (initial CSS and photon states) are those of Fig. 2.

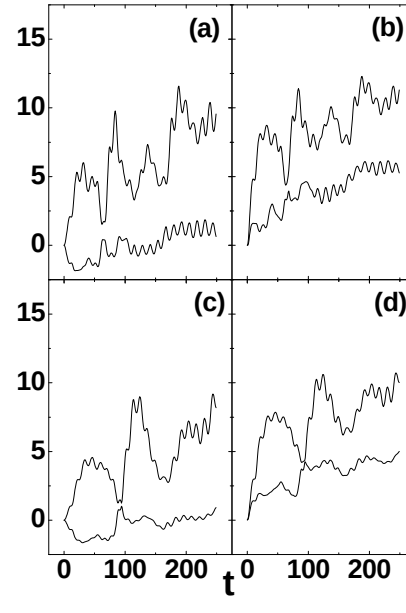

Fig. 8. Entropy squeezing, $E(S_{j'})$, and variance squeezing, $V_{j'}$ (atoms in the Λ-configuration). The initial atomic state correspond to a CSS with $\theta_0 = \pi/4$ and $\phi_0 = 0$. The rest of the parameters are those of Fig. 6.

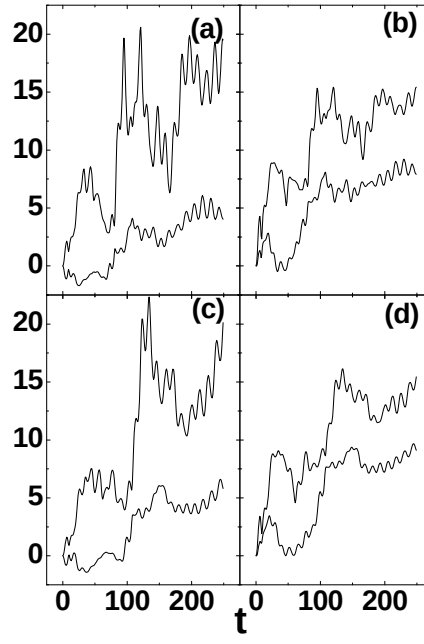

Fig. 9. Time dependence of the entropy-squeezing, $E(S_{j'})$, and variance-squeezing, $V_{j'}$ (Λ-configuration). The initial atomic state correspond to a CSS with $\theta_0 = \pi/2$ and $\phi_0 = 0$. The rest of the parameters are those of Fig. 6.

the evolution of an initial photon coherent state with $\langle n_a \rangle = 40$, and for insets (c) and (d) we have taken $\langle n_a \rangle = 20$. In the case of Fig. 7, we have chosen an initial CSS with $\theta_0 = \pi/4$ and $\phi_0 = 0$. Finally, for the case of Fig. 8, the initial atomic coherent state is oriented by the angles $\theta_0 = \pi/2$ and $\phi_0 = 0$, and for this state $\langle \mathbf{S} \rangle$ is orientated along the x axis. The results shown in Figs. 6–8 enforce the idea that, for the Λ-configuration, the entropy-squeezing parameter predicts smaller periods of atomic squeezing than the variance-squeezing parameter.

To study the dependence of the previously discussed results with the couplings, we have performed calculations varying them, both for the ladder- and Λ-configurations. The obtained values of the entropy-squeezing and variance-squeezing, for couplings in the range[31,32,36,37] ($0.002 \leq \lambda \leq 0.08$; $0.0 \leq \zeta \leq 0.10$), exhibit the same trend of the previous figures, though the oscillations, for large values of the time are washed-out and both observables reach nearly constant values. In the cases considered in the present calculations, the values of ζ are consistent with the normal phase, that is the energy domain where the gap between the first excited state and the ground state of the atomic configuration is nonvanishing. The behavior of the energy of the first excited state, as a function of the coupling between the photons and the atomic spin and for the ladder configuration, is shown in Fig. 10. It is seen that the larger value of ζ used in the calculations ($\zeta = 0.10$) is still at the left of the transition ($\zeta \approx 0.16$). For larger values of

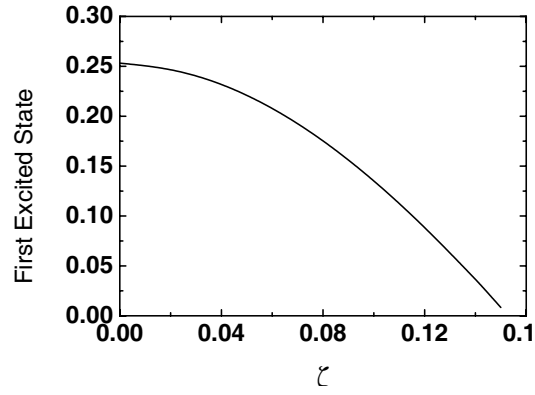

Fig. 10. Energy of the first excited state, for the ladder configuration, as a function of the coupling constant ζ. Both quantities are dimensionless.

the coupling ($\zeta > 0.16$), the photon-atom interaction dominates and the squeezing disappears.

4. Conclusions

In this work, we have studied the appearance of atomic squeezing for three-level ^{87}Rb-atoms in interaction with a radiation field. We have analyzed two possible configurations associated to the interactions between atoms and a radiation field. The interaction model for the ladder-configuration[25,26] favors the persistence of spin alignment. A more restricted condition was found for the atoms in the Λ-configuration scheme. In the two models (ladder and Λ-configurations) the interaction with the radiation field washes-out the atomic-squeezing. From the analysis of the numerical results, the choice of a CSS aligned close to the axial z axes seems to favor the appearance of spin squeezing. Both parameters, variance-squeezing and entropy-squeezing provide similar information when the atoms are interacting *via* the ladder-configuration scheme of Hamiltonian (1). For the model interaction (13) (Λ-configuration) the predictions about the persistence of atomic squeezing are more stringent that those obtained from the variance squeezing. However, from the overall analysis of the results it can be concluded that both parameters predict similar trends for the appearance of atomic squeezing.

Acknowledgments

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Appendix A

In this appendix we shall illustrate the steps leading to the condition (25). The density operator of the system is written

$$\rho(t) = |\Psi(t)\rangle\langle\Psi(t)| \,. \tag{A.1}$$

For the noninteracting system

$$|\Psi(t)\rangle = \mathrm{e}^{-iH_0 t}|\mathrm{I}\rangle \,, \tag{A.2}$$

where H_0 is the Hamiltonian which includes the non-interacting atomic and photon sectors of H. We shall consider for $|\mathrm{I}\rangle$ the initial state of Eq. (16), with the atomic coherent state, $|z_{\mathrm{at}}\rangle$, written in the basis $|n_0, n_2\rangle$, that is

$$\begin{aligned} |z_{\mathrm{at}}\rangle = \mathcal{N}_{\mathrm{at}} \sum_{l=0}^{N} z_{\mathrm{at}}^{l} 2^{l/2} \begin{pmatrix} N \\ l \end{pmatrix}^{1/2} \\ \times \sum_{k=0}^{N-l} z_{\mathrm{at}}^{2k} \begin{pmatrix} N-l \\ k \end{pmatrix}^{1/2} |N-l-k, l\rangle \,. \end{aligned} \tag{A.3}$$

The normalization of the atomic coherent state is given by

$$\mathcal{N}_{\mathrm{at}} = \frac{1}{(1+|z_{\mathrm{at}}|^2)^N} \,. \tag{A.4}$$

For the initial photon state we write

$$\begin{aligned} |z_{\mathrm{ph}}\rangle &= \mathcal{N}_{\mathrm{ph}} \mathrm{e}^{z_{\mathrm{ph}} a^\dagger} |0\rangle \\ &= \mathcal{N}_{\mathrm{ph}} \sum_{l=0}^{\infty} z_{\mathrm{ph}}^{l} \frac{1}{l!} a^{\dagger l} |\rangle \\ &= \mathcal{N}_{\mathrm{ph}} \sum_{l=0}^{\infty} \frac{z_{\mathrm{ph}}^{l}}{\sqrt{l!}} |l\rangle \,, \end{aligned} \tag{A.5}$$

with $\mathcal{N}_{\mathrm{ph}} = \mathrm{e}^{-|z_{\mathrm{ph}}|^2/2}$. By taking the trace on the photon degrees of freedom, we obtain the reduced atomic density matrix, $\mathrm{Tr}_{\mathrm{ph}}(\rho(t))$, which is given by the expression

$$\rho_{\mathrm{at}}(t) = \mathcal{N}_{\mathrm{at}}^2 \times \sum_{k\ k'} z_{\mathrm{at}}^{k} z_{\mathrm{at}}^{*\ k'} \begin{pmatrix} 2N \\ k \end{pmatrix}^{1/2} \begin{pmatrix} 2N \\ k' \end{pmatrix}^{1/2} |k\rangle\langle k'| \,. \tag{A.6}$$

In the above equation, the state $|k\rangle$ is given by applying k-times the operator S_+ on the state $|N, 0\rangle$

$$|k\rangle = \sqrt{\frac{(2N-k)!}{k!(2N)!}} S_+^k |N, 0\rangle \,. \tag{A.7}$$

Next, with this expression for the reduced density matrix, and with the eigenstates of the operators $S_{x'}$, $S_{y'}$ and $S_{z'}$, in the system where z' is oriented in the direction of $\langle \mathbf{S} \rangle$, we have evaluated the associated probabilities as

$$P_j(\sigma) = \langle \sigma, j | \rho_{\rm at} | \sigma, j \rangle \,, \quad \sigma = x', y', z' \,, \quad j = 0, \ldots, 2N \,,$$

and obtained the results

$$P_j(z') = \begin{cases} 1 \,, & j = 0 \\ 0 \,, & 1 < j < 2N \end{cases} ,$$

$$P_j(x') = P_j(y') = \frac{1}{2^{2N}} \binom{2N}{j} . \tag{A.8}$$

From these,

$$\begin{aligned} H(S_{z'}) &= 0 \,, \\ H(S_{x'}) &= H(S_{y'}) \\ &= \log(2^{2N}) - \frac{1}{2^{2N}} \sum_k \binom{2N}{k} \log \binom{2N}{k} , \end{aligned} \tag{A.9}$$

with variations given by

$$\begin{aligned} \delta H(S_{z'}) &= 1 \,, \\ \delta H(S_{x'}) = \delta H(S_{y'}) &= 2^{2N} \prod_k \binom{2N}{k}^{-\frac{1}{2^{2N}} \binom{2N}{k}} . \end{aligned} \tag{A.10}$$

which is the limit appearing in (25) and (26).

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