# A Categorical Equivalence for Tense Pseudocomplemented Distributive Lattice

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**Abstract**

In this paper, we introduce the concept of tense operators on pseudocomplemented distributive lattices. Specifically, we utilize the Kalman construction to establish a categorical equivalence between the algebraic category of tense KAN-algebras and a category whose objects are pairs $(\mathbf{A}, S)$, where $\mathbf{A}$ is a tense pseudocomplemented distributive lattice, and $S$ is a tense Boolean filter of $\mathbf{A}$.

## 1 Introduction

The investigation of tense operators emerged in the 1980s, with notable contributions by Burges (see [4]). Classical tense logic is a logical system that extends bivalent logic by incorporating the tense operators $G$ (indicating that something will always be the case) and $H$ (indicating that something has always been the case) (see [12]). These operators allow us to express statements that hold consistently in the future or have always been true in the past. Tense logic provides a formal framework for reasoning about time-dependent propositions and has applications in various fields, including computer science, artificial intelligence, and philosophy of time.

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By incorporating appropriate tense operators, we can expand upon existing logical systems, such as intuitionistic calculus and many-valued logics, to create new tense logics (see [10, 7]). This extension enhances the expressiveness of the logical systems, enabling a more nuanced analysis of the tense dimension in statements. The study of tense logics has led to the development of various variants, each with its own unique features and applications across different fields of study. Two other operators $F$ and $P$ are usually defined via $G$ and $H$ by $F(x) = -G(-x)$ and $P(x) = -H(-x)$, where $-x$ denotes negation of the proposition $x$. In a classical propositional calculus, which is represented using a Boolean algebra $\mathcal{B} = \langle B, \vee, \wedge, \neg, 0, 1\rangle$, the axioms for tense operators were established in [12] as follows:

(B1) $G(1) = 1$ and $H(1) = 1$;

(B2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$;

(B3) $x \leq GP(x)$ and $x \leq HF(x)$.

In order to introduce tense operators in non-classical logics, it is necessary to add additional axioms for $G$ and $H$ to establish their connections with other operations or logical connectives. Tense operators have been extensively investigated by various authors across different classes of algebras (see [1, 3, 5, 8, 9, 10, 14]), and the notion of tense operators on bounded distributive lattices was introduced by Chajda and Paseka in [5]. More precisely, a tense distributive lattice is a structure $\mathcal{A} = \langle A, G, H, F, P\rangle$ where $A = \langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice, and $G$, $H$, $F$, and $P$ are tense operators defined on $A$. In particular, these operators satisfy:

(T1) $P(x) \leq y$ if and only if $x \leq G(y)$,

(T2) $F(x) \leq y$ if and only if $x \leq H(y)$,

(T3) $G(x) \wedge F(y) \leq F(x \wedge y)$ and $H(x) \wedge P(y) \leq P(x \wedge y)$,

(T4) $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$.

Notice that, from the perspective of Universal Algebra, the class of tense distributive lattices constitutes a variety (see [5]).

## 2 Preliminaries

In this section, we will summarize some definitions and necessary results for what follows. We assume that the reader is familiar with bounded distributive lattices, De Morgan algebras, pseudocomplemented distributive lattices, and Kleene algebras (see [2]).

A pseudocomplemented distributive lattice (or distributive $p$-algebra) is an algebra $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, and for every $a, b \in A$, it holds that $a \wedge b = 0$ if and only if $a \leq b^*$. This means that for every $a \in A$, there is a largest member of $A$ that is disjoint with $a$, namely $a^*$. The class of distributive $p$-algebras is a variety (see [2]). Also, note that in a distributive $p$-algebra, the conditions $1 = 0^*$ and $0 = 1^*$ necessarily hold.

Recall that if $A$ is a distributive $p$-algebra, a non-empty subset $S \subseteq A$ is said to be a filter of $A$ if $S$ is an upset, and $x \wedge y \in S$ for all $x, y \in S$.

An element $a$ of $A$ is called a dense element if $a^* = 0$, and the set $D(A)$ of all dense elements of $A$ forms a filter in $A$.

If $A$ is a distributive $p$-algebra and $R$ an equivalence relation on $A$, we adopt the notation $[a]_R$ for the equivalence class of $a$ modulo $R$, and also $A/R$ for the set of equivalence classes. The definitions of Boolean filter and Boolean congruence on a given distributive $p$-algebra will be used throughout the paper, so we choose to introduce these definitions and the link between them in the present section (for more details, see [15]).

**Definition 2.1.** Let $A$ be a distributive $p$-algebra. We say that a congruence $R$ on $A$ is a Boolean congruence if $A/R$ is a Boolean algebra, or equivalently, if $a \vee a^* \in [1]_R$ for every $a \in A$.

**Definition 2.2.** Let $A$ be a distributive $p$-algebra. A filter $S$ of $A$ is called a Boolean filter if $x \vee x^* \in S$ for each $x$ in $A$.

Since $x \vee x^* \in D(A)$ for all $x$ in $A$, it is evident that $D(A)$ is a Boolean filter of $A$. In fact, it is the smallest Boolean filter of $A$.

The following three lemmas are well-known in the field of distributive $p$-algebras and are frequently cited, as exemplified in [11, 15].

**Lemma 2.1.** *Let $A$ be a distributive $p$-algebra. The following conditions are equivalent:*

1. *$A$ is a Boolean algebra.*
2. *$D(A) = \{1\}$.*

**Lemma 2.2.** *Let $A$ be a distributive p-algebra and $R$ be a congruence on $A$. The following conditions are equivalent:*

1. *$R$ is a Boolean congruence.*
2. *$[1]_R$ is a Boolean filter.*

**Lemma 2.3.** *Let $A$ be a distributive p-algebra. If $R$ is a Boolean congruence, then $[1]_R$ is a Boolean filter. If $S$ is a Boolean filter, then the set*

$$\Theta(S) = \{(a, b) \in A \times A : a \wedge s = b \wedge s \text{ for some } s \in S\}$$

*is a Boolean congruence. Moreover, the assignments $R \mapsto [1]_R$ and $S \mapsto \Theta(S)$ define an order isomorphism between the poset of Boolean congruences of $A$ and the poset of Boolean filters of $A$.*

Recall that a Kleene algebra is an algebra $\langle T, \vee, \wedge, \sim, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ satisfying that $\langle T, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\sim$ is an involution (i.e., $\sim\sim x = x$ for every $x \in T$) such that

1. $\sim (x \vee y) =\sim x\wedge \sim y$ and
2. $x\wedge \sim x \leq y\vee \sim y$.

hold for every $x, y \in T$.

In [11], the authors extend Kleene algebras with a unary operation $\neg$, referred to as intuitionistic negation, and define the variety of Kleene algebras with intuitionistic negation, abbreviated as KAN-algebras. More precisely, a KAN-algebra is an algebra $\langle T, \wedge, \vee, \sim, \neg, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that $\langle T, \wedge, \vee, \sim, 0, 1\rangle$ is a Kleene algebra and the following conditions are satisfied for every $x, y \in T$:

(N1) $\neg(x \wedge \neg(x \wedge y)) = \neg(x \wedge \neg y)$,

(N2) $\neg(x \vee y) = \neg x \wedge \neg y$,

(N3) $x\wedge \sim x = x \wedge \neg x$,

(N4) $\sim x \leq \neg x$,

(N5) $\neg(x \wedge y) = \neg((\sim \neg x) \wedge y)$.

If $\langle T, \vee, \wedge, \sim, \neg, 0, 1\rangle$ is a KAN-algebra, an application of (N3) yields $\neg 1 = 1 \wedge \neg 1 = 1\wedge \sim 1 = 1 \wedge 0 = 0$. Taking $x = 0$ in (N4) we obtain that $\neg 0 = 1$. In addition, if $x \leq y$, then $\neg y \leq \neg x$ by (N2).

# 3 Tense operators on distributive $p$-algebras

In this section, we will define the variety of tense pseudocomplemented distributive lattices and prove some basic properties. Additionally, we will introduce the tense version of Boolean filter and Boolean congruence in the subsequent discussion.

**Definition 3.1.** An algebra $\mathbf{A} = (A, G, H, F, P)$ is a tense pseudocomplemented distributive lattice, or tense $p$-algebra, if $\langle A, \vee, \wedge, ^*, 0, 1\rangle$ is a distributive $p$-algebra, and $G, H, F, P$ are unary operations on $A$ that satisfy the following conditions:

(T1) $P(x) \leq y$ if and only if $x \leq G(y)$,

(T2) $F(x) \leq y$ if and only if $x \leq H(y)$,

(T3) $G(x) \wedge F(y) \leq F(x \wedge y)$ and $H(x) \wedge P(y) \leq P(x \wedge y)$,

(T4) $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$,

(T5) $F(x)^* \leq G(x^*)$ and $P(x)^* \leq H(x^*)$,

(T6) $G(x)^* \leq F(x^*)$ and $H(x)^* \leq P(x^*)$.

***Example* 3.1.** Given a distributive $p$-algebra $A$, there are two extreme examples of tense operators:

(1) Define $G, H, F,$ y $P$ as the identity function $\mathrm{id}_A$.

(2) Define $G$ and $H$ as the constant function $1_A$ (i.e., $G(x) = 1 = H(x)$ for all $x \in A$), and $F$ and $P$ as the constant function $0_A$ (i.e., $F(x) = 0 = P(x)$ for all $x \in A$).

**Remark 3.1.** Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. Then, according to properties (T1) to (T4), we can conclude that the reduct $\langle A, \vee, \wedge, G, H, F, P\rangle$ forms a tense distributive lattice (see [5, 13]).

We will list several fundamental properties that hold in tense $p$-algebras and provide proofs for some of them.

**Proposition 3.1.** *Let $(A, G, H, F, P)$ be a tense $p$-algebra. Then*

(T7) $G(1) = 1$ *and* $H(1) = 1$,

(T8) $G(x \wedge y) = G(x) \wedge G(y)$ *and* $H(x \wedge y) = H(x) \wedge H(y)$, (T9) $x \leq GP(x)$ *and* $x \leq HF(x)$,

(T10) $F(0) = 0$ *and* $P(0) = 0$,

(T11) $F(x \vee y) = F(x) \vee F(y)$ *and* $P(x \vee y) = P(x) \vee P(y)$,

(T12) $FH(x) \leq x$ *and* $PG(x) \leq x$,

(T13) $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$,

(T14) $x \leq y$ *implies* $F(x) \leq F(y)$ *and* $P(x) \leq P(y)$,

(T15) $x \wedge F(y) \leq F(P(x) \wedge y)$ *and* $x \wedge P(y) \leq P(F(x) \wedge y)$,

(T16) $F(x) \wedge y = 0$ *if and only if* $x \wedge P(y) = 0$,

(T17) $G(x \vee H(y)) \leq G(x) \vee y$ *and* $H(x \vee G(y)) \leq H(x) \vee y$,

(T18) $x \vee H(y) = 1$ *if and only if* $G(x) \vee y = 1$,

(T19) $G(x^*) \leq F(x)^*$ *and* $H(x^*) \leq P(x)^*$,

(T20) $F(x^*) \leq G(x)^*$ *and* $P(x^*) \leq H(x)^*$.

*Proof.* Note that (T7) to (T12) follow from (T1) and (T2). Axioms (T13) and (T14) are consequences of axioms (T8) and (T11), respectively. Next, let's prove (T15). From (T9), we have $x \wedge F(y) \leq GP(x) \wedge F(y)$. Using this statement and (T3), we obtain $x \wedge F(y) \leq F(P(x) \wedge y)$. The reverse inequality can be proven similarly. Now, let's verify (T16). Suppose $F(x) \wedge y = 0$. Using (T10) and (T15), we get $x \wedge P(y) \leq P(F(x) \wedge y) = P(0) = 0$. Hence, $x \wedge P(y) = 0$. Similarly, we can prove the reverse direction. Moreover, axioms (T17) and (T18) can be proven using a similar technique as in the proof of (T15) and (T16), respectively. Finally, let's prove (T19) and (T20). Using (T3) and (T10), we have $G(x^*) \wedge F(x) \leq F(x^* \wedge x) = F(0) = 0$. Thus, $G(x^*) \leq F(x)^*$. Similarly, $H(x^*) \leq P(x)^*$. Additionally, (T20) can be proven using a similar technique.

□

**Remark 3.2.** If $\mathbf{A} = (A, G, H, F, P)$ is a tense $p$-algebra, and $A$ is a Boolean algebra, it is easy to see that $(A, G, H)$ is a tense Boolean algebra.

**Definition 3.2.** Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. A filter $S$ of $A$ is called a *tense filter* if it satisfies the following condition:

(tf) $x \in S$ implies $G(x) \in S$ and $H(x) \in S$.

**Definition 3.3.** Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. A tense filter $S \subseteq A$ is called a *tense Boolean filter* if it contains all dense elements, i.e., $D(A) \subseteq S$.

***Example* 3.2.** Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. The set $D(A)$ is a tense Boolean filter. It is evident that $D(A)$ forms a Boolean filter. Let's prove that $D(A)$ is closed under both $G$ and $H$. Suppose $x \in D(A)$. From this assertion, applying axioms (T6), (T20), and (T10), we have $G(x)^* = F(x^*) = F(0) = 0$. Consequently, it follows that $G(x) \in D(A)$. Similarly, we can verify that $H(x) \in D(A)$.

**Definition 3.4.** Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. A *tense congruence* on $\mathbf{A}$ is a $p$-congruence $\theta$ which is compatible with every tense operators, i.e. if $(x, y) \in \theta$, then $(T(x), T(y)) \in \theta$, for every $T \in \{G, H, F, P\}$.

**Definition 3.5.** Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. A tense congruence $\theta$ is a *tense Boolean congruence* of $\mathbf{A}$ if the quotient algebra $\mathbf{A}/\theta = (A/\theta, G, H)$ is a tense Boolean algebra.

**Remark 3.3.** Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. The set of all tense Boolean congruences forms a lattice.

**Lemma 3.1.** *Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra. If $\theta$ is a tense Boolean congruence, then $[1]_\theta$ is a tense Boolean filter of $\mathbf{A}$.*

*Proof.* It is known that $[1]_\theta$ is a Boolean filter (see [11, Lemma 1.2]). Let $x \in [1]_\theta$ . Then, $(x, 1) \in \theta$. Since $\theta$ is a tense congruence, we have $(G(x), G(1)) \in \theta$. Applying property (T8), we conclude that $G(1) = 1$. Therefore, $G(x) \in [1]_\theta$. Similarly, we can deduce that $H(x) \in [1]_\theta$ using a similar approach. Therefore, $[1]_\theta$ is a tense Boolean filter. □

From the established results in [11] and Lemma 3.1, the following result is obtained.

**Lemma 3.2.** *Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra and $S$ a tense Boolean filter. Then, the set*

$$\Theta(S) = \{(a, b) \in A \times A : a \wedge s = b \wedge s \text{ for some } s \in S\}$$

*is a tense Boolean congruence. Moreover, the assignments $\theta \mapsto [1]_\theta$ and $S \mapsto \Theta(S)$ define an order isomorphism between the poset of tense Boolean congruences of $\mathbf{A}$ and the poset of tense Boolean filters of $\mathbf{A}$.*

**Remark 3.4.** Upon examining the assignments from the previous lemma, it can be proven that a correspondence exists between the set of all tense filters and the set of all tense congruences of a tense $p$-algebra $\mathbf{A}$.

## 4 Tense operators on KAN-algebras

In this section we will introduce the notion of tense operators on the variety of KAN-algebras.

Let $\langle T, \vee, \wedge, \sim, \neg, 0, 1\rangle$ be a KAN-algebra, and let $G$ and $H$ be two unary operators on $T$. We define the operators $P(x) :=\sim H(\sim x)$ and $F(x) :=\sim G(\sim x)$.

**Definition 4.1.** An algebra $\mathbf{T} = (T, G, H)$ is a tense KAN-algebra if $\langle T, \vee, \wedge, \sim, \neg, 0, 1\rangle$ is a KAN-algebra, and $G$ and $H$ are unary operations on $T$ that satisfy the following conditions:

(t1) $G(1) = 1$ and $H(1) = 1$,

(t2) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$,

(t3) $x \leq GP(x)$ and $x \leq HF(x)$,

(t4) $G(x \vee y) \leq G(x) \vee F(y)$ and $H(x \vee y) \leq H(x) \vee P(y)$,

(t5) $G(\neg x) = \neg F(x)$ and $P(\neg x) = \neg H(x)$,

(t6) $\neg G(x) = F(\neg x)$ and $H(\neg x) = \neg P(x)$.

***Example*** **4.1.** Let $\mathbf{B} = (B, G, H)$ be a tense Boolean algebra, and let the unary operation $\sim$ be defined as $\sim x := \neg x$. According to Example 2.3 in [11], it is stated that $(B, \wedge, \vee, \sim, \neg, 0, 1)$ is a KAN-algebra. By checking that $G$ and $H$ satisfy the axioms (t1) to (t6), we can conclude that $\mathbf{B}$, with this additional operation $\sim$, is a tense KAN-algebra.

The following proposition contains some properties of tense KAN-algebras that will be useful throughout the paper. The proof is analogous to that of Lemma 3.1, so we omit it.

**Proposition 4.1.** *Let* $\mathbf{T} = (T, G, H)$ *be a tense KAN-algebra. Then,*

(t7) $F(0) = 0$ *and* $P(0) = 0$,

(t8) $F(x \vee y) = F(x) \vee F(y)$ *and* $P(x \vee y) = P(x) \vee P(y)$,

(t9) $PG(x) \leq x$ *and* $FH(x) \leq x$,

(t10) $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$,

(t11) $x \leq y$ *implies* $F(x) \leq F(y)$ *and* $P(x) \leq P(y)$,

(t12) $G(x) \wedge F(y) \leq F(x \wedge y)$ *and* $H(x) \wedge P(y) \leq P(x \wedge y)$, (t13) $x \wedge F(y) \leq F(P(x) \wedge y)$ *and* $x \wedge P(y) \leq P(F(x) \wedge y)$,

(t14) $F(x) \wedge y = 0$ *if and only if* $x \wedge P(y) = 0$,

(t15) $G(x \vee H(y)) \leq G(x) \vee y$ *and* $H(x \vee G(y)) \leq H(x) \vee y$,

(t16) $x \vee H(y) = 1$ *if and only if* $G(x) \vee y = 1$.

# 5 Kalman's Construction

In this section, we prove some results that establish the connection between tense $p$-algebras and tense KAN-algebras.

Let $\mathbf{A} = (A, G, H, F, P)$ be a tense $p$-algebra and let us consider

$$K(A) := \{(a, b) \in A \times A : a \wedge b = 0\}.$$

As established in the well-known [11, Lemma 2.4], by defining:

$$\begin{array}{rcl} (a,b) \vee (x,y) & := & (a \vee x, b \wedge y), \\ (a,b) \wedge (x,y) & := & (a \wedge x, b \vee y), \\ \neg(a,b) & := & (a^*, a), \\ \sim (a,b) & = & (b,a), \\ 0 & = & (0,1), \\ 1 & = & (1,0), \end{array}$$

we get that the algebra $\mathbb{K}(A) = \langle K(A), \vee, \wedge, \sim, \neg, 0, 1\rangle$ is a KAN-algebra.

Now, let us define the following unary operators on $K(A)$:

$$\begin{array}{rcl} G_K((a,b)) & := & (G(a), F(b)), \\ H_K((a,b)) & := & (H(a), P(b)), \\ F_K((a,b)) & := & (F(a), G(b)), \\ P_K((a,b)) & := & (P(a), H(b)). \end{array}$$

**Lemma 5.1.** *Let* $\mathbf{A} = (A, G, H, F, P)$ *be a tense $p$-algebra and let* $(a, b) \in K(A)$*. Then, the following hold:*

(a) $G_K(a, b) \in K(A)$ *and* $H_K(a, b) \in K(A)$,

(b) $F_K(a,b) = \sim G_K(\sim (a,b))$ *and* $P_K(a,b) = \sim H_K(\sim (a,b))$,

(c) $F_K(a,b) \in K(A)$ *and* $P_K(a,b) \in K(A)$.

*Proof.* We will focus on proving property (a), leaving the remaining properties for the reader to verify. Let $(a,b) \in K(A)$. Hence, $a \wedge b = 0$. Then, from (T3) and (T10), $G(a) \wedge F(b) \leq F(a \wedge b) = F(0) = 0$. Therefore, $(G(a), F(b)) \in K(A)$. In a similar way, we can prove $H_K(a,b) \in K(A)$. □

**Lemma 5.2.** *Let* $\mathbf{A} = (A, G, H, F, P)$ *be a tense p-algebra. Then, the structure* $K(\mathbf{A}) = (\mathbb{K}(A), G_K, H_K)$ *is a tense KAN-algebra. Furthermore, for* $\mathbf{B} = (B, G, H, F, P)$ *as a tense p-algebra and a morphism* $f : \mathbf{A} \longrightarrow \mathbf{B}$*, the map* $K(f) : K(\mathbf{A}) \longrightarrow K(\mathbf{B})$*, defined by* $K(f)(a,b) = (f(a), f(b))$*, is a functor from the category of tense p-algebras to the category of tense KAN-algebras.*

*Proof.* Based on [11, Lemma 2.4], we are aware that $\mathbb{K}(A)$ is a KAN-algebra, and from Lemma 5.1 we know that $G_K$ and $H_K$ are well-defined. Therefore, our focus will be on proving that $K(\mathbf{A})$ satisfies axioms (t1) to (t6). Due to the symmetry of tense operators $G$ and $H$, we will only prove the axioms for the operator $G$.

Let $(a,b)$ and $(x,y)$ be elements of $K(\mathbf{A})$.

(t1): $G_K(1,0) = (G(1), F(0)) = (1,0)$.

(t2): $G_K((a,b) \wedge (x,y)) = G_K(a \wedge x, b \vee y) = (G(a \wedge x), F(b \vee y))$. Using (T8) and (T11), we have $(G(a \wedge x), F(b \vee y)) = (G(a) \wedge G(x), F(b) \vee F(y)) = (G(a), F(b)) \wedge (G(x), F(y))$. Therefore, $G_K((a,b) \wedge (x,y)) = G_K(a,b) \wedge G_K(x,y)$.

(t3): $G_K(P_K(a,b)) = G_K(P(a), H(b)) = (G(P(a)), F(H(b)))$. Using (T9) and (T12), we have $(a,b) \leq (G(P(a)), F(H(b)))$, hence $(a,b) \leq G_K P_K(a,b)$.

(t4): $G_K((a,b) \vee (x,y)) = G_K(a \vee x, b \wedge y) = (G(a \vee x), F(b \wedge y))$. Using (T3) and (T4), we have $(G(a \vee x), F(b \wedge y)) \leq (G(a) \vee F(x), G(y) \wedge F(b))$, and $(G(a) \vee F(x), G(y) \wedge F(b)) = (G(a), F(b)) \vee (F(x), G(y)) = G_K(a,b) \vee F_K(x,y)$. Therefore, $G_K((a,b) \vee (x,y)) \leq G_K(a,b) \vee F_K(x,y)$.

(t5): $\neg F_K(a,b) = \neg(F(a), G(b)) = (F(a)^*, F(a))$. From (T5) and (T19), we have $F(a)^* = G(a^*)$, thus $\neg F_K(a,b) = (G(a^*), F(a)) = G_K(a^*, a) = G_K(\neg(a,b))$.

(t6): $F_K(\neg(a,b)) = F_K(a^*, a) = (F(a^*), G(a))$. Using (T6) and (T20), we have $F(a^*) = G(a)^*$, therefore $F_K(\neg(a,b)) = (G(a)^*, G(a)) = \neg G_K(a,b)$.

Now, according to [11, Lemma 2.4], we know that $K$ is a functor from the category of distributive $p$-algebras to the category of KAN-algebras. We will prove that $K$ preserves the tense operator $G$: $K(f)(G_K(a,b)) = K(f)(G(a),F(b)) = (f(G(a)), f(F(b))) = K(f)(G(a),F(b)) = K(f)(G_K(a,b))$. Similarly, we can observe that $K$ preserves $H, F$, and $P$. $\square$

Let $(T, \wedge, \vee, \sim, \neg, 0, 1)$ be a KAN-algebra, and let $\theta \subseteq T^2$ be defined as

$$(x,y) \in \theta \iff \neg x = \neg y \tag{1}$$

The relation $\theta$ is an equivalence relation that will play a crucial role in establishing a categorical equivalence for the class of tense KAN-algebras.

Recall that $[x]_\theta$ denotes the set $\{y \in T : (x,y) \in \theta\}$, and the set $\{[x]_\theta : x \in T\}$ is denoted by $T/\theta$.

**Lemma 5.3.** *Let $\mathbf{T} = (T, G, H)$ be a tense KAN-algebra, and let $\theta \subseteq T^2$ be defined as specified in 1. Then, the equivalence relation $\theta$ is compatible with the operations $\wedge, \vee, \neg$, as well as the tense operators $G$ and $H$.*

*Proof.* From [11, Lemma 2.7], we know that $\theta$ is compatible with $\wedge, \vee$, and $\neg$. We will now prove that $\theta$ is also compatible with the tense operators $G$ and $H$. Let $(x,y) \in \theta$. We have $\neg x = \neg y$, which implies $F(\neg x) = F(\neg y)$. By applying property (t6), we have $\neg G(x) = \neg G(y)$, and therefore $(G(x), G(y)) \in \theta$. Similarly, we can show that $(H(x), H(y)) \in \theta$. This confirms that $\theta$ is compatible with the tense operators. $\square$

Let $\mathbf{T} = (T, G, H)$ be a tense KAN-algebra. By applying Lemma 5.3 and [11, Lemma 1.8], we deduce that $(T/\theta, \wedge, \vee, \neg, [0]_\theta, [1]_\theta)$ forms a distributive $p$-algebra, and the order $\leq$ in $T/\theta$ can be characterized as $[x]_\theta \leq [y]_\theta$ if and only if $\neg y \leq \neg x$.

**Lemma 5.4.** *Let $\mathbf{T} = (T, G, H)$ be a tense KAN-algebra, and consider the relation $\theta$ defined in 1. By defining $G_\theta([x]_\theta) = [G(x)]_\theta$, $H_\theta([x]_\theta) = [H(x)]_\theta$, $F_\theta([x]_\theta) = [F(x)]_\theta$, and $P_\theta([x]_\theta) = [P(x)]_\theta$, we have that $(T/\theta, G_\theta, H_\theta, F_\theta, P_\theta)$ forms a tense $p$-algebra.*

*Proof.*

(T1): Let's assume $P_\theta([x]_\theta) \leq [y]_\theta$. Due to the characterization of the order in $T/\theta$, it follows that $\neg y \leq \neg P(x)$. Using property (t11), we have $F(\neg y) \leq F(\neg P(x))$, and by applying properties (t6) and (t3), we obtain $\neg G(y) \leq \neg x$. Consequently, $[x]_\theta \leq G_\theta([y]_\theta)$. Now, let's assume $[x]_\theta \leq G_\theta([y]_\theta)$, which implies $\neg G(y) \leq \neg x$. From property (t10), we have $H(\neg G(y)) \leq H(\neg x)$, and by using properties (t6) and (t11), we obtain $\neg y \leq \neg P(x)$. Hence, $P_\theta([x]_\theta) \leq [y]_\theta$.

(T2): The proof is analogous to (T1).

(T3): It is immediate from property (t12) and the fact that $x \leq y$ implies $\neg y \leq \neg x$.

(T4): It is immediate from property (t4) and the fact that $x \leq y$ implies $\neg y \leq \neg x$.

(T5): From property (t5), we have $\neg G(\neg x) = \neg\neg F(x)$, which implies $\neg F_\theta(x) \leq G_\theta(\neg x)$. Similarly, we have $\neg P_\theta(x) \leq H_\theta(\neg x)$.

(T6): The proof is similar to the proof of (T5).

□

**Lemma 5.5.** *Let* $\mathbf{A} = (A, G, H, F, P)$ *be a tense p-algebra. Then the mapping* $g : K(\mathbf{A})/\theta \longrightarrow \mathbf{A}$ *defined as follows:*

$$g([(a,b)]_\theta) = a,$$

*is an isomorphism of tense p-algebras.*

*Proof.* We will only prove that $g$ preserves the tense operators. We have that $g(G_\theta([(a,b)]_\theta)) = g([G_K(a,b)]_\theta) = g([(G(a), F(b))]_\theta) = G(a)$, and $G([g(a,b)]_\theta) = G(a)$. Similarly, we can prove that $g$ preserves $H_\theta$, $F_\theta$, and $P_\theta$. Hence, $g$ is an isomorphism of tense $p$-algebras. □

**Lemma 5.6.** *Let* $\mathbf{T} = (T, G, H)$ *be a tense KAN-algebra. Then the mapping* $\rho : \mathbf{T} \longrightarrow K(\mathbf{T}/\theta)$ *defined as* $\rho(x) = ([x]_\theta, ([\sim x]_\theta)$ *is an injective morphism of tense KAN-algebras.*

*Proof.* We will show that $\rho$ preserves the tense operators $G$ and $H$. We have that $\rho(G(x)) = ([G(x)]_\theta, [\sim G(x)]_\theta)$, and

$$G_K(\rho(x)) = G_K([x]_\theta, [\sim x]_\theta) = (G_\theta([x]_\theta), F(([\sim x)]_\theta) = ([G(x)]_\theta, [F(\sim x)]_\theta).$$

Since $F(\sim x) =\sim G(x)$, we can conclude that $\rho$ preserves $G$. Similarly, we can prove that $\rho$ preserves $H$. Therefore, $\rho$ is an injective morphism of tense KAN-algebras. □

## 6 A categorical equivalence for tense KAN-algebras

In this section, we will prove that tense $p$-algebras and tense Boolean filters provide a characterization of tense KAN-algebras.

If $A$ is a $p$-algebra and $S$ a Boolean filter of $A$, we can define the set

$$K(A, S) := \{(a, b) \in A \times A : a \wedge b = 0 \text{ and } a \vee b \in S\}.$$

From [11, Theorem 2.11], we know that for two $p$-algebras $A$ and $A'$, and two Boolean filters $S$ and $S'$, there exists a well-defined function $K(f) : K(A,S) \longrightarrow K(A',S')$ given by $K(f)(a,b) = (f(a), f(b))$. Furthermore, it is known that the set $K(A,S)$ is the universe of a subalgebra of $\mathbb{K}(A)$, making it a KAN-algebra.

**Proposition 6.1.** *If $\mathbf{A} = (A, G, H, F, P)$ is a tense $p$-algebra and $S$ is a tense Boolean filter of $A$, then the set $K(\mathbf{A},\mathbf{S}) := \{(a,b) \in A \times A : a \wedge b = 0$ and $a \vee b \in S\}$ is a tense KAN-algebra.*

*Proof.* We know that $K(A,S)$ is a subalgebra of $\mathbb{K}(A)$. Therefore, we only need to prove that $K(\mathbf{A},\mathbf{S})$ is closed under the tense operators $G_K$ and $H_K$. Let $(a,b) \in K(\mathbf{A},\mathbf{S})$. We have $G_K(a,b) = (G(a), F(b))$. By using (t12) and (t7), we have $G(a) \wedge F(b) \leq F(a \wedge b) = 0$. Additionally, using (t4) and the fact that $S$ is a tense filter, we have that $G(a) \vee F(b) \in S$. Therefore, $G_K(a,b) \in K(\mathbf{A},\mathbf{S})$. The proof for $H_K$ follows a similar argument. □

**Proposition 6.2.** *If $\mathbf{A}$ and $\mathbf{A}'$ are two tense $p$-algebras, and $S$ and $S'$ are two tense Boolean filters of $\mathbf{A}$ and $\mathbf{A}'$ respectively, let $f : \mathbf{A} \longrightarrow \mathbf{A}'$ be a morphism of tense $p$-algebras such that $f(S) \subseteq S'$. We can define the morphism $K(f) : K(\mathbf{A},\mathbf{S}) \longrightarrow K(\mathbf{A}',\mathbf{S}')$ of tense KAN-algebras as $K(f)(a,b) = (f(a), f(b))$.*

*Proof.* From [11, Theorem 2.11], we know that $K(f)$ is a morphism of KAN-algebras. By Proposition 6.1, we have that $K(\mathbf{A},\mathbf{S})$ is a tense KAN-algebra, so we only need to prove that $K(f)$ preserves the tense operators $G_K$ and $H_K$. For $G_K$, we have $K(f)(G_K(a,b)) = K(f)(G(a), F(b)) = (f(G(a)), f(F(b)))$. Since $f$ is a morphism of tense $p$-algebras, it preserves $G$ and $F$. Therefore, $(f(G(a)), f(F(b))) = (G(f(a)), F(f(b))) = G_K(f(a), f(b)) = G_K(K(f)(a,b))$. Hence, $K(f)$ preserves $G_K$. The proof for $H_K$ follows a similar argument. □

The proof of the following theorem can be obtained by combining the results from Lemma 5.2, Lemma 5.5, and [11, Theorem 2.11].

**Theorem 6.1.** *If $\mathbf{A}$ is a tense $p$-algebra and $S$ is a tense Boolean filter of $\mathbf{A}$, then the quotient algebra $K(\mathbf{A},\mathbf{S})/\theta$ is isomorphic to $\mathbf{A}$. Furthermore, if $\mathbf{A}'$ is a tense $p$-algebra and $S'$ is a tense Boolean filter of $\mathbf{A}'$, and $f : \mathbf{A} \longrightarrow \mathbf{A}'$ is a morphism of tense $p$-algebras such that $f(S) \subseteq S'$, then $K(f)$ is a morphism of tense KAN-algebras.*

**Lemma 6.1.** *Let $\mathbf{T} = (T, G, H)$ be a tense KAN-algebra . Then the positive part $\mathbf{T}^+ := \{x \in T : \neg \sim x = 1\}$ is a tense filter of $\mathbf{T}$ that includes all elements $x \in \mathbf{T}$ satisfying $\neg\neg x = 1$. Consequently, $\mathbf{T}^+/\theta$ is a tense Boolean filter of $\mathbf{T}/\theta$.*

*Proof.* From [11, Lemma 2.14], we only need to prove that $\mathbf{T}^+$ is closed under $G$ and $H$. Suppose $x \in \mathbf{T}^+$. Then, we have that $\neg \sim x = 1$. Therefore, $G(\neg \sim x) = 1$. By using the definition of $F$ and property (t5), we have $\neg F(\sim x) = \neg \sim G(x) = G(\neg \sim x) = 1$. Consequently, $\neg \sim G(x) = 1$, which implies $G(x) \in T^+$. The proof for $H$ follows a similar reasoning. Hence, $\mathbf{T}^+$ is a tense filter of $\mathbf{T}$, and consequently, $\mathbf{T}^+/\theta$ forms a tense filter of $\mathbf{T}/\theta$. $\square$

**Theorem 6.2.** *Let $\mathbf{T} = (T, G, H)$ be a tense KAN-algebra. Then $\mathbf{T}$ is isomorphic to $K(\mathbf{T}/\theta; \mathbf{T}^+/\theta)$. Furthermore, if $\mathbf{T}'$ is also a tense KAN-algebra and $f : \mathbf{T} \longrightarrow \mathbf{T}'$ is a morphism between tense KAN-algebras, then the mapping $f_\theta : \mathbf{T}/\theta \longrightarrow \mathbf{T}'/\theta$ defined as $f_\theta([x]_\theta) = [f(x)]_\theta$ is a morphism of tense p-algebras. It is worth noting that $f_\theta(\mathbf{T}^+/\theta) \subseteq (\mathbf{T}')^+/\theta$ holds.*

*Proof.* We know from [11, Theorem] that $\rho : T \longrightarrow K(T/\theta; T^+/\theta)$ is an isomorphism of KAN-algebras, and Lemma 5.6 establishes that $\rho$ is a tense morphism. Therefore, $\rho$ is an isomorphism for tense KAN-algebras. Furthermore, according to [11, Theorem 2.15], we only need to proof that $f_\theta$ preserves the tense operators. Let's consider $f_\theta(G_\theta([x]_\theta)) = f_\theta([G(x)]_\theta) = [f(G(x))]_\theta$. Since $f$ is a morphism of tense $p$-algebras, we have $[f(G(x))]_\theta = [G(f(x))]_\theta = G_\theta([f(x)]_\theta) = G_\theta(f_\theta([x]_\theta))$. Hence, we conclude that $f_\theta(G_\theta([x]_\theta)) = G_\theta(f_\theta([x]_\theta))$. Similar reasoning can be applied to prove the preservation of tense operators for $H_\theta$, $F_\theta$, and $P_\theta$. $\square$

We denote by **tPDL** the category whose objects are pairs $(\mathbf{A}, S)$, where $\mathbf{A}$ is a tense $p$-algebra and $S$ is a tense Boolean filter of $A$, and whose arrows $f : (\mathbf{A}, S) \longrightarrow (\mathbf{A}', S')$ are morphisms $f : \mathbf{A} \longrightarrow \mathbf{A}'$ such that $f(S) \subseteq S'$.

Based on the previous results, we can conclude that if $\mathbf{T} = (T, G, H)$ is a tense KAN-algebra, then $K(\mathbf{T}/\theta, \mathbf{T}^+/\theta) \in$ **tPDL**. Moreover, when $f : \mathbf{T} \longrightarrow \mathbf{T}'$ is a morphism between tense KAN-algebras, it follows that $f_\theta$ is a morphism in **tPDL**. Consequently, we can observe that the aforementioned assignments establish a functor from the algebraic category of tense KAN-algebras to the category **tPDL**.

The proof of the following theorem is a direct consequence of Theorem 6.1 and Theorem 6.2.

**Theorem 6.3.** *The functor $K$ establishes a categorical equivalence between the category of tense KAN-algebras and the category* **tPDL***.*

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