

BICLIQUE IMMERSIONS IN GRAPHS WITH INDEPENDENCE NUMBER 2

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ABSTRACT. The analog of Hadwiger’s conjecture for the immersion relation states that every graph G contains an immersion of $K_{\chi(G)}$. For graphs with independence number 2, this is equivalent to stating that every such n -vertex graph contains an immersion of $K_{\lceil n/2 \rceil}$. We show that every n -vertex graph with independence number 2 contains every complete bipartite graph on $\lceil n/2 \rceil$ vertices as an immersion.

§1. INTRODUCTION

A central problem in graph theory is guaranteeing dense substructures in graphs with a given chromatic number. Hadwiger’s Conjecture [12] is one of the most important examples of this pursuit, stating that every loopless graph G contains the complete graph $K_{\chi(G)}$ as a minor (where $\chi(G)$ is the chromatic number of G), thus aiming to generalize the Four Color Theorem. This difficult conjecture is known to hold whenever $\chi(G) \leq 6$ [26], and it is open for the remaining values. Thus, a natural approach is to study whether it holds whenever G is restricted to a particular class of graphs. A class of graphs that has received particular attention (and yet remains open) is that of graphs with independence number 2. A recent survey of Seymour [27] emphasizes the importance of this case, which was first remarked by Mader (see [23]). Plummer, Stiebitz, and Toft [23] gave an equivalent formulation of Hadwiger’s Conjecture for such graphs: every n -vertex graph with independence number 2 contains a minor of $K_{\lceil n/2 \rceil}$. Before that, in 1982, Duchet and Meyniel [7] had shown a result that implies that every such graph contains a minor of $K_{\lceil n/3 \rceil}$. Despite much work (see e.g. [10, 13, 14, 31]), it is still open whether there is a constant $c > 1/3$ such that every graph with independence number 2 contains a minor of $K_{\lceil cn \rceil}$. Given the difficulty to obtain a clique minor on $\lceil n/2 \rceil$ vertices, Norin and Seymour [22] recently turned into finding dense minors on this amount of vertices. They proved that every

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n -vertex graph with independence number 2 contains a (simple) minor of a graph H on $\lceil n/2 \rceil$ vertices and $0.98688 \cdot \binom{|V(H)|}{2} - o(n^2)$ edges.

The focus of this paper is a conjecture related to Hadwiger's, concerned with finding graph immersions in graphs with a given chromatic number; this type of substructure is defined as follows. To *split off* a pair of adjacent edges uv, vw amounts to deleting those two edges and adding the edge uw . A graph G is said to contain an *immersion* of another graph H if H can be obtained from a subgraph of G by splitting off pairs of edges and deleting isolated vertices. Notice then that if G contains H as a subdivision, it contains H as an immersion (and as a minor). Immersions have received increased attention in recent years [5, 8, 19–21, 30], particularly since Robertson and Seymour [25] proved that graphs are well-quasi-ordered by the immersion relation. Much of this attention has been centered around the following conjecture of Abu-Khzam and Langston [1], which is the immersion-analog of Hadwiger's Conjecture.

Conjecture 1 (Abu-Khzam and Langston [1]). *Every loopless graph G contains an immersion of $K_{\chi(G)}$.*

The above conjecture holds whenever $\chi(G) \leq 4$ because Hajós' subdivision conjecture holds in this case, actually giving a subdivision of $K_{\chi(G)}$ [6]. The cases where $\chi(G) \in \{5, 6, 7\}$ were verified independently by Lescure and Meyniel [18] and by DeVos, Kawarabayashi, Mohar, and Okamura [4]. In general, a result of Gauthier, Le, and Wollan [11] guarantees that every graph G contains an immersion of a clique on $\lceil \frac{\chi(G)-4}{3.54} \rceil$ vertices. This result improves on theorems due to Dvořák and Yepremyan [9] and DeVos, Dvořák, Fox, McDonald, Mohar, and Scheide [3].

The case of graphs with independence number 2 has also received attention in regard to Conjecture 1. In particular, Vergara [29] showed that, for such graphs, Conjecture 1 is equivalent to the following conjecture.

Conjecture 2 (Vergara [29]). *Every n -vertex graph with independence number 2 contains an immersion of $K_{\lceil n/2 \rceil}$.*

As evidence for her conjecture, Vergara proved that every n -vertex graph with independence number 2 contains an immersion of $K_{\lceil n/3 \rceil}$. This was later improved by Gauthier *et al.* [11], who showed that every such graph contains an immersion of $K_{2\lceil n/5 \rceil}$. This last result was extended to graphs with arbitrary independence number [2]. Additionally, Vergara's Conjecture has been verified for graphs with small forbidden subgraphs [24]. The main contribution of this paper is the following result, which states that graphs with independence number 2 contain an immersion of every complete bipartite graph on $\lceil n/2 \rceil$ vertices.

Theorem 3. *Let G be an n -vertex graph with independence number 2, and $\ell \leq \lceil n/2 \rceil - 1$ be a positive integer. Then G contains an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$.*

Using an argument due to Plummer *et al.* [23] we show (see Lemma 17) that this implies the following.

Corollary 4. *Let G be an n -vertex graph with independence number 2, and $1 \leq \ell \leq \chi(G) - 1$. Then G contains an immersion of $K_{\ell, \chi(G) - \ell}$.*

This result leads us to make the following conjecture, which holds trivially when $\ell = 1$.

Conjecture 5. *If $1 \leq \ell \leq \chi(G) - 1$, then G contains an immersion of $K_{\ell, \chi(G) - \ell}$.*

We denote by $K_{a,b,c}$ the graph that admits a partition into parts of sizes a , b , and c such that any pair of these parts induces a complete bipartite graph. In addition to Corollary 4, as evidence for Conjecture 5, we give a short proof of the following strengthening of the case $\ell = 2$.

Proposition 6. *If $\chi(G) \geq 3$, then G contains $K_{1,1, \chi(G) - 2}$ as an immersion.*

We note that Conjecture 5 has its parallel in the minor order. Woodall [32] and, independently, Seymour (see [17]), proposed the following conjecture: every graph G with $\ell \leq \chi(G) - 1$ contains a minor of $K_{\ell, \chi(G) - \ell}$. In [32], Woodall showed that (the list-coloring strengthening of) his conjecture holds whenever $\ell \leq 2$. Kostochka and Prince [17] showed that the case $\ell = 3$ holds as long as $\chi(G) \geq 6503$. Kostochka [15] proved it for every ℓ as long as $\chi(G)$ is very large in comparison to ℓ , and later [16] improved this so that $\chi(G)$ could be polynomial in ℓ , namely, whenever $\chi(G) > 5(200\ell \log_2(200\ell))^3 + \ell$. In fact, the results in [15–17] obtain the full join $K_{\ell, \chi(G) - \ell}^*$ which is the graph obtained from the disjoint union of a K_ℓ and an independent set on $\chi(G) - \ell$ vertices by adding all of the possible edges between them. This and the above-cited result of Norin and Seymour leads us to make the following conjecture.

Conjecture 7. *Let G be an n -vertex graph with independence number 2, and $1 \leq \ell \leq \lceil n/2 \rceil - 1$. Then G contains a minor of $K_{\ell, \lceil n/2 \rceil - \ell}$.*

Notice that since $\chi(G) \geq \lceil n/\alpha \rceil$ for every n -vertex graph with independence number α , the conjecture of Woodall and Seymour would imply Conjecture 7. However, the result of Kostochka leaves open the balanced case, thus not implying Conjecture 7. Moreover, it is not hard to build a graph that is denser than the minor obtained by the result of Norin and Seymour, and yet does not contain $K_{\lceil n/4 \rceil, \lceil n/4 \rceil}$: take a complete graph on $\lceil n/2 \rceil$ vertices and delete $\lfloor n/4 \rfloor + 1$ edges incident to the same vertex. Thus Conjecture 7 is not implied by this result either.

The rest of the paper is organized as follows. In Section 2 we give some definitions and prove an interesting lemma that is used to prove Theorem 3, in Section 3. In Section 4 we prove Corollary 4 and Proposition 6. Finally, in Section 5 we present a related result together with an open problem.

§2. PRELIMINARIES AND NOTATION

All the graphs considered in this work are finite and loopless. For a graph G , we denote by $V(G)$ and $E(G)$ its set of vertices and edges, respectively. For every positive integer n , let $[n] = \{1, \dots, n\}$. Let G be a graph. For $v \in V(G)$ and $S \subseteq V(G)$, we define $E(v, S) = \{vu \in E(G) : u \in S\}$. For a set of vertices $W \subseteq V(G)$ (resp. set of edges $F \subseteq E(G)$), we write $G[W]$ (resp.

$G[F]$ to denote the subgraph induced by W (resp. F). We also denote by $N_G(v)$ and $N_G[v]$ the neighborhood and the closed neighborhood, respectively, of v in $V(G)$. The degree of a vertex v of a graph G is denoted by $d_G(v)$. We simply write $N(v)$, $N[v]$, and $d(v)$ when G is clear from the context. Moreover, when convenient, given a set $F \subseteq E(G)$ and a vertex $v \in V(G)$, we write $N_F(v)$, $N_F[v]$, and $d_F(v)$ to denote, respectively, $N_{G[F]}(v)$, $N_{G[F]}[v]$, and $d_{G[F]}(v)$.

Given an edge e of a graph G , the *multiplicity of (the edge) e in G* , denoted by $m_G(e)$, is the number of edges in G having the same pair of vertices as the edge e . For sets $A, B \subseteq V(G)$, we say that a path P of G is an *A, B -path* if P has an endvertex in A and the other in B . We denote by $\alpha(G)$ the maximum size of an independent set of G , which is the *independence number* of G .

The *complete graph* on n vertices is denoted by K_n and the *complete bipartite graph* with parts of size a and b is denoted by $K_{a,b}$. If A and B are disjoint sets, we let $K_{A,B}$ be the complete bipartite graph with bipartition (A, B) . A substructure (subgraph, immersion, minor, etc.) is a *clique* if it is a complete graph, and is a *biclique* if it is a complete bipartite graph.

In a manner that is equivalent to the definition given in the introduction, we say that a graph G *contains an immersion of H* if there exists an injection $f: V(H) \rightarrow V(G)$ and a collection of edge-disjoint paths in G , one for each edge of H , such that the path P_{uv} corresponding to the edge uv has endvertices $f(u)$ and $f(v)$.

In our proof, we make use of the following lemma, which we believe to be interesting by itself, and that we could not find in the literature. The lemma below may be rephrased in terms of (disjoint) systems of distinct representatives.

Lemma 8. *Let $j \leq k$ be two positive integers, and let $C_1, \dots, C_j \subseteq [n]$ be sets of size k . Let A be a set of size k disjoint from $[n]$. Then there are disjoint matchings M_1, \dots, M_j in $K_{A,[n]}$ such that M_i matches A with C_i for every $i \in [j]$.*

Proof. We start building an auxiliary bipartite graph H such that $V(H) = \{u_1, \dots, u_j\} \cup [n]$ and $E(H) = \{u_i x : x \in C_i, i \in [j]\}$. Since $|C_i| = k$, we have $d(u_i) = k$. Moreover, the degree of a vertex $x \in [n]$ is precisely the number of C_i 's that contain x , which is at most $j \leq k$. Therefore, H is a bipartite graph of maximum degree precisely k . Then, by König's line coloring theorem, we can color the edges of H with k colors, say $D_1, \dots, D_k \subseteq E(H)$.

Let $A = \{v_1, \dots, v_k\}$ be disjoint from $[n]$. Now we construct the matchings of $K_{A,[n]}$. For each $i \in \{1, \dots, k\}$, let $u_i w_1^i, \dots, u_i w_k^i$ be the edges of H incident to u_i , where $u_i w_j^i \in D_j$ (for an example, see Figure 1). Note that, by the definition of H , we have $w_1^i, \dots, w_k^i \in C_i$. Then we create the matching $M_i = \{v_1 w_1^i, \dots, v_k w_k^i\}$, which matches A with C_i .

We claim that $M_i \cap M_j = \emptyset$ for every $i \neq j$. Suppose, without loss of generality, that $v_1 w \in M_i \cap M_j$. By the construction of M_i , we have that $u_i w \in D_1$. Analogously, $u_j w \in D_1$. Then, since D_1 is a matching, $u_i = u_j$, which means $i = j$. Thus, $M_i \cap M_j = \emptyset$ when $i \neq j$. \square

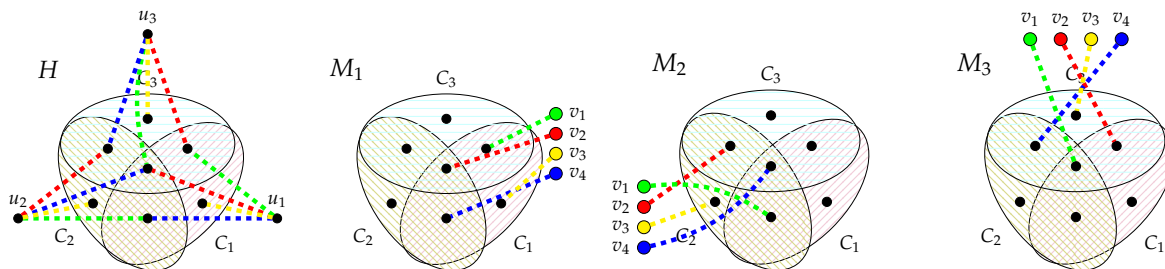


FIGURE 1. An example of Lemma 8.

§3. PROOF OF THEOREM 3

Indeed we can consider graphs with independence number at most 2. The proof follows by induction on $n + \ell$. Let G be an n -vertex graph with $\alpha(G) \leq 2$ and let $\ell \leq \lceil n/2 \rceil - 1$ be a positive integer. It is not hard to check that the result holds when $n \leq 4$, and thus we may assume that $n \geq 5$. Note also that it suffices to prove the statement in the case G is *edge-critical*, i.e., that the removal of any edge of G increases its independence number. Now, if $n \leq 4\ell - 2$, then $\lceil n/2 \rceil - \ell \leq 2\ell - 1 - \ell < \ell$. Thus, by induction, there is an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$ in G , where $\ell' = \lceil n/2 \rceil - \ell$. But this is the desired immersion because $K_{\lceil n/2 \rceil - \ell, \lceil n/2 \rceil - \lceil n/2 \rceil + \ell} = K_{\lceil n/2 \rceil - \ell, \ell}$. Thus, from now on, we may assume that

$$n \geq 4\ell - 1. \quad (1)$$

Our proof is divided into four parts, and the following part is the only step of the proof in which we directly use induction.

3.1. Non-adjacent vertices with at least $\ell - 1$ common neighbors. Our first task is to show that we can assume that

$$|N(u) \cap N(v)| \leq \ell - 2 \text{ for every pair of non-adjacent vertices } u, v, \quad (2)$$

as, otherwise, we have the desired immersion. For that, suppose that G contains two non-adjacent vertices, say x and y , with at least $\ell - 1$ common neighbors, and let $G' = G - x - y$. If $\ell \leq \lceil n/2 \rceil - 2 = \lceil (n-2)/2 \rceil - 1$ the induction hypothesis guarantees that G' contains an immersion of $K_{\ell, \lceil (n-2)/2 \rceil - \ell}$, which we call H' . Otherwise, if we have $\ell = \lceil n/2 \rceil - 1$ we let H' be an arbitrary set of ℓ vertices. Let L and B be the parts of H' having size ℓ and $\lceil n/2 \rceil - 1 - \ell$, respectively, and let $R = V(G') \setminus (L \cup B)$ (see Figure 2a). As $\alpha(G) = 2$, every vertex in G' is either adjacent to x or to y in G . This is true in particular for the vertices in L . In what follows, we add either x or y to B , in order to obtain the desired immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$. This is immediate if x or y is adjacent to every vertex in L . Thus we may assume that $|E(y, L)|, |E(x, L)| < \ell$. Now, consider the following sets.

- ▷ L_x , the set of vertices in L adjacent to x but not adjacent to y ;
- ▷ L_y , the set of vertices in L adjacent to y but not adjacent to x ;

- ▷ L_c , the set of vertices in L adjacent to both x and y ; and
- ▷ O_c , the set of vertices adjacent to both x and y that are not in L .

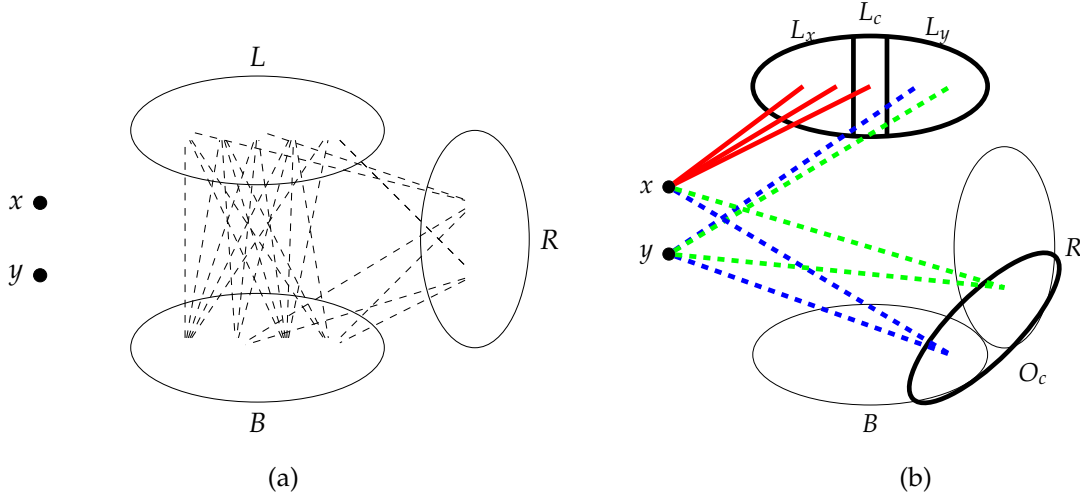


FIGURE 2. (2a) Partition of $V(G)$ with respect to H' ; (2b) Paths from x to L_y using O_c and y depicted in dashed lines.

As x is adjacent to every vertex in $L_x \cup L_c$, it is enough to find (edge-disjoint) paths from x to L_y without using edges of H' . Notice that $|L_y| + |L_c| = |E(y, L)| \leq \ell - 1$ and that, by hypothesis, we have $|L_c| + |O_c| \geq \ell - 1$. Thus $|O_c| \geq |L_y|$. Let $O_c = \{o_1, o_2, \dots, o_{|O_c|}\}$ and $L_y = \{\ell_1, \ell_2, \dots, \ell_{|L_y|}\}$. For $1 \leq i \leq |L_y|$, we take the path $x o_i y \ell_i$ (see Figure 2b). These paths are clearly pairwise edge-disjoint and do not use edges of H' because the edges $x o_i$, $o_i y$, and $y \ell_i$ are not in G' . This concludes the proof of (2).

3.2. Direct consequences of edge-criticality. Recall that G is edge-critical, meaning that the removal of any edge $uv \in E(G)$ creates an independent set of size 3. Hence, for such an edge there is a vertex w that is not adjacent to both u and v . We formalize this argument in the following claim.

Claim 9. For any $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $N[u] \cup N[v] \neq V(G)$.

For the rest of the proof, we fix two non-adjacent vertices x and y , and partition the vertices of G into the following three sets:

- ▷ $C = N(x) \cap N(y)$, the set of common neighbors of x and y ;
- ▷ $X = \overline{N[y]}$, the set of non-neighbors of y excluding y , which contains x ; and
- ▷ $Y = \overline{N[x]}$, the set of non-neighbors of x excluding x , which contains y .

Note that by (2), we have $|C| \leq \ell - 2$. Moreover, both X and Y induce complete subgraphs of G , otherwise we could find an independent set of size 3. Moreover, the edge-criticality of G yields the following claim.

Claim 10. For every vertex $a \in C$, we have $X, Y \not\subseteq N(a)$.

Proof. First, note that $N[x] = X \cup C$. Thus, since $a \in N(x)$, if $Y \subseteq N(a)$, then xa is an edge of G for which $N[x] \cup N[a] = V(G)$, a contradiction to Claim 9. Analogously, we have $X \not\subseteq N(a)$. \square

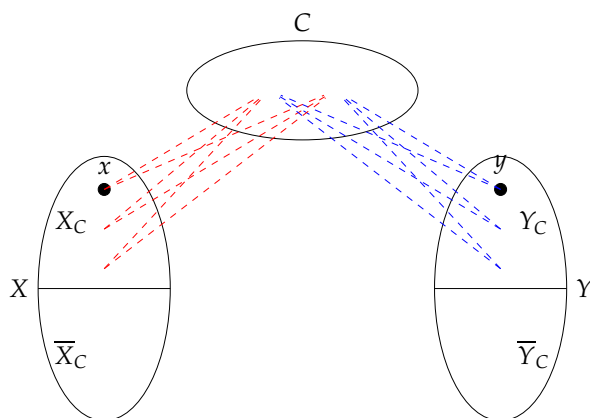


FIGURE 3. Partition of G according to non-adjacent vertices x and y and their common neighborhood C .

3.3. Key vertex sets and their sizes. We partition X and Y in different ways. Let $X_C \subseteq X$ (resp. $Y_C \subseteq Y$) be the set containing vertices $v \in X$ (resp. $v \in Y$) for which $C \subset N(v)$, and put $\bar{X}_C = X \setminus X_C$ (resp. $\bar{Y}_C = Y \setminus Y_C$) (see Figure 3). Similarly, given a vertex a in C , we denote by X_a (resp. Y_a) the set of vertices in X (resp. in Y) that are adjacent to a , and put $\bar{X}_a = X \setminus X_a$ and $\bar{Y}_a = Y \setminus Y_a$. Notice that if $v \in \bar{X}_a$ and $w \in \bar{Y}_a$, then v and w must be adjacent, as the independence number of G is 2. Thus we get $K_{\bar{X}_a, \bar{Y}_a}$ as a subgraph of G . Note that $X_C \subseteq X_a$ and $\bar{X}_C \supseteq \bar{X}_a$ (resp. $Y_C \subseteq Y_a$ and $\bar{Y}_C \supseteq \bar{Y}_a$) for every $a \in C$. Indeed, we have $X_C = \bigcap_{a \in C} X_a$ (resp. $Y_C = \bigcap_{a \in C} Y_a$) and $\bar{X}_C = \bigcup_{a \in C} \bar{X}_a$ (resp. $\bar{Y}_C = \bigcup_{a \in C} \bar{Y}_a$). The following claims give bounds on the sizes of some of these sets. This control is the key to build the desired immersion.

Claim 11. Given $a \in C$, we have that $|X_C| \leq |X_a| \leq \ell - 2$ and $|Y_C| \leq |Y_a| \leq \ell - 2$. Furthermore, we have $|\bar{X}_a| \geq \lceil n/2 \rceil - |Y| + 3$ and $|\bar{Y}_a| \geq \lceil n/2 \rceil - |X| + 3$.

Proof. We prove the bounds for X_a and \bar{X}_a , as the remaining bounds follow analogously. By Claim 10, there is a vertex $w \in \bar{X}_C$ such that $wa \notin E(G)$. By the definition of X_a we have $X_C \subseteq X_a \subseteq N(a)$, which gives us $|X_C| \leq |X_a|$. Moreover, since X induces a complete graph, we have $X_a \subseteq N(w)$, and hence $X_a \subseteq N(a) \cap N(w)$. Therefore, (2) implies $|X_a| \leq \ell - 2$.

Now, we use (1) in the following to obtain the claim:

$$\begin{aligned}
|Y| + |\overline{X}_a| &= n - |C| - |X_a| \\
&\geq n - \ell + 2 - \ell + 2 \\
&= \lceil n/2 \rceil + \lfloor n/2 \rfloor - 2\ell + 4 \\
&\geq \lceil n/2 \rceil + \lfloor (4\ell - 1)/2 \rfloor - 2\ell + 4 \\
&= \lceil n/2 \rceil + \lfloor 2\ell - (1/2) \rfloor - 2\ell + 4 \\
&= \lceil n/2 \rceil + 2\ell - 1 - 2\ell + 4 \\
&= \lceil n/2 \rceil + 3. \quad \square
\end{aligned}$$

The claim above is used to prove the following three claims. In fact, the “+3” term in its bounds is almost necessary for the proof of the next claim.

Claim 12. For every $a \in C$, we have either $|X \setminus \overline{X}_a| \geq 2(\ell - |\overline{Y}_a|)$ or $|Y \setminus \overline{Y}_a| \geq 2(\ell - |\overline{X}_a|)$.

Proof. For a contradiction, assume $|X \setminus \overline{X}_a| < 2(\ell - |\overline{Y}_a|)$ and $|Y \setminus \overline{Y}_a| < 2(\ell - |\overline{X}_a|)$. Then we have

$$\begin{aligned}
|X| + |Y| &= |X| - |\overline{X}_a| + |\overline{X}_a| + |Y| - |\overline{Y}_a| + |\overline{Y}_a| \\
&< 2(\ell - |\overline{Y}_a|) + |\overline{X}_a| + 2(\ell - |\overline{X}_a|) + |\overline{Y}_a| \\
&= 4\ell - |\overline{Y}_a| - |\overline{X}_a|.
\end{aligned}$$

Also, by Claim 11, we have $|\overline{Y}_a| \geq \lceil n/2 \rceil - |X| + 3$ and $|\overline{X}_a| \geq \lceil n/2 \rceil - |Y| + 3$, which implies

$$\begin{aligned}
4\ell &> |X| + |Y| + |\overline{Y}_a| + |\overline{X}_a| \\
&\geq |X| + |Y| + \lceil n/2 \rceil - |X| + 3 + \lceil n/2 \rceil - |Y| + 3 \\
&\geq n + 6.
\end{aligned}$$

So we have $n < 4\ell - 6$, which contradicts (1). □

Claim 13. For every $v \in \overline{X}_C$ (resp. $w \in \overline{Y}_C$), we have $|N(v) \cap \overline{Y}_C| > \lceil n/2 \rceil - |X|$ (resp. $|N(w) \cap \overline{X}_C| > \lceil n/2 \rceil - |Y|$).

Proof. We prove the bound for $|N(v) \cap \overline{Y}_C|$, as the bound on $|N(w) \cap \overline{X}_C|$ follows analogously. By the definition of \overline{X}_C , there is a vertex $a \in C$ such that $va \notin E(G)$. Note that $\overline{Y}_a \subseteq N(v)$, otherwise G would contain an independent set of size 3. Since $\overline{Y}_a \subseteq \overline{Y}_C$, we have $\overline{Y}_a \subseteq N(v) \cap \overline{Y}_C$, and the desired bound follows from Claim 11. □

Finally, we need the following relation between the sizes of \overline{X}_C and X_C (resp. \overline{Y}_C and Y_C).

Claim 14. For every $a \in C$ we have the following:

- i) If $|\overline{X}_C| < \ell$, then $|X_C| > \lceil n/2 \rceil - \ell - |\overline{Y}_a| \geq \ell - |\overline{Y}_a|$; and
- ii) If $|\overline{Y}_C| < \ell$, then $|Y_C| > \lceil n/2 \rceil - \ell - |\overline{X}_a| \geq \ell - |\overline{X}_a|$.

Proof. We only prove (i), as (ii) follows analogously. By Claim 11, if $|\overline{X}_C| < \ell$, then

$$\lceil n/2 \rceil + 3 \leq |\overline{Y}_a| + |X| = |\overline{Y}_a| + |\overline{X}_C| + |X_C| < |\overline{Y}_a| + \ell + |X_C|.$$

By (1) we have $\lceil n/2 \rceil \geq 2\ell$, which implies

$$|X_C| > \lceil n/2 \rceil - \ell - |\overline{Y}_a| \geq 2\ell - \ell - |\overline{Y}_a| = \ell - |\overline{Y}_a|. \quad \square$$

3.4. Constructing the immersion. The rest of the proof is divided into two cases which depend on the sizes of \overline{X}_C and \overline{Y}_C . The sets that form the bipartition of the immersion depend on which case we are dealing with. The construction requires more care in Case 2, where one of $\overline{X}_C, \overline{Y}_C$ is large. In that case, we apply Lemma 8 and obtain a set of paths that are not necessarily edge-disjoint. Nevertheless, the intersections are relatively few and with a combination of different techniques, we are able to fix them and obtain the desired immersion. For the rest of the proof, we fix $a \in C$.

Case 1, $|\overline{X}_C|, |\overline{Y}_C| < \ell$. By Claim 14, we can choose $X^* \subset X_C$ and $Y^* \subset Y_C$ such that $|X^*| = \lceil n/2 \rceil - \ell - |\overline{Y}_a|$ and $|Y^*| = \ell - |\overline{X}_a|$. In what follows, we find an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$ with bipartition $(Y^* \cup \overline{X}_a, X^* \cup \overline{Y}_a)$ (see Figure 4).

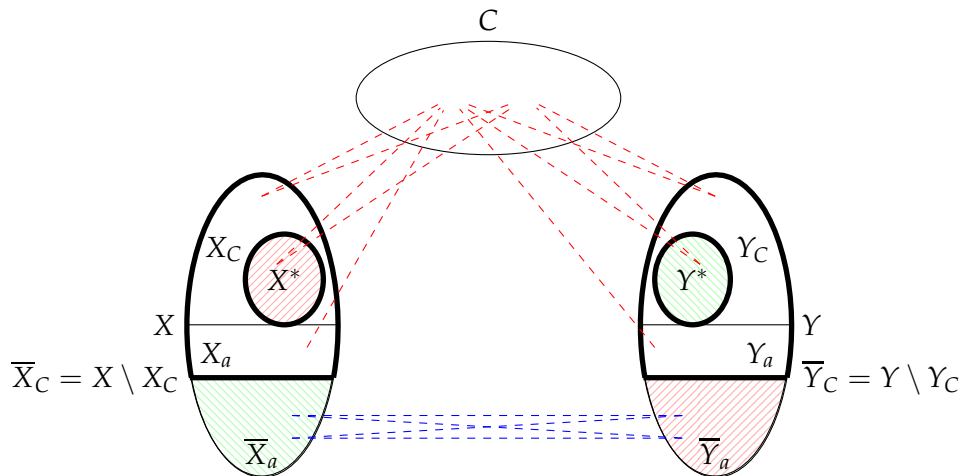
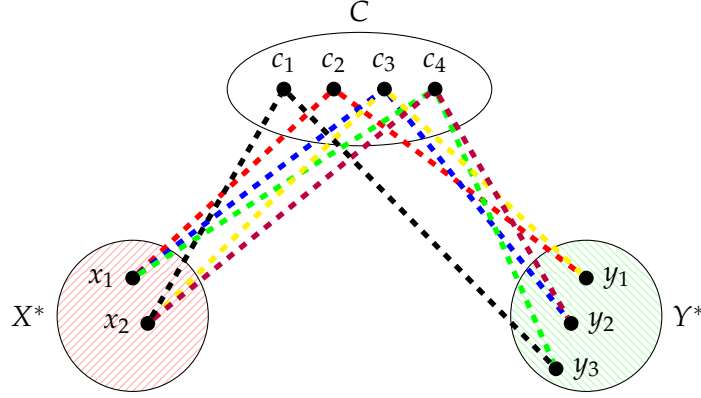


FIGURE 4. Immersion for Case 1, when $|\overline{X}_C|, |\overline{Y}_C| < \ell$: between $Y^* \cup \overline{X}_a$ and $X^* \cup \overline{Y}_a$.

Recall that X, Y , and $\overline{X}_a \cup \overline{Y}_a$ each induce a clique (each is contained in the non-neighborhood of a vertex and G has independence number 2), and thus G contains all edges joining (i) vertices in \overline{X}_a to vertices in \overline{Y}_a ; (ii) vertices in \overline{X}_a to vertices in X^* ; and (iii) vertices of \overline{Y}_a to vertices in Y^* . It remains to find edge-disjoint paths joining vertices in X^* to vertices in Y^* . For that, we use only edges incident to vertices in C , which guarantees that our new paths do not use any previously used edge. Let $X^* = \{x_1, \dots, x_{|X^*|}\}$, $Y^* = \{y_1, \dots, y_{|Y^*|}\}$, and $C = \{c_1, \dots, c_{|C|}\}$. For each pair of vertices x_i, y_j , consider the path $x_i c_{i+j} y_j$, where the additions in the subscript are taken modulo $|C|$ (see Figure 5).

FIGURE 5. Construction for Case 1, connecting X^* to Y^* through C .

It is easy to see that these paths are edge-disjoint if and only if $|C| \geq |X^*|, |Y^*|$. Since, Claim 11 gives us $|X_C|, |Y_a| \leq \ell - 2$, and since in Case 1 we assume $|\overline{X}_C| < \ell$, we have

$$\begin{aligned}
 |C| &= n - |\overline{X}_C| - |X_C| - |\overline{Y}_a| - |Y_a| \\
 &> n - \ell - (\ell - 2) - |\overline{Y}_a| - (\ell - 2) \\
 &= n - 3\ell - |\overline{Y}_a| + 4 \\
 &= \lceil n/2 \rceil + \lfloor n/2 \rfloor - 3\ell - |\overline{Y}_a| + 4 \\
 &\geq \lceil n/2 \rceil + 2\ell - 1 - 3\ell - |\overline{Y}_a| + 4 \quad \text{by (1)} \\
 &= \lceil n/2 \rceil - \ell - |\overline{Y}_a| + 3.
 \end{aligned} \tag{3}$$

Hence, $|C| > \lceil n/2 \rceil - \ell - |\overline{Y}_a| + 3 > |X^*|$. Analogously to (3), we can show that $|C| \geq \lceil n/2 \rceil - \ell - |\overline{X}_a| + 3$, and by (1), that $\lceil n/2 \rceil - \ell - |\overline{X}_a| + 3 > |Y^*|$, as desired. Since these X^*, Y^* -paths are mutually edge-disjoint and edge-disjoint from all previous paths, we get the desired immersion.

Case 2, $|\overline{X}_C| \geq \ell$ or $|\overline{Y}_C| \geq \ell$. Without loss of generality, we assume $|\overline{X}_C| \geq \ell$. By Claim 11, we can choose $Y^* \subset \overline{Y}_a$ with $|Y^*| = \lceil n/2 \rceil - |X|$, and since $|\overline{X}_C| \geq \ell$, we can choose $X^* \subset \overline{X}_C \setminus \overline{X}_a$ with $|X^*| = \ell - |\overline{X}_a|$. Note that $|X_a| - |X^*| + |Y^*| = |X_a| - \ell + |\overline{X}_a| + |Y^*| = |X| + |Y^*| - \ell = \lceil n/2 \rceil - \ell$. Now, we show that

$$|X^*| \leq |Y^*|. \tag{4}$$

Indeed, by Claim 11, we have $|X_a| \leq \ell - 2$, and hence

$$\begin{aligned}
 |Y^*| &= \lceil n/2 \rceil - |X| \\
 &= \lceil n/2 \rceil - |X_a| - |\overline{X}_a| \\
 &\geq 2\ell - (\ell - 2) - |\overline{X}_a| \\
 &= \ell + 2 - |\overline{X}_a| \\
 &> \ell - |\overline{X}_a| \\
 &= |X^*|.
 \end{aligned}$$

In what follows, we find an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$ with bipartition $(X^* \cup \bar{X}_a, (X_a \setminus X^*) \cup Y^*)$ (see Figure 6).

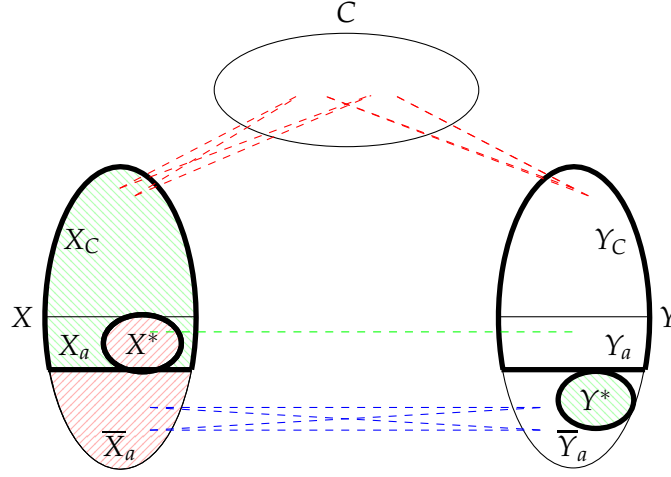


FIGURE 6. Immersion for Case 2, when $|\bar{X}_C| \geq \ell$: between $X^* \cup \bar{X}_a$ and $(X_a \setminus X^*) \cup Y^*$.

As before, since X , Y , and $\bar{X}_a \cup \bar{Y}_a$ induce cliques, G contains all edges joining (i) vertices in \bar{X}_a to vertices in Y^* ; (ii) vertices in \bar{X}_a to vertices in $X_a \setminus X^*$; and (iii) vertices in X^* to vertices in $X_a \setminus X^*$. It remains to find edge-disjoint paths joining vertices in X^* to vertices in Y^* . For these paths, we only use edges that are incident to vertices in Y and not to vertices in \bar{X}_a ; this assures that they are disjoint from the edges already used. Let $X^* = \{v_1, v_2, \dots, v_{|X^*|}\}$ and $Y^* = \{y_1, y_2, \dots, y_{|Y^*|}\}$. The first step is to use Lemma 8 to find paths joining each vertex v_i to all vertices in Y^* allowing edges between vertices of Y^* to be used at most twice.

Claim 15. *For each $i \in \{1, 2, \dots, |X^*|\}$, there is a subgraph $K(v_i)$ which contains an immersion of K_{v_i, Y^*} and satisfies that:*

- i) *each path of $K(v_i)$ with an endvertex in v_i has length at most 2;*
- ii) *for each path $v_i z y_j$ in $K(v_i)$ we have $z \in \bar{Y}_C$; and*
- iii) *if $i \neq j$ and $uw \in E(K(v_i)) \cap E(K(v_j))$, then there is no r with $r \neq i$ and $r \neq j$ such that $uw \in E(K(v_r))$, and one path containing uw ends at u while the other ends at w .*

Proof. Note that, since $X^* \subseteq \bar{X}_C$, Claim 13 assures that for each $i \in \{1, 2, \dots, |X^*|\}$, we have $|N(v_i) \cap \bar{Y}_C| > \lceil n/2 \rceil - |X| = |Y^*|$. In order to use Lemma 8, we define, for each such i , a set $N_i \subset N(v_i) \cap \bar{Y}_C$ with $|N_i| = |Y^*|$, and also a set of auxiliary vertices $A = \{a_1, a_2, \dots, a_{|Y^*|}\}$ with $N(a_j) = \bar{Y}_C$. By (4), we can apply Lemma 8 to $N_1, \dots, N_{|X^*|}$ together with A to obtain disjoint matchings $M_1, M_2, \dots, M_{|X^*|}$ such that M_i matches A to N_i , for $i \in \{1, \dots, |X^*|\}$. Let $M_i = \{z_{i,1}a_1, \dots, z_{i,|Y^*|}a_{|Y^*|}\}$ where $z_{i,j} \in N_i$ for every i, j (see Figure 7).

For each $v_i \in X^*$, we obtain $K(v_i)$ by using y_j whenever a_j is used in a matching. In other words, for every $1 \leq j \leq |Y^*|$, if $z_{i,j}a_j \in M_i$, then we use the path $v_i z_{i,j} y_j$. Notice that $z_{i,j}$ could

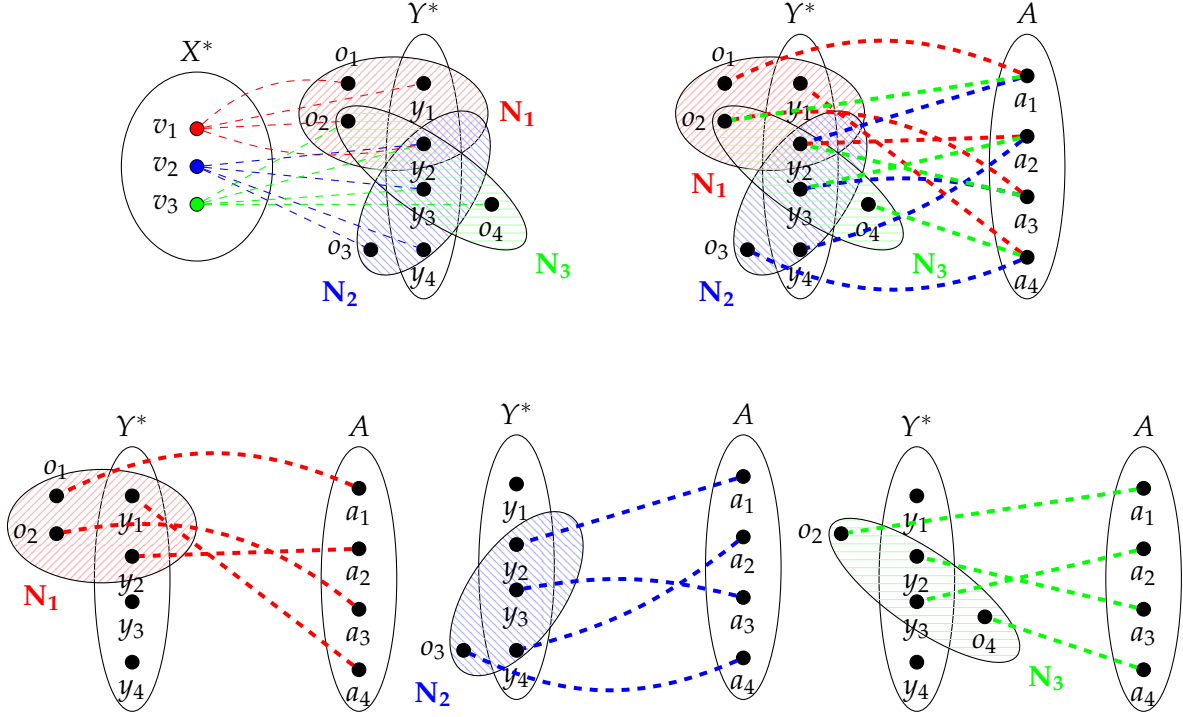


FIGURE 7. The setup for an example of Claim 15.

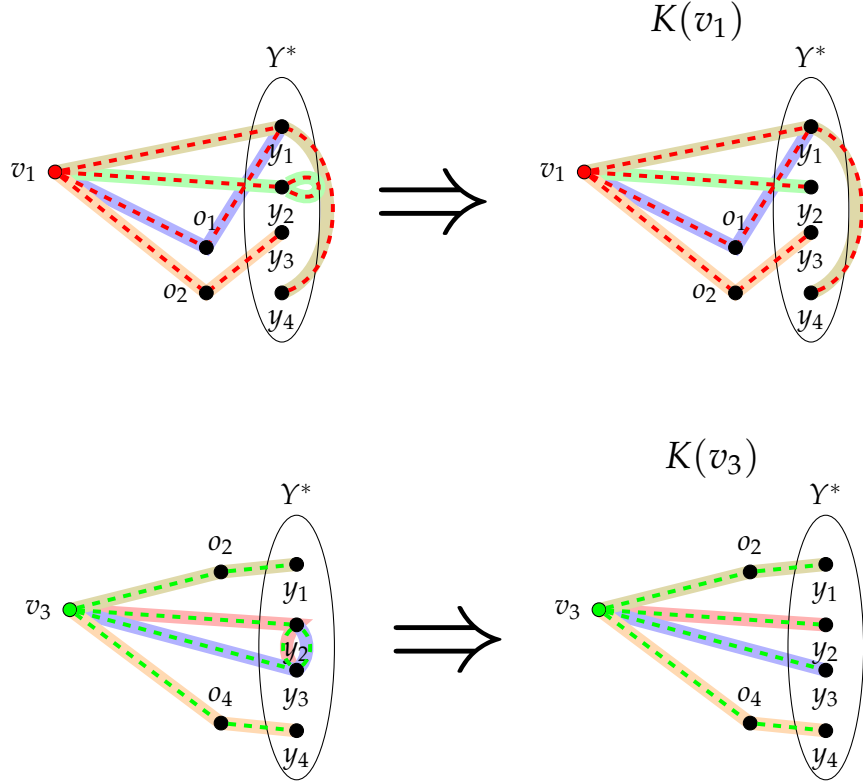
be y_j itself. When that is the case, we use the path $v_i y_j$. Formally, we define

$$P(i, j) = \begin{cases} v_i z_{i,j} y_j & \text{if } y_j \neq z_{i,j} \\ v_i y_j & \text{if } y_j = z_{i,j}. \end{cases}$$

Notice that $P(i, j)$ may not be edge-disjoint from $P(i, k)$ if $k \neq j$, but this can only happen if $P(i, j) = v_i y_k y_j$ and $P(i, k) = v_i y_j y_k$. If that is the case, we redefine $P(i, j)$ as $v_i y_j$ and $P(i, k)$ as $v_i y_k$. Thus, after doing all the necessary changes, we can assume that $P(i, j)$ is disjoint of $P(i, k)$ whenever $j \neq k$ (see Figure 8). Finally we define $K(v_i) = \bigcup_{j=1}^{|Y^*|} P(i, j)$ and since the $P(i, j)$'s are edge disjoint each $K(v_i)$ contains an immersion of K_{v_i, Y^*} and clearly satisfies items *i*) and *ii*).

Furthermore, as $M_1, \dots, M_{|X^*|}$ are disjoint matchings, if $uw \in E(K(v_i)) \cap E(K(v_j))$ for some pair $i \neq j$, then it must be that $u, w \in Y^*$. Let $u = y_h$ and $w = y_k$. Then either $z_{i,h} = y_k$ or $z_{i,k} = y_h$. Assume, without loss of generality, that $z_{i,h} = y_k$. This means that $z_{i,h} a_h = y_k a_h \in M_i$. Thus $y_k a_h \notin M_r$ for $r \neq i$. This, in turn, implies that $z_{j,k} = y_h$, which means that $y_h a_k \in M_j$ and $y_h a_k \notin M_r$ for $r \neq j$. This proves *iii*). \square

From now on, let $K(v_1), \dots, K(v_{|X^*|})$ be the subgraphs given by Claim 15. We would like the v_i, Y^* -paths on these subgraphs to be the X^*, Y^* -paths in our immersion. Yet, if $i \neq j$, $K(v_i)$ might not be edge disjoint from $K(v_j)$. Fortunately, by Claim 15 *iii*) these intersections are restricted, and, in what follows, we fix them.


 FIGURE 8. Defining $K(v_1)$ and $K(v_3)$ for the graph in Figure 7.

Let H be the multigraph with vertex set $V(G)$, and whose edge set is the disjoint union of $E(K(v_1)), \dots, E(K(v_{|X^*|}))$, where the multiplicity of an edge e is $|\{i : e \in E(K(v_i))\}|$. Notice that by Claim 15 iii), the multiplicity of every edge of H is at most 2, and the only edges with multiplicity 2 are between vertices of Y^* . To fix these repetitions, we take detours through other vertices in Y . We now need to divide Case 2 into two subcases, depending on whether $|\overline{Y}_C| \geq \ell$.

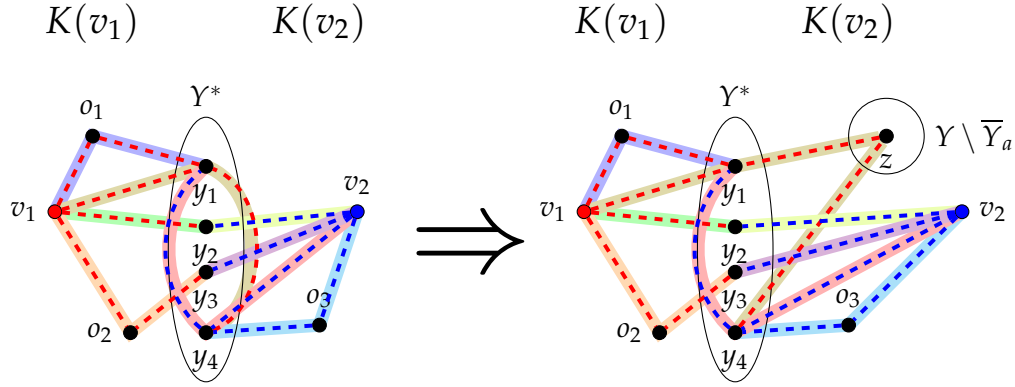
Case 2.1, $|\overline{Y}_C| \geq \ell$. Note that in this case, we recover the symmetry, i.e., now we have both $|\overline{X}_C|, |\overline{Y}_C| \geq \ell$, and due to Claim 12 we can assume, without loss of generality, that

$$|Y \setminus \overline{Y}_a| \geq 2(\ell - |\overline{X}_a|). \quad (5)$$

By Claim 15 iii) we know that if u is incident to an edge of multiplicity 2, say uw , then there is a path in some $K(v_i)$ that contains uw and ends at u . Recall that $m_H(e)$ is the multiplicity of e in H , and, for a given vertex $u \in Y^*$, let p_u be defined as

$$p_u = \sum_{e=uv, v \in Y^*} (m_H(e) - 1),$$

and let $q_u = |N_H(u) \cap (Y \setminus Y^*)|$. Let us see that $q_u + p_u \leq |X^*| = \ell - |\overline{X}_a|$. Indeed, there are precisely $|X^*|$ paths having u as an endpoint, where p_u of these paths use edges with multiplicity 2 and q_u use edges outside Y^* . Now we show that if uw is an edge with multiplicity 2 in H ,

FIGURE 9. Removing a multiple edge through $Y \setminus \bar{Y}_a$.

then

$$|Y \setminus Y^*| > q_u + q_w. \quad (6)$$

As uw has multiplicity 2, we have $p_u, p_w \geq 1$. So using (5) and that $Y^* \subseteq \bar{Y}_a$, we indeed have

$$\begin{aligned} q_u + q_w &\leq \ell - |\bar{X}_a| - p_u + \ell - |\bar{X}_a| - p_w \\ &\leq \ell - |\bar{X}_a| - 1 + \ell - |\bar{X}_a| - 1 \\ &= 2(\ell - |\bar{X}_a| - 1) \\ &\leq |Y \setminus \bar{Y}_a| - 2 \\ &\leq |Y \setminus Y^*| - 1. \end{aligned}$$

We now describe a process that at each step picks an edge uw with multiplicity 2 of H , reduces p_u and p_w by 1, increases q_u and q_w by 1, and leaves p_t and q_t unchanged for every vertex $t \in Y^* \setminus \{u, w\}$. In other words, the function $I: X^* \rightarrow \mathbb{N}$ such that $I(u) = p_u + q_u$ is an invariant of this process, while the number of edges with multiplicity 2 decreases.

Let $v_1, v_2 \in X^*$ be such that $uw \in E(K(v_1)) \cap E(K(v_2))$. By (6), there is a vertex $z \in Y \setminus Y^*$ such that z is neither adjacent to u nor w in H . Modify H by replacing the path v_1uw in $K(v_1)$ with the path v_1uzw , and leave v_2wu unchanged (see Figure 9). Then this replacement does not increase the multiplicity of any edge, reduces the multiplicity of edge uv by 1, and increases the sizes of $N_H(u) \cap Y \setminus Y^*$ and $N_H(w) \cap Y \setminus Y^*$ by 1. Furthermore, as $z \in Y \setminus Y^*$, it does not change the values of p_t and q_t for any vertex $t \in Y^* \setminus \{u, w\}$. Hence after each step we still have $q_u + p_u \leq |X^*| = \ell - |\bar{X}_a|$ for every $u \in Y^*$, and thus we can continue to assume (6). Repeating this process until H becomes a simple graph ends up giving us that H contains an immersion of K_{X^*, Y^*} and therefore G contains an immersion of $K_{\ell, \lfloor n/2 \rfloor - \ell}$.

Case 2.2, $|\bar{Y}_C| < \ell$. We first note that this case is asymmetric and we do not use Claim 12. Instead, we modify the subgraphs given by Claim 15 so that each $K(v_i)$ induces a matching in Y^* . For each $1 \leq i \leq |X^*|$ let $Q(v_i)$ be the subgraph of $K(v_i)$ induced by the vertices of Y^* .

Notice that for each vertex $y_j \in Y^*$, the degree of y_j in $Q(v_i)$ is the number of times y_j is in the middle of some path $v_i z_{i,k} y_k$ (i.e., $y_j = z_{i,k}$), which is at most 1 because $v_i y_j$ can be used only once, plus the number of times y_j is the end of some path $v_i z_{i,k} y_k$, which is 1 because there is precisely one path in $K(v_i)$ joining v_i to y_j . Thus we have $d_{Q(v_i)}(y_j) \leq 2$, and so each component of $Q(v_i)$ is either a path or a cycle.

If $t_1 t_2 \cdots t_s t_1$ is a cycle in $Q(v_i)$, then we replace, in $K(v_i)$, the path $v_i t_j t_{j+1}$ with the path $v_i t_{j+1}$, for every $1 \leq j \leq s$, where $t_{s+1} = t_1$. If $t_1 t_2 \cdots t_s$ is a component of $Q(v_i)$ that is a path, then we replace, in $K(v_i)$, the path $v_i t_j t_{j+1}$ with the path $v_i t_{j+1}$, for each $1 \leq j \leq s-2$, and replace the path $v_i t_{s-1} t_s$ with the path $v_i t_1 t_s$ (see Figure 10). After these operations $Q(v_i)$ is a matching.

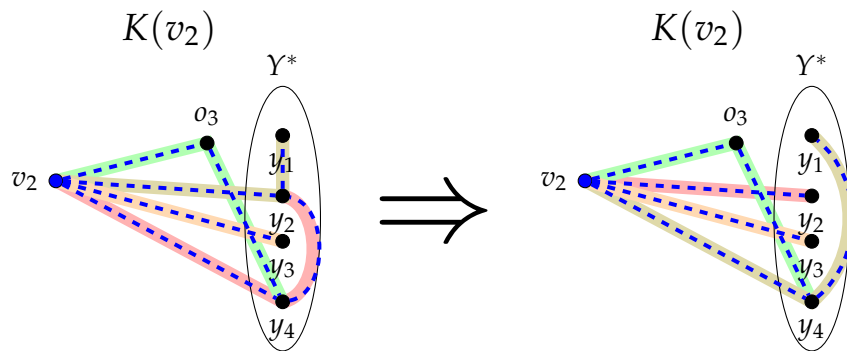


FIGURE 10. Removing a path in Y^* .

Now, Claim 14 implies $|Y_C| > \ell - |\bar{X}_a| = |X^*|$. Thus, there is an injection $f: X^* \rightarrow Y_C$, and we can replace in H every path $vu w \in K(v)$ for which $uw \in Y^*$ with the path $vuf(v)w$ (see Figure 11). Notice that no edge is used more than once because $K(v)$ induces a matching in Y^* . Applying this replacement for every $v \in X^*$ yields that H contains an immersion of K_{X^*, Y^*} and G contains the desired immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$. This concludes the proof of Theorem 3.

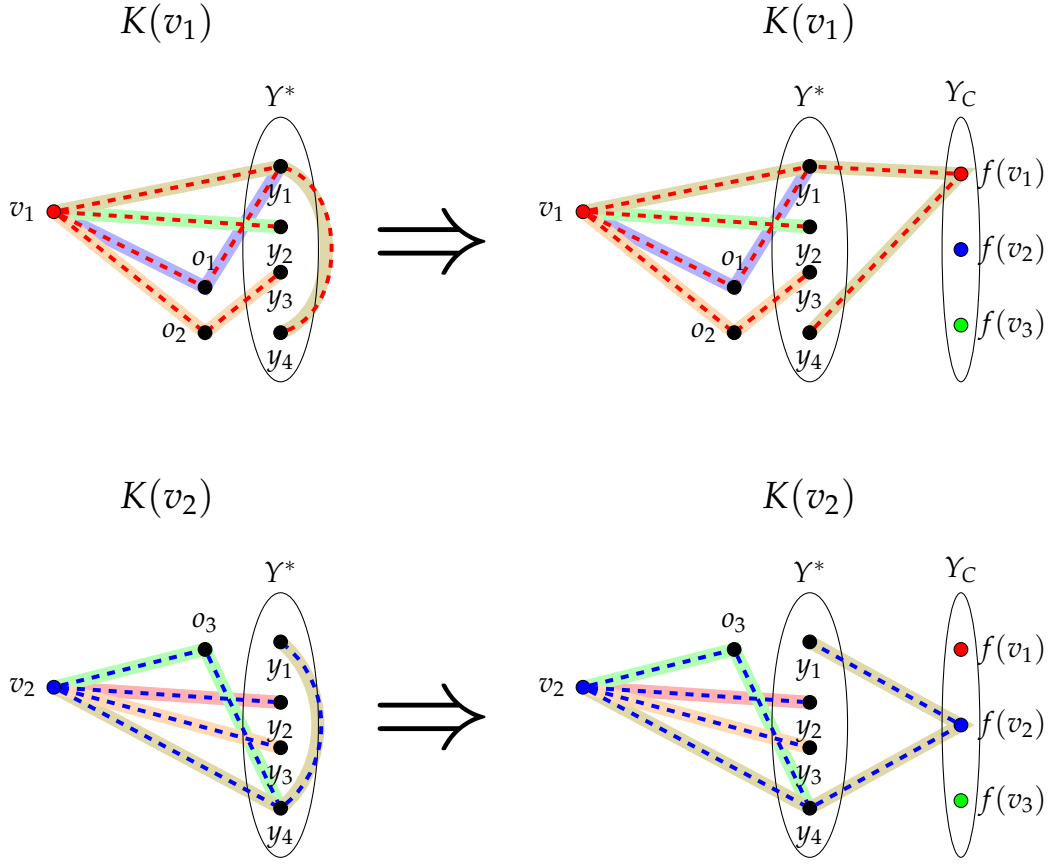
§4. EVIDENCE FOR CONJECTURE 5

In this section we show that Theorem 3 implies Corollary 4 (see Lemma 17), and prove Proposition 6. For the first proof, we use arguments of Plummer *et al.* [23]. We need the following theorem and, for that, a definition: a graph G is said to be *k-critical* if $\chi(G) = k$ and $\chi(G - v) < k$, for every $v \in V(G)$.

Theorem 16 (Stehlík [28]). *Let G be a k -critical graph such that its complement is connected. For every $v \in V(G)$, $G - v$ has a $(k - 1)$ -coloring in which every color class contains at least two vertices.*

Lemma 17. *Let k, n, ℓ be positive integers with $k \geq \ell + 1$, and let G be a k -chromatic n -vertex graph with independence number 2. The following are equivalent:*

- (a) G contains an immersion of $K_{\ell, k-\ell}$.
- (b) G contains an immersion of $K_{\ell, \lceil n/2 \rceil - \ell}$.

FIGURE 11. Removing edges in Y^* using Y_C .

Proof. It is well-known that any n -vertex graph with chromatic number k and independence number α satisfies $k \geq \lceil \frac{n}{\alpha} \rceil$. Therefore (a) implies (b). In the rest of the proof, we show that (b) implies (a). Assume that G is a counterexample to (a) that minimizes the number of vertices. It suffices to show that it is also a counterexample to (b). In order to apply Theorem 16, we show that G is k -critical and that its complement is connected. If $k = \ell + 1$, we use that any graph with chromatic number k has $K_{1,k-1} = K_{k-\ell,\ell}$ as a subgraph. Thus, we can assume that $k \geq \ell + 2$.

Let us first show that G is k -critical. Indeed, if G is not k -critical, then there would be some $v \in V(G)$ such that $\chi(G - v) = k$. Then $G - v$ has at least k vertices and is either a clique or has independence number 2, which, by choice of G , means it contains an immersion of $K_{\ell,k-\ell}$, a contradiction.

Next, we show that the complement of G is connected. If it were not, $V(G)$ could be partitioned into two sets V_1, V_2 such that $uv \in E(G)$ whenever $u \in V_1, v \in V_2$. Let k_1, k_2 be the chromatic numbers of $G[V_1]$ and $G[V_2]$, respectively, and note that $k = k_1 + k_2$. As $k \geq \ell + 2$, let ℓ_1, ℓ_2 be such that $\ell_1 + \ell_2 = \ell$, $k_1 \geq \ell_1 + 1$, and $k_2 \geq \ell_2 + 1$. Then, $G[V_1]$ and $G[V_2]$ must contain

an immersion of $K_{\ell_1, k_1 - \ell_1}$ and an immersion of $K_{\ell_2, k_2 - \ell_2}$, respectively, because G is a counterexample that minimizes the number of vertices. But every vertex in V_1 is adjacent in G to every vertex in V_2 . Thus, G must contain an immersion of $K_{\ell_1, k_1 - \ell_1, \ell_2, k_2 - \ell_2}$, which in turn implies an immersion of $K_{\ell_1 + \ell_2, k_1 - \ell_1 + k_2 - \ell_2} = K_{\ell, k - \ell}$. This contradicts the fact that G is a counterexample. Therefore, the complement of G is connected.

Finally, by Theorem 16, $G - v$ has a $(k - 1)$ -coloring such that each color class has at least two vertices. But since G has independence number 2, each color class has size exactly 2. Thus we have $|V(G)| = 2k - 1$. In particular, $\lceil \frac{|V(G)|}{2} \rceil = \frac{|V(G)| + 1}{2} = k$, meaning that G does not contain an immersion of $K_{\ell, \lceil |V(G)|/2 \rceil - \ell}$ and is a counterexample to (b), as desired. \square

Finally, we give a proof of Proposition 6, which claims that every graph with chromatic number at least 3 contains an immersion of $K_{1,1,\chi(G)-2}$.

Proof of Proposition 6. Assume $\chi(G) = s + 2 \geq 3$, and let $f: V(G) \rightarrow [s + 2]$ be a proper coloring of G that minimizes the number of vertices colored with $s + 2$. Let v be a vertex with $f(v) = s + 2$. By the minimality condition, v has at least one neighbor of each color. Furthermore, assume that f minimizes the number of vertices in $N(v)$ colored with $s + 1$.

Let $vw \in E(G)$ with $f(w) = s + 1$. For $1 \leq i \leq s$, consider the subgraph induced by vertices colored with i or $s + 1$, and let K_{wi} be the connected component containing vertex w . If K_{wi} does not contain neighbors of v colored with i , then we can switch the colors of K_{wi} (from $s + 1$ to i and vice versa), obtaining a new coloring of G with less neighbors of v colored with $s + 1$. Hence K_{wi} must contain at least one neighbor of v colored with i . Label one such vertex w_i .

For $1 \leq i \leq s$, let P_i be a path in K_{wi} joining w to w_i . Notice that if $1 \leq i \neq j \leq s$, then the path P_i is edge-disjoint from the path P_j , since P_i uses edges with endpoints colored i and $s + 1$ while P_j uses edges with endpoints colored j and $s + 1$. Further notice that these paths do not use any edge with v as one of its endpoints. Hence

$$F = \bigcup_{i=1}^s E(P_i) \cup \{vw_i : 1 \leq i \leq s + 1\}$$

induces an immersion of $K_{1,1,s}$. \square

§5. IMMERSIONS OF MATCHINGS

In this section, we discuss a related result that we obtained when proving Theorem 3, and that we believe could have further applications. Let G be a graph with a matching M , and let $A, B \subseteq V(G)$ be disjoint sets of vertices of G . We say that M is an (A, B) -*matching* if every edge in M has a vertex in A and another vertex in B . We prove the following.

Theorem 18. *Let k be a positive integer, let $G = K_{2k}$, and let $(A_1, B_1), \dots, (A_k, B_k)$ be pairs of sets of vertices of G such that for every $i \in \{1, \dots, k\}$ we have*

- (a) $k \geq |A_i| = |B_i| \geq 2(i - 1)$; and
- (b) $A_i \cap B_i = \emptyset$.

Then G contains edge-disjoint matchings M_1, \dots, M_k such that M_i is an (A_i, B_i) -matching for $i \in \{1, \dots, k\}$.

Proof. Assume M_1, \dots, M_{i-1} were found. Let $U = M_1 \cup \dots \cup M_{i-1}$ be the set of used edges and let $F = E(G[A_i \cup B_i]) \setminus U$ be the set of free edges. We use Hall's Theorem. For that, let $X \subseteq A_i$. We prove that $|N_F(X) \cap B_i| \geq |X|$. Suppose, for a contradiction, that $|N_F(X) \cap B_i| < |X|$. Since $d_U(u) \leq i - 1$ for every $u \in V(G)$, we have $|N_F(u) \cap B_i| \geq |B_i| - (i - 1)$, and hence

$$|X| > |N_F(X) \cap B_i| \geq |N_F(u) \cap B_i| \geq |B_i| - (i - 1). \quad (7)$$

On the other hand, if $u \in B_i \setminus N(X)$, then

$$|A_i| - |X| \geq d_F(u) \geq |A_i| - (i - 1),$$

from where we get $|X| \leq i - 1$, which, together with (7), gives $|B_i| < 2(i - 1)$, a contradiction. \square

As we conclude this paper, we raise the following question, which if true, would be a generalization of Theorem 18.

Question. Let k be a positive integer, let $G = K_{2k}$, and let $(A_1, B_1), \dots, (A_k, B_k)$ be pairs of sets of vertices of G such that for every $i \in \{1, \dots, k\}$ we have $|A_i| = |B_i| \leq k$ and $A_i \cap B_i = \emptyset$. Does G contain edge-disjoint immersions M_1, \dots, M_k such that M_i is an immersion of an (A_i, B_i) -matching for $i \in \{1, \dots, k\}$?

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