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Neighborhood-based argumental community support in the context of multi-topic debates

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ABSTRACT

The formal characterization of abstract argumentation has allowed the study of many exciting characteristics of the argumentation process. Nevertheless, while helpful in many aspects, abstraction diminishes the knowledge representation capabilities available to describe naturally occurring features of argumentative dialogues; one of these elements is the consideration of the topics involved in a discussion. In studying dialogical processes, participants recognize that some topics are closely related to the original issue; in contrast, others are more distant from the central subject or refer to unrelated matters. Consequently, it is reasonable to study different argumentation semantics that considers a discussion's focus to evaluate acceptability. In this work, we introduce the necessary representational elements required to reflect the focus of a discussion. We propose a novel extension of the semantics for *multi-topic abstract argumentation frameworks*, acknowledging that every argument has its own *zone of relevance* in the argumentation framework, leading to the concepts of neighborhoods and communities of legitimate defenses. Furthermore, other semantic elaborations are defined and discussed around this structure.

1. Introduction

Computational modeling of arguments is at the heart of many real-world scenarios given that they are especially suited for dealing with conflicting information since they are based on the study of reasons for and against specific claims. Arguments capture different ways in which available information or points of view may be leveraged towards reaching certain conclusions, which captures not only applications that are suited for this kind of knowledge representation and reasoning—such as political debates or discussions on social media [16,25]—but also other applications that need to handle conflicting points of view. Examples of the latter are recommendation systems [22], cybersecurity applications [27,23], health information systems [19,24], legal reasoning [2,28,26], and other decision-support systems [1,10].

Argumentation has been studied from a computational point of view for quite some time, with two main branches emerging with respect to the level of detail they consider: abstract and structured. In this paper, we focus on the former, whose main goal is to

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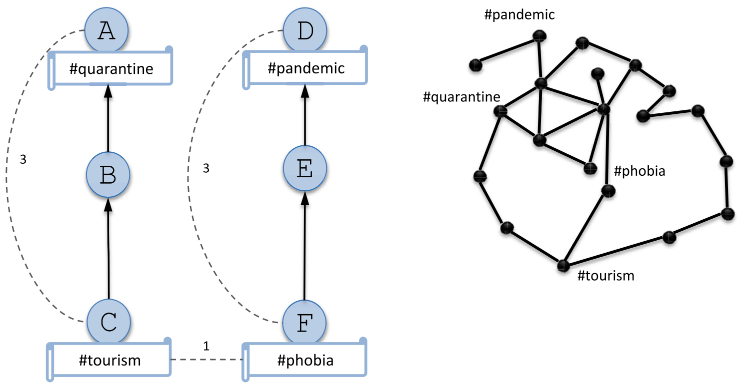


Fig. 1. Same-distance defenders under irregular network.

characterize one or more *argumentation semantics* that indicate which arguments are accepted and which ones are not. In order to do so, such approaches abstract away the underlying structure of arguments, focusing only on the relationships between them. Therefore, the intrinsic characteristics of different proposals are not taken into account, and arguments are simply treated as abstract entities involved in several relationships like attack, defeat, support and weakening, among others [12,3,5,29]. In this conceptual direction, Dung [12] introduced *Abstract Argumentation Frameworks* (*AFs*) where arguments are atomic entities, and a binary attack relation is defined. The focus is put on argument interaction and the possible outcomes of the argumentation process. Since then, and due to the highest level of abstraction adopted in this seminal work, many authors introduced extensions enriching the representational capabilities of the basic framework by attaching additional features such as preference, support relations, weights, probability, and values [3,5,29,21,7,9].

In this work, we are interested in the study of argument scenarios where semantic information about the content of arguments is taken into account. Thus, an argument is associated with a set of *abstract topics*, denoting concepts somehow referenced by that argument. This model was introduced by Budán et al. in [6], where Hashtagged Argumentation Frameworks were presented. In this framework, arguments are associated with abstract labels, called *topics* or simply *hashtags*, denoting concepts possibly referred to by arguments. A formalization of *distance* between two topics is also introduced, based on a semantic network. Then, arguments may refer to different topics of varied proximity, inducing notions of distance also between arguments. In this way, an argument may defend another argument, which may be close or distant according to their hashtags, providing a pathway to distinguishing important features in argument extensions. For instance, an argument A may refer to *pandemic*, to *quarantine*, and to the *welfare state*, while another argument B may refer to *pandemic* and *online music streaming*. Although both arguments are related to one same issue, argument B may not be as helpful while discussing health topics as argument A.

In Hashtagged Argumentation Frameworks, the notion of admissibility semantics based on proximity is defined, where only *close* arguments can be involved in argument defense—an argument is considered close enough to defend another argument if the distance between them is smaller than a particular threshold τ, and any potential defender *beyond* τ will not be considered. This is the basis of a proximity-based semantics as introduced in [6]; however, this approach can be considered to be restrictive since the threshold is the same for the entire framework. This can be revised in order to capture a more refined version of proximity semantics. Consider the framework of Fig. 1, where only the relevant hashtags are mentioned and, for simplicity, there is only one hashtag per argument. Consider the distance between arguments as the length of the shortest path between their single hashtags. In this case, the distance between A and C and between D and F is 3. If the threshold of proximity for the framework is 3, both A and D are defended by C and F, respectively. However, there seem to be more closely related topics in the upper area of the semantic graph, where health issues are addressed, than around *#tourism*. Since the semantic network of hashtags is not necessarily uniform, a universal threshold may not be adequate in all cases. An individual threshold for A may thus drop arguments about tourism, while a different threshold for D may enable all the arguments as defenders as long as they are both related to health issues.

In this work, we refine the proximity semantics of Hashtagged Frameworks by defining an individual scope of proximity for every argument, which may vary in range from argument to argument. As discussed above, individual thresholds are reasonable since the closeness depends on the intrinsic set of topics of both arguments and the topology of the semantic network. Hence, an argument A has an associated *neighborhood* formed by all the arguments close enough to A. Neighborhoods vary in size since they are induced by the topics of every individual argument. This leads to a new notion of proximity-based admissibility of arguments, where only defenders in the neighborhood are allowed, and out of range arguments are not considered relevant for interaction.

There exists, however, a particular effect that deserves further analysis: an argument may belong to different neighborhoods and it may then influence the acceptance or rejection of arguments not sharing the same neighborhood space. Suppose A and B are two arguments that are not in the same neighborhood, but share a common neighbor C. This is possible since the neighbor relation is not transitive, so A and B are not necessarily neighbors. Hence, argument C is close enough to influence the status of arguments A and B, even when neither A nor B identify each other as neighbors. This has some semantic consequences that ought to be studied, such as the fact that neighborhoods may form a network, linked by common arguments; we refer to this collection of neighborhoods as "communities", and it is a novel formalization presented in this work. Intuitively, an argument X in a community of Y may be allowed to defend Y since, although not a neighbor, it is not completely disconnected. Argument X is considered a "citizen" of an expanded, more flexible, notion of neighborhood for Y. We define an admissibility semantics based on communities, and then inquire about the relation with previous semantic elaborations.

An early version of the neighborhood-based semantics is presented in [13]; however, in the present work, the concept of neighborhood has been significantly refined, incorporating mathematical tools to establish the threshold of each argument based on its local and global topological features. As a result, the selected threshold more accurately represents the scope of arguments based on the semantic network of topics. Furthermore, as was mentioned before, here we introduce a novel notion of community, illustrating how neighborhoods can yield an improved set of accepted arguments. This concept of community constitutes a comprehensive refinement of the acceptability notions previously proposed for individual argument neighborhoods. In this direction, we explore the connections between the semantic notions of neighborhood and community, as well as their relevance to the semantics presented in the basic proximity formalism within the domain of argumentation. We also provide a full case study in which we analyze the argumentative model from various perspectives as outlined in the paper. This includes a comparative analysis of the semantics employed in each of these perspectives.

These two modifications of the original proximity-based semantics lead to novel admissible semantic notions that are addressed in this work. This paper is thus organized as follows. In Section 2, the multi-topic abstract argumentation frameworks are reviewed. In Section 3, we first introduce the intuitions behind a topological analysis over a hashcloud and the argumentation domain, formalizing later how we can compute topological properties associated with an argument, and then address the notion of neighborhood of an argument. Next, in Section 4, new proximity semantic concepts are introduced. In Section 5, we restrict defenses considering neighborhoods and community notions, which preserves Dung's original admissibility semantics. Furthermore, we present the properties that the proposed semantics satisfies, and we analyze the existing relationships between them. Finally, Sections 6 and 7 are devoted to discussion of related work and concluding remarks, respectively. Additionally, there are two appendices: Appendix A presents a case study that provides an application example in a real-world scenario, while Appendix B contains detailed formal proofs of the results presented in this work.

2. Background

In the realm of abstract argumentation, Dung introduced in [12] a conceptual framework that laid the foundation for analyzing and understanding complex reasoning structures. Building upon Dung's pioneering work, Budán et al. in [6] extended this framework to incorporate a nuanced perspective, delving into the thematic facets inherent to argumentative discourse. Dung's original model provided a structured approach to abstract argumentation, while Budán's extension enriched this paradigm by elucidating the thematic dimensions that arguments traverse within the discourse landscape. This section explores the theoretical underpinnings of Dung's abstract argumentation and Budán's augmentation, elucidating the thematic topics integral to the articulation and analysis of arguments in the context of argumentative discussions.

2.1. Abstract argumentation framework

In [12] the notion of *Abstract Argumentation Frameworks* (*AF*) was introduced as the simplest model for argumentation. In these frameworks, an argument is considered as an abstract entity with unspecified internal structure, which is related to other arguments through a conflict relation called *attack*.

Definition 1 *(Argumentation framework [12]).* An argumentation framework (AF) is a pair $\langle \mathtt{Arg}, \mathtt{Attacks} \rangle$, where $\mathtt{Arg}$ is a set of arguments, and $\mathtt{Attacks}$ is a binary relation defined over $\mathtt{Arg}$ (representing attack), that is, $\mathtt{Attacks} \subseteq \mathtt{Arg} \times \mathtt{Arg}$.

An argument A may be attacked by another argument B. If B is attacked by a third argument C, then it is said that C defends A. This is the main idea behind acceptability semantics, defined as follows.

Definition 2 *(Semantic notions [12]).* Let $AF = \langle \mathtt{Arg}, \mathtt{Attacks} \rangle$ be an argumentation framework, then:

- A set $\mathtt{S} \subseteq \mathtt{Arg}$ is said to be conflict-free if there are no arguments $\mathtt{A}, \mathtt{B} \in \mathtt{S}$ such that $(\mathtt{A}, \mathtt{B}) \in \mathtt{Attacks}$.
- An argument $\mathtt{A} \in \mathtt{Arg}$ is acceptable with respect to a set $\mathtt{S} \subseteq \mathtt{Arg}$ iff for each $\mathtt{B} \in \mathtt{Arg}$ that attacks A there exists an argument $\mathtt{C} \in \mathtt{S}$ such that $(\mathtt{C}, \mathtt{B}) \in \mathtt{Attacks}$; it is also said that B is *attacked* by S.
- A conflict-free set $\mathtt{S} \subseteq \mathtt{Arg}$ is admissible iff each argument in S is acceptable with respect to S.
- An admissible set $\mathtt{S} \subseteq \mathtt{Arg}$ is a complete extension iff S contains each argument that is acceptable with respect to S.
- A set $\mathtt{S} \subseteq \mathtt{Arg}$ is the unique *grounded extension* of AF iff S is a $\subseteq$-minimal complete extension.
- A set $\mathtt{S} \subseteq \mathtt{Arg}$ is a *preferred extension* of AF iff S is a $\subseteq$-maximal complete extension.
- A set $\mathtt{S} \subseteq \mathtt{Arg}$ is a *stable extension* of AF iff S is a conflict free set and S attacks every argument in $\mathtt{Arg} \setminus \mathtt{S}$.

Next, we present an abstract example in order to clarify the concepts introduced in the previous definition. Note that in this context the topics associated with the arguments are not analyzed, since only the abstract entity with their relation (attacks) are involved in the argumentation process.

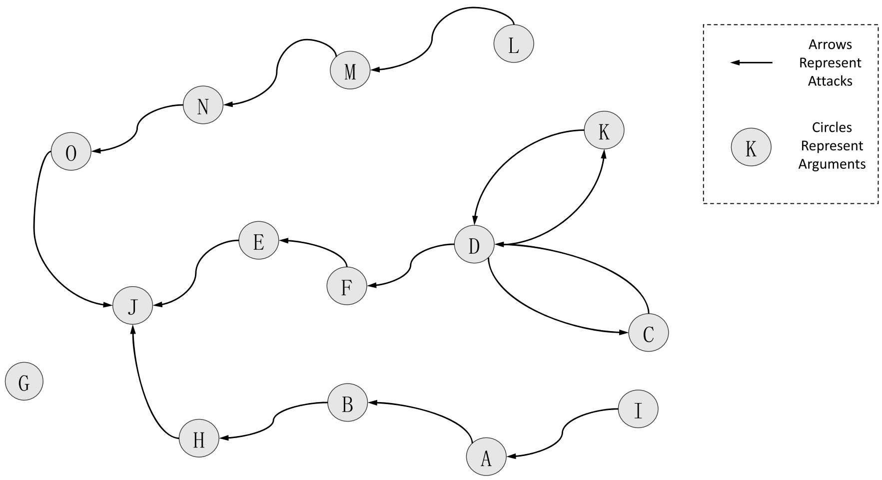


Fig. 2. Dung argumentation framework from Example 1.

Example 1. Consider the $AF = \langle \mathtt{Arg}, \mathtt{Attacks}\rangle$, graphically represented in Fig. 2, where:

$\mathtt{Args} = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}, \mathtt{I}, \mathtt{J}, \mathtt{K}, \mathtt{L}, \mathtt{M}, \mathtt{N}, \mathtt{O}\}$.
$\mathtt{Attacks} = \{(\mathtt{A}, \mathtt{B}), (\mathtt{B}, \mathtt{H}), (\mathtt{C}, \mathtt{D}), (\mathtt{D}, \mathtt{C}), (\mathtt{D}, \mathtt{F}), (\mathtt{E}, \mathtt{J}), (\mathtt{F}, \mathtt{E}), (\mathtt{H}, \mathtt{J}), (\mathtt{I}, \mathtt{A}), (\mathtt{K}, \mathtt{D}), (\mathtt{D}, \mathtt{K}), (\mathtt{L}, \mathtt{M}), (\mathtt{M}, \mathtt{N}), (\mathtt{N}, \mathtt{O}), (\mathtt{O}, \mathtt{J})\}$

The set $\mathtt{S}_1 = \{\mathtt{B}, \mathtt{G}, \mathtt{I}, \mathtt{L}, \mathtt{N}\}$ is a complete extension since it contains all the arguments that are defended by $\mathtt{S}_1$. Finally, it can be verified that $\mathtt{S}_1$ is a minimal set satisfying the previous conditions, and therefore it is the unique grounded extension of *AF*. Sets $\mathtt{S}_2 = \{\mathtt{B}, \mathtt{C}, \mathtt{F}, \mathtt{G}, \mathtt{I}, \mathtt{J}, \mathtt{K}, \mathtt{L}, \mathtt{N}\}$ and $\mathtt{S}_3 = \{\mathtt{B}, \mathtt{D}, \mathtt{E}, \mathtt{G}, \mathtt{I}, \mathtt{L}, \mathtt{N}\}$ are admissible and complete sets. Finally, $\mathtt{S}_2$ and $\mathtt{S}_3$ are the maximal sets verifying the previous conditions, and therefore they are preferred extensions of *AF*.

Dung's argumentation framework determines the status of arguments by considering only relations among them. Next, an extension of Dung's framework using an abstract model for argument topics is presented.

2.2. Hashtagged argumentation framework

A hashtagged framework [6] extends the expressive power of abstract frameworks by adding the notion of *topics* that are addressed, or referred to, by arguments. As it is usual in abstract frameworks, no reference to the underlying construction of the argument is made; however, Budán et al. [6] give relevance to *what* an argument refers to, not as a linguistic construction depending on its structure but as a whole piece of reasoning. Topics are also treated abstractly through labels called *hashtags*, denoted with the prefix #. A hashtag identifies subjects somehow referred to by the argument, either implicitly or explicitly. Without loss of generality, arguments are assumed to always have associated at least one hashtag.

Definition 3 *(Hashtagged argument [6]).* Given an argumentation framework $\Phi = \langle \mathtt{Args}, \mathtt{Attacks}\rangle$, let $\mathcal{H}$ be a finite non-empty set of hashtags. A *hashtagged argument structure* (or, when no confusion might arise, simply a *hashtagged argument*) is a pair $\langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle$, where $\mathtt{A} \in \mathtt{Args}$ and $\mathcal{H}_\mathtt{A} \subseteq \mathcal{H}$, $|\mathcal{H}_\mathtt{A}| > 0$. Then, given $\langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle$, it is said that $\mathtt{A}$ is tagged with $\mathcal{H}_\mathtt{A}$.

In the following, hashtagged arguments will generally be succinctly denoted with the letters $\mathbb{A}$, $\mathbb{B}$, ..., possibly with subscripts or superscripts.

Hashtags as topics attached to arguments usually are not isolated knowledge, and they might be related to other hashtags, leading to a semantic network of concepts.[1] The resulting network can be characterized as follows.

Definition 4 *(Hashtag graph/hashcloud [6]).* A *hashtag graph* is a graph $\mathcal{G} = \langle \mathcal{H}, E\rangle$, where $\mathcal{H}$ is the set of vertices (hashtags) and E is a subset of $\mathcal{H} \times \mathcal{H}$ that represents a set of edges between the vertices (hashtag relations) in $\mathcal{H}$. Graph $\mathcal{G}$ will also be referred to as a *hashcloud*.

A *path* is a finite or infinite sequence of edges that connects a sequence of vertices, assuming that they are all distinct from one another. A path represents connections between topics and, consequently, connections between arguments. Another essential

[1] As Michel Foucault keenly pointed out in [14], "*the frontiers of a book are never clear-cut (...) it is a node within a network*"; our semantic network of concepts follows this idea.

component is the notion of distance in a graph, which is closely related to paths (see [4,15,11] for a comprehensive analysis). With these elements, not only the connections but also the distance between arguments can be analyzed.

Definition 5 *(Distance between hashtags [6]).* The (geodesic) distance $\mathsf{d}_{\mathcal{G}} : \mathcal{H}\times\mathcal{H} \rightarrow \mathbb{N}^0 \cup \{\infty\}$ between two vertices $\alpha, \beta \in \mathcal{H}$, denoted $\mathsf{d}_{\mathcal{G}}(\alpha,\beta)$, is the number of edges in a shortest path connecting them; additionally, if there is no path between α and β, we say that $\mathsf{d}_{\mathcal{G}}(\alpha,\beta) = \infty$, where ∞, conventionally, represents the greatest possible distance. For all $\alpha, \beta, \gamma \in \mathcal{H}$, $\mathsf{d}_{\mathcal{G}}(\cdot,\cdot)$ satisfies the following conditions:

1) $\mathsf{d}_{\mathcal{G}}(\alpha,\beta) = 0$ iff $\alpha = \beta$ (*identity of indiscernible*),
2) $\mathsf{d}_{\mathcal{G}}(\alpha,\beta) = \mathsf{d}_{\mathcal{G}}(\beta,\alpha)$ (*symmetry*), and
3) $\mathsf{d}_{\mathcal{G}}(\alpha,\gamma) \leq \mathsf{d}_{\mathcal{G}}(\alpha,\beta) + \mathsf{d}_{\mathcal{G}}(\beta,\gamma)$ (*subadditivity or triangle inequality*).

The hashtag network is *independent of the argumentation graph*, as it merely captures concepts and their semantic relations. From the point of view of knowledge representation, the topological structure $\mathcal{G}$ implies that the closer the hashtags (vertices) are in the graph, the closer the topics they stand for are in the represented domain. Thus, given a pair of hashtagged arguments, a notion of proximity between these arguments can be induced by the *distance* existing between the referred topics in the hashtagged graph $\mathcal{G}$ [6].

Definition 6 *(Distance between hashtagged arguments [6]).* A distance function on $\mathtt{Args}$ is defined as $\mathsf{d}_\Omega : \mathtt{Args} \times \mathtt{Args} \rightarrow \mathbb{N}^0 \cup \{\infty\}$, where for all $\langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle, \langle \mathtt{B}, \mathcal{H}_\mathtt{B}\rangle, \langle \mathtt{C}, \mathcal{H}_\mathtt{C}\rangle \in \mathtt{Args}$, the following conditions are satisfied:

1) $\mathsf{d}_\Omega(\langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle, \langle \mathtt{B}, \mathcal{H}_\mathtt{B}\rangle) = 0$ iff $\mathcal{H}_\mathtt{A} = \mathcal{H}_\mathtt{B}$ (identity of indiscernible),
2) $\mathsf{d}_\Omega(\langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle, \langle \mathtt{B}, \mathcal{H}_\mathtt{B}\rangle) = \mathsf{d}_\Omega(\langle \mathtt{B}, \mathcal{H}_\mathtt{B}\rangle, \langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle)$ (symmetry), and
3) $\mathsf{d}_\Omega(\langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle, \langle \mathtt{C}, \mathcal{H}_\mathtt{C}\rangle) \leq \mathsf{d}_\Omega(\langle \mathtt{A}, \mathcal{H}_\mathtt{A}\rangle, \langle \mathtt{B}, \mathcal{H}_\mathtt{B}\rangle) + \mathsf{d}_\Omega(\langle \mathtt{B}, \mathcal{H}_\mathtt{B}\rangle, \langle \mathtt{C}, \mathcal{H}_\mathtt{C}\rangle)$ (triangle inequality).

This formalization of a set of hashtags as a graph, representing both the topics and their abstract connections, enables the examination of interesting semantic issues emerging from the abstract notion of *closeness or proximity*. The following definition provides a formal framework for hashtagged argumentation.

Definition 7 *(Hashtagged argumentation framework [6]).* A hashtagged argumentation framework Ω is defined as a 3-tuple $\langle \Phi, \mathcal{G}_\Omega, \mathsf{d}_\Omega\rangle$, where $\Phi = \langle \mathtt{Args}, \mathtt{Attacks}\rangle$ is an abstract argumentation framework in which $\mathtt{Args}$ is a set of hashtagged arguments and $\mathtt{Attacks}$ is a subset of $\mathtt{Args} \times \mathtt{Args}$ representing an attack relation defined on $\mathtt{Args}$, $\mathcal{G}_\Omega = \langle \mathcal{H}, E\rangle$ is a hashtag graph, and $\mathsf{d}_\Omega(\cdot,\cdot)$ is a distance function between hashtagged arguments.

Given this formalization, an admissibility semantics is defined in [6] that uses the distance as a measure of relevance for defenses in the argumentation process. This is called *proximity-based semantics*, as recalled next.

2.3. Proximity-based semantics

The use of hashtags as abstract notions related to arguments enables the evaluation of argument extensions according to a semantic distance between their elements. In particular, *proximity-based semantics* as defined in [6] applies the intuition that, for any argument, *a closer defender is preferred over a distant one*. Therefore, acceptability for hashtagged arguments considering a threshold τ of proximity is introduced. Under this interpretation of close defense, a potential defender that is beyond the threshold will not be considered as such.

Definition 8 *(Basic proximity-based semantics [6]).* Let $\Omega = \langle \Phi, \mathcal{G}_\Omega, \mathsf{d}_\Omega\rangle$ be a hashtagged framework, $\mathtt{S} \subseteq \mathtt{Args}$, and $\tau \in \mathbb{N}^0$ be a threshold. Then:

– A set $\mathtt{S}$ is said to be conflict free if there are no hashtagged arguments $\mathbb{A}, \mathbb{B} \in \mathtt{S}$ such that $\mathbb{B}$ attacks $\mathbb{A}$.
– A hashtagged argument $\mathbb{A} \in \mathtt{Args}$ is τ*-acceptable* with respect to $\mathtt{S}$ when for every argument $\mathbb{B} \in \mathtt{Args}$ that attacks $\mathbb{A}$ there is a hashtagged argument $\mathbb{C} \in \mathtt{S}$ such that $\mathbb{C}$ attacks $\mathbb{B}$ and $\mathsf{d}_\Omega(\mathbb{A}, \mathbb{C}) \leq \tau$.
– $\mathtt{S}$ is said to be τ*-admissible* if every hashtagged argument in $\mathtt{S}$ is τ*-acceptable* with respect to $\mathtt{S}$.
– A τ*-admissible* set $\mathtt{S}$ is a τ*-complete* extension iff $\mathtt{S}$ contains each argument that is τ*-acceptable* with respect to $\mathtt{S}$.
– A set $\mathtt{S}$ is the τ*-grounded* extension of Ω iff $\mathtt{S}$ is a $\subseteq$-minimal τ*-complete* extension.
– A set $\mathtt{S}$ is an τ*-preferred* extension of Ω iff $\mathtt{S}$ is a $\subseteq$-maximal τ*-complete* extension.

Proximity-based semantics mimic classical admissibility by restricting defenders according to its semantic distance.

Example 2. Consider the hashtagged argumentation framework $\Omega = \langle \Phi, \mathcal{G}_\Omega, \mathsf{d}_\Omega\rangle$, graphically represented in Fig. 3, where:

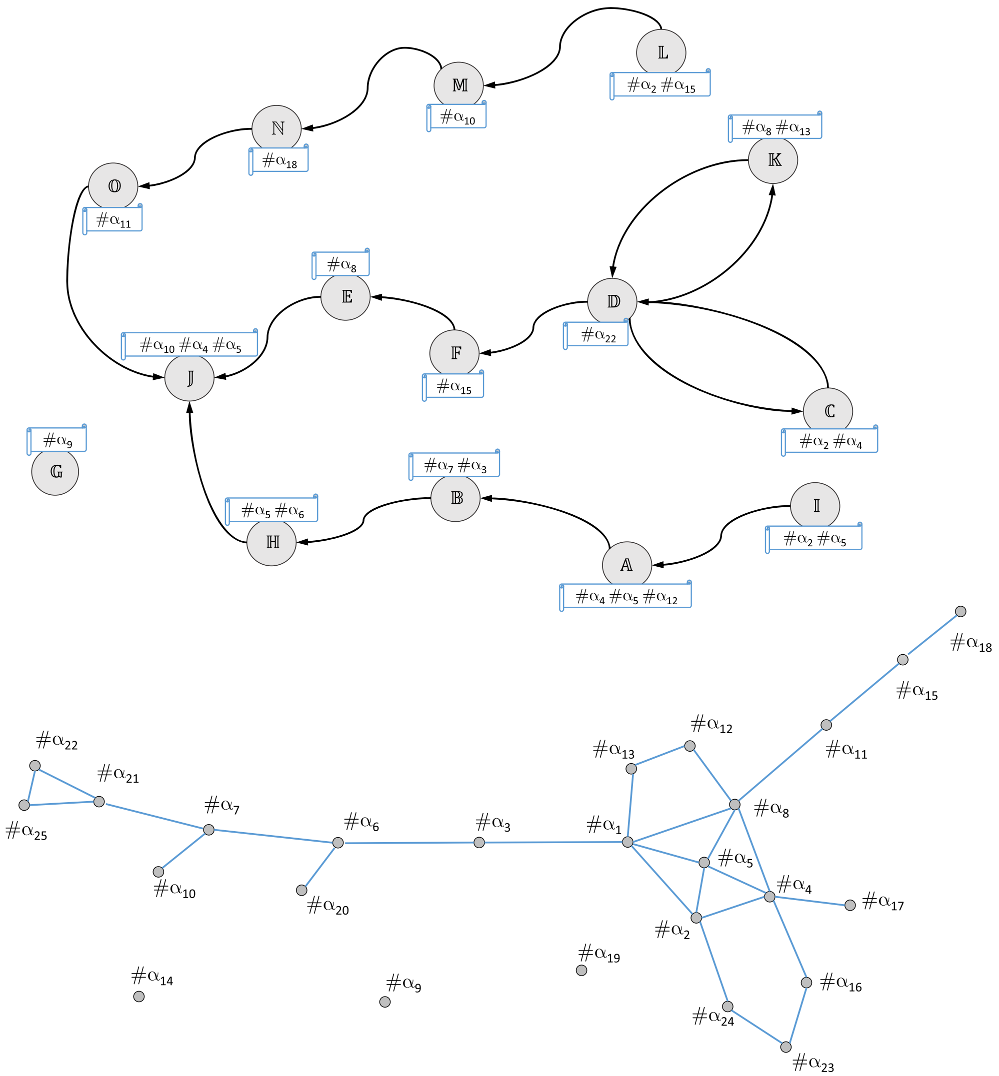


Fig. 3. Hashtagged argumentation framework and hashcloud for Ω.

$\mathcal{H} = \{\#\alpha_1, \#\alpha_2, \dots, \#\alpha_{25}\}$.
$E = \{(\#\alpha_1,\#\alpha_2),(\#\alpha_1,\#\alpha_8),(\#\alpha_1,\#\alpha_5),(\#\alpha_1,\#\alpha_{13}),(\#\alpha_2,\#\alpha_4),(\#\alpha_2,\#\alpha_5),(\#\alpha_2,\#\alpha_{24}),(\#\alpha_3,\#\alpha_6),(\#\alpha_4,\#\alpha_5),(\#\alpha_4,\#\alpha_8),(\#\alpha_4,\#\alpha_{16}),(\#\alpha_4,\#\alpha_{17}),$
$(\#\alpha_5,\#\alpha_8),(\#\alpha_6,\#\alpha_7),(\#\alpha_6,\#\alpha_{20}),(\#\alpha_7,\#\alpha_{21}),(\#\alpha_7,\#\alpha_{10}),(\#\alpha_8,\#\alpha_{12}),(\#\alpha_8,\#\alpha_{11}),(\#\alpha_{11},\#\alpha_{15}),(\#\alpha_{12},\#\alpha_{13}),(\#\alpha_{15},\#\alpha_{18}),(\#\alpha_{16},\#\alpha_{23}),$
$(\#\alpha_{21},\#\alpha_{25}),(\#\alpha_{21},\#\alpha_{22}),(\#\alpha_{22},\#\alpha_{25}),(\#\alpha_{23},\#\alpha_{24}),$
$\texttt{Args} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{I}, \mathbb{J}, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{N}, \mathbb{O}\}$.
$\texttt{Attacks} = \{(\mathbb{A},\mathbb{B}),(\mathbb{B},\mathbb{H}),(\mathbb{C},\mathbb{D}),(\mathbb{D},\mathbb{C}),(\mathbb{D},\mathbb{F}),(\mathbb{E},\mathbb{J}),(\mathbb{F},\mathbb{E}),(\mathbb{H},\mathbb{J}),(\mathbb{I},\mathbb{A}),(\mathbb{K},\mathbb{D}),(\mathbb{D},\mathbb{K}),(\mathbb{L},\mathbb{M}),(\mathbb{M},\mathbb{N}),(\mathbb{N},\mathbb{O})\}$

Then, consider the following non-intersection distance:

$$\mathrm{d}_{\Omega}(\mathbb{A},\mathbb{B}) = \begin{cases} 0 & \textit{If } \mathcal{H}_{\mathbb{A}} = \mathcal{H}_{\mathbb{B}}, \\ \infty & \textit{if there exist } \alpha \in \mathcal{H}_{\mathbb{A}} \textit{ and } \beta \in \mathcal{H}_{\mathbb{B}}, \\ & \textit{such that there is no path between them, and} \\ max(\mathrm{d}_{\mathcal{G}}(\alpha,\beta)) & \textit{otherwise.} \end{cases}$$

Table 1 shows the distances between arguments, where "–" means that the distance is ∞. Considering a threshold $\tau = 4$, we obtain the following extensions. The set $\mathrm{S}_1 = \{\mathbb{I}, \mathbb{B}, \mathbb{G}, \mathbb{L}\}$ is τ-*complete* extension since it contains all the arguments that are defended by S_1 (see Fig. 4 green marks). Furthermore, S_1 is a minimal set satisfying the previous conditions, and therefore it is the τ-*grounded* extension of Ω. Note that arguments $\mathbb{G}$, $\mathbb{L}$, and $\mathbb{I}$ are free of attackers, while $\mathbb{I}$ is a proper defender for $\mathbb{B}$ since the distance between these arguments is 4, and $4 \leq \tau$. However, $\mathbb{B}$ is not a proper defender for argument $\mathbb{J}$, because $\mathrm{d}_{\Omega}(\mathbb{B},\mathbb{J}) = 5$ (greater than τ, the same

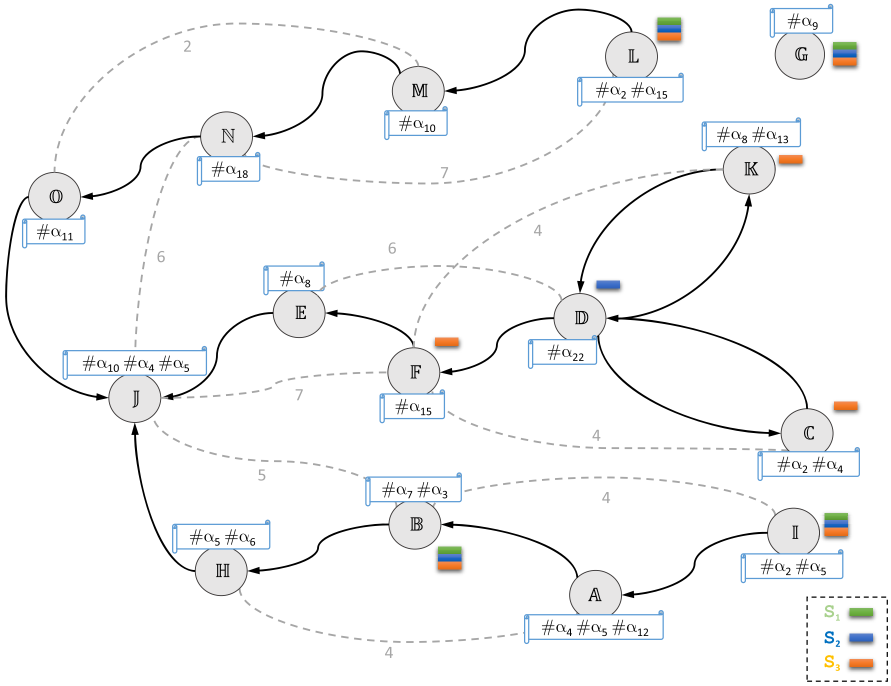


Fig. 4. Extensions of Ω from the running example (colors denote different extensions).

Table 1
Distances between the hashtagged arguments in Ω from the running example.

	$\mathbb{A}$	$\mathbb{B}$	$\mathbb{C}$	$\mathbb{D}$	$\mathbb{E}$	$\mathbb{F}$	$\mathbb{G}$	$\mathbb{H}$	$\mathbb{I}$	$\mathbb{J}$	$\mathbb{K}$	$\mathbb{L}$	$\mathbb{M}$	$\mathbb{N}$	$\mathbb{O}$
$\mathbb{A}$	0	5	3	7	1	3	-	4	3	6	3	3	4	6	2
$\mathbb{B}$	5	0	5	4	4	6	-	4	4	5	4	6	3	7	5
$\mathbb{C}$	3	5	0	7	2	4	-	4	1	6	3	4	5	6	3
$\mathbb{D}$	7	4	7	0	6	8	-	6	6	7	6	8	9	3	7
$\mathbb{E}$	1	4	2	6	0	2	-	3	2	5	2	2	3	5	1
$\mathbb{F}$	3	6	4	8	2	0	-	5	4	7	4	4	1	7	1
$\mathbb{G}$	-	-	-	-	-	-	0	-	-	-	-	-	-	-	-
$\mathbb{H}$	4	4	4	6	3	5	-	0	3	5	3	5	5	4	4
$\mathbb{I}$	3	4	1	6	2	4	-	3	0	5	2	4	5	3	3
$\mathbb{J}$	6	5	6	7	5	7	-	5	5	0	5	7	8	6	6
$\mathbb{K}$	3	4	3	6	2	4	-	3	2	5	0	4	5	5	3
$\mathbb{L}$	3	6	4	8	2	4	-	5	4	7	4	0	5	7	3
$\mathbb{M}$	4	3	5	9	3	1	-	5	5	8	5	5	0	8	2
$\mathbb{N}$	6	7	6	3	5	7	-	4	5	6	5	7	8	0	6
$\mathbb{O}$	2	5	3	7	1	1	-	4	3	6	3	3	2	6	0

analysis can be made for the failed defense between $\mathbb{L}$ and $\mathbb{N}$). On the other hand, the even attack cycle between the arguments $\mathbb{D}$ and $\mathbb{C}$, and $\mathbb{K}$ and $\mathbb{D}$, limit the acceptance of other arguments in the discussion, under a skeptical position. The set $S_2 = \{\mathbb{I}, \mathbb{B}, \mathbb{D}, \mathbb{G}, \mathbb{L}\}$ (see Fig. 4 blue marks) and $S_3 = \{\mathbb{I}, \mathbb{B}, \mathbb{C}, \mathbb{K}, \mathbb{F}, \mathbb{G}, \mathbb{L}\}$ (see Fig. 4 orange marks) are *τ-admissible* and *τ-complete* since they contain all the arguments that are defended by S_2 and S_3, respectively. In particular, in S_2, $\mathbb{C}$ and $\mathbb{K}$ are proper defenders of $\mathbb{F}$ (this is because $d_\Omega(\mathbb{C}, \mathbb{F}) = 4 = \tau$ and $d_\Omega(\mathbb{K}, \mathbb{F}) = \tau = 4$). However, $\mathbb{D}$ is not a proper defender argument for $\mathbb{E}$ because $d_\Omega(\mathbb{D}, \mathbb{E}) = 6$ (greater than τ), while $\mathbb{L}$ is not a proper defender argument for $\mathbb{N}$, because $d_\Omega(\mathbb{L}, \mathbb{N}) = 7$ (also greater than τ). Finally, S_2 and S_3 are the maximal sets verifying the previous conditions; thus, they are both *τ-preferred* extensions of Ω.

It is worth mentioning that in [6], a single threshold is fixed in the semantic analysis, which means that the same defense-enabling distance is applied to each argument. As stated in the motivating discussion in the introduction, a more general approach is to establish an individual threshold for *every* argument; thus, for a given argument, the range of defenses available becomes an essential attribute. This move requires a more detailed analysis of the hashcloud and its semantic elaborations; we address these issues in the rest of this work.

3. Topological analysis: unveiling argument properties derived from the hashcloud

A hashcloud, serving as a visual representation of topic relations, plays a central role in exploring not only various argument distances but also the topological properties of arguments derived from the hashtags that they possess. At the heart of this inquiry are essential topological metrics that shed light on the complex relationships among hashtags, influencing the topological characteristics associated with arguments.

Within the realm of hashcloud analysis, we will investigate two categories of metrics that will be useful in analyzing the structure and dynamics of interconnection among hashtags: local metrics and global metrics. Local metrics, at their core, focus on the individuality of hashtags. Hashtag degree, a fundamental metric, counts the number of direct connections a specific hashtag has. In a similar vein, the clustering coefficient is an indicator of the tendency of similar hashtags to form clusters, and thus measures thematic cohesion within the hashcloud. Expanding our focus to global metrics assumes a higher-level perspective. The hashcloud diameter emerges as a pivotal metric, shedding light on the maximum distance between two hashtags, providing insights into the semantic breadth of the hashcloud. Betweenness centrality measures the significance of specific hashtags as essential connectors between hashtags, serving as bridges between elements that might appear, at first glance, distant.

This combined analysis of local and global metrics within the context of hashclouds yields a more nuanced understanding of hashcloud topology and the nature of relationships among hashtags. In essence, these measures offer a better understanding of the hierarchical importance and interconnected influence of elements within the hashcloud [15]. Furthermore, it is worth noting that such hashcloud properties can be translated into argument properties, as will be explored in the following.

3.1. Characterizing a hashcloud

As described above, local metrics result in local properties that enable the analysis of hashtags individually, revealing their intrinsic characteristics and how they relate to neighboring elements. This is fundamental for understanding the specific features of each concept represented by a hashtag; furthermore, local metrics assist in identifying specific patterns and relationships in the proximity of each hashtag. This is critical for discovering more precise semantic connections and understanding how close interactions influence the interpretation of a particular hashtag. To achieve this, we assign m local metrics to each hashtag α that is part of hashcloud $\mathcal{G}_\Omega$. In practical terms, this involves applying an operator across $\mathcal{G}_\Omega$, resulting in a $k \times m$-matrix, where k represents the total number of hashtags in $\mathcal{G}_\Omega$ and m is the number of metrics applied to $\mathcal{G}_\Omega$. This matrix provides a comprehensive characterization of each hashtag, offering insights from m distinct perspectives.

Definition 9 *(Local topological metrics associated with a hashcloud).* Let $\mathcal{G}_\Omega = [\mathcal{H}, E]$ be a hashcloud, and $\{\alpha_1, \alpha_2, \dots, \alpha_k\} = \mathcal{H}$. A local topological metric associated with $\mathcal{G}_\Omega$, denoted $\mathcal{M}^{\downarrow}_{\Omega}(\mathcal{G}_\Omega)$, is defined as:

$$\mathcal{M}^{\downarrow}_{\Omega}(\mathcal{G}_\Omega) = \begin{bmatrix} m_1(\alpha_1) & m_1(\alpha_2) & \dots & m_1(\alpha_k) \\ m_2(\alpha_1) & m_2(\alpha_2) & \dots & m_2(\alpha_k) \\ \vdots & \vdots & \dots & \vdots \\ m_m(\alpha_1) & m_m(\alpha_2) & \dots & m_m(\alpha_k) \end{bmatrix},$$

where $m_i(.)$ are local metric functions applied to $\mathcal{H}$ with $1 \leq i \leq m$.

Next, we study the hashcloud $\mathcal{G}_\Omega$ as a whole, but considering t global metrics that give us different perspectives on $\mathcal{G}_\Omega$. This operator thus characterizes $\mathcal{G}_\Omega$ via a column vector of t elements, where each element gives us information related to the type of distribution and the type of relations existing in $\mathcal{G}_\Omega$.

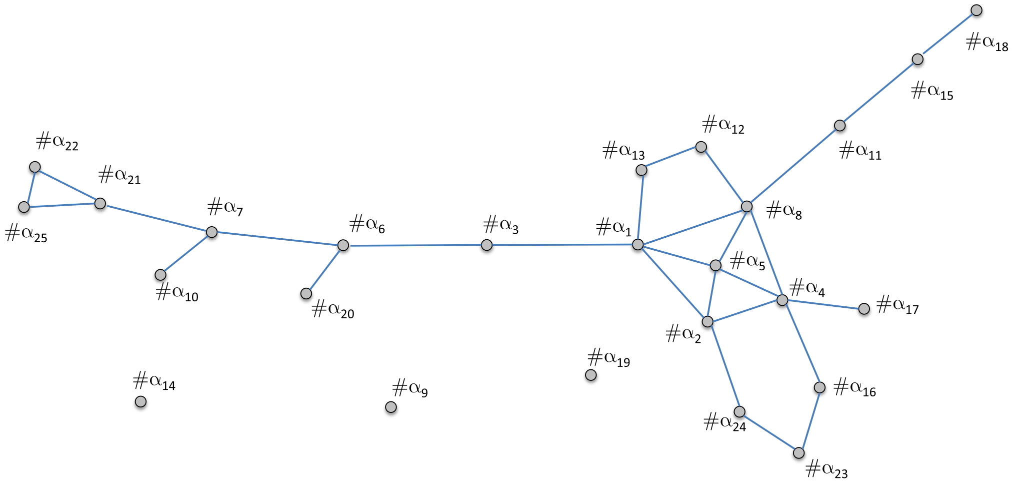
Definition 10 *(Global topological metrics assoc. with a hashcloud).* Let $\mathcal{G}_\Omega = [\mathcal{H}, E]$ be a hashcloud. A global topological metric associated with $\mathcal{G}_\Omega$, denoted $\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_\Omega)$, is defined:

$$\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_\Omega) = \begin{bmatrix} M_1(\mathcal{G}_\Omega) \\ M_2(\mathcal{G}_\Omega) \\ \vdots \\ M_t(\mathcal{G}_\Omega) \end{bmatrix},$$

where $M_i(.)$ are global metric functions applied to $\mathcal{G}_\Omega$ with $1 \leq i \leq t$.

In a practical case, we may wish to have a series of measurements associated with our hashcloud in order to analyze the scenario with different instruments. In the following example, we analyze the hashcloud proposed in the prior section.

Example 3. In this example, we employ the *Degree centrality* (see Equation (1)) and *Mean Neighbor Degree* (see Equation (2)) metrics to underscore the significance and impact of various topics, respectively. Specifically, Degree centrality is a metric indicating the centrality of a hashtag in $\mathcal{G}$, defined as the number of edges connected to the node—in other words, it gauges the influence of other hashtags within its immediate neighborhood. Conversely, Mean Neighbor Degree is the average degree of the neighbors of hashtag



Centrality degree

#α1	#α2	#α3	#α4	#α5	#α6	#α7	#α8	#α9	#α10	#α11	#α12	#α13	#α14	#α15	#α16	#α17	#α18	#α19	#α20	#α21	#α22	#α23	#α24	#α25
5	4	2	5	4	3	3	5	0	1	2	2	2	0	2	2	1	1	0	1	3	2	2	2	2

Neighbor degree

#α1	#α2	#α3	#α4	#α5	#α6	#α7	#α8	#α9	#α10	#α11	#α12	#α13	#α14	#α15	#α16	#α17	#α18	#α19	#α20	#α21	#α22	#α23	#α24	#α25
17/5	4	4	16/5	19/4	2	7/3	8/5	0	3	7/2	7/2	7/2	0	3/2	7/2	5	2	0	3	7/3	5/2	2	3	5/2

Fig. 5. Centrality and neighbor degrees for hashcloud $\mathcal{G}$.

$\#\alpha$, calculated as the mean of the degrees of all neighbors connected to $\#\alpha$. In Fig. 5 we present the values of these metrics for each hashtag in $\mathcal{G}$. In the following, we define:

$$InOut(\#\alpha) = \{\#\beta \in \mathcal{H} \mid (\#\beta, \#\alpha) \in E \text{ or } (\#\alpha, \#\beta) \in E\}.$$

$$D_{Centrality}(\#\alpha) = |\, InOut(\#\alpha)\,| \tag{1}$$

$$D_{Neighbor}(\#\alpha) = \frac{1}{D_{Centrality}(\#\alpha)} \sum_{i=1}^{n} D_{Centrality}(\beta_i), \text{ where } \beta_i \in InOut(\#\alpha) \tag{2}$$

On the other hand, we use the *Radius* and *Diameter* metrics (Equations (4) and (3), respectively) to identify how dispersed the discussion is. The *radius* of $\mathcal{G}$ is the minimum eccentricity among all hashtags in $\mathcal{G}$, while the *diameter* is the maximum eccentricity among all hashtags.[2] To find the diameter of a graph, we first find the shortest path between each pair of vertices; the greatest length of any of these paths is then the diameter.

$$Radius(\mathcal{G}) = \min_{\alpha \in \mathcal{G}} eccentricity(\alpha), \text{ where } eccentricity(\alpha) = \max_{\beta \in \mathcal{G}} \mathrm{d}_{\mathcal{G}}(\alpha, \beta) \tag{3}$$

$$Diameter(\mathcal{G}) = \max_{\alpha \in \mathcal{I}} eccentricity(\alpha) \tag{4}$$

Under these metrics, certain hashtags, namely $\#\alpha_1$, $\#\alpha_2$, $\#\alpha_8$, $\#\alpha_5$, and $\#\alpha_4$, emerge as pivotal within the ongoing discussion, representing the focal points where the most central issues converge within the hashcloud. In contrast, there are topics occupying a more peripheral role, exemplified by $\#\alpha_{18}$, $\#\alpha_{20}$, and $\#\alpha_{10}$. Additionally, there are topics that appear interconnected, potentially sparking a new line of discussion, such as $\#\alpha_{21}$, $\#\alpha_6$, and $\#\alpha_7$. Conversely, some topics remain entirely disconnected from the current discourse, as observed in $\#\alpha_{14}$, $\#\alpha_9$, and $\#\alpha_{19}$. This prominence extends beyond individual degree centrality, manifesting in the conceivable formation of cohesive thematic clusters. Such clusters not only underscore the significance of these central hashtags, but also contribute to thematic consistency within associated arguments. This interconnectedness facilitates engagement and holds the potential to propel these arguments into trending discussions, thereby amplifying their impact and relevance within the broader discourse.

The radius and diameter for $\mathcal{G}$ are infinite, signaling a highly dispersed discussion where certain issues lack connections with others in the cloud. Essentially, the global metrics are influenced by topics that are entirely unrelated to other topics within the cloud. If we exclude these isolated topics from the analysis, the diameter becomes 9 and the radius 5, indicating a more cohesive

[2] The eccentricity associated with a graph is defined as $eccentricity(\alpha) = \max_{\beta \in \mathcal{G}} \mathrm{d}_{\mathcal{G}}(\alpha, \beta)$. Furthermore, if $\alpha \in \mathcal{G}$ is not connected to any other hashtag in $\mathcal{G}$, its eccentricity is associated with the constant ∞, which, conventionally, represents the greatest possible distance.

argumentative landscape. In this scenario, the discussion is not very dispersed, emphasizing the significance of considering the interconnectedness of topics to obtain a more accurate representation of the argumentative structure.

We will now focus on how these tools can be applied to determine the topological properties associated with the arguments based on their topics. Note that metrics can be defined in different ranges or number spaces; therefore, it is necessary to normalize them to work on a single numerical measure. Next, we define how the metrics used on hashtags are translated to the arguments, representing, in this sense, specific topological properties. Thus, based on the arguments' topics, we compose the corresponding degrees to calculate the topological property associated with an argument.

3.2. Argument properties: obtaining the neighborhood of arguments

As we mentioned before, every argument abstracting a unit of reasoning addresses certain specific topics and every topic is naturally associated with many others in varied degrees of "closeness". Consequently, there is an underlying perception of *distance* also between arguments when considering their topics. In a mathematical sense, a *topological space* may be described as a set of points along with a set of *neighborhoods* for each point. Any metric space will also be a topological space because, given a set, any properly specified distance function defined on it induces a topology on that set. The pair $(\mathtt{Args}, \mathsf{d}_\Omega)$ associated with a Hashtagged Framework Ω can be regarded as a *metric space* in the topology sense, where a distance may be defined.

In this paper we are especially interested in neighborhoods. Intuitively speaking, a *neighborhood of a point p* is a set of points containing p and the points that can be reached within a given distance from p. It is not unique, since a point p may have several neighborhoods of different sizes by considering different distances. In Topology, a *ball* is the space bounded by a sphere—it may be a *closed* ball, including the boundary points of the sphere, or an *open* ball by excluding them. Thus, a neighborhood associated with a point $x \in S$ with radius ε_x is the closed ball defined as

$$B(x;\varepsilon_x) = \{y \in S : distance(x, y) \le \varepsilon_x\}.$$

It is important to remark that the property of being a "neighbor" is relative to an individual point since the threshold distance is not necessarily the same for every point.

Notation: Although both are related to distances, in order to highlight the contextual difference we will use τ to denote a general threshold and ε a local threshold for an argument in some semantic consideration.

Definition 11 *(Argument neighborhood [13]).* Let $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathsf{d}_\Omega \rangle$ be a hashtagged framework. Then, the neighborhood of an argument $\mathbb{A} \in \mathtt{Args}$ with radius $\varepsilon_{\mathbb{A}} \in \mathbb{N}^0$ under the metric $(\mathtt{Args}, \mathsf{d}_\Omega)$ is defined as the set

$$\mathfrak{N}_{\mathbb{A}}^{\varepsilon_{\mathbb{A}}} = \{\mathbb{X} \in \mathtt{Args} : \mathsf{d}_\Omega(\mathbb{A}, \mathbb{X}) \le \varepsilon_{\mathbb{A}}\}.$$

The set of neighborhoods associated with arguments in Ω will be denoted $\mathfrak{N}_\Omega$.

A neighborhood is defined by a threshold based on a measure of distance, as discussed above; since several notions of distance could be used, we will focus on those influenced by the topics referred to by the arguments. Our primary purpose is to only allow defenses for an argument that are *close enough* to the topics represented by the set of hashtags associated with this argument. Since these hashtags may be closely related or widely dispersed, we need a measure of argument semantic coverage that will provide a reference regarding its centrality property (always from the semantic point of view). Next, we formalize two notions based on hashcloud characterization, local and global properties associated with arguments involved in a specific discussion.

First, to obtain the local properties associated with an argument $\mathtt{A}$, we need to look at $\mathcal{H}_{\mathtt{A}}$ and analyze the local topological metrics assigned to each of its elements. As we formalized previously (cf. Definition 9), each hashtag α is characterized by m local metrics. To determine a specific local topological property associated with $\mathtt{A}$, with respect to a local topological metric m, we need to combine the corresponding value assigned to each element of $\mathcal{H}_{\mathtt{A}}$ with respect to m. That is, performing this process for each $\alpha \in \mathcal{H}_{\mathtt{A}}$ considering each metric m computed in $\mathcal{M}_\Omega^{\downarrow}(\mathscr{G}_\Omega)$, we will obtain the local metric property associated with $\mathtt{A}$.

Definition 12 *(Local topological metrics for an argument).* Let $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathsf{d}_\Omega \rangle$ be a hashtagged framework, $\mathbb{A} \in \mathtt{Args}$ be a hashtagged argument, $\{\alpha_1, \alpha_2, \dots, \alpha_l\} = \mathcal{H}_{\mathtt{A}}$ be the set of hashtags associated with $\mathbb{A}$, and $m_i(.)$ be local metric functions applied to $\mathcal{H}$ with $1 \le i \le m$. We will define the local topological property for $\mathbb{A} = \langle \mathtt{A}, \mathcal{H}_{\mathtt{A}} \rangle$, associated with the local topological metric $\mathcal{M}_\Omega^{\downarrow}(\mathscr{G}_\Omega)$, as:

$$\mathcal{F}^{\downarrow\Omega}(\mathcal{M}_\Omega^{\downarrow}(\mathbb{A})) = P_{\mathbb{A}}^{\downarrow\Omega}$$

where $\mathcal{F}^{\downarrow\Omega}(\mathcal{M}_\Omega^{\downarrow}(\mathbb{A}))$ is an abstract function $\mathbb{R}^{m\times l} \to \mathbb{R}$ applied over the matrix $\mathcal{M}_\Omega^{\downarrow}(\mathbb{A})$, and $P_{\mathbb{A}}^{\downarrow\Omega} \in \mathbb{R}$ is the local metric value associated with $\mathbb{A}$.

As mentioned in Example 3, the global clustering coefficient, radius, and diameter are global measures applicable to a specific hashcloud. However, each argument encompasses a subcloud of hashtags formed by the minimal graph containing all the hashtags associated with the argument. That is, the subcloud associated with an argument $\mathtt{A}$, denoted $\mathscr{G}_{\mathbb{A}}$, is composed of all elements that

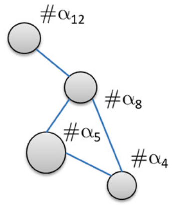


Fig. 6. Subgraph corresponding to the hashtags of argument $\mathbb{A}$ in the running example.

are found along the shortest paths between the hashtags belonging to $\mathcal{H}_{\mathtt{A}}$. Thus, we need to perform the analysis described in Definition 10 over $\mathcal{G}_{\mathbb{A}}$ to obtain the global topological properties for $\mathtt{A}$, as follows:

Definition 13 *(Global topological metrics for an argument).* Let $\mathcal{G}_\Omega = [\mathcal{H}, E]$ be a hashcloud, and $d_{\mathcal{G}} : \mathcal{H} \times \mathcal{H} \to \mathbb{N}^0 \cup \{\infty\}$ a geodesic distance, $\mathbb{A} \subseteq \mathtt{Args}$ be a hashtag argument, $\mathcal{G}_{\mathbb{A}}$ the subgraph generated by $\mathcal{H}_{\mathtt{A}}$, and $M_i(.)$ global metric functions applied to $\mathcal{G}_\Omega$ with $1 \le i \le t$. We will define the global topological property for $\mathbb{A}$, associated with the general topological measure $\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_\Omega)$, as:

$$\mathcal{F}^{\uparrow\Omega}(\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_{\mathbb{A}})) = P^{\uparrow\Omega}_{\mathbb{A}}$$

where $\mathcal{F}^{\uparrow\Omega}(\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_{\mathbb{A}}))$ is an abstract function $R^{t\times 1} \to R$ applied over the vector $\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_{\mathbb{A}})$ and $P^{\uparrow\Omega}_{\mathbb{A}} \in R$ is the global metric value associated with $\mathbb{A}$.

Local measures computed for an argument are based on the valuations obtained on the hashcloud, while the global ones are analyzed from the cloud associated with the argument. The main reason is that we use a local level to gauge the importance of the participating hashtags throughout the domain, identifying what role they play in the discussion, while global measures give us information on dispersion of the cloud associated with the argument itself.

Next, we determine the neighborhood threshold for an argument. The intuition is that an argument threshold is defined by the topics that it addresses, but without neglecting the relationship of these topics in the hashcloud, which reflect the semantics of the specific domain. The following definition combines the topological properties to explore this intuition.

Definition 14 *(Neighborhoods threshold for an argument).* Let $\Omega = \langle \Phi, \mathcal{G}_\Omega, \mathrm{d}_\Omega \rangle$ be a hashtagged framework, $\mathbb{A} \subseteq \mathtt{Args}$ be a hashtagged argument, $\mathcal{G}_{\mathbb{A}}$ be the subgraph generated by $\mathcal{H}_{\mathtt{A}}$, and $\mathcal{M}^{\downarrow}_{\Omega}(\mathbb{A})$ and $\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_{\mathbb{A}})$ be the local and global properties associated with $\mathbb{A}$ respectively. We will define the neighborhoods threshold for $\mathbb{A}$, denoted $\varepsilon_{\mathbb{A}}$, as

$$\varepsilon_{\mathbb{A}} = P^{\downarrow\Omega}_{\mathbb{A}} \otimes P^{\uparrow\Omega}_{\mathbb{A}}$$

where $\otimes$ must satisfy the commutativity, associativity, and monotonicity (monotonically increasing) properties. We will denote with $\mathfrak{N}^{\varepsilon\Omega}_{\Omega} = \{\mathfrak{N}^{\varepsilon_{\mathbb{X}}}_{\mathbb{X}} : \mathbb{X} \in \mathtt{Args}\}$ the set of neighborhoods associated with the arguments of Ω.

In practice, the neighborhood threshold is obtained as the conjunction of the local and global topological measures associated with an argument; the $\otimes$ operator is used to combine the local and global argument property.

This definition provides a helpful characterization of the *influence of an argument* in the framework. The most natural way of doing this would be to directly add these, considering the conditions imposed by the domain. This is why the $\otimes$ operation is required to satisfy some of the same properties as addition of real numbers—it is commutative and associative. Furthermore, it must satisfy monotonicity to ensure that the neighborhoods threshold of an argument does not decrease if the topological properties associated with such argument increase. It is important to remark that this influence is evaluated according to the topics and not the underlying linguistic structure, which is not relevant here given our abstract approach. Topics are therefore our formal clues of what an argument is about.

Example 4. We now analyze hashcloud $\mathcal{G}_\Omega$ corresponding to the hashtagged argumentation framework Ω represented in Fig. 3. We wish to determine the neighborhood of the hashtagged arguments in order to allow only those defenses that are close enough to the topics represented by the set of hashtags associated with each argument. Thus, one possible characterization is to take the centrality degree and Neighbor Degree as local topological measures (represented in Fig. 5); and radius and diameter as general topological measures.

Then, for the hashtagged argument $\mathbb{A} = \langle \mathcal{A}, \mathcal{H}_{\mathcal{A}} \rangle$ where $\mathcal{H}_{\mathcal{A}} = \{\#\alpha_4, \#\alpha_5, \#\alpha_{12}\}$, we obtain the subcloud derived from $\mathcal{H}_{\mathcal{A}}$ considering those hashtags that are found along the shortest paths between $\#\alpha_{12}$, $\#\alpha_4$, and $\#\alpha_5$. Here, for argument $\mathbb{A}$ the hashtag $\#\alpha_8$ plays the semantic role of connecting the hashtags $\#\alpha_{12}$ with $\#\alpha_4$ and $\#\alpha_5$, which belong to $\mathcal{H}_{\mathcal{A}}$. That is, $\#\alpha_8$ is present along the shortest path between the hashtags $\#\alpha_{12}$ and $\#\alpha_4$, and the hashtags $\#\alpha_{12}$ and $\#\alpha_5$, as shown in Fig. 6.

So, to analyze the local topological measures we consider the centrality and eigenvector degrees obtained from the previous analysis. Then, applying Definition 12, the local topological measure is obtained as follows:

$$\mathcal{M}^{\downarrow}_{\Omega}(\mathbb{A}) = \begin{bmatrix} m_1(\alpha_4) & m_1(\alpha_5) & m_1(\alpha_{12}) \\ m_2(\alpha_4) & m_2(\alpha_5) & m_2(\alpha_{12}) \end{bmatrix} = \begin{bmatrix} 5 & 4 & 2 \\ 16/5 & 19/4 & 7/2 \end{bmatrix},$$

where $m_1(.)$ is the degree centrality metric and $m_2(.)$ is the neighbor degree metric that are applied to each element of $\mathcal{H}_{\mathbb{A}}$ considering the entire hashcloud. Thus, we obtain a matrix $R^{2\times 3}$, which is the input for function $\mathcal{F}^{\downarrow\Omega}(\mathcal{M}^{\downarrow}_{\Omega}(\mathbb{A}))$, where $\mathcal{F}^{\downarrow\Omega}$ is instantiated with the following function:

$$\mathcal{F}^{\downarrow\Omega}(\mathcal{M}^{\downarrow}_{\Omega}(\mathbb{A})) = \max_{i=1\ldots 2} \min_{j=1\ldots 3}(m_{ij}(\alpha)).$$

Then, we obtain that:

$$\boldsymbol{P}^{\downarrow\Omega}_{\mathbb{A}} = 3.2 \approx 3.$$

Now, in order to perform a global topological analysis, we consider the radius and diameter measures obtained from the subgraph corresponding to the argument involved ($\mathbb{A}$). Then, applying Definition 13, the global topological measure is obtained as follows:

$$\mathcal{M}^{\uparrow}_{\Omega}(\mathcal{G}_{\mathbb{A}}) = \begin{bmatrix} \boldsymbol{M}_1(\mathcal{G}_{\mathbb{A}}) \\ \boldsymbol{M}_2(\mathcal{G}_{\mathbb{A}}) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

where $\boldsymbol{M}_1(.)$ is the radius and $\boldsymbol{M}_2(.)$ is the diameter applied to $\mathcal{G}_{\mathbb{A}}$, representing the dispersion associated with the minimal subgraph that connects the hashtags that belong to $\mathcal{H}_{\mathbb{A}}$. Thus, we obtain a vector $R^{2\times 1}$, which is the input for function $\mathcal{F}^{\uparrow\Omega}(\mathcal{M}^{\uparrow}_{\Omega}(\mathbb{A}))$, where $\mathcal{F}^{\uparrow\Omega}$ is instantiated with the following function:

$$\mathcal{F}^{\uparrow\Omega}(\mathcal{M}^{\uparrow}_{\Omega}(\mathbb{A})) = \min_{i=1\ldots 2}(\boldsymbol{M}_i(\mathcal{G}_{\mathbb{A}})).$$

Then, we obtain:

$$\boldsymbol{P}^{\uparrow\Omega}_{\mathbb{A}} = 2.$$

Finally, in order to define the neighborhood for $\mathbb{A}$, we calculate the threshold composing the local and global property established in Definition 14 as follows:

$$\varepsilon_{\mathbb{A}} = \boldsymbol{P}^{\downarrow\Omega}_{\mathbb{A}} \otimes \boldsymbol{P}^{\uparrow\Omega}_{\mathbb{A}},$$

where $\otimes$ is instantiated as:

$$\frac{\boldsymbol{P}^{\downarrow\Omega}_{\mathbb{A}} + \boldsymbol{P}^{\uparrow\Omega}_{\mathbb{A}}}{2}$$

Thus, the neighborhood threshold for $\mathbb{A}$ is $\varepsilon_{\mathbb{A}} = \frac{3+2}{2} = \frac{5}{2} = 2.5 \approx 3$, yielding the neighborhood $\{\mathbb{A}, \mathbb{C}, \mathbb{E}, \mathbb{F}, \mathbb{I}, \mathbb{K}, \mathbb{L}, \mathbb{O}\}$.

Naturally, the same model can be applied to every argument that participates in the discussion, as shown in Table 2. The whole argumentation scenario is depicted in Fig. 7.

Once the neighborhoods of arguments have been defined, it is necessary to establish certain bounds that will allow us to find links between the classical argumentation semantics and earlier proximity-based extensions.

Definition 15 *(Metric associated with neighborhoods [13]).* Consider a hashtagged framework $\Omega = \langle \Phi, \mathcal{G}_{\Omega}, \mathsf{d}_{\Omega} \rangle$, let $\mathsf{d}_{\Omega}(\cdot,\cdot)$ be a distance function over the set `Args`, $(\texttt{Args}, \mathsf{d}_{\Omega})$ be the metric space associated with the tagged framework Ω, and $\mathfrak{N}^{\varepsilon_\Omega}_{\Omega}$ be the set of neighborhoods associated with the arguments of Ω. Then:

- A neighborhood $\mathfrak{N}^{\varepsilon_{\mathbb{A}}}_{\mathbb{A}} \in \mathfrak{N}_{\Omega}$ is the *Greatest Neighborhood* iff there is no $\mathfrak{N}^{\varepsilon_{\mathbb{B}}}_{\mathbb{B}} \in \mathfrak{N}_{\Omega}$ such that $\varepsilon_{\mathbb{B}} > \varepsilon_{\mathbb{A}}$. We will use $\mathcal{T}^{\mathsf{g}}_{\Omega}$ to denote the greatest radius associated with the greatest neighborhood of $\mathfrak{N}_{\Omega}$.
- A neighborhood $\mathfrak{N}^{\varepsilon_{\mathbb{A}}}_{\mathbb{A}} \in \mathfrak{N}_{\Omega}$ is the *Smallest Neighborhood* iff there is no $\mathfrak{N}^{\varepsilon_{\mathbb{B}}}_{\mathbb{B}} \in \mathfrak{N}_{\Omega}$ such that $\varepsilon_{\mathbb{B}} \leq \varepsilon_{\mathbb{A}}$. We will use $\mathcal{T}^{\mathsf{s}}_{\Omega}$ to denote the smallest radius associated with the smallest neighborhood of $\mathfrak{N}_{\Omega}$.

In the next section, we analyze the proximity-based semantics as in [6] with this new context of individual thresholds. We consider first the classical proximity approach where restrictions are applied only to defenses.

4. Neighborhood-bounded admissibility

Since an argument $\mathbb{X}$ now has a defense range $\varepsilon_{\mathbb{X}}$ defined by its neighborhood, it is necessary to provide a notion of *admissibility* that is restricted to these spaces. A set of arguments will be admissible if every argument is acceptable with respect to that set, but only by using defenses inside every internal neighborhood. Thus, the following definitions and propositions characterize the semantics notions in our framework. In the following, we assume a framework $\Omega = \langle \Phi, \mathcal{G}_{\Omega}, \mathsf{d}_{\Omega} \rangle$, where $\mathsf{d}_{\Omega}(\cdot,\cdot)$ is a distance function on the set `Args`, and $(\texttt{Args}, \mathsf{d}_{\Omega})$ is the metric space associated with Ω.

Table 2
Argument neighborhoods for Example 4.

Argument	Radius	Diameter	Centrality Degree	Neighbor Degree	Global Topological Property	Local Topological Property	Threshold	Neighborhoods
$\mathbb{A}$	2	2	$[5 \quad 4 \quad 2]$	$\left[\frac{16}{5} \quad \frac{19}{4} \quad \frac{7}{2}\right]$	2	$\frac{16}{5}$	3	$\mathbb{A}, \mathbb{C}, \mathbb{E}, \mathbb{F}, \mathbb{I}, \mathbb{K}, \mathbb{L}, \mathbb{O}$
$\mathbb{B}$	2	2	$[2 \quad 3]$	$\left[4 \quad \frac{7}{3}\right]$	2	$\frac{7}{3}$	2	$\mathbb{B}$
$\mathbb{C}$	1	1	$[4 \quad 5]$	$\left[4 \quad \frac{16}{5}\right]$	1	4	3	$\mathbb{A}, \mathbb{C}, \mathbb{E}, \mathbb{I}, \mathbb{K}, \mathbb{O}$
$\mathbb{D}$	0	0	$[2]$	$\left[\frac{5}{2}\right]$	0	$\frac{5}{2}$	1	$\mathbb{D}$
$\mathbb{E}$	0	0	$[5]$	$\left[\frac{8}{5}\right]$	0	5	3	$\mathbb{A}, \mathbb{C}, \mathbb{E}, \mathbb{F}, \mathbb{H}, \mathbb{I}, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{O}$
$\mathbb{F}$	0	0	$[2]$	$\left[\frac{3}{2}\right]$	0	2	1	$\mathbb{F}, \mathbb{M}, \mathbb{O}$
$\mathbb{G}$	0	0	$[0]$	$[0]$	0	0	0	$\mathbb{G}$
$\mathbb{H}$	3	3	$[4 \quad 3]$	$\left[\frac{19}{4} \quad 2\right]$	3	3	3	$\mathbb{E}, \mathbb{H}, \mathbb{I}, \mathbb{K}$
$\mathbb{I}$	1	1	$[4 \quad 4]$	$\left[4 \quad \frac{19}{4}\right]$	1	4	3	$\mathbb{A}, \mathbb{C}, \mathbb{E}, \mathbb{H}, \mathbb{I}, \mathbb{K}, \mathbb{N}, \mathbb{O}$
$\mathbb{J}$	5	6	$[5 \quad 4 \quad 1]$	$\left[\frac{16}{5} \quad \frac{19}{4} \quad 3\right]$	5	3	4	$\mathbb{J}$
$\mathbb{K}$	2	2	$[5 \quad 2]$	$\left[\frac{8}{5} \quad \frac{7}{2}\right]$	2	2	2	$\mathbb{E}, \mathbb{I}, \mathbb{K}$
$\mathbb{L}$	4	4	$[4 \quad 2]$	$\left[4 \quad \frac{3}{2}\right]$	4	2	3	$\mathbb{A}, \mathbb{E}, \mathbb{L}, \mathbb{O}$
$\mathbb{M}$	0	0	$[1]$	$[3]$	0	3	2	$\mathbb{F}, \mathbb{M}, \mathbb{O}$
$\mathbb{N}$	0	0	$[1]$	$[2]$	0	2	1	$\mathbb{N}$
$\mathbb{O}$	0	0	$[2]$	$\left[\frac{7}{2}\right]$	0	$\frac{7}{2}$	2	$\mathbb{A}, \mathbb{E}, \mathbb{F}, \mathbb{M}, \mathbb{O}$

Definition 16 *(Conflict-freeness, acceptability and admissibility [13]).* Let $\mathfrak{N}_{\Omega}^{\varepsilon_\Omega}$ be the set of neighborhoods associated with the arguments of Ω. Then:

- A set $\mathbb{S} \subseteq \mathtt{Args}$ is said to be conflict free if there are no hashtagged arguments $\mathbb{A}, \mathbb{B} \in \mathbb{S}$ such that $\mathbb{B}$ attacks $\mathbb{A}$.
- A hashtagged argument $\mathbb{A} \in \mathtt{Args}$ is *η-acceptable* with respect to $\mathbb{S}$ if for every argument $\mathbb{B} \in \mathtt{Args}$, if $\mathbb{B}$ attacks $\mathbb{A}$ then there is a hashtagged argument $\mathbb{C} \in \mathbb{S}$ such that $\mathbb{C} \in \mathfrak{N}_{\mathbb{A}}^{\varepsilon_\mathbb{A}}$ and $\mathbb{C}$ attacks $\mathbb{B}$.
- $\mathbb{S}$ is said to be *η-admissible* if every hashtagged argument in $\mathbb{S}$ is *η-acceptable* with respect to $\mathbb{S}$.

Note that we are considering a defense as valid only if it occurs within the neighborhood of an attacked argument.

Proposition 1 *([13]). Given an argument $\mathbb{A} \in \mathtt{Args}$, $\mathbb{A}$ is not attacked in Ω iff $\mathbb{A}$ is not attacked in Φ.*

Under this notion of distance-bounded defense, an argument that may be a defender according to classical acceptability may no longer be a defender; however, the quality of a set being (classically) admissible is preserved because, as will be shown later, attacks are not restricted by distance, only defenses. Admissibility semantics is focused on the characterization of sets of arguments that provide mutual defenses in the set. By restricting defenses within a particular neighborhood, we are *reshaping* the original notion of admissible sets (and also changing the notion of a focused, rational position) while respecting argument conflicts in the whole scenario.

Proposition 2 *([13]). Let $\mathfrak{N}_{\Omega}^{\varepsilon_\Omega}$ be the set of neighborhoods associated with the arguments of Ω and Φ be the underlying abstract argumentation framework. Then:*

i) *If $\mathbb{A} \in \mathtt{Args}$ is η-acceptable w.r.t. a set $\mathbb{S}$ in Ω, then it is acceptable w.r.t. $\mathbb{S}$ in Φ.*

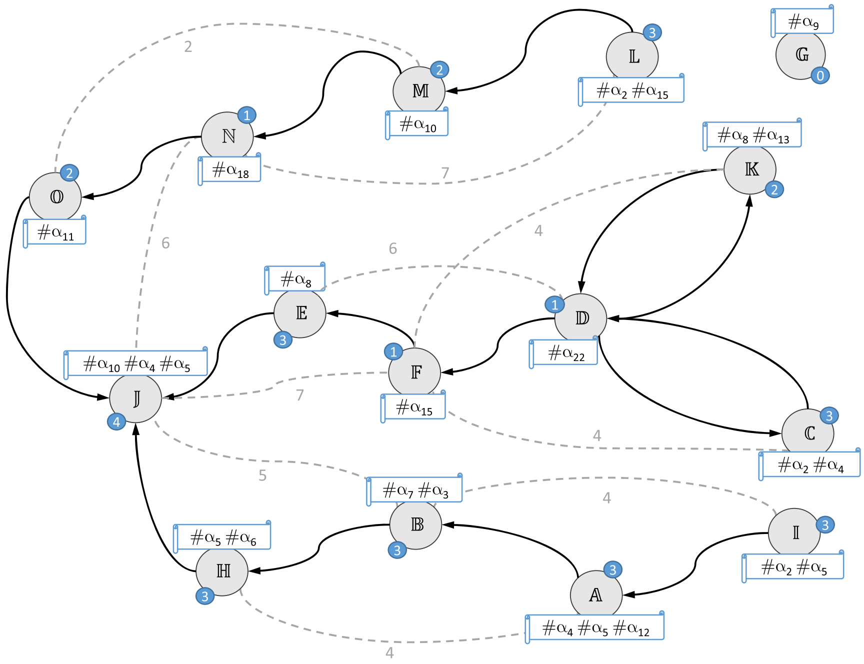


Fig. 7. Argumentation framework considering neighborhood thresholds.

ii) *If a set* $\mathbb{S}$ *is η-admissible in* Ω *then it is admissible in* Φ.

The converses of the statements in Proposition 2 do not hold. For instance, an argument $\mathbb{A}$ may be acceptable with respect to the set $\{\mathbb{B}\}$, but not η-acceptable if $\mathbb{B} \notin \mathfrak{N}_{\mathbb{A}}^{\epsilon_{\mathbb{A}}}$. Therefore, an admissible set may not be η-admissible. Furthermore, as we mentioned, these semantic notions are a refined version of the semantics proposed in Section 2.3. Thus, we have the following proposition.

Proposition 3 *([13]). Let* $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathrm{d}_\Omega \rangle$ *be the underlying a hashtagged framework,* $\tau \in \mathbb{N}^0$ *be a threshold,* $\mathfrak{N}_\Omega^{\epsilon_\Omega}$ *be the set of neighborhoods associated with the arguments of* Ω*,* $\mathfrak{T}_\Omega^{s}$ *and* $\mathfrak{T}_\Omega^{g}$ *be the radius associated with the smallest and greatest neighborhoods of* $\mathfrak{N}_\Omega$*, respectively; let* $\mathbb{S} \subseteq \mathtt{Args}$ *be a set of hashtagged arguments. Then, we have:*

i) *If* $\mathbb{A} \in \mathtt{Args}$ *is* τ*-acceptable w.r.t.* $\mathbb{S}$ *with* $\tau = \mathfrak{T}_\Omega^{s}$*, then it is* η*-acceptable w.r.t.* $\mathbb{S}$*;*
ii) *If* $\mathbb{A} \in \mathtt{Args}$ *is* η*-acceptable w.r.t.* $\mathbb{S}$ *then it is* τ*-acceptable w.r.t.* $\mathbb{S}$ *with* $\tau = \mathfrak{T}_\Omega^{g}$*;*
iii) *If* $\mathbb{S}$ *is* τ*-admissible with* $\tau = \mathfrak{T}_\Omega^{s}$*, then it is* η*-admissible; and*
iv) *If* $\mathbb{S}$ *is* η*-admissible then it is* τ*-admissible with* $\tau = \mathfrak{T}_\Omega^{g}$*.*

As usual in abstract argumentation, Definition 16 leads to different notions providing a new proximity-based interpretation of classical admissibility. In this version, we propose a more refined analysis considering the admitted interaction field associated with the hashtagged arguments. This new notion allows the analysis of the argumentation process from a new point of view, where the scope associated with the hashtagged arguments is taken into account.

Definition 17 *(η-*complete*, η-*grounded*, η-*preferred *ext. [13]).* Let $\mathfrak{N}_\Omega^{\epsilon_\Omega}$ be the set of neighborhoods associated with the arguments of Ω, and $\mathbb{S} \subseteq \mathtt{Args}$ be a set of hashtags arguments. Then:

i) An *η-admissible* set $\mathbb{S}$ is an *η-complete extension* iff $\mathbb{S}$ contains each argument that is *η-acceptable* with respect to $\mathbb{S}$.
ii) Set $\mathbb{S}$ is the *η-grounded extension* of Ω iff $\mathbb{S}$ is an $\subseteq$-minimal *η-complete* extension.
iii) Set $\mathbb{S}$ is an *η-preferred extension* of Ω iff $\mathbb{S}$ is an $\subseteq$-maximal *η-complete* extension.

Next, we present an example to illustrate these new acceptability concepts.

Example 5. Continuing with the hashtagged argumentation framework depicted in Example 1, and based on the distance between hashtagged arguments presented in Table 1, the neighborhoods associated with each argument presented in Table 2, and represented in Fig. 6, we have that:

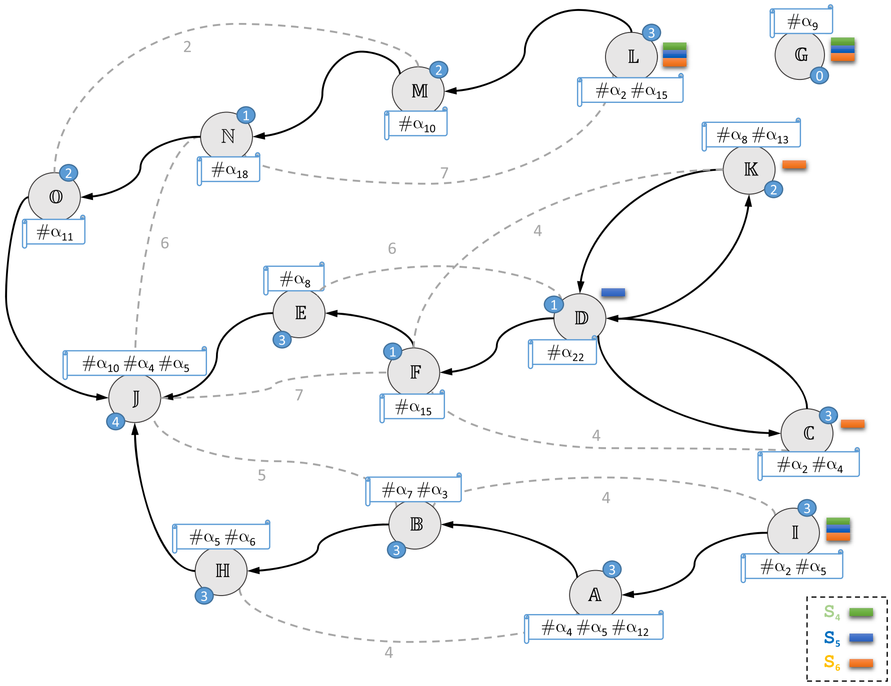


Fig. 8. *η-extensions* from Example 5.

- $\mathbb{I}$ is not a defender of $\mathbb{B}$ since $\mathbb{I}$ is not a neighbor of $\mathbb{B}$ (the distance between $\mathbb{I}$ and $\mathbb{B}$ is 4, where $\mathbb{B}$ has a neighborhood range of 3). Furthermore, $\mathbb{B}$ is not a defender of $\mathbb{J}$ since the distance between $\mathbb{J}$ and $\mathbb{B}$ is 5, and $\mathbb{J}$ has a neighborhood range of 4; thus, $\mathbb{B}$ is not a neighbor of $\mathbb{J}$. Also, $\mathbb{A}$ is not a defender of $\mathbb{H}$ since $\mathbb{A}$ is not a neighbor of $\mathbb{H}$ (the distance between $\mathbb{H}$ and $\mathbb{A}$ is 4, where $\mathbb{H}$ has a neighborhood range of 3)
- On the other hand, $\mathbb{C}$ is not a defender of $\mathbb{F}$ since the distance between $\mathbb{C}$ and $\mathbb{F}$ is 4, while $\mathbb{F}$ has a neighborhood range of 1. In addition, $\mathbb{K}$ is not a defender of $\mathbb{F}$ since $\mathbb{K}$ is not a neighbor of $\mathbb{F}$ (the distance between $\mathbb{K}$ and $\mathbb{F}$ is 4, where $\mathbb{F}$ has a neighborhood range of 1).
- $\mathbb{D}$ is not a defender of $\mathbb{E}$ since $\mathbb{D}$ is not a neighbor of $\mathbb{E}$ (the distance between $\mathbb{D}$ and $\mathbb{E}$ is 6, where $\mathbb{E}$ has a neighborhood range of 3). Also, $\mathbb{F}$ is not a defender of $\mathbb{J}$ since the distance between $\mathbb{J}$ and $\mathbb{F}$ is 7, and $\mathbb{J}$ has a neighborhood range of 4. Thus, $\mathbb{F}$ is not a neighbor of $\mathbb{J}$.
- $\mathbb{L}$ is not a defender of $\mathbb{N}$ since $\mathbb{L}$ is not a neighbor of $\mathbb{N}$ (the distance between $\mathbb{L}$ and $\mathbb{N}$ is 7, where $\mathbb{N}$ has a neighborhood range of 1). Also, $\mathbb{N}$ is not a defender of $\mathbb{J}$ since the distance between $\mathbb{J}$ and $\mathbb{N}$ is 6, and $\mathbb{J}$ has a neighborhood range of 4. Thus, $\mathbb{N}$ is not a neighbor of $\mathbb{J}$. Finally, $\mathbb{M}$ is a defender of $\mathbb{O}$ since $\mathbb{M}$ is a neighbor of $\mathbb{O}$ (the distance between $\mathbb{O}$ and $\mathbb{M}$ is 2, where $\mathbb{O}$ has a neighborhood range of 2)

Thus, analyzing the acceptability notions presented in Definition 16, the sets $S_4 = \{\mathbb{I}, \mathbb{G}, \mathbb{L}\}$, $S_5 = \{\mathbb{I}, \mathbb{D}, \mathbb{G}, \mathbb{L}\}$ and $S_6 = \{\mathbb{I}, \mathbb{C}, \mathbb{K}, \mathbb{G}, \mathbb{L}\}$ are the $\subseteq$-maximal η-admissible extensions. Furthermore, S_4 is the *η-grounded extension* (see Fig. 8) while S_5 and S_6 are the *η-preferred extensions*, under the conditions established in Definition 17.

Admissibility discards argument defenders that do not belong to the neighborhood associated with the attacked argument in this proximity semantics version. If the argument has associated topics covering a particular thematic field, it will be defended by those arguments that are related to the same field. On the other hand, an opinion cannot be defended by formulations or assertions out of its spectrum of discussion.

As expected, if the smallest neighborhood is large enough, Dung's admissibility and η-admissibility coincide. Thus, the next results hold between the classical and the proximity-acceptable sets of arguments. In the following, we will consider a framework $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathrm{d}_\Omega \rangle$, where $\mathrm{d}_\Omega(\cdot,\cdot)$ is a distance function on the set $\mathtt{Args}$, $(\mathtt{Args}, \mathrm{d}_\Omega)$ is the metric space associated with Ω, and $\mathfrak{N}_\Omega^{\varepsilon_\Omega}$ is the set of neighborhoods associated with the arguments of Ω.

Proposition 4 *([13]). Let $\mathfrak{N}_\Omega^{\varepsilon_\Omega}$ be the set of neighborhoods associated with the arguments of Ω. If the smallest neighborhood $\mathcal{T}_\Omega^{\mathrm{s}}$ is such that $\mathcal{T}_\Omega^{\mathrm{s}} \geq diameter(\mathcal{H})$, then every η-{admissible, complete, grounded, preferred} extension is an {admissible, complete, grounded, preferred} extension, respectively.*

In abstract argumentation, a grounded extension is the skeptical position of acceptance, and it is unique. In our definition of proximity-based semantics, the skeptical position is related to the set of neighborhoods associated with the metric space, and different sets of neighborhoods lead to different η-grounded extensions; however, as in classical frameworks, the extension always exists.

Proposition 5 *([13]). There always exists a unique η-grounded extension.*

As we said before, hashtagged argumentation frameworks are an extension of abstract frameworks in the sense that we are considering additional elements; thus, if hashtag-related information is discarded, a classical abstract framework remains. The new proximity-based and the classical abstract semantics are related, as the following theorem establishes by showing a link between this redefined proximity-based semantics and its corresponding abstract semantics counterpart, observing that the former is a refinement of the latter.

Theorem 1 *([13]). Let $\mathfrak{N}_{\Omega}^{\varepsilon_{\Omega}}$ be the set of neighborhoods associated with the arguments of Ω, and $\Phi = \langle \mathtt{Args}, \mathtt{Attacks} \rangle$ be the underlying abstract argumentation framework. Then, the following properties hold:*

i) *If S_{Ω} is η-complete extension in Ω, then there exists a complete extension S_{Φ} in Φ such that $\mathrm{S}_{\Omega} \subseteq \mathrm{S}_{\Phi}$;*
ii) *If S_{Ω} is an η-grounded extension in Ω, then there exists a grounded extension S_{Φ} in Φ such that $\mathrm{S}_{\Omega} \subseteq \mathrm{S}_{\Phi}$; and,*
iii) *If S_{Ω} is an η-preferred extension in Ω, then there exists a preferred extension S_{Φ} in Φ such that $\mathrm{S}_{\Omega} \subseteq \mathrm{S}_{\Phi}$.*

Thus, the rationale of classic argumentation semantics is preserved. The addition of the concept neighborhood improves the argumentation model by introducing a new view on valid defenses for an individual argument; this idea is compelling because it leads to a new family of semantics, possibly parameterized with various metrics. Since the notion of neighborhood can be defined by considering different metrics associated with the hashcloud, different conceptualizations of the notion of neighborhood clearly influence the general outcome of the argumentation scenario. The relation of proximity between arguments is now relevant for the argumentation process.

Finally, considering the intuitions presented above, the following result establishes a connection between the proximity-based semantics and the proximity-based semantics based on the definition of neighborhood.

Theorem 2 *([13]). Let $\Omega = \langle \Phi, \mathscr{G}_{\Omega}, \mathrm{d}_{\Omega} \rangle$ be the underlying a hashtagged framework, $\tau \in \mathbb{N}^0$ be a threshold, $\mathfrak{N}_{\Omega}^{\varepsilon_{\Omega}}$ be the set of neighborhoods associated with the arguments of Ω. Then, we have:*

i) If $\tau = \mathfrak{I}_{\Omega}^{s}$ is the threshold associated with the smallest neighborhood of Ω, $\mathrm{S}_{\Omega}^{\eta}$ is an η-complete (respectively, η-grounded, and η-preferred) extension, and $\mathrm{S}_{\Omega}^{\tau}$ is an τ-complete (respectively, τ-grounded, and τ-preferred) extension, then it holds that $\mathrm{S}_{\Omega}^{\tau} \subseteq \mathrm{S}_{\Omega}^{\eta}$.
ii) If $\tau = \mathfrak{I}_{\Omega}^{g}$ is the threshold associated with the greatest neighborhood of Ω, $\mathrm{S}_{\Omega}^{\eta}$ is an η-complete (respectively, η-grounded, and η-preferred) extension, and $\mathrm{S}_{\Omega}^{\tau}$ is an τ-complete (respectively, τ-grounded, and τ-preferred) extension, then it holds that $\mathrm{S}_{\Omega}^{\eta} \subseteq \mathrm{S}_{\Omega}^{\tau}$.

As we postulated in the original proximity-based semantics, under this new interpretation of "defense", where we consider an admitted defense field associated with each argument, a potential defender argument may no longer be considered as such. However, the interrelation of argument neighborhoods, considering the concept of neighborhood from Definition 11, can form an even more extensive defense field, allowing the emergence of new defending arguments. We will analyze this proposal in more detail below.

5. A more inclusive notion: communities

As mentioned before, when neighborhoods overlap, it is possible to consider a new range of defense based on a sort of *familiar* closeness of arguments. Here, the neighbors of an argument can serve as connections to other neighborhoods, expanding then the borders of a "society" of arguments. We refer to this set of expanded neighbors as a *community*. First, we introduce the notion of close connection, where we formalize how the neighborhood of an argument is extended; then, based on this sequence, we define the concept of community.

Definition 18 *(Semantic path between arguments)*. Let $\Omega = \langle \Phi, \mathscr{G}_{\Omega}, \mathrm{d}_{\Omega} \rangle$ be a hashtagged framework, $\mathrm{d}_{\Omega}(\cdot,\cdot)$ be a distance function on the set $\mathtt{Args}$, $(\mathtt{Args}, \mathrm{d}_{\Omega})$ be the metric space associated with the tagged framework Ω, and $\mathfrak{N}_{\Omega}$ be the set of neighborhoods associated with arguments in Ω. We say that there is a semantic path from an argument $\mathbb{X}_1$ to an argument $\mathbb{X}_n$ if and only if there exists a sequence of arguments $[\mathbb{X}_1, \dots, \mathbb{X}_i, \dots, \mathbb{X}_n]$ where $\mathfrak{N}_{\mathbb{X}_i}^{\varepsilon_{\mathbb{X}_i}} \cup \mathfrak{N}_{\mathbb{X}_{i+1}}^{\varepsilon_{\mathbb{X}_{i+1}}} \neq \emptyset$, with $1 \leq i \leq n-1$. We will use $\mathbb{X}_1 \rightsquigarrow \mathbb{X}_n$ to denote a path between $\mathbb{X}_1$ and $\mathbb{X}_n$.

The key idea behind the notion of community is to consider the common neighbors. Hence, two arguments, $\mathbb{A}$ and $\mathbb{B}$, are in the same community if they belong to the same neighborhood or if there exists a semantic path between them. Formally, we define this as follows.

Definition 19 *(Community for an argument).* Let $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathsf{d}_\Omega \rangle$ be a hashtagged framework, $\mathsf{d}_\Omega(\cdot,\cdot)$ be a distance function on the set $\texttt{Args}$, $(\texttt{Args}, \mathsf{d}_\Omega)$ be the metric space associated with the tagged framework Ω, and $\mathfrak{N}_\Omega$ be the set of neighborhoods associated with arguments in Ω. A community for an argument $\mathbb{A} \in \texttt{Args}$, denoted as $\mathfrak{C}_\mathbb{A}$, is defined as:

$$\mathfrak{C}_\mathbb{A} = \{\mathbb{B} \in \texttt{Args} \mid \mathbb{A} \rightsquigarrow \mathbb{B} \vee \mathbb{B} \rightsquigarrow \mathbb{A}\}.$$

We denote with $\mathfrak{C}_\Omega$ the set of communities associated with the arguments of Ω.

Different communities are always disjoint, since a common argument denotes two overlapping neighborhoods that, by definition, must be in the same community.

Proposition 6. *Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω. Given any $\mathfrak{C}_\mathbb{A}, \mathfrak{C}_\mathbb{B} \in \mathfrak{C}_\Omega$, either $\mathfrak{C}_\mathbb{A} \cap \mathfrak{C}_\mathbb{B} = \emptyset$ or $\mathfrak{C}_\mathbb{A} = \mathfrak{C}_\mathbb{B}$.*

On the other hand, every argument belongs to a community since every argument has a neighborhood, which can be a community by itself if it is entirely isolated. Hence, the union of communities comprises the entire set of arguments in the framework. Then, based on the result presented in the previous proposition, the concept of community naturally induces a partition of $\texttt{Args}$. In this case, we denote with $\mathfrak{C}_\Omega = \{\mathfrak{C}_1, ..., \mathfrak{C}_n\}$ the set of communities associated with the arguments of Ω without referring to the arguments belonging to a community.

Proposition 7. *Let $\mathfrak{C}_\Omega = \{\mathfrak{C}_1, ..., \mathfrak{C}_n\}$ be the set of communities associated with the arguments of Ω. Then:*

$$\bigcup_{i=1}^{n} \mathfrak{C}_i = \texttt{Args}.$$

Thus, in the following, we formalize the notions introduced above using now the notion of community and their impact on the semantics in our argumentation formalism. In the following, consider framework $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathsf{d}_\Omega \rangle$, where $\mathsf{d}_\Omega(\cdot,\cdot)$ is a distance function on the set $\texttt{Args}$, $(\texttt{Args}, \mathsf{d}_\Omega)$ is the metric space associated with Ω, and $\mathfrak{N}_\Omega^{\varepsilon_\Omega}$ the set of neighborhoods associated with the arguments of Ω.

Proposition 8. *Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω. Then, if $\mathfrak{C}_\Omega$ is a singleton set then $\mathfrak{C}_\Omega = \texttt{Args}$.*

In other words, if there is only one community in the framework, then there are no isolated neighborhoods. This means that for any argument $\mathtt{A}$ it is possible to *reach* another argument $\mathtt{B}$ in $\texttt{Args}$ simply by "*jumping*" through a sequence of overlapping neighborhoods. Hence, the community is itself the set of all arguments $\texttt{Args}$.

Example 6. Consider the framework depicted in Fig. 7, with the neighborhoods associated with each argument as detailed in Table 2. Considering Definition 19, we can obtain the following four communities:

- $\mathfrak{C}_1 = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{E}, \mathbb{F}, \mathbb{H}, \mathbb{I}, \mathbb{K}, \mathbb{L}, \mathbb{M}, \mathbb{N}, \mathbb{O}\}$ is the largest community. To obtain this set, we consider the neighborhoods associated with arguments $\mathbb{A}$, $\mathbb{I}$, $\mathbb{B}$, and $\mathbb{F}$, highlighted with blue, green, gray, and orange, respectively in Fig. 9. Thus, we find a sequence to arrive from $\mathbb{A}$ to $\mathbb{N}$ navigating through the neighborhood of $\mathbb{A}$ and $\mathbb{I}$ ($\mathbb{A}$ has $\mathbb{I}$ as neighbor and from $\mathbb{I}$ we have a sequence to reach $\mathbb{N}$ – $\mathbb{I} \rightsquigarrow \mathbb{N}$, since $\mathbb{I}$ has $\mathbb{N}$ as neighbor), a sequence to arrive from $\mathbb{I}$ to $\mathbb{M}$ ($\mathbb{I}$ has $\mathbb{F}$ as neighbor and from $\mathbb{F}$ we have a sequence to reach $\mathbb{M}$ – $\mathbb{F} \rightsquigarrow \mathbb{M}$, since $\mathbb{F}$ has $\mathbb{M}$ as neighbor), and a sequence to arrive from $\mathbb{M}$ to $\mathbb{B}$ ($\mathbb{B}$ has $\mathbb{M}$ as neighbor – $\mathbb{B} \rightsquigarrow \mathbb{M}$);
- $\mathfrak{C}_2 = \{\mathbb{D}\}$ is a single community;
- $\mathfrak{C}_3 = \{\mathbb{G}\}$ is a single community; and
- $\mathfrak{C}_4 = \{\mathbb{J}\}$ is a single community.

In Fig. 10 we represent the communities associated with Ω, where $\mathfrak{C}_1$ is marked with purple dots, $\mathfrak{C}_2$ with red, $\mathfrak{C}_3$ with yellow, and $\mathfrak{C}_4$ with blue.

We have thus established an expanded notion of *familiarity* between arguments neighborhoods with common members form a larger society of arguments ready to defend themselves. We next characterize admissibility semantics over this expanded range of defense.

Now that arguments have a defense community range, it is necessary to provide a notion of *admissibility* that is restricted to these argument-intrinsic spaces. Here, a set of arguments is said to be admissible if every argument is acceptable with respect to that set, but only by using defenses inside every internal community range. This is formalized as follows.

Definition 20 *(Conflict-freeness, acceptability, admissibility).* Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω, and $\mathfrak{C} \in \mathfrak{C}_\Omega$. Then, we have:

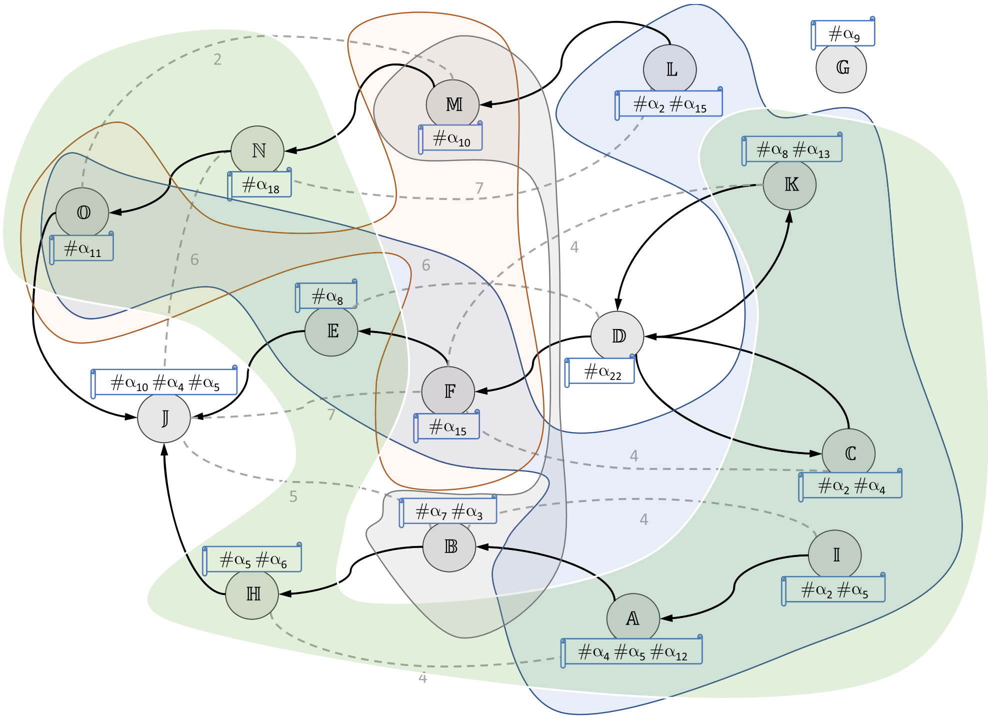


Fig. 9. Towards communities in the Proximity-based argumentation framework from the running example.

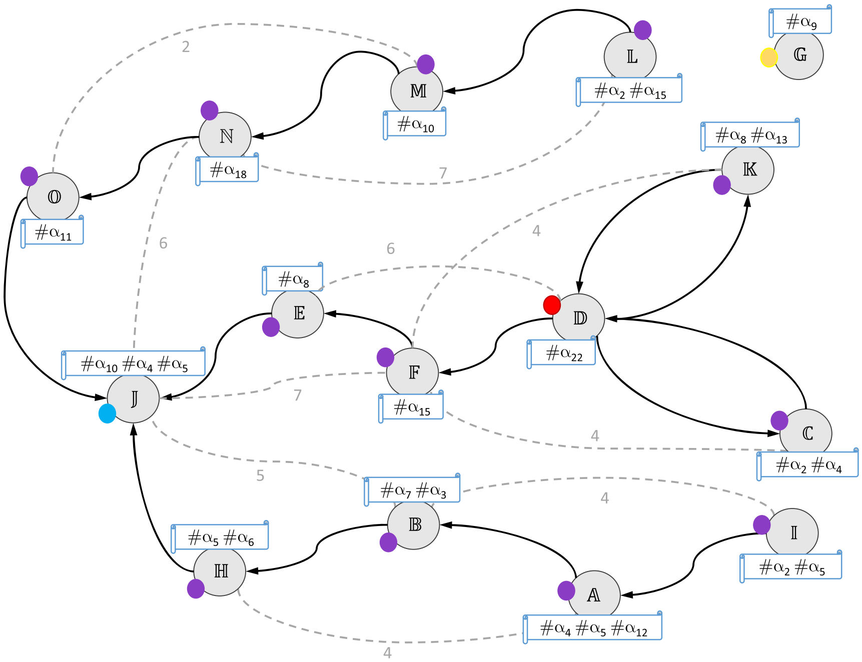


Fig. 10. Communities in the framework from the running example.

i) A set $\mathbb{S} \subseteq \texttt{Args}$ is said to be conflict free if there are no hashtagged arguments $\mathbb{A}, \mathbb{B} \in \mathbb{S}$ such that $\mathbb{B}$ attacks $\mathbb{A}$.

ii) A hashtagged argument $\mathbb{A} \in \texttt{Args}$ and $\mathbb{A} \in \mathfrak{C}$ is ζ*-acceptable* with respect to $\mathfrak{C}_\Omega$ if for every argument $\mathbb{B} \in \texttt{Args}$, if $\mathbb{B}$ attacks $\mathbb{A}$ then there is a hashtagged argument $\mathbb{C} \in \mathfrak{C}$ such that $\mathbb{C}$ attacks $\mathbb{B}$.

iii) $\mathbb{S}$ is said to be ζ*-admissible* if every hashtagged argument that belongs to a community of $\mathfrak{C}_\Omega$ is ζ*-acceptable* with respect to $\mathfrak{C}_\Omega$.

Note that, as before, we are considering a defense as valid only if it occurs within the community of an attacked argument.

Proposition 9. *Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω. Then, $\mathbb{A} \in \mathtt{Args}$ is not attacked in Ω iff $\mathbb{A} \in \mathtt{Args}$ is not attacked in Φ.*

Proposition 10. *Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω such that $\mathfrak{C}_\Omega$ is a singleton set. Then, we have:*

i) *$\mathbb{A} \in \mathtt{Args}$ is ζ-acceptable w.r.t. a set $\mathtt{S}$ iff it is acceptable w.r.t. $\mathtt{S}$; and*
ii) *A set $\mathtt{S}$ is ζ-admissible iff it is admissible.*

By confining defenses to a particular community, we are relaxing the concept of admissible sets (and also adjusting the idea of a focused, rational position), particularly by expanding neighborhoods of arguments through the transitive closure.

Proposition 11. *Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω. Then:*

i) *If $\mathbb{A} \in \mathtt{Args}$ is η-acceptable w.r.t. a set $\mathtt{S}$ then it is ζ-acceptable w.r.t. $\mathtt{S}$.*
ii) *If a set $\mathtt{S}$ is η-admissible then it is ζ-admissible.*

The converse of the statements in Proposition 11 do not hold. For instance, an argument $\mathbb{A}$ may be ζ-acceptable with respect to the set $\{\mathbb{B}\}$ because $\mathbb{B} \in \mathfrak{C}_{\mathbb{A}}$, but not η-acceptable if $\mathbb{B} \notin \mathfrak{N}_{\mathbb{A}}^{\varepsilon_{\mathbb{A}}}$. Therefore, a ζ-admissible set may not be η-admissible.

Lemma 1. *Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω. Then, we have:*

i) *If $\mathbb{A} \in \mathtt{Args}$ is ζ-acceptable w.r.t. a set $\mathtt{S}$ then it is acceptable w.r.t. $\mathtt{S}$.*
ii) *If a set $\mathtt{S}$ is ζ-admissible then it is admissible.*

As mentioned, these semantic notions are an alternative version of the semantics proposed in Section 2.3. Thus, the following properties hold.

Proposition 12. *Let $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathrm{d}_\Omega \rangle$ be the underlying a hashtagged framework, $\tau \in \mathbb{N}^0$ be a threshold, $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω, $\mathfrak{T}_\Omega^{\mathrm{s}}$ and $\mathfrak{T}_\Omega^{\mathrm{g}}$ be the radius associated with the smallest and greatest communities of $\mathfrak{C}_\Omega$, respectively, and $\mathtt{S} \subseteq \mathtt{Args}$ be a set of hashtagged arguments. Then:*

i) *If $\mathbb{A} \in \mathtt{Args}$ is τ-acceptable w.r.t. $\mathtt{S}$ with $\tau = \mathfrak{T}_\Omega^{\mathrm{s}}$, then it is ζ-acceptable w.r.t. $\mathtt{S}$;*
ii) *If $\mathbb{A} \in \mathtt{Args}$ is η-acceptable w.r.t. $\mathtt{S}$ then it is τ-acceptable w.r.t. $\mathtt{S}$ with $\tau = \mathfrak{T}_\Omega^{\mathrm{g}}$;*
iii) *If $\mathtt{S}$ is τ-admissible with $\tau = \mathfrak{T}_\Omega^{\mathrm{s}}$, then it is ζ-admissible; and*
iv) *If $\mathtt{S}$ is ζ-admissible then it is τ-admissible with $\tau = \mathfrak{T}_\Omega^{\mathrm{g}}$.*

Now, Definition 20 leads to an extended proximity-based interpretation of neighborhood admissibility, where a collective defense is considered. That is, each argument belongs to a community depending on how their neighborhoods are connected.

Definition 21. Let $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω and $\mathtt{S} \subseteq \mathtt{Args}$ be a set of hashtags arguments. Then:

i) A *ζ-admissible* set $\mathtt{S}$ is a *ζ-complete extension* iff $\mathtt{S}$ contains each argument that is *ζ-acceptable* with respect to $\mathtt{S}$.
ii) Set $\mathtt{S}$ is the *ζ-grounded extension* of Ω iff $\mathtt{S}$ is a $\subseteq$-minimal *ζ-complete* extension.
iii) Set $\mathtt{S}$ is a *ζ-preferred extension* of Ω iff $\mathtt{S}$ is a $\subseteq$-maximal *ζ-complete* extension.

Next, we present an example to shed light on these new concepts.

Example 7. Now, continuing with Example 6 and analyzing the acceptability notions presented in Definition 21, the sets: $\mathtt{S}_7 = \{\mathbb{I}, \mathbb{B}, \mathbb{G}, \mathbb{L}, \mathbb{N}\}$, $\mathtt{S}_8 = \{\mathbb{I}, \mathbb{B}, \mathbb{D}, \mathbb{G}, \mathbb{L}, \mathbb{N}\}$ and $\mathtt{S}_9 = \{\mathbb{I}, \mathbb{B}, \mathbb{C}, \mathbb{F}, \mathbb{G}, \mathbb{K}, \mathbb{L}, \mathbb{N}\}$ are the maximal ζ-admissible extensions. Fig. 11 depicts arguments with colored dots indicating their neighborhood's range, using the same color of the communities to which they belong, as detailed in Example 6 (cf. Fig. 10). Furthermore, $\mathtt{S}_7$ is the *ζ-grounded extension*, while $\mathtt{S}_8$ and $\mathtt{S}_9$ are the *ζ-preferred extensions*, under the conditions established in Definition 17 (cf. Fig. 11).

Note that admissibility discards argument defenders that do not belong to the community associated with the attacked argument in this proximity semantics version. If the argument has associated topics covering a particular thematic field, it will be defended by

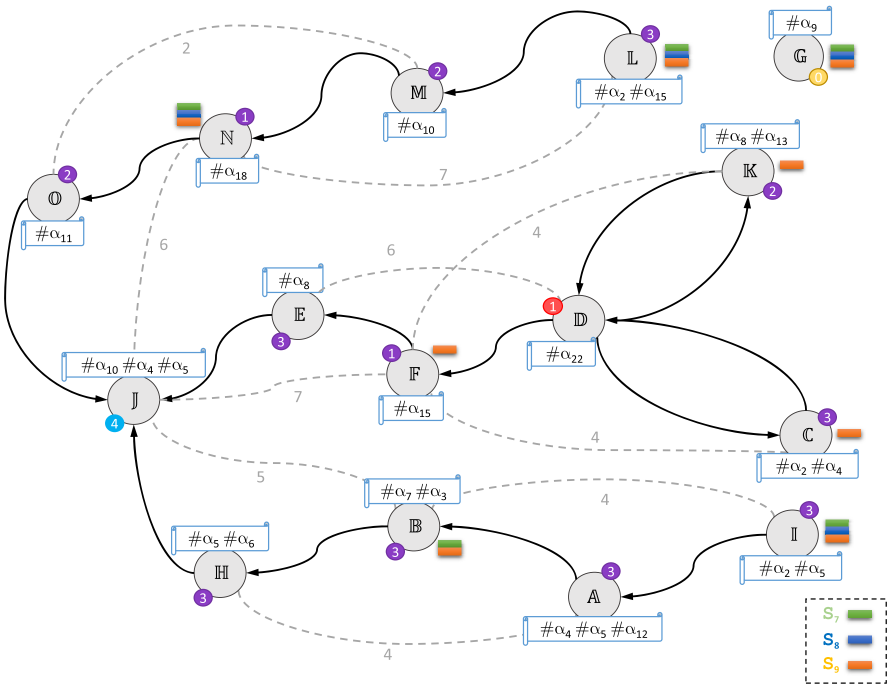


Fig. 11. ζ-*extensions* in Ω from the running example.

those arguments that are related to the same field. On the other hand, an opinion cannot be defended by formulations or assertions out of its spectrum of discussion.

As expected, and as before, if the smallest community is big enough, Dung's admissibility and ζ-admissibility coincide. Thus, we have the following connection between classical and proximity-acceptable sets of arguments.

Proposition 13. *Let* $\mathfrak{C}_\Omega$ *be the set of communities associated with the arguments of* Ω. *If the smallest community* $\mathcal{T}^s_\Omega$ *is such that* $\mathcal{T}^s_\Omega \geq diameter(\mathcal{H})$, *then it holds that every* ζ-$\{admissible, complete, grounded, preferred\}$ *extension is an* $\{admissible, complete, grounded, preferred\}$ *extension, respectively.*

In abstract argumentation, a grounded extension is a single skeptical stance on acceptance. Our proximity-based semantics limits the concept of defense to a threshold linked to each argument labeled with a hashtag and its corresponding community, meaning that the skeptical stance is linked to the collection of communities, and distinct sets of communities result in different ζ-grounded extensions. Nevertheless, as in traditional frameworks, the ζ-grounded extension always exists (although it could be empty). The following result then holds.

Proposition 14. *Let* $\mathfrak{C}_\Omega$ *be the set of communities associated with the arguments of* Ω. *Then, there always exists a unique* ζ-*grounded extension.*

As we said before, hashtagged argumentation frameworks extend the scope of abstract frameworks by including additional elements. We are left with a classical abstract framework if we remove the hashtag information. The redefined proximity-based and traditional abstract semantics are connected, as demonstrated by the following theorem, which establishes a link between the two and shows that the former is a more precise version of the latter.

Theorem 3. *Let* $\mathfrak{C}_\Omega$ *be the set of communities associated with the arguments of* Ω *and* $\Phi = \langle \mathtt{Args}, \mathtt{Attacks} \rangle$ *be the underlying abstract argumentation framework. Then, the following properties hold:*

i) *If* S_Ω *is a* ζ-*complete extension in* Ω, *then there exists a complete extension* S_Φ *in* Φ *such that* $S_\Omega \subseteq S_\Phi$;
ii) *If* S_Ω *is a* ζ-*grounded extension in* Ω, *then there exists a grounded extension* S_Φ *in* Φ *such that* $S_\Omega \subseteq S_\Phi$; *and*
iii) *If* S_Ω *is a* ζ-*preferred extension in* Ω, *then there exists a preferred extension* S_Φ *in* Φ *satisfying that* $S_\Omega \subseteq S_\Phi$.

Thus, the rationale behind classical argumentation semantics is preserved.

The inclusion of the concept of community in argumentation models represents a significant improvement as it offers a fresh perspective on valid defenses for individual arguments and leads to a new set of semantics. Note that the definition of community

Table 3
Summary of results for proximity-based semantics in Ω from the running example.

	Grounded Extension	Preferred Extension I	Preferred Extension II
Basic Proximity	$\{\mathbb{I}, \mathbb{B}, \mathbb{G}, \mathbb{L}\}$	$\{\mathbb{I}, \mathbb{B}, \mathbb{C}, \mathbb{F}, \mathbb{G}, \mathbb{K}, \mathbb{L}\}$	$\{\mathbb{I}, \mathbb{B}, \mathbb{D}, \mathbb{G}, \mathbb{L}\}$
Neighborhood	$\{\mathbb{I}, \mathbb{G}, \mathbb{L}\}$	$\{\mathbb{I}, \mathbb{C}, \mathbb{G}, \mathbb{K}, \mathbb{L}\}$	$\{\mathbb{I}, \mathbb{D}, \mathbb{G}, \mathbb{L}\}$
Communities	$\{\mathbb{I}, \mathbb{B}, \mathbb{G}, \mathbb{L}, \mathbb{N}\}$	$\{\mathbb{I}, \mathbb{B}, \mathbb{C}, \mathbb{F}, \mathbb{G}, \mathbb{K}, \mathbb{L}, \mathbb{N}\}$	$\{\mathbb{I}, \mathbb{B}, \mathbb{D}, \mathbb{G}, \mathbb{L}, \mathbb{N}\}$

can vary depending on the characterization used within the hashcloud. Thus, different conceptualizations of communities will have a noticeable impact on the overall outcome of the argumentation scenario—this highlights the significance of the proximity relationship between arguments. By incorporating the notion of community, argumentation-based models become a more capable representation tool, allowing for a more accurate model of the complex interplay between arguments. Using this representational device can ultimately lead to better decision-making and problem-solving in various domains, considering a more refined analysis based on how argument topics are related in an argumentative dispute.

After considering the intuitions presented above, we can establish a connection between the basic and community-centered proximity-based semantics. This connection has significant implications for the field of argumentation as a whole; by bridging these three approaches to argumentation, we can gain a more comprehensive understanding of the underlying principles and mechanisms at play, which allows us to refine and improve our methods for modeling and analyzing complex real-world scenarios.

Theorem 4. *Let $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathrm{d}_\Omega \rangle$ be a hashtagged framework, $\tau \in \mathbb{N}^0$ be a threshold, and $\mathfrak{C}_\Omega$ be the set of communities associated with the arguments of Ω. Then, the following properties hold:*

i) *If $\tau = \mathfrak{T}^{s}_{\Omega}$ is the threshold associated with the smallest community of Ω, $\mathrm{S}^{\zeta}_{\Omega}$ is ζ-complete (respectively, ζ-grounded, and ζ-preferred) extension and $\mathrm{S}^{\tau}_{\Omega}$ is τ-complete (respectively, τ-grounded, and τ-preferred) extension, then it holds that $\mathrm{S}^{\tau}_{\Omega} \subseteq \mathrm{S}^{\zeta}_{\Omega}$.*
ii) *If $\tau = \mathfrak{T}^{g}_{\Omega}$ is the threshold associated with the greatest community of Ω, $\mathrm{S}^{\zeta}_{\Omega}$ is ζ-complete (respectively, ζ-grounded, and ζ-preferred) extension and $\mathrm{S}^{\tau}_{\Omega}$ is τ-complete (respectively, τ-grounded, and τ-preferred) extension, then it holds that $\mathrm{S}^{\zeta}_{\Omega} \subseteq \mathrm{S}^{\varepsilon}_{\Omega}$.*

As proposed in the original proximity-based framework, this work analyzes the defenses between arguments using a proximity-based framework. Two perspectives are considered: setting ranges of individual defenses through neighboring arguments and establishing communities of arguments for collective defense. To highlight the variations resulting from the proposed semantics, Table 3 summarizes the results of the different extensions.

Note that our analysis has only been conducted within the context of the defense relation. We decided to stay within the framework of solutions originally proposed in abstract frameworks and refine them. If we consider the attack relation and dismiss some of these attacks based on distance, it would introduce a significant change in the argumentative process, potentially resulting in entirely different solutions than the original ones. In Appendix A, we present a case study that provides an application example in a real-world scenario that is limited in scope but complex enough to highlight the essential aspects of the framework. In future work we plan to address this issue and explore the implications of incorporating distance into the entire argumentation process.

6. Related work

In the context of classical abstract argumentation frameworks, there have been several proposals where other elements are added to the theoretical, abstract representational structure extending the possibility of representing more characteristics of the application domain. Some of these proposals bring new ways to identify the *quality* of attacks between arguments, a subject of interest in our work. In particular, several approaches provide mechanisms for discriminating the entire consideration of attacks, making some of those irrelevant and ignored under specific semantics. How an attack comes into play, and what the semantic consequences of this sort of filtering of attacks are, has been the focus of a few works through different argumentation models.

Bench-Capon [3] argues in his research that oftentimes it is impossible to conclusively demonstrate in the context of disagreement that either party is wrong, particularly in situations involving practical reasoning. The fundamental role of arguments in such cases is to persuade rather than to prove, demonstrate, or refute. In his own words: "*The point is that in many contexts the soundness of an argument is not the only consideration: arguments also have a force which derives from the value they advance or protect.*" Based on this intuition, the authors propose a formalism, called *Valued-Based Argumentation Framework* (*VAF*), extending Dung's model to consider the strength of arguments and, through these assessments, reflect the preference of the audience to which the arguments are directed. Specifically, in *VAF*, an argument has associated a value from some set that has an ordering based on a specific audience. Then, from the valuations assigned to the arguments and the preferences defined by the audience, it is possible to specify when an argument is strong enough to attack and defeat another argument.

Therefore, different audiences specify different orders over the set of values, determining different defeat relations between arguments, leading to the distinction of arguments accepted by all the audiences (arguments accepted objectively), and another containing those arguments that are accepted by at least one audience (arguments accepted subjectively). In our work we do not have audiences, but we also provide a distinctive way to identify the force or relevance of an attack. For this we do not rely on the perception of value of arguments, but in the semantic references that an argument can hold. In our model hashtags are not prohibited from denoting values, but since we use graph distance notions to evaluate the relevance of defenses, then our model determines just how far values fall apart. Nevertheless, since we do not model audiences, there is no notion of subjective acceptance.

We associate an abstract argument with a set of hashtags. In [18], S. Kaci and L. van der Torre generalize Bench-Capon's value-based argumentation theory such that arguments can promote multiple values, and preferences among values or arguments can be specified in various ways. Each value can be associated with one or more arguments, and vice versa. Then, once the different values are mapped to each argument involved in the discussion, the existing conflict relations are analyzed with the intention of identifying the successful attacks. In Bench-Capon's value-based framework, the attack of an argument A over an argument B is successful if and only if A attacks B and the value promoted by B is not preferred over that promoted by A. However, in this new proposal, the arguments can promote more than one value—this increases the difficulty of determining when an argument is preferred to another based on their valuations. Again, here the preference among values is used to determine the success (somehow the validation) of an attack. The plurality of values requires an analysis that integrates these values into a single decision about the attack. There is a similar problem in our model, where several hashtags must lead to a distance measure that finally establishes the relevance of a defense. In [18], the authors provide a different solution based on the principles of minimal/maximal specificity, that allow to establish a unique possible ordering (total ordering) over the set of values associated with the arguments. This is possible since preferences are present in the formalization.

The enrichment of individual arguments with some form of meta-information is also present in [17]. There, Hunter addresses the idea that attacks might have attached some uncertainty about whether these attacks hold, *i.e.*, some attacks might be believed, some might be disbelieved, and some might be unknown. To investigate how the attachment of probability to attacks influences the semantic analysis in the abstract framework, the author considers a probability distribution over the spanning subgraphs of an argumentation graph. From this distribution, the probability that a set of arguments is admissible or included in an extension can be determined. Therefore, adding probabilities to attacks in abstract argumentation frameworks leads to a formalism where attacks might or might not be a part of the semantic analysis, choosing a direction that differs from our approach. In our model, hashtags are abstract decorations of a single argument that denotes a set of semantic references. Uncertainty is a numeric valuation of an argument; a hashtag may denote an uncertainty level, and then the induced hashcloud is a linear graph. A defense is then allowed if the gap of uncertainty between two arguments is larger than a threshold. This is clearly a different approach to the one in [17], but the interpretation of hashtags as uncertainty levels is interesting, since the intuition is sound.

In [8], Budán et al. presented a formalization of a bipolar abstract argumentation framework that incorporates a novel mechanism for identifying meta-structures (coalitions) based on the similarity between supported related arguments, which is used as a measure of the coalition's cohesion. The coherence of the coalition was determined by the similarity between supported related arguments, and this concept allowed for the evaluation of the coalition and identification of conversational trends. They also used the similarity degree to describe the attacks between coalitions, thereby advancing a measure of controversy. Additionally, a method for determining the level of weakening over a set of arguments was proposed by computing all the attacks received by an s-coalition. In our model, we use the classic argumentation framework proposed by Dung as a basis, which analyzes the effects of the distance between arguments in terms of defense. However, we incorporate a semantic topics network to conduct various metric analyses and determine how arguments can impact each other based on the topics they address. Additionally, we formalize two perspectives: a neighborhood and a community admissibility range, which enables us to obtain different families of semantic extensions. In another sense, in [8] the authors emphasize the importance of detecting communities in social networks by identifying clusters of individuals with similar tastes and preferences. Towards this end, they propose a novel mechanism for finding meta-structures (coalitions) based on the similarity between related arguments in discussions, considering their internal descriptors and their values (semantic internal structure). In order to establish a more informative relation to the present work using topic-based measures, one could explore methods to enhance the identification of these communities by incorporating topic modeling techniques. By associating arguments with specific topics or themes, similarity measures could be refined based on the relevance of arguments' content. That is, we can explore based on a semantic hashcloud how relevant the arguments are in a specific discussion, how the different topological measures affect the similarity notion between arguments, and which relation exists between them. This would provide a more nuanced understanding of the relationships between individuals in discussions, capturing not only the structural but also the thematic aspects of community formation.

Another proposal for allowing the consideration of attacks is [20], where Kontarinis et al. advance an idea in the context of modeling online multi-agent debates involving multi-party argumentation. The introduction of agents in a debate with expertise on specific areas opens an interesting perspective: when a debate is deemed unresolved in a "controversial" manner, calling an additional expert may be a natural way to help make a decision. The expert then can analyze the situation from a more informed point of view, and introduce a resolution. Different examples of application domains are studied, such as the construction of more interactive forums on the Web like Debate Graph. Some of these systems just provide a way to represent arguments, attacks (in a declarative or abstract way) and information about them. Furthermore, some systems include reasoning machinery, usually from argumentation theory, which provides a formal way to decide on the acceptability of statements supported by acceptable arguments. However, the conflicting relation between arguments presented in a debate has different importance levels according to the votes presented by experts. The quality of these experts can vary depending on the topics of each argument. Thus, the influence of an expert to judge an attack relation depends on the quality of such expert in the topics presented by the conflicting arguments. Furthermore, in some domains, an argumentation process can be controversial: one case is when several conflicting (in some sense) acceptable outcomes are returned after the argumentation process; another case arises when voting does not offer a clear majority to support the fact that an attack should be taken or not into account. In order to avoid these issues, the authors set up a model extending the classical argumentation framework, assuming that arguments are tagged with the topics they refer to. Although arguments are decorated with topics, there is no semantic network that allows an inter-argument analysis to evaluate a relative difference between arguments. The whole process is related to the context of multi-agent argumentation—briefly speaking, they propose a procedure composed by three phases: the first phase consists in the aggregation of the different opinions of the agents, and allows to obtain an aggregated weighted argumentation system (WAS); then comes an evaluation phase, which allows to determine how controversial the aggregation is; finally, in the third phase, they choose an expert to make the aggregated WAS less controversial.

7. Conclusions

In essence, abstract argumentation is the study of arguments and their relationships, transcending certain details of the underlying structural logic. Arguments are treated as abstract entities linked through specific relations like attack and defeat, and later support and weakening, among others. Since the seminal work of Dung, some authors elaborated new semantic notions on the same abstract framework, while others proposed enriched formalizations by adding new details towards different models of argumentation. The addition of these elements is needed because a wide range of expressions of arguments and dialogues with intrinsic properties cannot be captured by only two simple elements of a graph: nodes and attacks.

In this paper we have developed a formalization of an enriched abstract argumentation framework that aims to include, in the semantic analysis, information about *what* the arguments are referring to. This is important because abstract frameworks and their corresponding semantics are based on an exhaustive consideration of arguments and attacks while ignoring the fact that, when considering the acceptance of an argument, some arguments may be more relevant than others. Usually, when humans argue about a subject they tend to include, sooner or later, arguments that refer to minor, almost unrelated subjects that are not close to the original subject. In our proposal, arguments are decorated with abstract topics in the form of *hashtags* as it is commonly done in social media. These topics provide abstract information about what the arguments are addressing, and provide a supporting structure for the analysis of multi-topic argumentation. Basic admissibility semantics were presented in previous papers, where the consideration of argument defenses is restricted to close (in the sense of topics) arguments. In this work we elaborate new semantic notions by the characterization of individual areas of closeness for every argument, called *neighborhoods*. Topics are related to each other, configuring a graph structure as a semantic network in which a notion of distance between topics is introduced naturally and is used to identify closeness between topics, leading to the study of proximity-based semantics.

The central aspect of these argumentation semantics is the initial idea that an argument should be defended by closely related arguments linked to the addressed topics. We explore this idea by defining here new elements such as neighborhood-bounded admissible sets and the connections between neighborhoods. A community is a chain of neighborhoods sharing arguments—such shared arguments serve as a bridge for expanding the notion of proximate defenders. Hence, an argument may find defenders that although are not neighbors, belong to a somehow familiar set of indirect neighbors. A community is then a connected, second range of proximity-based defenses. We analyzed the relation between these new formalizations, an earlier version of proximity-based admissibility, and the classical admissibility semantics. These semantic notions provide new characterizations for the idea of *focused argumentation*, where extensions are defined only by arguments that relate to a subset of interconnected topics, close enough to each other, avoiding an analysis of arguments that digress from certain subjects. We believe that the semantic extensions presented constitute an important step in the search for novel models of concentrated argumentation.

The addition of topics to abstract argumentation suggests several directions for future work. There is room for a wide study of centrality notions applied to topics and their arguments; for instance, *central* and *peripheral* arguments can be identified in a similar way as central and peripheral nodes are in a graph, and varied centrality-inspired semantic extensions can be defined. If more detail about the underlying logical structure is added, like rules or literals, more precise forms of focused extensions can be explored. We are also interested in the construction and analysis of the semantic network of hashtags. We believe that the areas of information retrieval and social network analysis may contribute to establish the importance of arguments according to their own set of hashtags in a given hashcloud, leading to new characterizations of focused argumentation.

CRediT authorship contribution statement

Irene M. Coronel: Writing – review & editing, Writing – original draft, Investigation, Formal analysis. **Melisa G. Escañuela Gonzalez:** Writing – review & editing, Validation, Methodology, Formal analysis. **Diego C. Martinez:** Writing – review & editing, Conceptualization. **Gerardo I. Simari:** Writing – review & editing, Supervision, Funding acquisition. **Maximiliano C.D. Budán:** Writing – review & editing, Writing – original draft, Project administration, Methodology, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Gerardo I. Simari reports financial support was provided by Universidad Nacional de Entre Ríos under grant PDTS-UNER 7066. Gerardo I. Simari reports financial support was provided by Agencia Nacional de Promoción Científica y Tecnológica under grant PICT-2018-0475 - PRH-2014-0007.

Data availability

No data was used for the research described in the article.

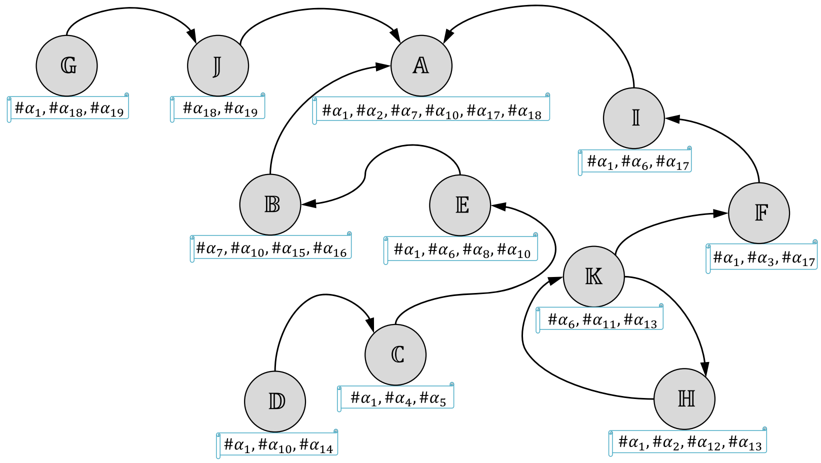


Fig. A.12. Proximity-based argumentation framework – case study.

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Appendix A. A case study for proximity-based semantics based on neighborhoods and communities

Higher education has been a topic of debate for decades, with many people believing it is crucial for success in life and career, while others question its value. In this discussion, we will present arguments for and against higher education, offering a variety of perspectives. By considering these arguments, we can gain a more comprehensive and critical understanding of the impact of higher education on society.

- 𝔸 Higher education is a valuable investment that offers numerous benefits, including higher incomes, improved job security, and enhanced critical thinking skills. Education also promotes social mobility, personal growth, and self-esteem. Graduates can make positive contributions to society by addressing environmental challenges and healthcare disparities. Additionally, education can improve overall health and well-being. In summary, higher education is crucial for personal and societal development and creating a more sustainable future for all.
- 𝔹 College graduates tend to earn more money over the course of their careers than non-graduates, which can improve their quality of life and financial stability.
- ℂ Higher education provides opportunities to develop social and emotional skills, as well as to build valuable professional and personal networks.
- 𝔻 Many jobs do not require higher education and can be equally rewarding and well paid, especially in trades and practical fields.
- 𝔼 The cost of higher education has increased significantly in recent decades, making access to higher education more difficult for many low-income students.
- 𝔽 Higher education can improve people's health, as health professionals require specialized and detailed training, leading to better overall health care.
- 𝔾 Higher education can have a positive impact on transport, as educated people tend to be more environmentally aware and can develop innovative solutions to reduce congestion and pollution.
- ℍ Higher education can help strengthen democracy and politics, as educated citizens are more informed, critical thinkers and can make more informed and responsible decisions at the ballot box.
- 𝕀 Higher education can have a negative impact on mental health, as university students experience high levels of stress, anxiety, and depression due to workload, pressure, and competition.
- 𝕁 The concentration of universities in urban areas can lead to traffic problems and congestion, which can have negative effects on city transport, leading to environmental problems.
- 𝕂 University students may be less critical and more conformist in their political thinking, which may affect their ability to challenge the status quo and contribute to positive societal changes.

Thus, based on the previous argumentation discussion, we consider the Hashtagged Argumentation Framework $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathsf{d}_\Omega \rangle$, graphically represented in Fig. A.12, where:

$\#\alpha_1$	High education	$\#\alpha_{10}$	Income
$\#\alpha_2$	Critical thinking	$\#\alpha_{11}$	Society
$\#\alpha_3$	Specialized training	$\#\alpha_{12}$	Democracy
$\#\alpha_4$	Professional networks	$\#\alpha_{13}$	Politics
$\#\alpha_5$	Social skills	$\#\alpha_{14}$	Jobs
$\#\alpha_6$	Students	$\#\alpha_{15}$	Finantial stability
$\#\alpha_7$	College graduated	$\#\alpha_{16}$	Quality of life
$\#\alpha_8$	Cost	$\#\alpha_{17}$	Health
$\#\alpha_9$	Inequality	$\#\alpha_{18}$	Environment
	$\#\alpha_{19}$	Transport	

Fig. A.13. Topics comprising the argumentation discussion – case study.

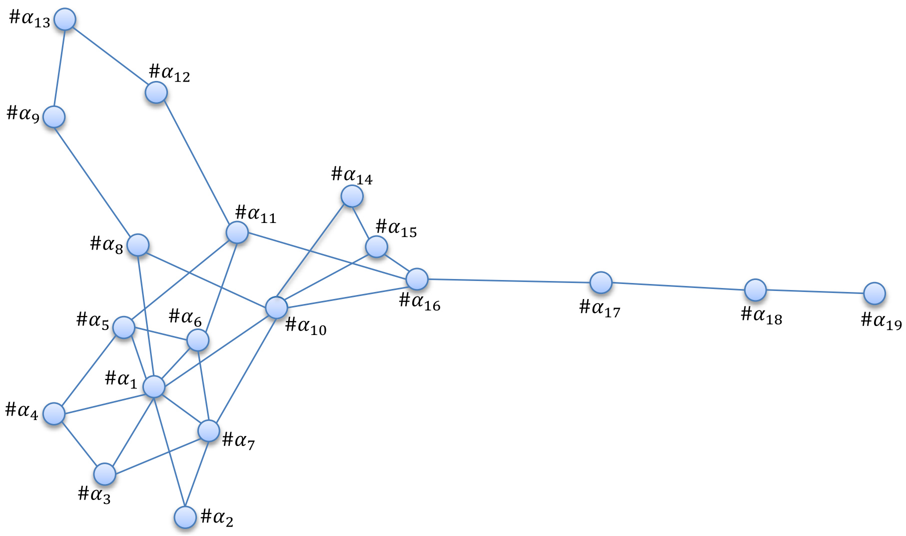

Fig. A.14. Hashcloud associated with the argumentation discussion – case study.

$\mathcal{H} = \{\#\alpha_1, \#\alpha_2, \dots, \#\alpha_{19}\}$ (detailed in Fig. A.13).
$E = \{(\#\alpha_1,\#\alpha_2),(\#\alpha_1,\#\alpha_3),(\#\alpha_1,\#\alpha_4),(\#\alpha_1,\#\alpha_5),(\#\alpha_1,\#\alpha_6),(\#\alpha_1,\#\alpha_7),(\#\alpha_1,\#\alpha_8),(\#\alpha_1,\#\alpha_{10}),(\#\alpha_2,\#\alpha_7),(\#\alpha_3,\#\alpha_4),(\#\alpha_3,\#\alpha_7),(\#\alpha_4,\#\alpha_5),(\#\alpha_5,\#\alpha_6),(\#\alpha_5,\#\alpha_{11}),(\#\alpha_6,\#\alpha_7),(\#\alpha_6,\#\alpha_{11}),(\#\alpha_7,\#\alpha_{10}),(\#\alpha_8,\#\alpha_9),(\#\alpha_8,\#\alpha_{10}),(\#\alpha_9,\#\alpha_{13}),(\#\alpha_{10},\#\alpha_{14}),(\#\alpha_{10},\#\alpha_{15}),(\#\alpha_{10},\#\alpha_{16}),(\#\alpha_{11},\#\alpha_{12}),(\#\alpha_{11},\#\alpha_{16}),(\#\alpha_{12},\#\alpha_{13}),(\#\alpha_{14},\#\alpha_{15}),(\#\alpha_{15},\#\alpha_{16}),(\#\alpha_{16},\#\alpha_{17}),(\#\alpha_{17},\#\alpha_{18}),(\#\alpha_{18},\#\alpha_{19})\}$ (cf. Fig. A.14).
$\texttt{Args} = \{\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}, \mathbb{I}, \mathbb{J}, \mathbb{K}\}$.
$\texttt{Attacks} = \{(\mathbb{D},\mathbb{C}),(\mathbb{C},\mathbb{E}),(\mathbb{E},\mathbb{B}),(\mathbb{B},\mathbb{A}),(\mathbb{G},\mathbb{J}),(\mathbb{J},\mathbb{A}),(\mathbb{A},\mathbb{I}),(\mathbb{I},\mathbb{F}),(\mathbb{K},\mathbb{F}),(\mathbb{K},\mathbb{H}),(\mathbb{H},\mathbb{K})\}$

Then, consider the following non-intersection distance:

$$d_\Omega(\mathbb{A},\mathbb{B}) = \begin{cases} max(d_{\mathscr{G}}(\alpha,\beta)) & \textit{where } \alpha \in \mathcal{H}_{\mathbb{A}} \setminus \mathcal{H}_{\mathbb{B}} \textit{ and } \beta \in \mathcal{H}_{\mathbb{B}} \setminus \mathcal{H}_{\mathbb{A}}, \\ 0 & \textit{when } \mathcal{H}_{\mathbb{A}} = \mathcal{H}_{\mathbb{B}}, \\ \infty & \textit{if for all } \alpha \in \mathcal{H}_{\mathbb{A}} \textit{and } \beta \in \mathcal{H}_{\mathbb{B}}, \\ & \textit{there is no path between them.} \end{cases}$$

We calculate the distances between arguments detailed in Table A.4, and analyze in depth the hashcloud $\mathscr{G}_\Omega$ corresponding to Ω represented in Fig. A.14. Now, in this state, we determine the neighborhood of the hashtagged arguments in order to allow only those defenses that are close enough to the topics represented by the set of hashtags associated with each argument. We take, as we established in the previous example, the centrality degree and neighbor degree as local topological measures, and radius and diameter as general topological measures. Table A.5 shows the neighborhood associated with each argument that belongs to the argumentation discussion.

Table A.4
Distances between the hashtagged arguments in Ω – case study.

	𝔸	𝔹	ℂ	𝔻	𝔼	𝔽	𝔾	ℍ	𝕀	𝕁	𝕂
𝔸	0	4	5	4	4	5	6	5	4	6	5
𝔹	4	0	3	2	3	3	5	4	3	5	4
ℂ	5	3	0	3	2	4	6	4	4	6	4
𝔻	4	2	3	0	3	3	5	4	3	5	4
𝔼	4	3	2	3	0	3	6	3	4	5	3
𝔽	5	3	4	3	3	0	6	4	4	6	4
𝔾	6	5	6	5	6	6	0	6	5	5	6
ℍ	5	4	4	4	3	4	6	0	4	6	4
𝕀	4	3	4	3	4	4	5	4	0	5	4
𝕁	6	5	6	5	5	6	5	6	5	0	6
𝕂	5	4	4	4	3	4	6	4	4	6	0

Table A.5
Argument neighborhoods from the case study.

Argument	Radius	Diameter	Centrality Degree	Neighbor Degree	Local Topological Propery	Global Topological Property	Treshold	Neighborhoods
𝔸	3	5	[8,2,5,6,2,2]	$\left[\frac{11}{17}\frac{2}{3}\frac{3}{5}\frac{7}{11}\frac{5}{8}\frac{1}{2}\right]$	2	5	3	𝔸
𝔹	1	2	[5,6,3,4]	$\left[\frac{3}{5}\frac{7}{11}\frac{2}{3}\frac{4}{7}\right]$	3	2	2	𝔹
ℂ	1	1	[8,3,4]	$\left[\frac{11}{17}\frac{2}{3}\frac{4}{7}\right]$	3	1	2	ℂ, 𝔼
𝔻	1	2	[8,6,2]	$\left[\frac{11}{17}\frac{7}{11}\frac{7}{11}\right]$	2	2	2	𝔹, 𝔻
𝔼	1	2	[8,4,3,6]	$\left[\frac{11}{17}\frac{7}{12}\frac{1}{3}\frac{7}{11}\right]$	3	2	2	ℂ, 𝔼
𝔽	3	4	[8,3,2]	$\left[\frac{11}{17}\frac{1}{2}\frac{5}{8}\right]$	2	4	3	𝔹, 𝔻, 𝔼, 𝔽
𝔾	4	5	[8,2,1]	$\left[\frac{11}{17}\frac{1}{2}0\right]$	1	5	3	𝔾
ℍ	3	4	[8,2,2,2]	$\left[\frac{11}{17}\frac{2}{3}\frac{3}{5}\frac{1}{3}\right]$	2	4	3	𝔼, ℍ
𝕀	3	3	[8,4,2]	$\left[\frac{11}{17}\frac{7}{12}\frac{5}{8}\right]$	2	3	2	𝕀
𝕁	1	1	[2,1]	$\left[\frac{1}{2}0\right]$	1	1	1	𝕁
𝕂	2	3	[4,4,2]	$\left[\frac{7}{12}\frac{4}{7}\frac{1}{3}\right]$	2	3	2	𝕂

Continuing with the Hashtagged Argumentation Framework depicted in Fig. A.15 based on the neighborhoods associated with each argument presented in Table A.5, we have that:

- 𝔾 is not a defender of 𝔸 since the distance between 𝔾 and 𝔸 is 6, while 𝔸 has a neighborhood range of 3.
- On the other hand, 𝔻 is not a defender of 𝔼 since 𝔻 is not a neighbor of 𝔼 (the distance between 𝔻 and 𝔼 is 3, where 𝔼 has a neighborhood range of 2). Furthermore, 𝔼 is not a defender of 𝔸 since the distance between 𝔼 and 𝔸 is 4 and 𝔸 has a neighborhood range of 3.
- ℍ is not a defender of 𝔽 since ℍ is not a neighbor of 𝔽 (the distance between ℍ and 𝔽 is 4, where 𝔽 has a neighborhood range of 3). Also, 𝕂 is not a defender of 𝕀 since the distance between 𝕂 and 𝕀 is 4 and 𝕀 has a neighborhood range of 2. Thus, 𝕂 is not a neighbor of 𝕀.
- Finally, 𝔽 is not a defender of 𝔸 since 𝔽 is not a neighbor of 𝔸 (the distance between 𝔸 and 𝔽 is 5, where 𝔸 has a neighborhood range of 3).

Thus, analyzing the acceptability notions presented in Definition 16, the sets $\mathbb{S}_1 = \{\mathbb{D}, \mathbb{G}\}$, $\mathbb{S}_2 = \{\mathbb{D}, \mathbb{G}, \mathbb{H}\}$, and $\mathbb{S}_3 = \{\mathbb{D}, \mathbb{G}, \mathbb{K}\}$ are the maximal η-admissible extensions. Furthermore, $\mathbb{S}_1$ is the *η-grounded extension*, while $\mathbb{S}_2$ and $\mathbb{S}_3$ are the *η-preferred extensions* under the conditions established in Definition 17 (see Fig. A.16).

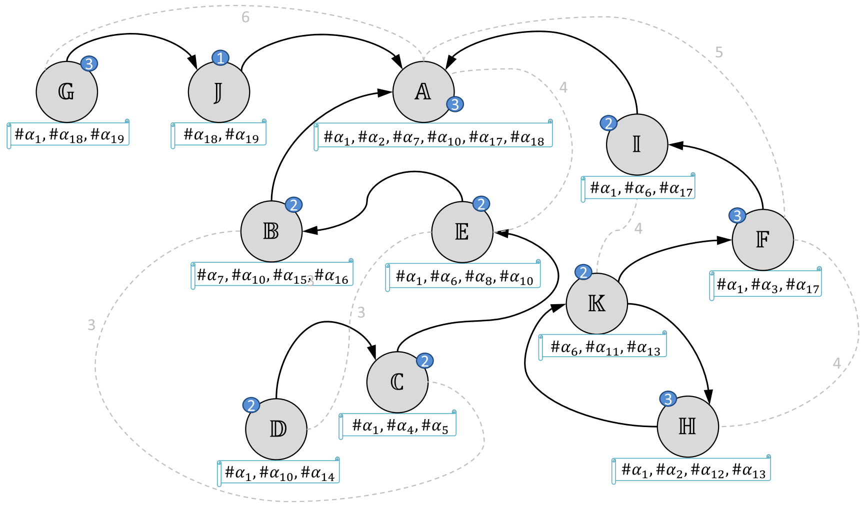

Fig. A.15. Framework from the case study considering neighborhood parameters.

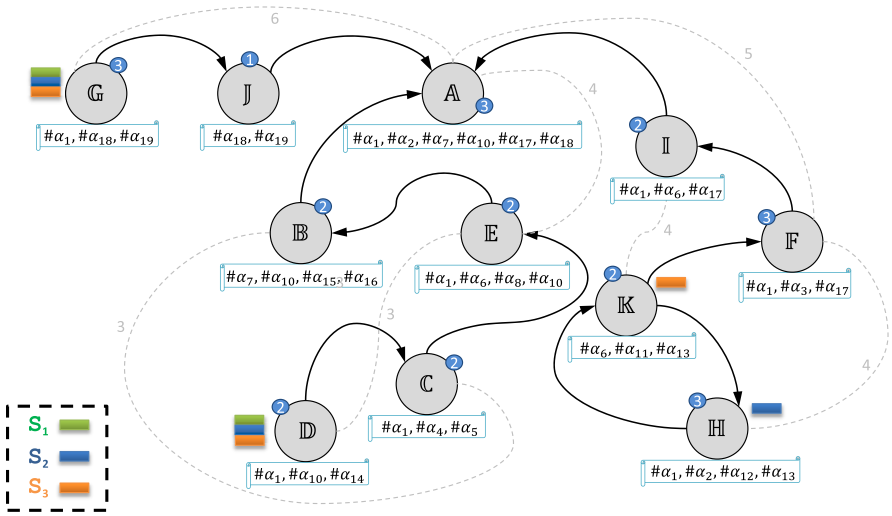

Fig. A.16. The *η-extensions* in Ω from the case study.

Now, taking as basis the Hashtagged Argumentation Framework depicted in Fig. A.15, the neighborhoods associated with each argument detailed in Table A.5 and considering the Definition 19, we can obtain the following four communities:

- $\mathfrak{C}_1 = \{\mathbb{A}\}$, $\mathfrak{C}_2 = \{\mathbb{G}\}$, $\mathfrak{C}_3 = \{\mathbb{I}\}$, $\mathfrak{C}_4 = \{\mathbb{J}\}$ and $\mathfrak{C}_5 = \{\mathbb{J}\}$ are single communities; and
- $\mathfrak{C}_6 = \{\mathbb{B}, \mathbb{C}, \mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{H}\}$, where from $\mathbb{F}$ we find a sequence to arrive to $\mathbb{C}$ ($\mathbb{F}$ has $\mathbb{E}$ as neighbor, which is at the same time neighbor of $\mathbb{C}$); Furthermore, since we have $\mathbb{E}$ in the community we can incorporate $\mathbb{H}$ (by Definition 16, we have a sequence between arguments iff $\mathbb{H} \rightsquigarrow \mathbb{E}$ or $\mathbb{E} \rightsquigarrow \mathbb{H}$).

These communities are non-empty, disjoint, and their union is the set $\mathtt{Args}$.

Now, analyzing the acceptability notions presented in Definition 21, the sets: $\mathbb{S}_4 = \{\mathbb{D}, \mathbb{E}, \mathbb{G}\}$, $\mathbb{S}_5 = \{\mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}\}$ and $\mathbb{S}_6 = \{\mathbb{D}, \mathbb{G}, \mathbb{E}, \mathbb{K}\}$ are the maximal ζ-admissible extensions. Note that the color of the spheres accompanying the arguments indicating their neighborhood's range differentiates the color of the communities to which they belong. Furthermore, $\mathbb{S}_4$ is the *ζ-grounded extension*, while $\mathbb{S}_5$ and $\mathbb{S}_6$ are the *ζ-preferred extensions*, under the conditions established in Definition 17 (see Fig. A.16 and Fig. A.17). In Table A.6, we summarize the extensions obtained from the different proposed semantics to highlight the differences between them.

We have analyzed a case study to evaluate the effects of Higher Education on graduates' lives and other related concerns. We considered arguments from various perspectives, both in favor and against the arguments involved. Thus, considering the Neighborhood point of view, we can arrive at the following conclusions:

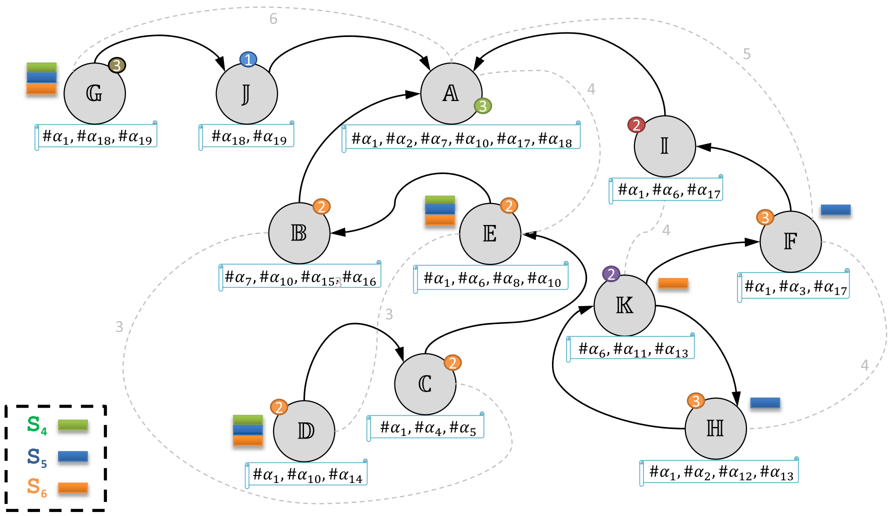

Fig. A.17. The ζ-*extensions* in Ω from the case study.

Table A.6
Summary of proximity-based argumentation semantics in Ω from the case study.

	Grounded Extension	Preferred Extension I	Preferred Extension II
Neighborhood	$\{\mathbb{D}, \mathbb{G}\}$	$\{\mathbb{D}, \mathbb{G}, \mathbb{H}\}$	$\{\mathbb{D}, \mathbb{G}, \mathbb{K}\}$
Communities	$\{\mathbb{D}, \mathbb{E}, \mathbb{G}\}$	$\{\mathbb{D}, \mathbb{E}, \mathbb{F}, \mathbb{G}, \mathbb{H}\}$	$\{\mathbb{D}, \mathbb{E}, \mathbb{G}, \mathbb{K}\}$

– Considering the accepted arguments $\mathbb{D}$, $\mathbb{G}$, and $\mathbb{H}$, we can conclude the complex relationship between higher education and various domains of society. While many jobs do not require higher education, it can have a positive impact on transport by creating environmentally aware individuals who can develop innovative solutions to reduce congestion and pollution. Moreover, higher education can strengthen democracy and politics by producing more informed and critical thinkers who can make better decisions at the ballot box. However, the potential for university students to become less critical and more conformist in their political thinking suggests that higher education may only sometimes lead to positive societal changes. Ultimately, the value of higher education depends on its ability to prepare individuals for their roles in society, whether in practical fields or in shaping the future of our democracy.
– Based on $\mathbb{D}$, $\mathbb{G}$, and $\mathbb{K}$, the accepted arguments highlight the importance of considering the role of higher education concerning different domains of society. While many jobs do not require higher education, it can positively impact areas such as transportation by creating environmentally aware individuals who can develop innovative solutions to address challenges such as congestion and pollution. However, the potential for university students to become less critical and more conformist in their political thinking suggests that higher education may not always lead to positive societal changes. Therefore, it is crucial to consider the potential drawbacks of higher education as well as its benefits, particularly in promoting critical thinking and a willingness to challenge the status quo. Ultimately, a well-rounded approach to education that balances practical skills and critical thinking may be the most effective way to promote positive societal changes.

Now, taking into account the communities point of view, we can conclude the following:

– Based on $\mathbb{D}$, $\mathbb{E}$, $\mathbb{F}$, $\mathbb{G}$, and $\mathbb{H}$, we can conclude that there exist both benefits and challenges of higher education in different domains of society. While many jobs do not require higher education, they can positively impact areas such as transportation, democracy, and health care. Educated individuals tend to be more environmentally aware and can develop innovative solutions to address transportation issues such as congestion and pollution. They also have the potential to make more informed and responsible decisions at the ballot box, ultimately strengthening democracy and politics. Additionally, specialized training in health care professions can lead to better overall health care. However, the rising cost of higher education has made access to higher education more difficult for many low-income students, presenting a challenge to the potential benefits of higher education. Therefore, it is essential to consider both the benefits and challenges of higher education in order to ensure that it remains accessible and beneficial to individuals and society as a whole.
– From these arguments $\mathbb{D}$, $\mathbb{G}$, and $\mathbb{K}$, it can be concluded that while higher education can have benefits such as improving environmental awareness and critical thinking skills, it is only sometimes necessary for a successful and fulfilling career. However, the increasing cost of higher education may limit access to those who could benefit from it the most. Additionally, higher education may lead to conformity in political thinking, potentially limiting the ability of graduates to challenge established systems

and contribute to positive change. Ultimately, the decision to pursue higher education should be weighed carefully, considering both personal goals and the potential societal impact.

We aimed to demonstrate how including specific characteristics in abstract argumentation frameworks can enhance the representation of topological elements associated with an argument's proximity to other framework members, underscoring the value of our approach as a reliable knowledge representation tool. To demonstrate these capabilities more comprehensively, we plan to conduct a more elaborate experiment in a forthcoming paper.

Appendix B. Proofs

Proposition 1. *Given an argument* $\mathbb{A} \in \mathtt{Args}$*,* $\mathbb{A}$ *is not attacked in* Ω *iff* $\mathbb{A}$ *is not attacked in* Φ*.*

Proof. This result follows from the definitions of Φ and Ω. If $\mathbb{A} \in \mathtt{Args}$ is not attacked in Ω, then there is no hashtagged argument $\mathbb{B} \in \mathtt{Args}$ such that $\mathbb{B}$ attacks $\mathbb{A}$ in Ω. Since Φ is an extension of Ω, there is no hashtagged argument $\mathbb{B} \in \mathtt{Args}$ such that $\mathbb{B}$ attacks $\mathbb{A}$ in Φ. □

Proposition 2. *Let* $\mathfrak{N}_\Omega^{\varepsilon_\Omega}$ *be the set of neighborhoods associated with the arguments of* Ω *and* Φ *be the underlying abstract argumentation framework. Then:*

i) *If* $\mathbb{A} \in \mathtt{Args}$ *is* η*-acceptable* w.r.t. *a set* $\mathtt{S}$ *in* Ω*, then it is acceptable* w.r.t. $\mathtt{S}$ *in* Φ*.*
ii) *If a set* $\mathtt{S}$ *is* η*-admissible in* Ω *then it is admissible in* Φ*.*

Proof. This demonstration follows directly from the definitions, and it will be done in two parts:

i) If $\langle \mathtt{A}, \mathcal{H}_\mathtt{A} \rangle \in \mathtt{Args}$ is η-acceptable *w.r.t.* a set $\mathtt{S}$ in Ω, then $\mathtt{A}$ is acceptable *w.r.t.* $\mathtt{S}$ in Φ. By hypothesis, $\langle \mathtt{A}, \mathcal{H}_\mathtt{A} \rangle$ is *η-acceptable w.r.t.* $\mathtt{S}$, then for every attacker hashtagged argument $\langle \mathtt{B}, \mathcal{H}_\mathtt{B} \rangle \in \mathtt{Args}$ there exists a defender $\langle \mathtt{C}, \mathcal{H}_\mathtt{C} \rangle \in \mathtt{S}$. Thus, we can say that $\mathtt{A}$ is *acceptable w.r.t.* $\mathtt{S}$ since there exists a defender $\mathtt{C}$ for $\mathtt{A}$ from $\mathtt{B}$ attack's.

ii) If a set $\mathtt{S}$ is η-admissible in Ω then it is admissible in Φ. By hypothesis, $\mathtt{S}$ is *η-admissible* in Ω. Thus, every hashtagged argument in $\mathtt{S}$ is *η-acceptable w.r.t.* $\mathtt{S}$. Furthermore, by consequence of i) if a hashtagged argument $\mathbb{A} \in \mathtt{Args}$ is *η-acceptable w.r.t.* a set $\mathtt{S}$ in Ω then it is acceptable *w.r.t.* $\mathtt{S}$ in Φ. Then, we can deduce that $\mathtt{S}$ is admissible in Φ. □

Proposition 3 *([13]). Let* $\Omega = \langle \Phi, \mathscr{G}_\Omega, \mathrm{d}_\Omega \rangle$ *be the underlying a hashtagged framework,* $\tau \in \mathbb{N}^0$ *be a threshold,* $\mathfrak{N}_\Omega^{\varepsilon_\Omega}$ *be the set of neighborhoods associated with the arguments of* Ω*,* $\mathfrak{T}_\Omega^{\mathtt{s}}$ *and* $\mathfrak{T}_\Omega^{\mathtt{g}}$ *be the radius associated with the smallest and greatest neighborhoods of* $\mathfrak{N}_\Omega$*, respectively; let* $\mathtt{S} \subseteq \mathtt{Args}$ *be a set of hashtagged arguments. Then, we have:*

i) *If* $\mathbb{A} \in \mathtt{Args}$ *is* τ*-acceptable* w.r.t. $\mathtt{S}$ *with* $\tau = \mathfrak{T}_\Omega^{\mathtt{s}}$*, then it is* η*-acceptable* w.r.t. $\mathtt{S}$*;*
ii) *If* $\mathbb{A} \in \mathtt{Args}$ *is* η*-acceptable* w.r.t. $\mathtt{S}$ *then it is* τ*-acceptable* w.r.t. $\mathtt{S}$ *with* $\tau = \mathfrak{T}_\Omega^{\mathtt{g}}$*;*
iii) *If* $\mathtt{S}$ *is* τ*-admissible with* $\tau = \mathfrak{T}_\Omega^{\mathtt{s}}$*, then it is* η*-admissible; and*
iv) *If* $\mathtt{S}$ *is* η*-admissible then it is* τ*-admissible with* $\tau = \mathfrak{T}_\Omega^{\mathtt{g}}$*.*

Proof. This demonstration will be done in four parts:

1) If $\mathbb{A} \in \mathtt{Args}$ is τ-acceptable *w.r.t.* a set $\mathtt{S}$ with $\tau = \mathfrak{T}_\Omega^{\mathtt{s}}$, then it is η-acceptable *w.r.t.* $\mathtt{S}$. By hypothesis, $\mathbb{A}$ is *τ-acceptable w.r.t.* $\mathtt{S}$ with $\tau = \mathfrak{T}_\Omega^{\mathtt{s}}$. Thus, the threshold applied to obtain the acceptable set of arguments is equal to the radius associated with the lowest neighborhood of $\mathfrak{N}_\Omega$. Then, for every attacked hashtagged argument $\mathbb{A} \in \mathtt{S}$ there exists a defender $\mathbb{C} \in \mathtt{S}$ such that $\mathbb{C} \in \mathfrak{N}_\mathbb{A}^{\varepsilon_\mathbb{A}}$. Then, $\mathbb{A}$ is η-acceptable *w.r.t.* $\mathtt{S}$.

2) If $\mathbb{A} \in \mathtt{Args}$ is η-acceptable *w.r.t.* a set $\mathtt{S}$, then it is τ-acceptable *w.r.t.* $\mathtt{S}$ with $\tau = \mathfrak{T}_\Omega^{\mathtt{g}}$. By hypothesis, $\mathbb{A}$ is *η-acceptable w.r.t.* $\mathtt{S}$. Thus, for every attacker hashtagged argument $\mathbb{B} \in \mathtt{S}$ there exists a defender $\mathbb{C} \in \mathtt{S}$ such that $\mathbb{C} \in \mathfrak{N}_\mathbb{A}^{\varepsilon_\mathbb{A}}$ where the radius of $\mathfrak{N}_\mathbb{A}^{\varepsilon_\mathbb{A}}$ is lower or equal than τ. Then, $\mathbb{A}$ is τ-acceptable *w.r.t.* $\mathtt{S}$ with $\tau = \mathfrak{T}_\Omega^{\mathtt{g}}$.

3) If a set $\mathtt{S}$ is *τ-admissible* with $\tau = \mathfrak{T}_\Omega^{\mathtt{s}}$, then it is η-admissible. By hypothesis, $\mathtt{S}$ is *τ-admissible*; then, every hashtagged argument in $\mathtt{S}$ is *τ-acceptable w.r.t.* $\mathtt{S}$. Furthermore, as a consequence of *i*), if a hashtagged argument $\mathbb{A} \in \mathtt{Args}$ is *τ-acceptable w.r.t.* a set $\mathtt{S}$ then it is η-acceptable *w.r.t.* $\mathtt{S}$.

4) If a set $\mathtt{S}$ is η-admissible then it is τ-admissible with $\tau = \mathfrak{T}_\Omega^{\mathtt{g}}$. By hypothesis, $\mathtt{S}$ is *η-admissible*; thus, every hashtagged argument in $\mathtt{S}$ is *η-acceptable w.r.t.* $\mathtt{S}$. Furthermore, by consequence of *i*) if a hashtagged argument $\mathbb{A} \in \mathtt{Args}$ is *η-acceptable w.r.t.* a set $\mathtt{S}$ then it is τ-acceptable *w.r.t.* $\mathtt{S}$. □

Proposition 4. *Let* $\mathfrak{N}_\Omega^{\varepsilon_\Omega}$ *be the set of neighborhoods associated with the arguments of* Ω*. If the smallest neighborhood* $\mathfrak{T}_\Omega^{\mathtt{s}}$ *is such that* $\mathfrak{T}_\Omega^{\mathtt{s}} \geq diameter(\mathcal{H})$*, then every* η*-{admissible, complete, grounded, preferred} extension is an {admissible, complete, grounded, preferred} extension, respectively.*

Proof. For the acceptability-based semantics, it is sufficient to prove that the hashtagged defender arguments in Ω are also present as defenders in the underlying abstract argumentation framework, for the η-admissible, complete, grounded, preferred extension. Suppose that an argument $\mathtt{A}$ is defended by an argument $\mathtt{C}$ from the attacks of $\mathtt{B}$ in the underlying abstract argumentation framework, but the counterpart hashtagged argument $\mathbb{A}$ is not defended in Ω by the hashtagged version of $\mathbb{C}$. This means that, by Definition 16, $\mathrm{d}_{\Omega}(\mathbb{A},\mathbb{C}) \notin \mathfrak{N}_{\mathbb{A}}^{\varepsilon_{\mathbb{A}}}$ (†). However, the radius associated with the smallest neighborhood $\mathfrak{T}_{\Omega}^{s} \geq diameter(\mathcal{H})$ where $diameter(\mathcal{H})$ is the maximum eccentricity of the hashtags in $\mathcal{H}$. Nevertheless, the relation (†) is not possible since $diameter(\mathcal{H})$ is the maximum of the distances to all other hashtags in $\mathcal{H}$. Contradiction. □

Proposition 5. *There always exists a unique η-grounded extension.*

Proof. Suppose there are two distinct sets $\mathtt{S}$ and $\mathtt{S}'$, both serving as η-grounded extensions. Let $\mathbb{A} \in \mathtt{S}$ and $\mathbb{A} \notin \mathtt{S}'$. This implies that $\mathbb{A}$ is not η-acceptable with respect to $\mathtt{S}'$. Therefore, for any argument $\mathbb{B} \in \mathtt{Args}$ attacking $\mathbb{A}$, there does not exist a hashtagged argument $\mathbb{C} \in \mathtt{S}'$ such that $\mathbb{C} \in \mathfrak{N}_{\mathbb{A}}^{\varepsilon_{\mathbb{A}}}$ and $\mathbb{C}$ attacks $\mathbb{B}$. However, according to our hypothesis, $\mathbb{A} \in \mathtt{S}$ is an η-complete extension. Thus, $\mathbb{A} \in \mathtt{Args}$ is η-acceptable with respect to $\mathtt{S}$. This means that for any argument $\mathbb{B} \in \mathtt{Args}$, if $\mathbb{B}$ attacks $\mathbb{A}$, there exists a hashtagged argument $\mathbb{C} \in \mathtt{S}$ such that $\mathbb{C} \in \mathfrak{N}_{\mathbb{A}}^{\varepsilon_{\mathbb{A}}}$ and $\mathbb{C}$ attacks $\mathbb{B}$. Furthermore, if $\mathtt{S}'$ is a $\subseteq$-minimal η-complete extension, and $\mathbb{A} \notin \mathtt{S}'$, then $\mathbb{A} \in \mathtt{Args}$ is not η-acceptable with respect to $\mathtt{S}'$. However, as mentioned earlier, there exists a set of arguments $\mathtt{S}$ defending $\mathbb{A}$ in ϕ. Consequently, $\mathtt{S}'$ cannot be a $\subseteq$-minimal η-complete extension. Thus, we arrive to a contradiction. □

Theorem 1 *([13]). Let $\mathfrak{N}_{\Omega}^{\varepsilon_{\Omega}}$ be the set of neighborhoods associated with the arguments of Ω, and $\Phi = \langle \mathtt{Args}, \mathtt{Attacks} \rangle$ be the underlying abstract argumentation framework. Then, the following properties hold:*

i) *If $\mathtt{S}_{\Omega}$ is η-complete extension in Ω, then there exists a complete extension $\mathtt{S}_{\Phi}$ in Φ such that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$;*
ii) *If $\mathtt{S}_{\Omega}$ is an η-grounded extension in Ω, then there exists a grounded extension $\mathtt{S}_{\Phi}$ in Φ such that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$; and,*
iii) *If $\mathtt{S}_{\Omega}$ is an η-preferred extension in Ω, then there exists a preferred extension $\mathtt{S}_{\Phi}$ in Φ such that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$.*

Proof. We separate the proof into three parts:

1) If $\mathtt{S}_{\Omega}$ is a η-complete extension in Ω, then there exists a complete extension $\mathtt{S}_{\Phi}$ in Φ satisfying that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$. Suppose that $\mathtt{S}_{\Omega}$ is a η-complete extension in Ω, but there is no complete extension $\mathtt{S}_{\Phi}$ in Φ satisfying that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$. Then, there exists the hashtagged argument $\mathbb{A}$ which is η-acceptable *w.r.t.* the η-admissible extension $\mathtt{S}_{\Omega}$ but the underlying argument (no hashtags) $\mathtt{A}$ is not acceptable *w.r.t.* $\mathtt{S}$. Thus, $\mathbb{A}$ is defended by $\mathtt{S}_{\Omega}$ but it is not defended by $\mathtt{S}$ in the underlying abstract argumentation framework. However, by Proposition 2, if $\mathbb{A} \in \mathtt{Args}$ is *η-acceptable w.r.t.* a set $\mathtt{S}$ then it is acceptable *w.r.t.* $\mathtt{S}$, and if a set $\mathtt{S}$ is *η-admissible* then $\mathtt{S}$ is admissible. Contradiction.

2) If $\mathtt{S}_{\Omega}$ is a η-grounded extension in Ω, then there exists a grounded extension $\mathtt{S}_{\Phi}$ in Φ satisfying that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$. This holds trivially, since because of *i*) if $\mathtt{S}_{\Omega}$ is a η-complete extension in Ω, there exists a complete extension $\mathtt{S}_{\Phi}$ in Φ satisfying that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$. Thus, the proof of this point is a special case where $\mathtt{S}_{\Omega}$ is the minimal η-complete extension in Ω and $\mathtt{S}_{\Phi}$ is the minimal complete extension in Φ.

3) If $\mathtt{S}_{\Omega}$ is η-preferred extension in Ω, then there exists a preferred extension $\mathtt{S}_{\Phi}$ in Φ satisfying that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$. This holds trivially, since because of *i*) if $\mathtt{S}_{\Omega}$ is η-complete extension in Ω, there exists a complete extension $\mathtt{S}_{\Phi}$ in Φ satisfying that $\mathtt{S}_{\Omega} \subseteq \mathtt{S}_{\Phi}$. Thus, the proof of this point is a special case where $\mathtt{S}_{\Omega}$ is a maximal η-complete extension in Ω and $\mathtt{S}_{\Phi}$ is a maximal complete extension in Φ where the inclusion condition is satisfied. □

Theorem 2 *([13]). Let $\Omega = \langle \Phi, \mathscr{G}_{\Omega}, \mathrm{d}_{\Omega} \rangle$ be the underlying a hashtagged framework, $\tau \in \mathbb{N}^0$ be a threshold, $\mathfrak{N}_{\Omega}^{\varepsilon_{\Omega}}$ be the set of neighborhoods associated with the arguments of Ω. Then, we have:*

i) *If $\tau = \mathfrak{T}_{\Omega}^{s}$ is the threshold associated with the smallest neighborhood of Ω, $\mathtt{S}_{\Omega}^{\eta}$ is an η-complete (respectively, η-grounded, and η-preferred) extension, and $\mathtt{S}_{\Omega}^{\tau}$ is a τ-complete (respectively, τ-grounded, and τ-preferred) extension, then it holds that $\mathtt{S}_{\Omega}^{\tau} \subseteq \mathtt{S}_{\Omega}^{\eta}$.*
ii) *If $\tau = \mathfrak{T}_{\Omega}^{g}$ is the threshold associated with the greatest neighborhood of Ω, $\mathtt{S}_{\Omega}^{\eta}$ is an η-complete (respectively, η-grounded, and η-preferred) extension, and $\mathtt{S}_{\Omega}^{\tau}$ is a τ-complete (respectively, τ-grounded, and τ-preferred) extension, then it holds that $\mathtt{S}_{\Omega}^{\eta} \subseteq \mathtt{S}_{\Omega}^{\tau}$.*

Proof. We separate the proof in two parts, each one divided into three items:

i) *a*) If $\tau = \mathfrak{T}_{\Omega}^{s}$ is the threshold associated with the smallest neighborhood of Ω, $\mathtt{S}_{\Omega}^{\eta}$ is an η-complete extension and $\mathtt{S}_{\Omega}^{\tau}$ is a τ-complete extension, then it holds that $\mathtt{S}_{\Omega}^{\tau} \subseteq \mathtt{S}_{\Omega}^{\eta}$. Suppose that $\mathtt{S}_{\Omega}^{\eta}$ is an η-complete extension and $\mathtt{S}_{\Omega}^{\tau}$ is not a τ-complete extension Then, there exists a hashtagged argument $\mathbb{A}$ which is τ-acceptable *w.r.t.* the *τ-admissible* extension $\mathtt{S}_{\Omega}^{\tau}$ but $\mathbb{A}$ is not η-acceptable *w.r.t.* $\mathtt{S}_{\Omega}^{\eta}$. Thus, $\mathbb{A}$ is defended by $\mathtt{S}_{\Omega}^{\tau}$ but it is not defended by $\mathtt{S}_{\Omega}^{\eta}$. However, by Proposition 3, if $\mathbb{A} \in \mathtt{Args}$ is τ-acceptable *w.r.t.* a set $\mathtt{S}_{\Omega}^{\tau}$ with $\tau = \mathfrak{T}_{\Omega}^{s}$, then it is η-acceptable, and if a set $\mathtt{S}_{\Omega}^{\varepsilon}$ is *τ-admissible* then it is η-admissible. Contradiction.
b) If $\tau = \mathfrak{T}_{\Omega}^{s}$ is the threshold associated with the smallest neighborhood of Ω, $\mathtt{S}_{\Omega}^{\eta}$ is a η-grounded extension and $\mathtt{S}_{\Omega}^{\tau}$ is a τ-grounded extension, then it holds that $\mathtt{S}_{\Omega}^{\tau} \subseteq \mathtt{S}_{\Omega}^{\eta}$. Trivial, since by *a*) if $\mathtt{S}_{\Omega}^{\eta}$ is η-complete extension and $\mathtt{S}_{\Omega}^{\tau}$ is τ-complete

extension, then it holds that $\mathtt{S}^{\tau}_{\Omega} \subseteq \mathtt{S}^{\eta}_{\Omega}$. Thus, the proof of this point is a special case where $\mathtt{S}^{\tau}_{\Omega}$ is the minimal τ-complete extension in Ω and $\mathtt{S}^{\eta}_{\Omega}$ is the minimal η-complete extension in Ω.

c) If $\tau = \mathfrak{T}^{s}_{\Omega}$ is the threshold associated with the smallest neighborhood of Ω, $\mathtt{S}^{\eta}_{\Omega}$ is a η-grounded extension and $\mathtt{S}^{\tau}_{\Omega}$ is a τ-grounded extension, then it holds that $\mathtt{S}^{\tau}_{\Omega} \subseteq \mathtt{S}^{\eta}_{\Omega}$. Trivial, since by *a*) if $\mathtt{S}^{\eta}_{\Omega}$ is a η-preferred extension and $\mathtt{S}^{\tau}_{\Omega}$ is a τ-preferred extension, then it holds that $\mathtt{S}^{\tau}_{\Omega} \subseteq \mathtt{S}^{\eta}_{\Omega}$. Thus, the proof of this point is a special case where $\mathtt{S}^{\tau}_{\Omega}$ is the maximal τ-complete extension in Ω and $\mathtt{S}^{\eta}_{\Omega}$ is the maximal η-complete extension in Ω.

ii) *a*) Let $\tau = \mathfrak{T}^{g}_{\Omega}$ be the threshold associated with the greatest neighborhood of Ω, $\mathtt{S}^{\eta}_{\Omega}$ is η-complete extension and $\mathtt{S}^{\tau}_{\Omega}$ is τ-complete extension, then it holds $\mathtt{S}^{\eta}_{\Omega} \subseteq \mathtt{S}^{\tau}_{\Omega}$. $\mathtt{S}^{\eta}_{\Omega}$ is η-complete extension and $\mathtt{S}^{\tau}_{\Omega}$ is not a τ-complete extension Then, there exists a hashtagged argument $\mathbb{A}$ which is η-acceptable *w.r.t.* the η-admissible extension $\mathtt{S}^{\eta}_{\Omega}$ but $\mathbb{A}$ is not τ-acceptable *w.r.t.* $\mathtt{S}^{\tau}_{\Omega}$. Thus, $\mathbb{A}$ is defended by $\mathtt{S}^{\eta}_{\Omega}$ but it is not defended by $\mathtt{S}^{\tau}_{\Omega}$. However, by Proposition 3, if $\mathbb{A} \in \mathtt{Args}$ is η-acceptable *w.r.t.* a set $\mathtt{S}^{\eta}_{\Omega}$, then it is τ-acceptable, and if a set $\mathtt{S}^{\eta}_{\Omega}$ is η-*admissible* then it is τ-admissible. Contradiction.

b) If $\tau = \mathfrak{T}^{g}_{\Omega}$ is the threshold associated with the smallest neighborhood of Ω, $\mathtt{S}^{\eta}_{\Omega}$ is a η-grounded extension and $\mathtt{S}^{\tau}_{\Omega}$ is a τ-grounded extension, then it holds that $\mathtt{S}^{\eta}_{\Omega} \subseteq \mathtt{S}^{\tau}_{\Omega}$. Trivial, since by *a*) if $\mathtt{S}^{\eta}_{\Omega}$ is η-complete extension and $\mathtt{S}^{\tau}_{\Omega}$ is τ-complete extension, then it holds that $\mathtt{S}^{\eta}_{\Omega} \subseteq \mathtt{S}^{\tau}_{\Omega}$. Thus, the proof of this point is a special case where $\mathtt{S}^{\eta}_{\Omega}$ is the minimal η-complete extension in Ω and $\mathtt{S}^{\tau}_{\Omega}$ is the minimal τ-complete extension in Ω.

c) If $\tau = \mathfrak{T}^{g}_{\Omega}$ is the threshold associated with the smallest neighborhood of Ω, $\mathtt{S}^{\eta}_{\Omega}$ is a η-grounded extension and $\mathtt{S}^{\tau}_{\Omega}$ is a τ-grounded extension, then it holds that $\mathtt{S}^{\eta}_{\Omega} \subseteq \mathtt{S}^{\tau}_{\Omega}$. Trivial, since by a) if $\mathtt{S}^{\eta}_{\Omega}$ is η-preferred extension and $\mathtt{S}^{\tau}_{\Omega}$ is τ-preferred extension, then it holds that $\mathtt{S}^{\tau}_{\Omega} \subseteq \mathtt{S}^{\eta}_{\Omega}$. Thus, the proof of this point is a special case where $\mathtt{S}^{\eta}_{\Omega}$ is the maximal η-complete extension in Ω and $\mathtt{S}^{\tau}_{\Omega}$ is the maximal τ-complete extension in Ω. □

Proposition 6. *Let $\mathfrak{C}_{\Omega}$ be the set of communities associated with the arguments of Ω. Given any $\mathfrak{C}_{\mathbb{A}}, \mathfrak{C}_{\mathbb{B}} \in \mathfrak{C}_{\Omega}$, either $\mathfrak{C}_{\mathbb{A}} \cap \mathfrak{C}_{\mathbb{B}} = \emptyset$ or $\mathfrak{C}_{\mathbb{A}} = \mathfrak{C}_{\mathbb{B}}$.*

Proof. Suppose $\mathfrak{C}_{\mathbb{A}}$ and $\mathfrak{C}_{\mathbb{B}}$ are two communities whose intersection is non-empty and are not equal. This means that there is an argument $\mathbb{X}$ that is in both communities, but there is at least one argument $\mathbb{Y}$ that is in $\mathfrak{C}_{\mathbb{A}}$ but not in $\mathfrak{C}_{\mathbb{B}}$, or vice versa. If there is a semantic path from $\mathbb{A}$ to $\mathbb{Y}$ and there is a semantic path from $\mathbb{A}$ to $\mathbb{X}$, then there is a semantic path from $\mathbb{X}$ to $\mathbb{Y}$, which means that $\mathbb{Y}$ should be in $\mathfrak{C}_{\mathbb{B}}$ as well. The same is true if there is a semantic path from $\mathbb{B}$ to $\mathbb{Y}$, which implies that $\mathbb{Y}$ should also be in $\mathfrak{C}_{\mathbb{A}}$. This contradicts our initial assumption that $\mathbb{Y}$ is in one community but not in the other. Therefore, our initial assumption that there are two communities whose intersection is non-empty, but they are not equal, leads to a contradiction. This means that, if the intersection between two communities is non-empty, then the two communities are equal. □

Proposition 7. *Let $\mathfrak{C}_{\Omega} = \{\mathfrak{C}_1, ..., \mathfrak{C}_n\}$ be the set of communities associated with the arguments of Ω. Then:*

$$\bigcup_{i=1}^{n} \mathfrak{C}_i = \mathtt{Args}.$$

Proof. It follows directly from Definition 19. □

Proposition 8. *Let $\mathfrak{C}_{\Omega}$ be the set of communities associated with the arguments of Ω. Then, if $\mathfrak{C}_{\Omega}$ is a singleton set then $\mathfrak{C}_{\Omega} = \mathtt{Args}$.*

Proof. If $\mathfrak{C}_{\Omega}$ is a singleton set, that is $\mathfrak{C}_{\Omega} = \{\mathfrak{C}_1\}$, then $\mathfrak{C}_{\Omega}$ consists of all arguments $\mathbb{B} \in \mathtt{Args}$ such that there is a hashtagged argument $\mathbb{A} \in \mathfrak{C}_1$ that verifies $\mathbb{B} \in \mathfrak{N}^{\epsilon_{\mathbb{A}}}_{\mathbb{A}}$. □

Lemma 1. *Let $\mathfrak{C}_{\Omega}$ be the set of communities associated with the arguments of Ω. Then, we have:*

i) *If $\mathbb{A} \in \mathtt{Args}$ is ζ-acceptable w.r.t. a set $\mathtt{S}$ then it is acceptable w.r.t. $\mathtt{S}$.*
ii) *If a set $\mathtt{S}$ is ζ-admissible then it is admissible.*

Proof. This demonstration follows directly from the definitions; we show this in two parts:

i) If $\mathbb{A} \in \mathtt{Args}$ is ζ-*acceptable w.r.t.* a set $\mathtt{S}$ then it is *acceptable w.r.t.* $\mathtt{S}$. Suppose that $\mathbb{A} \in \mathtt{Args}$ is ζ-*acceptable w.r.t.* a set $\mathtt{S}$ but $\mathbb{A} \in \mathtt{Args}$ is not *acceptable w.r.t.* a set $\mathtt{S}$. Then, there exists a hashtagged argument $\mathbb{B} \in \mathtt{Args}$ such that $\mathbb{B}$ attacks $\mathbb{A}$ and there not exists a hashtagged argument $\mathbb{C} \in \mathtt{Args}$ such that $\mathbb{C}$ attacks $\mathbb{B}$. However, $\mathbb{A}$ is ζ-*acceptable w.r.t.* $\mathtt{S}$ and then for every attacker hashtagged argument $\mathbb{B} \in \mathtt{Args}$ there exists a defender $\mathbb{C} \in \mathtt{S}$. Contradiction.

ii) If a set $\mathtt{S}$ is ζ-*admissible* then it is *admissible*. Suppose that $\mathtt{S}$ is ζ-*admissible* but it is not *admissible*. Then, there exists a hashtagged argument $\mathbb{A} \in \mathtt{Args}$ such that $\mathbb{A}$ is not *acceptable w.r.t.* $\mathtt{S}$. However, $\mathtt{S}$ is ζ-*admissible*. Thus, every hashtagged argument in $\mathtt{S}$ is ζ-*acceptable w.r.t.* $\mathtt{S}$. Furthermore, by consequence of i) if a hashtagged argument $\mathbb{A} \in \mathtt{Args}$ is ζ-*acceptable w.r.t.* a set $\mathtt{S}$ then it is *acceptable w.r.t.* $\mathtt{S}$. Contradiction. □

Note regarding remaining proofs. Given the similarity between the remaining propositions and theorems that stem from the extension of the neighborhood-based framework and those corresponding to the community-based framework, their presentation is omitted here. Since the proofs addressed thus far lay the groundwork for understanding the underlying reasoning, it is evident that addressing the remaining propositions and theorems merely requires adapting the application of concept of neighborhood to that of community.

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