« FOOP 10701 layout: Small Extended v.1.1 file: foop9746.tex (Alapsin) class: spr-small-v1.3 v.2013/08/14 Prn:2013/09/17; 10:06 p. 1/20»
« doctopic: OriginalPaper numbering style: ContentOnly reference style: mathphys»

Found Phys
DOI 10.1007/s10701-013-9746-0

On the Concept of Entropy for Quantum Decaying Systems

O. Civitarese · M. Gadella

Received: 13 November 2012 / Accepted: 30 August 2013

Abstract The classical concept of entropy was successfully extended to quantum mechanics by the introduction of the density operator formalism. However, further extensions to quantum decaying states have been hampered by conceptual difficulties associated to the particular nature of these states. In this work we address this problem, by (i) pointing out the difficulties that appear when one tries a consistent definition for this entropy, and (ii) building up a plausible formalism for it, which is based on the use of coherent complex states in the context of a path integration.

Keywords Entropy · Gamow states · Complex coherent states

1 Introduction

The vast majority of states observed in quantum physics are unstable, yet most of studies in quantum physics deal with the properties of stable or stationary states. Recently, many publications have appeared dealing with the behavior of quantum unstable states, both from the non-relativistic to the relativistic point of view. These studies cover topics like exponential decay and its deviations, the existence of vectors or density operators for exponentially decaying states and the solution to the mathematical problems that these objects pose, the existence of good analytical models to describe resonances, etc. The bibliography at the end of this paper covers only a small portion of the vast literature on the subject.

O. Civitarese (✉)
Department of Physics, University of La Plata, c.c. 67, 1900 La Plata, Argentina
e-mail: osvaldo.civitarese@fisica.unlp.edu.ar

M. Gadella
Department of de FTAO, Facultad de Ciencias, University of Valladolid, Paseo Belén 7, 47011 Valladolid, Spain

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1

Here we use the acquired experience to adapt the existing physical theories for stable systems to the unstable ones. One field that remains essentially unexplored is the statistical mechanics for quantum unstable states or resonances. In this paper, we want to start a discussion that may open the way to this formalism.

One problem to be discussed is the suitable form of an unstable state, in order to be treated statistically. Thus, it is unavoidable to ask up to what extent the properties of quantum unstable states can be used in a statistical description. It seems evident that these states are not equilibrium states, instead they tend to equilibrium in the distant future. Central to it is the notion of entropy for quantum decaying states. In relation to this problem, and up to our knowledge, two approaches have been proposed. One by Prigogine and coworkers [1, 2] defines an entropy operator and gives an evolution of the entropy for all times. A second approach was given by Kobayashi and Shimbori [3, 4], in which some notion of entropy was defined for unstable states with a long lifetime so that in a short period they could be considered as equilibrium states.

Along this paper, we advocate the approximation of decaying states by the use of Gamow vectors, or state vectors that decay exponentially at all orders. This is, in general, a reasonable approximation, since most of experimental data are collected within the range of intermediate times, neither very short to observe the Zeno effect nor too long to observe the long tail deviations from the exponential law. However, Gamow states produce some difficulties due to the fact that they are not normalizable vectors in the usual sense and therefore some standard operations on vectors in Hilbert space, like scalar products or observable averages, are not defined for Gamow states in general. It is also noteworthy that Gamow densities, or densities describing purely exponential states in Liouville spaces, even the nonfactorizable ones, do not behave like dissipative systems, as we shall see.

In order to make our discussion more concrete we made use of the Friedrichs model which is the simplest mathematical model, albeit non-trivial, for resonance phenomena, which has the advantage of being exactly solvable. The presence of one resonance in the model is not a limitation, since its basic structure can be extended to the case of several non-overlapping resonances. After some elementary discussions on Gamow densities and related objects, we shall deal with the determination of the canonical entropy of a decaying system. As noted in [3], we may consider a decaying state as a state in thermal equilibrium, provided that its width Γ is sufficiently small. We recall that the canonical state of a system, defined by the operator ρ, makes extremal (or critical using the language of variational calculus) the Boltzmann entropy $S = -k\,\mathrm{tr}\{\rho\log\rho\}$, with the constrains that $\mathrm{tr}\,\rho = 1$ and $\mathrm{tr}\{H\rho\} = E$, E being a fixed quantity. Then, variational calculus gives the solution $\rho = \exp\{-\beta H\}$ with $\beta = (kT)^{-1}$, k being the Boltzmann constant and T the temperature.

In the case of a system with resonances, a consistent definition of averages has formal difficulties, so that this usual procedure cannot be applied as it does in the case of quantum statistical mechanics in Hilbert space, as it was first noted in [3]. There is a possible solution to this problem, which is the postulate that the state of critical entropy is also in this case given by $\rho \equiv \exp\{-\beta H\}$, provided the total Hamiltonian H is written in a form that depends explicitly on the resonance.

This paper is organized as follows: In Sect. 2, we review the notions of Gamow states and give its relation with a time asymmetric quantum mechanics. In Sect. 3,

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

we review the simplest form of the Friedrichs model. After a brief discussion on the possibility of defining scalar products and averages, we give a second quantization language that will be useful in the construction and operation with the canonical unstable state. In Sect. 3, we study the properties of Gamow densities and confirm the time evolution of the Misra-Prigogine entropy for a Gamow state. In Sect. 4 we discuss the Liouville representation of Gamow states. Sections 5 and 6 are devoted to the analysis and implementation of the approximations needed to define the entropy for quantum unstable systems. The conclusions are drawn in Sect. 7. All intermediate steps of the derivations are presented in the Appendix.

2 Resonances, Gamow Vectors and Time Asymmetric Quantum Mechanics

Resonances in non-relativistic quantum mechanics are defined in various ways, which are often equivalent [5–8]. Resonances are determined by resonance poles which are poles of certain analytic functions. Perhaps the most popular way of determining these resonance poles is the identification of them with the poles of the analytical continued S-matrix (usually in the energy representation) [5, 6]. However, we shall use here the following definition of resonance [7]:

Assume that we have a Hamiltonian pair $\{H_0, H\}$, where H_0 is the unperturbed or free Hamiltonian and that H is the total Hamiltonian, written in the form $H = H_0 + V$, where V is some potential. The respective resolvents of H_0 and H are $R_{H_0} := (H_0 - z)^{-1}$ and $R_H = (H - z)^{-1}$. These resolvents are defined for any complex number z not belonging to the spectrum of H_0 or to the spectrum of H, respectively. Assume that there is a dense subspace $\mathcal{D}$ of the Hilbert space $\mathcal{H}$, where H_0 and H act, such that both $R^0_\psi(z) := \langle\psi|R_{H_0}\psi\rangle$ and $R_\psi(z) = \langle\psi|R_H\psi\rangle$ admit analytic continuations as functions of z beyond the spectrum (often the positive half line). If for some $z_R = E_R - i\Gamma/2$, $R^0_\psi(z_R)$ is analytic but $R_\psi(z_R)$ has a pole, we say that the Hamiltonian pair $\{H_0, H\}$ has a resonance at z_R. Here, $E_R > 0$ and $\Gamma > 0$ are the resonance energy and the width.

This definition is motivated by the fact that a scattering experiment with free Hamiltonian H_0 and potential V usually has a channel in which the cross section, in the energy representation, shows a bump (usually of Lorenzian shape) centered at E_R with width Γ, which one identifies with a resonance.

For an interesting study of resonances and their relation with cross sections, poles, etc, see [9]. Therein it is shown that if we want to describe a decaying state as a normalizable vector in the Hilbert space $\mathcal{H}$, in which the Hamiltonian pair $\{H_0, H\}$ acts, this vector cannot decay exponentially at all times. For small and long times, the decay is not exponential, although it is exponential for a range of intermediate times. This is due to the fact that both H_0 and H are semi-bounded in standard non-relativistic quantum mechanics.

Usually the range of experimentation lies inside this range of exponential decay. Therefore, it is tantalizing to approximate the decaying state as a vector state which decays exponentially. However, this is contradictory with the request of normalizability for the vector state. In fact, if $|\psi^D\rangle$ is such a vector, it should be an eigenvector of H with a complex eigenvalue, which is contradictory with the self-adjointness of H.

The proper candidate for $|\psi^D\rangle$ would be an eigenvector of H with eigenvalue z_R, i.e., $H|\psi^D\rangle = z_R|\psi^D\rangle$, so that

$$e^{-itH}|\psi^D\rangle = e^{-itE_R}e^{-t\Gamma/2}|\psi^D\rangle. \tag{1}$$

This idea was discussed by Nakanishi [10]. The situation reminds that of the plane waves, which are non-normalizable eigenfunctions of H with eigenvalues in the continuous spectrum of H. Both problems have the same cure: extending the Hilbert space to a non-Hilbert topological vector space, in which H can be extended (with some notion of continuity) and has as eigenvectors plane waves and objects like $|\psi^D\rangle$. This can be done with the use of rigged Hilbert spaces or Gelfand triplets [11]

$$\Phi \subset \mathcal{H} \subset \Phi^\times. \tag{2}$$

Here, Φ is a dense subspace of the Hilbert space $\mathcal{H}$ with the property that $H\phi \in \Phi$, for all $\phi \in \Phi$, i.e, Φ is included in the domain of H. Φ is endowed with a topology stronger than the topology that it has inherited from $\mathcal{H}$, so that the canonical injection $i : \Phi \longmapsto \mathcal{H}$ $(i(\phi) = \phi)$ is continuous. In addition, H is continuous with the topology on Φ. The space $\Phi^\times$ is the antidual of Φ, i.e., the space of all continuous antilinear functionals on Φ. This means that for any $F \in \Phi^\times$, for any $\phi \in \Phi$ and for any complex number α, one has $F(\alpha\phi) = \alpha^* F(\phi)$, where the star denotes complex conjugation and $F(\phi)$ is the action of F on ϕ, which gives a complex number. The space $\Phi^\times$ includes the Hilbert space $\mathcal{H}$ as a subspace.

Now, it is possible to extend H to the space $\Phi^\times$ via the duality formula. In order to use a notation similar to the Dirac notation in quantum mechanics, we denote the action $F \in \Phi^\times$ in $\phi \in \Phi$ as $F(\phi) = \langle\phi|F\rangle$. Let us define the extension $H^\times$ of H to $\Phi^\times$ as the unique operator that for any $\phi \in \Phi$ and any $F \in \Phi^\times$ satisfies:

$$\langle H\phi|F\rangle = \langle\phi|H^\times F\rangle. \tag{3}$$

The operator $H^\times$ is continuous on $\Phi^\times$ in some weak sense. Whenever no cause of confusion be possible, we omit the cross in $H^\times$ and simply write H. This construction is always possible as proved by Gelfand [11]. See also [12–16]. Then, plane waves as well as $|\psi^D\rangle$ with the desired properties are properly defined as vectors in $\Phi^\times$.

The vector state $|\psi^D\rangle$ is called the decaying Gamow vector, honoring G. Gamow who introduced the notion in his pioneering work on alpha decay [17], a nuclear process which exhibits an exponential decay rate as in (1). As noted before, $|\psi^D\rangle$ should be a functional; i.e, an object mapping test vectors into complex numbers. In this case, we need to have spaces of analytic functions including the resonances in their analyticity domain. This was solved with the aid of test functions of Hardy class in a half plane [18, 19].

It is interesting to summarize this construction from a broad scope, leaving the details to the already published papers [20–22], monographs [18, 19] and conference proceedings [8, 23, 24]. With the aid of Schwarz functions, Hardy functions on the upper and lower half planes [25] and the spectral theorem for self adjoint operators

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

in Hilbert space, one can construct two Gelfand triplets (also called rigged Hilbert spaces (RHS)):

$$\Phi_{\pm} \subset \mathcal{H} \subset \Phi_{\pm}^{\times}, \tag{4}$$

so that:

(i) For any $\phi_{\pm} \in \Phi_{\pm}$, we have that $H\phi_{\pm} \in \Phi_{\pm}$. Then, we can extend H to the duals $\Phi_{\pm}^{\times}$ via the duality formula.

(ii) There exist a vector $|\psi^D\rangle$ in $\Phi_{+}^{\times}$ and a vector $|\psi^G\rangle$ in $\Phi_{-}^{\times}$ such that

$$H|\psi^D\rangle = z_R|\psi^D\rangle, \qquad H|\psi^G\rangle = z_R^*|\psi^G\rangle, \tag{5}$$

where $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$. Here, we have identified H with its extensions to the anti-duals $\Phi_{\pm}^{\times}$. The first equation in (5) is defined in $\Phi_{+}^{\times}$ and the second in $\Phi_{-}^{\times}$.

(iii) We have the following time behavior for $|\psi^D\rangle$ and $|\psi^G\rangle$ [18, 19]

$$e^{-itH}|\psi^D\rangle = e^{-itE_R}e^{-t\Gamma/2}|\psi^D\rangle \tag{6}$$

$$e^{-itH}|\psi^G\rangle = e^{-itE_R}e^{t\Gamma/2}|\psi^G\rangle. \tag{7}$$

According to a usual terminology [8, 18, 19], $|\psi^D\rangle$ is called the decaying Gamow vector and $|\psi^G\rangle$ the growing Gamow vector. Depending on the choice of the test spaces $\Phi_{\pm}$, this time behavior obeys either a semigroup decaying law (this means that (6) is well defined for $t \geq 0$ only and that (7) is well defined for $t \leq 0$ only) or a group law, in which case (6) and (7) are valid for all real values of t. In the first case, we have made use of spaces of Hardy functions on a half plane to construct $\Phi_{\pm}$. The second case is needed for our discussion on the time evolution of Gamow densities held in Sect. 4. In this case, the construction of $\Phi_{\pm}$ is given in the Appendix. In any case, the character of either semigroup or group on the time evolution does not affect our discussion on the entropy given in Sect. 6.

(iv) Let T be the time reversal operator. Then, $T\Phi_{\pm} = \Phi_{\mp}$, $T\Phi_{\pm}^{\times} = \Phi_{\mp}^{\times}$, $T|\psi^D\rangle = |\psi^G\rangle$, $T|\psi^G\rangle = |\psi^D\rangle$. This means that, in some sense, we have a dual representation of decaying process, one looking at the future and other to the past, but otherwise totally equivalent.

Another interpretation is possible, given by the formalism called Time Asymmetric Quantum Mechanics (TAQM), introduced by A. Bohm, in order to define a quantum arrow of time [24, 26–30]. Elements of the triplet $\Phi_{+} \subset \mathcal{H} \subset \Phi_{+}^{\times}$, the *in*-elements, are produced in the past by a *preparation apparatus*, before the resonance is produced. They are identified as states and considered for $t < 0$ only. Then, the resonance appears by the effect of scattering due to the Hamiltonian pair $\{H_0, H\}$. The products of the decaying of the resonance go far apart of the interaction region and are observed. Then, the elements of the triplet $\Phi_{-} \subset \mathcal{H} \subset \Phi_{-}^{\times}$ or *out*-elements are interpreted as observables, which are registered when the interaction are ceased by the *registration apparatus* and considered for $t > 0$ only. In the TAQM interpretation, the arrow of time goes from the preparation to the registration apparatus.

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

3 The Friedrichs Model

The most elementary version of the Friedrichs model [31] is an exactly solvable model for resonances having all the ingredients present in resonance phenomena. Due to this property, it is a very good laboratory to test physical quantities over resonances. These quantities may include thermodynamical variables that should be defined for this case. A survey of some applications of the Friedrichs model to the theory of resonances can be found in [32]

The Friedrichs model considers a Hamiltonian pair $\{H_0, H\}$, where H_0 is the free Hamiltonian and $H = H_0 + \lambda V$. Here:

$$H_0 = \omega_0 |1\rangle\langle 1| + \int_0^\infty \omega |\omega\rangle\langle\omega| d\omega, \tag{8}$$

$$V = \int_0^\infty f(\omega)\big[|\omega\rangle\langle 1| + |1\rangle\langle\omega|\big] d\omega. \tag{9}$$

The function $f(\omega)$ is called the form factor. It should be square integrable and can be chosen to be real provided that $f^2(\omega)$ admits an analytic continuation to an open set including $\mathbb{R}^+ \equiv [0, \infty)$, i.e., the positive semi-axis [33]. The constant λ, which is real, is taken as the strength of the coupling. H_0 has a bound state, $|1\rangle$, with eigenvalue ω_0 and an absolutely continuous spectrum, which is $\mathbb{R}^+$. Vectors $|\omega\rangle$ are the generalized eigenvectors of H_0 with respective eigenvalues $\omega \in \mathbb{R}^+$, i.e., $H_0|\omega\rangle = \omega|\omega\rangle$. We take $|1\rangle$ with norm one. Since discrete and continuous subspaces are mutually orthogonal and $\{|1\rangle, |\omega\rangle\}$ forms a complete system or "basis", we have that

$$\langle 1|1\rangle = 1; \qquad \langle 1|\omega\rangle = \langle\omega|1\rangle = 0; \qquad \langle\omega|\omega'\rangle = \delta\big(\omega - \omega'\big). \tag{10}$$

As a consequence of the interaction, the bound state $|1\rangle$ of H_0 disappears and is replaced by a resonance that depends analytically on the coupling constant λ. In the limit $\lambda \mapsto 0$, we recover the bound state.

Resonances are characterized by poles of the reduced resolvent of H, which are regular points in the resolvent of H_0 [34]. In the case of the Friedrichs model, we consider the following reduced resolvent:

$$\frac{1}{\eta(z)} := \langle 1|\frac{1}{z - H}|1\rangle. \tag{11}$$

Under weak conditions on the form factor $f(\omega)$ [33], the function $\eta(z)$ is analytic on the complex plane except for a branch cut on the positive semi-axis $\mathbb{R}^+$, and admits an analytic continuation through the cut from above to below. This analytic continuation admits one zero located at the point $z_R = E_R - i\Gamma/2$. Here, E_R and Γ are identified with the energy and the width of the resonance. The zero of $\eta(z)$ is a pole of the reduced resolvent (11) and can be identified with a resonance. Analogously, $\eta(z)$ admits an analytic continuation from below to above with zero at $z_R^* = E_R + i\Gamma/2$. This second resonance pole is nothing else that the time reversal of the former [35].

« FOOP 10701 layout: Small Extended v.1.1 file: foop9746.tex (Alapsin) class: spr-small-v1.3 v.2013/08/14 Prn:2013/09/17; 10:06 p. 7/20»
« doctopic: OriginalPaper numbering style: ContentOnly reference style: mathphys»

The value of the resonance pole z_R can be calculated explicitly and gives [36]:

$$z_R = \omega_0 + \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{[z_R - \omega]_+} := \omega_0 + \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{z_R - \omega} - 2\pi i \lambda^2 |f(z_R)|^2. \quad (12)$$

Gamow vectors or *Gamow states* are the eigenvectors of the total Hamiltonian H with eigenvalues z_R and z_R^*. They obey the following eigenvalue equations:

$$H|\psi^D\rangle = z_R|\psi^D\rangle; \qquad H|\psi^G\rangle = z_R^*|\psi^G\rangle. \quad (13)$$

Needless to say that, due to the self adjointness of H, both Gamow vectors cannot be normalized. Likewise the generalized eigenvectors of H_0, $|\omega\rangle$, the Gamow vectors are functionals on a certain space of test vectors (see the Appendix).

In terms of the basis $\{|1\rangle, |\omega\rangle\}$, the Gamow vectors admit the following explicit representation [36]:

$$|\psi^D\rangle \approx \left[|1\rangle + \int_0^\infty d\omega \frac{\lambda f(\omega)}{[z_R - \omega]_+}|\omega\rangle\right] \quad (14)$$

$$|\psi^G\rangle \approx \left[|1\rangle + \int_0^\infty d\omega \frac{\lambda f(\omega)}{[z_R^* - \omega]_-}|\omega\rangle\right], \quad (15)$$

Note that these are functionals on some test spaces, which are different [6, 18, 19, 36]. The meaning of the sign minus in the denominator of (15) is

$$\int_0^\infty d\omega \frac{\lambda F(\omega)}{[z_R^* - \omega]_-} = \int_0^\infty d\omega \frac{\lambda F(\omega)}{z_R^* - \omega} + 2\pi i \lambda^2 |F(z_R^*)|^2. \quad (16)$$

The formula for the integral in (14) is similar, we only have to use a minus sign before the second term of the right hand side of (16). For any test vector ϕ, $F(\omega) = f(\omega)\langle\phi|\omega\rangle$. Note that the space of test vectors has to be defined in such a way that $\langle\phi|\omega\rangle$ is analytically continuable so that its value at z_R^* has a meaning.

It can be demonstrated that the Gamow vector $|\psi^D\rangle$ decays exponentially with time and that $|\psi^G\rangle$ grows exponentially with time so that [18, 19]

$$\left|\psi^D(t)\right\rangle = e^{-iE_R t} e^{-\Gamma t}\left|\psi^D(t)\right\rangle; \qquad \left|\psi^G(t)\right\rangle = e^{-iE_R t} e^{\Gamma t}\left|\psi^G(t)\right\rangle. \quad (17)$$

In the discussion of the Gamow states in Liouville space proposed here (see below), we assume that (17) is valid for all time (see also the Appendix). The spectrum of the total Hamiltonian H is $\mathbb{R}^+$. For each value $\omega \in [0, \infty)$, it has two generalized eigenvectors $H|\omega^\pm\rangle = \omega|\omega^\pm\rangle$ with

$$|\omega^+\rangle = |\omega\rangle + \frac{\lambda f(\omega)}{\tilde{\eta}_+(\omega)}\left[|1\rangle + \int_0^\infty d\omega' \frac{\lambda f(\omega')}{\omega - \omega' + i0}|\omega'\rangle\right], \quad (18)$$

$$|\omega^-\rangle = |\omega\rangle + \frac{\lambda f(\omega)}{\eta_-(\omega)}\left[|1\rangle + \int_0^\infty d\omega' \frac{\lambda f(\omega')}{\omega - \omega' - i0}|\omega'\rangle\right]. \quad (19)$$

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

Here, $\eta_+(\omega)$ and $\eta_-(\omega)$ are the boundary values of the function $\eta(\omega)$ from above to below and from below to above, respectively, and

$$\frac{1}{\tilde{\eta}_+(\omega)} = \frac{1}{\eta_+(\omega)} + 2\pi i \frac{\delta(\omega - z_R)}{\eta'(z_R)}. \tag{20}$$

We need to add that $\{|\psi^D\rangle, |\omega\rangle^+\}$ and $\{|\psi^G\rangle, |\omega\rangle^-\}$ form two different basis of eigenvectors of H, the former are forward solutions and the second ones are backward solutions in the formalization of the Time Asymmetric Quantum Theory (TAQT) formulated by Arno Bohm [24, 26–30]. In terms of these two basis, the spectral representations of H are:

$$H = z_R|\psi^D\rangle\langle\psi^G| + \int_0^\infty \omega|\omega^+\rangle\langle\omega^-|d\omega, \tag{21}$$

which is the forward spectral representation, and

$$H = z_R^*|\psi^G\rangle\langle\psi^D| + \int_0^\infty \omega|\omega^-\rangle\langle\omega^+|d\omega, \tag{22}$$

which is the backward spectral representation. From a mathematical point of view, these operators act on different spaces as explained in the Appendix. Note also that

$$\langle\psi^G|H = z_R\langle\psi^G|; \qquad \langle\psi^D|H = z_R^*\langle\psi^D|. \tag{23}$$

This completes relations (13).

Next, we show that some products between objects introduced above can be defined and some others need regularization. Take the expression for the decaying Gamow vector (14) and construct its bra:

$$\langle\psi^D| \approx \left[\langle 1| + \int_0^\infty d\omega \frac{\lambda f(\omega)}{[z_R^* - \omega]_-}\langle\omega|\right]. \tag{24}$$

Then, perform the formal scalar product $\langle\psi^D|\psi^G\rangle$ and use relations (10). We obtain:

$$\langle\psi^D|\psi^G\rangle = N^2\left[1 + \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{[z_R^* - \omega]_-^2}\right], \tag{25}$$

where

$$\int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{[z_R^* - \omega]_-^2} \tag{26}$$

is a first derivative, and it is therefore well defined. The conclusion is that the bracket $\langle\psi^D|\psi^G\rangle$ makes sense, since it gives a proportionality factor:

$$N = \frac{1}{\sqrt{1 + \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{[z_R^* - \omega]_-^2}}}. \tag{27}$$

Note that this factor can be taken equal to one up to first order in the coupling constant λ. An analogous analysis can be done for the product $\langle\psi^G|\psi^D\rangle$, but it cannot be extended to brackets like $\langle\psi^G|\psi^G\rangle$ or $\langle\psi^D|\psi^D\rangle$, where from

$$\langle\psi^D|\psi^D\rangle \approx \left[1+\int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{[z_R^* - \omega]_-[z_R-\omega]_+}\right]. \tag{28}$$

the proportionality factor is not well defined as one can realize after the inspection of formulas (12) and (16). The only way out seems to be a choice of the form factor $f(\omega)$ such that $f(z_R)=0$. But *this is not possible*, because to fulfill that one has to find such a z_R for each value of λ. For general conditions that we are always admitting [33], z_R is analytic as a function of λ, so that all possible values of z_R form a curve with limit point ω_0. In this curve $f(z)$ vanishes. Therefore, by the principle of analytic continuation [37], we conclude that $f(z)$ must be identically zero everywhere, which is nonsense. This means that the product of distributions given by (28) is not normalizable, an the same is true for $\langle\psi^G|\psi^G\rangle$.

3.1 The Friedrichs Model in Second Quantization Language

The Friedrichs model can be written in an alternative way using creation and annihilation operators [38]. We start with a vacuum state $|0\rangle$ and then define the creation operator $a^\dagger$ for the bound state $|0\rangle$ and the creation operators $b_\omega^\dagger$ for the generalized states $|\omega\rangle$ with ω in the continuous spectrum of H_0 respectively as:

$$|1\rangle = a^\dagger|0\rangle; \qquad |\omega\rangle = b_\omega^\dagger|0\rangle. \tag{29}$$

The corresponding annihilation operators a and b_ω satisfies the following commutation relations with the creation operators:

$$[a, a^\dagger] = 0; \qquad [b_\omega, b_{\omega'}^\dagger] = \delta(\omega-\omega') \tag{30}$$

and have the following natural property:

$$a|0\rangle = b_\omega|0\rangle = 0. \tag{31}$$

Then, the form of H_0 and V in this language is respectively:

$$H_0 = \omega_0 a^\dagger a + \int_0^\infty d\omega\omega b_\omega^\dagger b_\omega, \tag{32}$$

$$V = \int_0^\infty d\omega f(\omega)\big(a^\dagger b_\omega + a b_\omega^\dagger\big). \tag{33}$$

In [38], the following creation and annihilation operators are defined:

$$A_{IN}^\dagger := \int_\gamma d\omega \frac{\lambda f(\omega)}{\omega - z_R} b_\omega^\dagger - a^\dagger, \tag{34}$$

$$A_{OUT} := \int_\gamma d\omega \frac{\lambda f(\omega)}{\omega - z_R} b_\omega - a, \tag{35}$$

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

$$B^\dagger_{\omega,IN} := b^\dagger_\omega + \frac{\lambda f(\omega)}{\tilde\eta^+(\omega)}\left\{\int_0^\infty d\omega' \frac{\lambda f(\omega')}{\omega'-\omega-i0} b^\dagger_{\omega'} - a^\dagger\right\}, \tag{36}$$

$$B_{\omega,OUT} := b_\omega + \frac{\lambda f(\omega)}{\tilde\eta^+(\omega)}\left\{\int_0^\infty d\omega' \frac{\lambda f(\omega')}{\omega'-\omega-i0} b_{\omega'} - a\right\}. \tag{37}$$

Operators (34)–(37) satisfy the commutation relations:

$$\left[A_{OUT}, A^\dagger_{IN}\right] = 1; \qquad \frac{\eta^+(\omega)}{\eta^-(\omega)}\left[B_{\omega,OUT}, B^\dagger_{\omega',IN}\right] = \delta(\omega-\omega'). \tag{38}$$

All other commutators vanish.

$$H = z_R A^\dagger_{IN} A_{OUT} + \int_0^\infty d\omega\omega \frac{\eta^+(\omega)}{\eta^-(\omega)} B^\dagger_{\omega,IN} B_{\omega,OUT}. \tag{39}$$

Note that $A^\dagger_{IN}$ and A_{OUT} are, respectively, the creation and annihilation operators [39] of the decaying Gamow vector $|\psi^D\rangle$:

$$|\psi^D\rangle = A^\dagger_{IN}|0\rangle, \qquad A_{OUT}|\psi^D\rangle = |0\rangle. \tag{40}$$

Although from the explicit expressions (8) and (9), $A^\dagger_{IN}$ and A_{OUT} are not formal adjoint of each other, in practise they should be considered as they are. Note that both operators are defined in terms of a distributional kernel with a pole in the lower half of the complex plane, which means that they can be multiplied by each other, which would not be the case if the distributional kernels would have been defined in different half planes. From the point of view of the time asymmetric quantum theory [24, 26–30], both operators are defined for $t > 0$, the "decaying part" of a resonance process.

4 Liouville Representation of Gamow States

The aim of the present section is to discuss the possibilities for a definition of a Gamow state in the Liouville space and to show that all of them fulfill an equation of motion of Liouville type. This indicates that the decaying states should not be treated as diffusion processes or any other that satisfies an equation of motion of Lindblad type. There are a few references concerning Gamows in Liouville spaces [2, 39, 40].

Gamow states in Liouville space have been introduced either in factorizable form, i.e., as a dyadic product of two Gamow vectors, or in non-factorizable form. Let us discuss first the nature of factorizable candidates for Liouville densities. These objects are functionals on a given space of test vectors as Gamow vectors are functionals over a space of test vectors. These particular functionals, for which time evolution should exist for any real value of time, are discussed in the Appendix.

There are two possibilities for defining a factorizable Gamow density. The most natural one is $\rho = |\psi^D\rangle\langle\psi^G|$ [39].

This choice has some inconveniences, as we describe here. Recall that the Liouvillian or Liouville operator is defined by means of the Hamiltonian H on the

« FOOP 10701 layout: Small Extended v.1.1 file: foop9746.tex (Alapsin) class: spr-small-v1.3 v.2013/08/14 Prn:2013/09/17; 10:06 p. 11/20»
« doctopic: OriginalPaper numbering style: ContentOnly reference style: mathphys»

tensor product of Hilbert spaces $\mathcal{H}\otimes\mathcal{H}^\times$, where $\mathcal{H}$ is the Hilbert space where H acts on ρ[1] is given by $L=H\otimes I-I\otimes H$, where I is the identity. Using (5) and the dual expressions given in (23), we have that $L|\psi^D\rangle\langle\psi^G|=0$. As a consequence, the state $|\psi^D\rangle\langle\psi^G|$ does not evolve. In fact, noting that the evolution operator in Liouville space is given by $e^{-itL}=e^{-itH}\otimes e^{itH}$, a simple calculation gives $e^{-itL}|\psi^D\rangle\langle\psi^G|=|\psi^D\rangle\langle\psi^G|$.

The conclusion that the Gamow density $|\psi^D\rangle\langle\psi^G|$ does not evolve is not logical. Gamow vectors grow or decay exponentially, so that a similar behavior should be expected from Gamow densities. Therefore, we must conclude that we should discard $|\psi^D\rangle\langle\psi^G|$ as an acceptable Gamow density.

A second candidate is

$$\rho:=|\psi^D\rangle\langle\psi^D|. \tag{41}$$

This looks to be a better candidate as it fulfills the following properties:

$$L\rho=-i\Gamma|\psi^D\rangle\langle\psi^D|;\qquad \rho(t)=e^{-itL}|\psi^D\rangle\langle\psi^D|=e^{-t\Gamma}|\psi^D\rangle\langle\psi^D|. \tag{42}$$

The proof of these two expressions is straightforward. If we take (1) as the Gamow density we have, after a straightforward calculation:

$$\big[H,\rho(t)\big]|f\rangle=-i\Gamma\rho(t)|f\rangle, \tag{43}$$

where $|f\rangle$ is an arbitrary test vector. Then, we arrive to the following result:

$$\frac{d}{dt}\rho(t)=i\big[H,\rho(t)\big], \tag{44}$$

which is a Liouville equation and not a Lindblad equation. The formal proof is trivial.

There is still a non-factorizable form of the Gamow density, which has been considered in [39]. Let H be a Hamiltonian on a separable infinite dimensional Hilbert space $\mathcal{H}$, L its Liouvillian and $R_L(z)=(z-L)^{-1}$ the resolvent of L. If $\mathcal{H}$ is a Hilbert space of functions on $\mathbb{R}$, the set of real numbers, let us consider the operators B of the type

$$(Bf)(x):=\int_{-\infty}^{\infty}B(x,y)f(y)dy, \tag{45}$$

for any $f(x)\in\mathcal{H}$, where $B(x,y)$ is some integral kernel. This type of operators include the Hilbert-Schmidt class, so that it makes sense to apply the Liouville operator L to B. The result is given by [39]:

$$(LB)f(x)=\int_{-\infty}^{\infty}\big\{H_xB(x,y)-H_yB(x,y)\big\}f(y)dy. \tag{46}$$

Here, H_x and H_y means that H acts with respect to the variable x and y respectively. Let $P(\omega)$, with ω in the spectrum of H (we assume that H has purely non-degenerate

[1] $\mathcal{H}^\times$ is the dual of the Hilbert space $\mathcal{H}$. We know that these two spaces can be identified via the Riesz theorem, but here it is convenient to distinguish them, as we often distinguish between the spaces of ket vectors (in $\mathcal{H}$) and spaces of bra vectors (in $\mathcal{H}^\times$).

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

absolutely continuous spectrum as it is the case in the Friedrichs model), be the spectral projections of H and correspondingly $P_x(\omega)$, $P_y(\omega)$ the spectral projections of H_x and H_y respectively (of course these two families are identical). We define the following extension of the resolvent of the Liouville operator [39]:

$$\tilde{R}_L(z) = R_L(z) + 2\pi i P_x(\zeta) \otimes \tilde{P}_y(\zeta - z), \tag{47}$$

Note that the argument of $P_y(\cdot)$ is complex due to an analytic continuation [39]. To stress that we are using this continuation, we have added a tilde to $P_y(\cdot)$. Then, if z_R is a singularity of the extended resolvent $\tilde{R}_L$, the non-factorizable Gamow density is given by

$$\Pi_L := -\frac{1}{2\pi i} \oint_\gamma \tilde{R}_L(\zeta) d\zeta, \tag{48}$$

where γ is a closed contour that encloses z_R as the only singularity. A rather straightforward calculation shows that this non-factorizable Gamow density has the following properties:

$$[H, \Pi_L] = L\Pi_L, \qquad \frac{d}{dt}\Pi_L(t) = -i[H, \Pi_L]. \tag{49}$$

Again, this is a Liouville equation. The conclusion is clear: any version of the Gamow density satisfies a Liouville equation and not a Lindblad equation, as one would expect if considers the Gamow state as a dissipative state.

5 Entropy of a Gamow Density

The idea of representing an entropy operator as a Lyapunov variable, i.e., a dynamical variable varying monotonically with time, was introduced by Prigogine and coworkers [1]. This means the existence of a positive operator M such that $\langle \rho(t)|M\rho(t)\rangle$ should vary monotonically with time for any initial state $\rho(0)$. In [1] the authors have established the following result: should an entropy operator M exist as an operator on the Hilbert space of pure states of a system, it should satisfy the following conditions:

$$i[H, M] = D \geq 0 \quad \text{and} \quad [M, D] = 0. \tag{50}$$

The authors conclude that such an operator M with these properties cannot exist due to the semiboundedness of the Hamiltonian H. Then, they suggest that M should be an operator on the Liouville space of densities and satisfy properties (50) with H replaced by the Liouvillian L. In this case, M cannot be a factorizable operator in the sense that there exists two operators A and B such that for any density ρ, one has $M\rho = A\rho B$, where A and B are operators acting on the Hilbert space of pure states.

The point is that this analysis is not longer valid for resonances as resonances are not described by objects in Hilbert space or in the ordinary Liouville space, but in suitable extensions of them. In a later paper of Prigogine and coworkers [2], the suggested entropy operator was defined as

$$M = |\psi^G\rangle\langle\psi^G|. \tag{51}$$

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

« FOOP 10701 layout: Small Extended v.1.1 file: foop9746.tex (Alapsin) class: spr-small-v1.3 v.2013/08/14 Prn:2013/09/17; 10:06 p. 13/20»
« doctopic: OriginalPaper numbering style: ContentOnly reference style: mathphys»

This is neither an operator in Hilbert or in Liouville spaces. We have already determined (see the Appendix) that M can be well defined as a functional. In addition, this M in (51) can be viewed as an operator between a dense subspace of a Hilbert space and its dual. This is exactly what happens with the Hamiltonian. Then, it results that from this point of view relations (50) have a meaning at least formally.

As an operator, the action of M on any test vector ϕ is

$$M|\phi\rangle = \big(\langle\psi^G|\phi\rangle\big)|\psi^G\rangle, \tag{52}$$

so that M acts as a projector, sending all the test vector space Ψ_- on the one dimensional space spanned by $|\psi^D\rangle$.

The properties of M as defined in (51) are the following:

(i) Positivity: For any test vector ϕ, we have

$$\langle\phi|M\phi\rangle = \langle\phi|\psi^G\rangle\langle\psi^G|\phi\rangle = |\langle\psi^G|\phi\rangle|^2 \geq 0. \tag{53}$$

(ii) M is a Lyapunov variable:

$$\begin{aligned}\langle\phi(t)|M\phi(t)\rangle &= \langle e^{-itH}\phi|\psi^G\rangle\langle\psi^G|e^{-itH}\phi\rangle\\ &= \langle\phi|e^{itH}|\psi^G\rangle\langle\psi^G|e^{-itH}\phi\rangle = e^{t\Gamma}|\langle\phi|\psi^G\rangle|^2,\end{aligned} \tag{54}$$

which is obviously well defined and monotonically increasing.

(iii) M satisfies relations:

$$\begin{aligned}&i[H,M] = i\big(H|\psi^G\rangle\langle\psi^G| - |\psi^G\rangle\langle\psi^G|H\big) = -\Gamma M = D,\\ &[D,M] = 0.\end{aligned} \tag{55}$$

Since M and Γ are positive, now D is a negative operator.

For a given state ρ the entropy is defined as [1]:

$$S := \mathrm{tr}\big\{\rho^\dagger M\rho\big\}, \tag{56}$$

whenever this expression be well defined. This definition includes a suitable choice of the notion of trace. Here $\rho^\dagger$ means formal adjoint of ρ, which is usually formally Hermitean.

Note that a Gamow state cannot be a pure state in the usual sense. First of all, a pure state in Hilbert space is pure if and only if its entropy vanishes. We can show that the entropy for a factorizable Gamow density of the type $\rho = |\psi^D\rangle\langle\psi^D|$ cannot be zero. In order to do it, we first construct the following object ($\rho = |\psi^D\rangle\langle\psi^D|$):

$$\rho^\dagger M\rho = |\psi^D\rangle\langle\psi^D|\psi^G\rangle\langle\psi^G|\psi^D\rangle\langle\psi^D|. \tag{57}$$

In the Friedrichs model, the formal product $\langle\psi^D|\psi^G\rangle$ is well defined, so that it can be normalized to one. In this case, we conclude that

$$\rho^\dagger M\rho = \rho. \tag{58}$$

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

Next, we need one reasonable notion of trace. For the Friedrichs model, we may use the eigenvector basis of H_0, which is $\{|1\rangle, |\omega\rangle\}$, so that

$$\operatorname{tr}\rho = \langle 1|\psi^D\rangle\langle\psi^D|1\rangle + \int_0^\infty \langle\psi^D|\omega\rangle\langle\omega|\psi^D\rangle d\omega. \tag{59}$$

The integral involves products of distributions which are not well defined. In fact, taken into account (14), we have

$$\langle\omega|\psi^D\rangle = \int_0^\infty d\omega' \frac{\lambda f(\omega')}{z_R-\omega'}\langle\omega|\omega'\rangle = \int_0^\infty d\omega' \frac{\lambda f(\omega')}{z_R-\omega'}\delta(\omega-\omega') = \frac{\lambda f(\omega)}{z_R-\omega}. \tag{60}$$

Then, let us use (60) into (59) and take into account (10). We obtain:

$$\operatorname{tr}\rho \approx \left[1 + \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{[z_R-\omega]_+[z_R^*-\omega]_-}\right]. \tag{61}$$

The integral term in (61) is ill defined and it cannot be regularized, in principle, as mentioned previously (see Sect. 3). However, we have here an interesting point. Since

$$\rho(t)M\rho(t) = e^{-itL}\rho M e^{-itL}\rho = e^{-2t\Gamma}\rho(0) = e^{-t\Gamma}\rho(t), \tag{62}$$

we have that

$$\begin{aligned} S(t) &= \operatorname{tr}\big[\rho(t)M\rho(t)\big] = e^{-2\Gamma t}\operatorname{tr}\rho(0) \\ &\approx e^{-2\Gamma t}\left[1 + \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{[z_R-\omega]_+[z_R^*-\omega]_-}\right]. \end{aligned} \tag{63}$$

Although the integral in (63) is not well defined, and furthermore cannot be regularized, we still can draw a couple of conclusions: the entropy of a factorizable Gamow density cannot be zero (which means that a Gamow state could not be considered a pure state) and that it has to decay exponentially. This latter result has been obtained in [2].

Note that (63) is again not well defined and that it cannot be regularized. Under the hypothesis that λ and $f(\omega_0)$ be small (although not vanishing), we may assume that expression (63) could be written as:

$$S(t) = e^{-2\Gamma t}|N|^2\left[1 + \int_0^\infty d\omega \frac{\lambda^2 f^2(\omega)}{(z_R-\omega)(z_R^*-\omega)}\right]. \tag{64}$$

Observe that (64) is a real number as the form factor $f(\omega)$ has been chosen as real.

6 An Approximation to the Canonical Entropy for Quantum Unstable States

In this section, we intend to give an approximate value for the entropy of the canonical state of a quasi stable system. By quasi stable, we mean an unstable system with long

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

mean life i.e., with small width Γ, so that it can be considered as an equilibrium system for short time intervals. This can be implemented by a Friedrichs model with the same hypothesis leading to formula (63), i.e., non-vanishing small values of λ and $f(\omega_0)$.

We have already mentioned the difficulties associated to the definition of the canonical state for unstable states. In standard quantum statistical mechanics, the canonical state is defined as quantum state with maximal entropy at a fixed energy. This is given by $\rho = e^{-\beta H}$, being H the Hamiltonian of the system and $\beta = (kT)^{-1}$, where k is the Boltzmann constant and T the temperature. It represents a system in thermodynamical equilibrium with a bath. In our case, the trace and mean value of the energy for a decaying state are not well defined. In spite of this nasty situation, we still define a canonical state for the Gamow state as the most immediate and economical generalization of the canonical state for a given Hamiltonian.

This generalization should be also given by $\rho = e^{-\beta H}$, were in our case H should be the total Hamiltonian for the Friedrichs model. However, H should be written in a basis showing explicitly the presence of the resonance. Thus, the expression for H that we shall use in the sequel is that one given by (39).

Our approach is based in the idea developed by ter Haar [41] of path integration [42] over coherent states. Coherent states are eigenvectors of the annihilation operator with arbitrary complex eigenvalue [43].

In our particular case, we shall construct coherent states that are eigenvectors of the annihilation operator A_{OUT}. For the formal adjoint of A_{OUT}, we shall take $A^\dagger_{IN}$. For any complex number α, we shall define the coherent state $|\alpha\rangle$ and its bra $\langle\alpha|$ as:

$$\begin{aligned} |\alpha\rangle &:= \exp\{\alpha A^\dagger_{IN} - \alpha^* A_{OUT}\}|0\rangle, \\ \langle\alpha| &:= \langle 0|\exp\{\alpha^* A_{OUT} - \alpha A^\dagger_{IN}\}, \end{aligned} \tag{65}$$

where $|0\rangle$ is the vacuum state. Making use of the commutation relations (65) it becomes evident that $|\alpha\rangle$ satisfies the standard properties of coherent states. In particular:

$$\begin{aligned} A_{OUT}|\alpha\rangle = \alpha|\alpha\rangle; \qquad \langle\alpha|A^\dagger_{IN} = \alpha^*\langle\alpha|; \\ \int_{\mathbb{C}} \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = 1; \quad d^2\alpha = (d\,\mathrm{Real}\,\alpha)(d\,\mathrm{Im}\,\alpha), \end{aligned} \tag{66}$$

where $\mathbb{C}$ denotes the field of complex numbers.

Take now the canonical density operator state $\rho = e^{-\beta H}$ with H as in (39). The mentioned strategy to calculate the entropy of ρ based on path integration over coherent states starts by defining for any pair of fixed complex numbers α_i and α_f the following expression:

$$\langle\alpha_i|\rho|\alpha_f\rangle = \lim_{N\mapsto\infty} \rho_N(\alpha_i, \alpha_f), \tag{67}$$

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

where [41]

$$\rho_N(\alpha_i,\alpha_f) = \int \prod_{k=1}^{N} \left(\frac{d^2\alpha_k}{\pi}\right) \exp\left\{-\tau\left[\sum_{n=1}^{N} H_+(\alpha_{n-1},\alpha_n) + \sum_{n=1}^{N+1}\left\{\left(\frac{\alpha_n^* - \alpha_{n-1}^*}{2\tau}\right)\alpha_n - \alpha_{n-1}^*\left(\frac{\alpha_n - \alpha_{n-1}}{2\tau}\right)\right\}\right]\right\}, \tag{68}$$

with $\alpha_0 = \alpha_i$, $\alpha_{N+1} = \alpha_f$ and $\tau = \beta/N$. We write $\alpha_i = x_i + iy_i$ and $d\alpha_i = dx_i dy_i$, so that we have $2N$ integrals in the variables $x_1, \dots, x_N$, $y_1, \dots, y_N$. The integration limits are $-\infty$ and ∞ in all cases, since there must be one coherent state for any complex number. The term

$$H_+(\alpha,\alpha') = \frac{\langle\alpha|H|\alpha'\rangle}{\langle\alpha|\alpha'\rangle}, \qquad \langle\alpha|\alpha'\rangle = \exp\left\{-\frac{|\alpha|^2}{2} - \frac{|\alpha'|^2}{2} + \alpha^*\alpha'\right\} \tag{69}$$

is called the normal expansion of the Hamiltonian H. In our case, we have:

$$H_+(\alpha,\alpha') = z_R\alpha^*\alpha'. \tag{70}$$

An explicit calculation of (68) with (69) is given in [44]. This calculation includes an elimination of infinities as is sometimes usual in the path integral formalism [42]. The final result gives

$$\rho_N(\alpha_i,\alpha_f) := \exp\{-\beta z_R \alpha_i^* \alpha_f\}, \tag{71}$$

which does not depend on N, so that we shall omit the subindex N in $\rho_N(\alpha_i,\alpha_f)$ from now on.

In order to calculate the entropy for $\rho = e^{-\beta H}$, we shall use the standard formula:

$$S = k\left(1 - \beta\frac{\partial}{\partial\beta}\right)\log Z, \tag{72}$$

where Z is the partition function:

$$Z = \int_{-\infty}^{\infty} \frac{d^2\alpha}{\pi}\rho(\alpha,\alpha) = \int_{-\infty}^{\infty}\frac{d^2\alpha}{\pi}\exp\{-\beta z_R|\alpha|^2\} = \frac{1}{\pi}\int_{-\infty}^{\infty} dx e^{-\beta z_R x^2}\int_{-\infty}^{\infty} dy e^{-\beta z_R y^2} = \frac{1}{\beta z_R}. \tag{73}$$

Then, using (73) into (72), one finally obtains:

$$S = k\bigl(1 - \log(\beta z_R)\bigr) = k\left[1 - \ln\left(\beta\sqrt{E_R^2 + \frac{\Gamma^2}{4}}\right) - i\arctan\left(\frac{\Gamma}{2E_R}\right)\right], \tag{74}$$

In (74) we have taken the principal branch of $\log z$. The presence of a complex entropy, for the case of Gamow vectors, requires of some interpretation on the meaning

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF

of its imaginary part. The situation is quite similar to the existence of complex energy for decaying states, where the imaginary part is interpreted as the inverse of the half life.

Note that the resonance in the Friedrichs model is caused by the interaction of the system with the background, which plays the role of the thermodynamical bath. Then, we suggest that the real part of the entropy (74) is the entropy of the system and that the imaginary part of it is the entropy transferred from the system to the background. Should the thermodynamical entropy be identified with the modulus of (74), one concludes that the total entropy for a decaying state is bigger than the entropy of a stable system.

A final comment: We have obtained an approximate value of the entropy of the canonical Gamow state assuming that under the hypothesis of small widths and short times of observation, the Gamow state is in thermodynamical equilibrium. On the other hand, Eq. (63) just show that the entropy of an arbitrary Gamow state should be different from zero and that it decays exponentially. Both results refer to different situations and should not be contradictory.

7 Conclusions

In this work we have revisited a long standing question about the thermodynamics of unstable systems. We have focussed on the concept of entropy for states which behave as decaying states in the conventional quantum mechanical sense. In dealing with this concept, we have found the same difficulties which are faced when one is dealing with the concept of probability for the same class of states. To overcome these difficulties we have resorted to the formulation in terms of Gamow vectors and written the associated density in an amenable mathematical form. The tools introduced to accomplish the task were essentially two, complex coherent states and path integrals. When these tools are realized in the context of the solvable model due to Friedrichs, the notion of entropy for decaying states acquires a deep physical significance, since it allows for a comprehensive interpretation of the role of the width of the resonance in the thermodynamics of the system. As a departure from preliminary attempts in the field, the present formulation does not rely upon the notion of a complex temperature. We summarize the present results by stating that the total entropy for a decaying state, as obtained in our formalism, is bigger than the entropy of a stable system.

Acknowledgements This work has been partially supported by the CONICET of Argentina (PIP 0740), and the ANPCYT (Argentina), and the Spanish Government Grants No. PR2011-0343, No. FPA2008-04772-E, and No. MTM2009-10751.

Appendix

In the present Appendix, we shall sketch some mathematical properties related to objects that have been introduced along the present paper.

To begin with, let us define the space $C_b^\infty(\mathbb{R})$ as the vector space of all functions $f(x):\mathbb{R}\longmapsto\mathbb{C}$ having the following properties: i.) They are continuously differentiable at all orders and ii.) they have compact support, i.e., they vanish outside a finite interval of the real line $\mathbb{R}$.

This space has the following properties:

Let us consider the Fourier transform $\mathcal{F}$. Then, $\mathcal{F}$ is an one to one onto mapping

$$\mathcal{F}:C_b^\infty(\mathbb{R})\longmapsto Z, \tag{75}$$

where Z are entire analytic functions of exponential type [45]. Let us call $\mathcal{Z}$ the vector space of restrictions of functions of Z on the positive semi-axis $\mathbb{R}^+\equiv[0,\infty)$. Some properties of the spaces $\mathcal{Z}$ are the following:

(i) Functions in Z are in $L^2(\mathbb{R})$.

(ii) $\mathcal{Z}$ is dense in $L^2(\mathbb{R}^+)$ with respect to the topology of the latter, so that

$$\mathcal{Z}\subset L^2\big(\mathbb{R}^+\big)\subset\mathcal{Z}^\times \tag{76}$$

are well defined Gelfand triplets or RHS. This can be proven as done for similar triplets of Hardy functions [21].

(iii) For any $t\in\mathbb{R}$ and any $f(E)\in\mathcal{Z}$ we have that

$$e^{itE}f(E)\in\mathcal{Z}. \tag{77}$$

Then, take $F\in\mathcal{Z}^\times$ and consider the duality formula

$$\langle e^{itE}f(E)|F\rangle=\langle f(E)|e^{-itE}F\rangle. \tag{78}$$

This means that for any fixed $t\in\mathbb{R}$ and any $F\in\mathcal{Z}^\times$, $e^{-itE}F\in\mathcal{Z}^\times$.

(iv) The total Hamiltonian H for the Friedrichs model has non-degenerate absolutely continuous spectrum $\sigma(H)\equiv[0,\infty)$. Then, there is a unitary operator $U:L^2(\mathbb{R}^+)\oplus\mathbb{C}\longmapsto L^2(\mathbb{R}^+)$ such that UHU^{-1} is the multiplication operator on $L^2(\mathbb{R}^+)$, i.e., $UHU^{-1}f(E)=Ef(E)$, for any $f(E)\in L^2(\mathbb{R}^+)$ such that $Ef(E)\in L^2(\mathbb{R}^+)$ [34]. Then, if $\Phi:=U^{-1}\mathcal{Z}$, we have a new Gelfand triplet [18, 19]:

$$\Phi\subset L^2\big(\mathbb{R}^+\big)\oplus\mathbb{C}\subset\Phi^\times. \tag{79}$$

Any test vector ϕ in Φ can be written as $\phi=U^{-1}\phi(E)$ for any function $\phi(E)\in\mathcal{Z}$.

The Gamow vectors $|\psi^D\rangle$ and $|\psi^G\rangle$ can be defined as functionals on $\Phi^\times$, as follows:

$$\langle\phi|\psi^D\rangle=\phi^*(z_R);\quad \langle\phi|\psi^G\rangle=\phi^*\big(z_R^*\big);\quad \forall\phi\in\Phi, \tag{80}$$

where $\phi^*(z_R)$ and $\phi^*(z_R^*)$ are the values on z_R and z_R^* respectively of the entire function $\phi(E)\in\mathcal{Z}$ with $U\phi=\phi(E)$ and the star denotes complex conjugation. Formula (78) and an analysis as given in [18, 19] shows that

$$e^{-itH}|\psi^D\rangle=e^{-itE_R}e^{-t\Gamma/2}|\psi^D\rangle;\qquad e^{-itH}|\psi^G\rangle=e^{-itE_R}e^{t\Gamma/2}|\psi^G\rangle,\quad t\in\mathbb{R}. \tag{81}$$

(v) Let us consider the following Gelfand triplet:

$$\Phi \otimes \Phi \subset \mathcal{H} \otimes \mathcal{H} \subset \Phi^{\times} \otimes {}^{\times}\Phi, \tag{82}$$

where $\mathcal{H} = L^2(\mathbb{R}^+) \oplus \mathbb{C}$, $\Phi^{\times}$ (${}^{\times}\Phi$) is the space of all continuous *anti*linear (linear) functionals on Φ. Then, the dyad

$$|\psi^D\rangle\langle\psi^D| \in \Phi^{\times} \otimes {}^{\times}\Phi. \tag{83}$$

satisfies (42) for all values of $t \in \mathbb{R}$.

This result is obvious. In fact, if $\phi \otimes \varphi \in \Phi \otimes \Phi$, we have for the Liouville operator $L = H \otimes I - I \otimes H$:

$$\langle \phi \otimes \varphi | e^{-itL} |\psi^D\rangle\langle\psi^D|\rangle = e^{-t\Gamma} \langle \phi \otimes \varphi || \psi^D\rangle\langle\psi^D|\rangle, \quad t \in \mathbb{R}. \tag{84}$$

Since the space spanned by vectors in factorizable form $\phi \otimes \varphi$ is dense in $\Phi \otimes \Phi$ [46], we have for all values of time:

$$e^{-itL}|\psi^D\rangle\langle\psi^D| = e^{-t\Gamma}|\psi^D\rangle\langle\psi^D|, \qquad e^{-itL}|\psi^G\rangle\langle\psi^G| = e^{t\Gamma}|\psi^G\rangle\langle\psi^G|. \tag{85}$$

We see that $\rho = |\psi^D\rangle\langle\psi^D|$ decays exponentially to the future and $\rho = |\psi^G\rangle\langle\psi^G|$ decays exponentially to the past for all values of time. Note that the latter coincides with the Prigogine's entropy operator (51). The price that we had to pay for this picture is to give up the formulation based in the Hardy spaces (compare to the results given in [40]) and therefore the standard presentation of the TAQM.

References

1. Misra, B., Prigogine, I., Courbage, M.: Proc. Natl. Acad. Sci. USA **76**, 4768 (1979)
2. Karpov, E., Ordóñez, G., Petrosky, T., Prigogine, I.: Int. J. Quant. Chem. **98**, 69 (2004)
3. Kobayhashi, T., Shimbori, T.: Phys. Rev. E **63**, 056101 (2001)
4. Kobayhashi, T.: Int. J. Theor. Phys. **42**, 2265–2283 (2003)
5. Nussenzveig, H.M.: Causality and Dispersions Relations. Academic Press, New York (1972)
6. Bohm, A.: Quantum Mechanics, Foundations and Applications. Springer, Berlin (2001)
7. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Volume IV: Analysis of Operators. Academic Press, New York (1978)
8. Antoniou, I.E., Gadella, M.: In: Benatti, F., Floreanini, R. (eds.) Irreversibility, Resonances and Rigged Hilbert Spaces, Irreversible Quantum Dynamics. Lecture Notes in Physics, vol. 622, pp. 245–302 (2003)
9. Fonda, L., Ghirardi, G.C., Rimini, A.: Rep. Prog. Phys. **4**, 587 (1978)
10. Nakanishi, N.: Prog. Theor. Phys. **19**, 607 (1958)
11. Gelfand, I.M., Vilenkin, N.Ya.: Generalized Functions, vol. IV. Academic Press, New York (1964)
12. Maurin, K.: General Eigenfunction Expansions and Unitary representation of Topological Groups. Polish Scientific Publishers, Warsaw (1968)
13. Roberts, J.E.: Commun. Math. Phys. **3**, 98 (1966)
14. Antoine, J.P.: J. Math. Phys. **10**, 53 (1969)
15. Melsheimer, O.: J. Math. Phys. **15**, 902 (1974)
16. Gadella, M., Gómez, F.: Found. Phys. **32**, 815–869 (2002)
17. Gamow, G.: Z. Phys. **51**, 204–212 (1928)
18. Bohm, A., Gadella, M., Kets, D.: Gamow Vectors and Gelfand Triplets. Lecture Notes in Physics, vol. 348. Springer, Berlin (1989)
19. Civitarese, O., Gadella, M.: Phys. Rep. **396**, 41–113 (2004)

« **FOOP 10701** layout: Small Extended v.1.1 file: foop9746.tex (Alapsin) class: spr-small-v1.3 v.2013/08/14 Prn:2013/09/17; 10:06 p. 20/20»
« doctopic: OriginalPaper numbering style: ContentOnly reference style: mathphys»

20. Bohm, A.: J. Math. Phys. **22**, 2813–2823 (1981)
21. Gadella, M.: J. Math. Phys. **24**, 1462–1469 (1983)
22. de la Madrid, R., Bohm, A., Gadella, M.: Fortschr. Phys. **50**, 185–216 (2002)
23. Bohm, A., Kaldass, H., Patuleanu, P.: Int. J. Theor. Phys. **38**, 115–130 (1999)
24. Bohm, A., Wickramasekara, S.: Irreversible Quantum Dynamics. Lect. Notes Phys. **622**, 315–326 (2002)
25. Koosis, P.: In: Introduction to H^p spaces, Cambridge, UK (1980)
26. Bohm, A., Scurrek, R., Wickramasekara, S.: Rev. Mex. Fis. **45**, 16–20 (1999)
27. Bohm, A., Kaldass, H., Wickramasekara, S.: Fortschr. Phys. **51**, 569–603 (2003)
28. Bohm, A., Kaldass, H., Wickramasekara, S.: Fortschr. Phys. **51**, 604–634 (2003)
29. Bohm, A., Kielanowski, P., Wickramasekara, S.: Ann. Phys. **321**, 2299–2317 (2006)
30. Bohm, A., Gadella, M., Kielanowski, P.: SIGMA **7**, 086 (2011)
31. Friedrichs, K.O.: Commun. Pure Appl. Math. **1**, 361 (1948)
32. Gadella, M., Pronko, G.P.: Fortschr. Phys. **59**, 795 (2011)
33. Exner, P.: Open Quantum Systems and Feynmann Integrals. Reidel, Dordrecht (1985)
34. Reed, M., Simon, B.: Functional Analysis. Academic Press, New York (1972)
35. Gadella, M., de la Madrid, R.: Int. J. Theor. Phys. **38**, 93–113 (1999)
36. Antoniou, I., Prigogine, I.: Physica A **192**, 443 (1993)
37. Marsden, J.E.: Basic Complex Analysis. Freeman, San Francisco (1973)
38. Antoniou, I.E., Gadella, M., Karpov, E., Prigogine, I., Pronko, G.: Chaos Solitons Fractals **12**, 2757–2775 (2001)
39. Antoniou, I., Melnikov, Y.: Chaos Solitons Fractals **12**, 2603 (2001)
40. Gadella, M., Laura, R.: Gamow Dyads and Expectation Values. Int. J. Quant. Chem. **81**, 307–320 (2001)
41. ter Haar, D.: Lectures on Selected Topics in Statistical Mechanics. Pergamon Press, Oxford (1977)
42. Feynman, R.P., Hibbs, A.R.: Quantum Mechanics and Path Integrals. McGraw-Hill, New York (1965)
43. Gazeau, J.-P.: Coherent States in Quantum Physics. Wiley-VCH, Berlin (2009)
44. Civitarese, O., Gadella, M.: (2013). Submitted
45. Gelfand, I.M., Shilov, G.E.: Generalized Functions vol. 1. Academic Press, New York (1964)
46. Pietsch, A.: Nuclear Locally Convex Spaces. Akademie-Verlag, Berlin (1972)

Journal ID: 10701, Article ID: 9746, Date: 2013-09-17, Proof No: 1, UNCORRECTED PROOF