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Norm and Numerical Radius Inequalities Related to the Selberg Operator

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Abstract: The main focus of this paper is the study of the Selberg operator. It aims to establish appropriate bounds for the norm and numerical radius of the product of three bounded operators, with one of them being a Selberg operator. Moreover, it offers several bounds involving the summation of operators, notably the Selberg operator. Through the examination of these properties and relationships, this study contributes to a better understanding of the Selberg operator and its influence on operator compositions. The paper also highlights the significance of symmetry in mathematics and its potential implications across various mathematical domains.

Keywords: Selberg operator; bounded operators; norm; numerical radius

MSC: 47A30; 47B47; 47A12

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1. Introduction

We let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space (over the real or complex number field). In this manuscript, $\mathcal{B}(\mathcal{H})$ denotes the C^*-algebra of bounded, linear operators defined on $\mathcal{H}$ and I the identity operator. For each operator $T \in \mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ denotes the algebra of all compact operators, we denote by $\{s_j(T)\}$ the sequence of singular values of T, i.e., the eigenvalues $\lambda_j(|T|)$, with $|T| = (T^*T)^{\frac{1}{2}}$, in a decreasing order and repeated according to multiplicity. If $\operatorname{rank}(T) = n$, we set $s_k(T) = 0$ for each $k > n$.

We consider the wide class of unitarily invariant norms $|||\cdot|||$ characterized by the invariance property $|||UTV||| = |||T|||$ for arbitrary unitary operators $U, V \in \mathcal{B}(\mathcal{H})$. The usual operator norm, Schatten p-norms for $1 \leq p < \infty$ and the Ky Fan norms defined by $\|T\|_{(k)} = \sum_{j=1}^{k} s_j(T)$ with $1 \leq k < \infty$ are special examples of such norms. Every unitarily invariant norm, denoted as $|||\cdot|||$, defines a two-sided ideal, denoted as $C_{|||\cdot|||}$, that is, a subset of $\mathcal{K}(\mathcal{H})$. The Ky Fan dominance Theorem states that given a unitarily invariant norm $|||\cdot|||$, $|||T||| \leq |||S|||$ if and only if $\|T\|_{(k)} \leq \|S\|_{(k)}$ for any $k \in \mathbb{N}$. The reader is referred to [1] for a detailed study of unitarily invariant norms.

For each $T \in \mathcal{B}(\mathcal{H})$, we let $\omega(T)$ be the numerical radius of T, where

$$\omega(T) = \{|\langle Tz, z\rangle| : z \in \mathcal{H}, \|z\| = 1\}.$$

It is obvious that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, equivalent to the operator norm $\|\cdot\|$ and, in fact, for any $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{1}$$

Moreover, the numerical radius of a normal operator T is the same as its typical operator norm. Understanding inequalities related to the norm and the numerical radius is crucial in mathematical analysis. This helps us gain valuable insights into how operators behave and how well they approximate. More details can be found in recent papers such as [2,3] and the sources cited in those papers.

As usual, for $T, S \in \mathcal{B}(\mathcal{H})$, $T \geq 0$ means that T satisfies $\langle Tz, z\rangle \geq 0$ for any $z \in \mathcal{H}$. The notion of positivity induces the order $T \geq S$ for self-adjoint operators if and only if $T - S \geq 0$.

A. Selberg determined the following inequality [4] for given nonzero vectors $\mathcal{Z} = \{z_i : i = 1, \cdots, n\} \subseteq \mathcal{H}$,

$$\sum_{i=1}^{n} \frac{|\langle x, z_i\rangle|^2}{\sum_{j=1}^{n} |\langle z_i, z_j\rangle|} \leq \|x\|^2, \tag{2}$$

which holds for all $x \in \mathcal{H}$. This inequality is called the Selberg inequality and we denote it by (SI). The equality in (2) holds if and only if $x = \sum_{i=1}^{n} a_i z_i$ for some complex scalars $a_1, \cdots, a_n$ such that for any $i \neq j$, $\langle z_i, z_j\rangle = 0$ or $|a_i| = |a_j|$ with $\langle a_i z_i, a_j z_j\rangle \geq 0$ (see Theorem 1 in [5]). It might be useful to observe that, from (2), one can derive other well-known inequalities, for example,

1. The Cauchy–Bunyakowsky–Schwarz inequality (CBSI),

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

for any $x, y \in \mathcal{H}$.

2. The Buzano inequality (BuI)

$$|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|)\|z\|^2,$$

for any x, y, z elements in $\mathcal{H}$.

3. The Bessel inequality (BeI),
If $\mathcal{E} = \{e_i : i = 1, \cdots, n\}$ are orthonormal in $\mathcal{H}$, i.e., $\langle e_i, e_j\rangle = \delta_{ij}$ for all $i, j \in \{1, \cdots, n\}$ where δ_{ij} is the Kronecker delta, then

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2,$$

for any $x \in \mathcal{H}$ (see [4]).

4. The Bombieri inequality ([6])

$$\sum_{i=1}^{n} |\langle x, z_i\rangle|^2 \leq \|x\|^2 \max_{1\leq i\leq n} \sum_{j=1}^{n} |\langle z_i, z_j\rangle|.$$

In our earlier work [7], we established the interrelation and derivability between (CBSI), (SI), and (BeI).

Given $\mathcal{Z} = \{z_i : i = 1, \cdots, n\} \subset \mathcal{H}$, we consider the Selberg operator associated to $\mathcal{Z}$ as follows:

$$S_{\mathcal{Z}} = \sum_{i=1}^{n} \frac{z_i \otimes z_i}{\sum_{j=1}^{n} |\langle z_i, z_j\rangle|} \in \mathcal{B}(\mathcal{H}),$$

where $T = x \otimes y$ denotes the rank one operator defined by $T(z) = \langle z, y\rangle x$ with $x, y, z \in \mathcal{H}$. Using such an operator, we can express (SI) in the following way:

$$0 \leq \langle S_{\mathcal{Z}} x, x\rangle = \sum_{i=1}^{n} \frac{|\langle x, z_i\rangle|^2}{\sum_{j=1}^{n} |\langle z_i, z_j\rangle|} \leq \langle x, x\rangle,$$

for any $x \in \mathcal{H}$. Then, the (SI) establishes $0 \leq S_{\mathcal{Z}} \leq I$, i.e., $S_{\mathcal{Z}}$ is a positive contraction. Moreover, we deduce from the previous operator inequality that $0 \leq I - S_{\mathcal{Z}} \leq I$, since

$$0 = \|x\|^2 - \|x\|^2 \leq \langle S_{\mathcal{Z}} x, x\rangle \Rightarrow 0 \leq \|x\|^2 - \langle S_{\mathcal{Z}} x, x\rangle \leq \|x\|^2,$$

for any $x \in \mathcal{H}$, and in particular that $\omega(I - S_{\mathcal{Z}}) = \|I - S_{\mathcal{Z}}\| \leq 1$.

In [8], the Selberg inequality is refined as follows: if $\langle z, z_i\rangle = 0$ for any $z_i \in \mathcal{Z}$, then

$$|\langle x, z\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle x, z_i\rangle|^2}{\sum_{j=1}^{n} |\langle z_i, z_j\rangle|} \|z\|^2 \leq \|x\|^2 \|z\|^2.$$

For a thorough understanding of CBSI and its associated inequalities, see [7] and the cited sources within that reference.

The paper is structured into two main sections. In Section 2, we focus on establishing appropriate bounds for the norm and numerical radius of the product of three bounded operators, one of them being a Selberg operator.

Moving on to Section 3, we shift our attention to the study of bounds involving the summation of operators, with special attention to the Selberg operator. We provide a comprehensive overview of the summation of operators and its importance in mathematical contexts. Building upon this foundation, we introduce and discuss several bounds involving the Selberg operator within the framework of operator summation. These bounds provide valuable insights and contribute to a deeper understanding of the role of the Selberg operator in operator compositions.

2. Some Norm and Numerical Radius Inequalities

In this section, we derive upper bounds for both the norm and the numerical radius of the product of three operators, one of which is the Selberg operator. This analysis applies to any subset $\mathcal{Z}$ within the Hilbert space $\mathcal{H}$. To prove the results presented in this section, we rely on the following lemma found in [7].

Lemma 1. *For any $x, y \in \mathcal{H}$, the following inequalities hold:*

$$|\langle S_{\mathcal{Z}} x, y\rangle| \leq \left|\langle S_{\mathcal{Z}} x, y\rangle - \frac{1}{2}\langle x, y\rangle\right| + \frac{1}{2}|\langle x, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|)$$

and

$$\left|\left\langle \left(S_{\mathcal{Z}} - \frac{1}{2}I\right) x, y\right\rangle\right| \leq \frac{1}{2}\|x\|\|y\|.$$

The first inequality in Lemma 1 validates the Buzano inequality for any Selberg operator.

Theorem 1. *We assume that $S_{\mathcal{Z}}$ is the Selberg operator defined above and $A, B \in \mathcal{B}(\mathcal{H})$; then, we have norm inequalities*

$$\|B S_{\mathcal{Z}} A\| \leq \frac{1}{2}(\|BA\| + \|A\|\|B\|) \tag{3}$$

and

$$\left\|B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right\| \leq \frac{1}{2}\|A\|\|B\|. \tag{4}$$

Also, we have the following numerical radius inequalities

$$\omega(BS_{\mathcal{Z}}A) \leq \frac{1}{2}\left[\omega(BA) + \frac{1}{2}\left\||A|^2 + |B^*|^2\right\|\right] \tag{5}$$

and

$$\omega\left(B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right) \leq \frac{1}{4}\left\||A|^2 + |B^*|^2\right\|. \tag{6}$$

Proof. From Lemma 1, we have the following inequalities for Selberg operators:

$$|\langle S_{\mathcal{Z}}x, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|)$$

and

$$\left|\left\langle\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)x, y\right\rangle\right| \leq \frac{1}{2}\|x\|\|y\|$$

for all $x, y \in \mathcal{H}$.

If we replace x by Ax and y by B^*y, we obtain

$$|\langle BS_{\mathcal{Z}}Ax, y\rangle| \leq \frac{1}{2}(|\langle BAx, y\rangle| + \|Ax\|\|B^*y\|) \tag{7}$$

and

$$\left|\left\langle B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)Ax, y\right\rangle\right| \leq \frac{1}{2}\|Ax\|\|B^*y\| \tag{8}$$

for all $x, y \in \mathcal{H}$.

Therefore, by taking the supremum over all x and y of norm one, we obtain

$$\begin{aligned}
\|BS_{\mathcal{Z}}A\| &= \sup_{\|x\|=\|y\|=1} |\langle BS_{\mathcal{Z}}Ax, y\rangle| \\
&\leq \frac{1}{2}\sup_{\|x\|=\|y\|=1}(|\langle BAx, y\rangle| + \|Ax\|\|B^*y\|) \\
&\leq \frac{1}{2}\left(\sup_{\|x\|=\|y\|=1}|\langle BAx, y\rangle| + \sup_{\|x\|=\|y\|=1}(\|Ax\|\|B^*y\|)\right) \\
&= \frac{1}{2}(\|BA\| + \|A\|\|B\|)
\end{aligned}$$

and

$$\begin{aligned}
\left\|B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right\| &= \sup_{\|x\|=\|y\|=1}\left|\left\langle B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)Ax, y\right\rangle\right| \\
&\leq \frac{1}{2}\sup_{\|x\|=\|y\|=1}(\|Ax\|\|B^*y\|) = \frac{1}{2}\|A\|\|B\|,
\end{aligned}$$

which prove (3) and (4).

From (7), for $y = x$, we obtain that

$$\begin{aligned}
\left|\left\langle BS_{\mathcal{Z}}Ax,x\right\rangle\right| &\le \frac{1}{2}\left(\left|\left\langle BAx,x\right\rangle\right| + \|Ax\|\|B^*x\|\right)\\
&\le \frac{1}{2}\left(\left|\left\langle BAx,x\right\rangle\right| + \frac{1}{2}\left(\|Ax\|^2 + \|B^*x\|^2\right)\right)\\
&= \frac{1}{2}\left(\left|\left\langle BAx,x\right\rangle\right| + \frac{1}{2}\left(\left\langle Ax,Ax\right\rangle + \left\langle B^*x,B^*x\right\rangle\right)\right)\\
&= \frac{1}{2}\left(\left|\left\langle BAx,x\right\rangle\right| + \frac{1}{2}\left(\left\langle A^*Ax,x\right\rangle + \left\langle BB^*x,x\right\rangle\right)\right)\\
&= \frac{1}{2}\left(\left|\left\langle BAx,x\right\rangle\right| + \frac{1}{2}\left(\left\langle |A|^2x,x\right\rangle + \left\langle |B^*|^2x,x\right\rangle\right)\right)
\end{aligned}$$

for all $x \in \mathcal{H}$. This implies that

$$\left|\left\langle BS_{\mathcal{Z}}Ax,x\right\rangle\right| \le \frac{1}{2}\left(\left|\left\langle BAx,x\right\rangle\right| + \frac{1}{2}\left\langle\left(|A|^2 + |B^*|^2\right)x,x\right\rangle\right) \tag{9}$$

for all $x \in \mathcal{H}$.

By taking the supremum over all x of norm one, we obtain

$$\begin{aligned}
\omega(BS_{\mathcal{Z}}A) &= \sup_{\|x\|=1}\left|\left\langle BS_{\mathcal{Z}}Ax,x\right\rangle\right|\\
&\le \frac{1}{2}\sup_{\|x\|=1}\left(\left|\left\langle BAx,x\right\rangle\right| + \left\langle\frac{1}{2}\left(|A|^2 + |B^*|^2\right)x,x\right\rangle\right)\\
&\le \frac{1}{2}\left(\sup_{\|x\|=1}\left|\left\langle BAx,x\right\rangle\right| + \frac{1}{2}\sup_{\|x\|=1}\left\langle\left(|A|^2 + |B^*|^2\right)x,x\right\rangle\right)\\
&= \frac{1}{2}\left[\omega(BA) + \frac{1}{2}\left\||A|^2 + |B^*|^2\right\|\right]
\end{aligned}$$

and Inequality (5) is proven.

From (8), we derive

$$\left|\left\langle B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)Ax,x\right\rangle\right| \le \frac{1}{2}\|Ax\|\|B^*x\| \le \frac{1}{4}\left(\|Ax\|^2 + \|B^*x\|^2\right).$$

Hence,

$$\left|\left\langle B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)Ax,x\right\rangle\right| \le \frac{1}{4}\left\langle\left(|A|^2 + |B^*|^2\right)x,x\right\rangle. \tag{10}$$

By taking the supremum over all x of norm one, we obtain the required Inequality (6). □

On the basis of Theorem 1, we can establish the following corollaries as direct applications:

Corollary 1. *We assume that $S_{\mathcal{Z}}$ is the Selberg operator defined above and $A, B \in \mathcal{B}(\mathcal{H})$; then, we have*

$$\begin{aligned}
\omega(BS_{\mathcal{Z}}A) &\le \frac{1}{2}\left[\omega(BA) + \frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|\right]\\
&\le \frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|\\
&\le \frac{1}{2}(\|BA\| + \|A\|\|B\|).
\end{aligned} \tag{11}$$

Proof. Replacing A by $\frac{A}{\|A\|}$ and B by $\frac{B}{\|B\|}$ in (5), respectively, we obtain the first inequality. On the other hand, as a consequence of a previous statement obtained in [9], we have

$$\omega(BA) \leq \frac{1}{2}\left\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right\|.$$

Thus,

$$\frac{1}{2}\left[\omega(BA) + \frac{1}{2}\left\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right\|\right] \leq \frac{1}{2}\left\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right\|. \quad (12)$$

Since $\frac{\|B\|}{\|A\|}|A|^2$ and $\frac{\|A\|}{\|B\|}|B^*|^2$ are positive operators, using the norm inequality for sums of two positive operators obtained in [10], we conclude that

$$\frac{1}{2}\left\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right\| \leq \frac{1}{2}(\|BA\| + \|A\|\|B\|). \quad (13)$$

Finally, if we combine Inequalities (12) and (13), we obtain the desired result. □

We note that Inequality (11) is a refinement of Lemma 1.2 in [11] in the particular case that X is a Selberg operator. Furthermore, from Corollary 1, we have

$$\omega(BS_{\mathcal{Z}}A) \leq \min\left\{ \frac{1}{2}\left[\omega(BA) + \frac{1}{2}\left\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right\|\right], \|BS_{\mathcal{Z}}A\| \right\},$$

for any $A, B \in \mathcal{B}(\mathcal{H})$ and Selberg operator $S_{\mathcal{Z}}$.

Applying Theorem 1 with the special case where $A = B$, we arrive at the following specific statement.

Corollary 2. *We assume that $S_{\mathcal{Z}}$ is the Selberg operator defined above and $A \in \mathcal{B}(\mathcal{H})$; then, we have norm inequalities*

$$\|AS_{\mathcal{Z}}A\| \leq \frac{1}{2}\left(\left\|A^2\right\| + \|A\|^2\right) \quad (14)$$

and

$$\left\|A\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right\| \leq \frac{1}{2}\|A\|^2.$$

Also, we have the following numerical radius inequalities

$$\omega(AS_{\mathcal{Z}}A) \leq \frac{1}{2}\left[\omega\left(A^2\right) + \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|\right] \quad (15)$$

and

$$\omega\left(A\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right) \leq \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|.$$

Remark 1. *From Inequalities (1) and (14), we conclude that*

$$\omega(AS_{\mathcal{Z}}A) \leq \frac{1}{2}\left(\left\|A^2\right\| + \|A\|^2\right),$$

with the Selberg operator associated to $\mathcal{Z}$ and $A \in \mathcal{B}(\mathcal{H})$. Otherwise, (15) provides a refinement of the previously inequality, since

$$\begin{aligned}
\frac{1}{2}\left[\omega\left(A^2\right) + \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|\right] &\leq \frac{1}{2}\|A^2\| + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\| \\
&\leq \frac{1}{2}\|A^2\| + \frac{1}{4}\left(\left\||A|^2\right\| + \left\||A^*|^2\right\|\right) \\
&\leq \frac{1}{2}\|A^2\| + \frac{1}{2}\|A\|^2.
\end{aligned}$$

Proposition 1. *We assume that $A \in \mathcal{B}(\mathcal{H})$ and $\mathcal{Z}$ are a finite subset contained in $\mathcal{H}$, then*

$$\max\{\omega^2(A), \omega(AS_{\mathcal{Z}}A)\} \le \frac{1}{2}\omega\left(A^2\right) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|.$$

Proof. In [12], Abu-Omar and Kittaneh obtained the following inequality:

$$\omega^2(A) \le \frac{1}{2}\omega\left(A^2\right) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|. \tag{16}$$

By combining Inequalities (15) and (16), we infer that

$$\max\{\omega^2(A), \omega(AS_{\mathcal{Z}}A)\} \le \frac{1}{2}\omega\left(A^2\right) + \frac{1}{4}\left\||A|^2 + |A^*|^2\right\|.$$

□

We proceed to generalize Inequalities (5) and (6) presented in Theorem 1.

Theorem 2. *We assume that $S_{\mathcal{Z}}$ is the Selberg operator defined above with $r \ge 1$ and $A, B \in \mathcal{B}(\mathcal{H})$; then, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\omega(BS_{\mathcal{Z}}A) \le \frac{1}{2^{\frac{1}{r}}}\left(\omega^r(BA) + \left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\|\right)^{\frac{1}{r}}, \tag{17}$$

provided that $rp \ge 2$, $rq \ge 2$; and for $s > 0$,

$$\omega\left(B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right) \le \frac{1}{2}\left\|\frac{1}{p}|A|^{sp} + \frac{1}{q}|B^*|^{sq}\right\|^{\frac{1}{s}} \tag{18}$$

for $sp \ge 2$ and $sq \ge 2$.

Proof. If we take the power $r \ge 1$ in (9), we obtain, by the convexity of power functions, that

$$|\langle BS_{\mathcal{Z}}Ax, x\rangle|^r \le \left(\frac{|\langle BAx, x\rangle| + \|Ax\|\|B^*x\|}{2}\right)^r,$$

for all $x \in \mathcal{H}$. Therefore, we infer that

$$|\langle BS_{\mathcal{Z}}Ax, x\rangle|^r \le \frac{|\langle BAx, x\rangle|^r + \|Ax\|^r\|B^*x\|^r}{2} \tag{19}$$

for every $x \in \mathcal{H}$.

From Young's inequality

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q,\ a,\ b \ge 0,\ p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$\begin{aligned}\|Ax\|^r\|B^*x\|^r &\le \frac{1}{p}\|Ax\|^{rp} + \frac{1}{q}\|B^*x\|^{rq} \\ &= \frac{1}{p}\|Ax\|^{2\frac{rp}{2}} + \frac{1}{q}\|B^*x\|^{2\frac{rq}{2}} \\ &= \frac{1}{p}\langle |A|^2x, x\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle |B^*|^2x, x\rangle^{\frac{rq}{2}}\end{aligned}$$

for all $x \in \mathcal{H}$.

By McCarthy's inequality [13],

$$\langle Ax, x\rangle^s \le \langle A^s x, x\rangle, \ s \ge 1$$

for $x \in \mathcal{H}$, $\|x\| = 1$, and since $rp \ge 2$, $rq \ge 2$, then

$$\frac{1}{p}\langle |A|^2 x, x\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle |B^*|^2 x, x\rangle^{\frac{rq}{2}} \le \frac{1}{p}\langle |A|^{rp} x, x\rangle + \frac{1}{q}\langle |B^*|^{rq} x, x\rangle$$

for $x \in \mathcal{H}$, $\|x\| = 1$. Thus, we deduce that

$$\frac{1}{p}\langle |A|^2 x, x\rangle^{\frac{rp}{2}} + \frac{1}{q}\langle |B^*|^2 x, x\rangle^{\frac{rq}{2}} \le \left\langle \left(\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right) x, x\right\rangle \tag{20}$$

for every $x \in \mathcal{H}$ with $\|x\| = 1$.

By utilizing (19) and (20), we obtain

$$|\langle BS_{\mathcal{Z}}Ax, x\rangle|^r \le \frac{1}{2}\left[|\langle BAx, x\rangle|^r + \left\langle \left(\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right) x, x\right\rangle\right]$$

for $x \in \mathcal{H}$, $\|x\| = 1$, and by taking the supremum over all x of norm one, we obtain

$$\omega^r(BS_{\mathcal{Z}}A) \le \frac{1}{2}\left[\omega^r(BA) + \left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|B^*|^{rq}\right\|\right]$$

which is equivalent to (17).

From (10), by taking the power $s > 0$, we get

$$\left|\left\langle B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right) Ax, x\right\rangle\right|^s \le \frac{1}{2^s}\|Ax\|^s \|B^*x\|^s \tag{21}$$

for $x \in \mathcal{H}$.

By Young's inequality and McCarthy's for $\frac{sp}{2} \ge 1$, $\frac{sq}{2} \ge 1$ we also have

$$\begin{aligned}\|Ax\|^s \|B^*x\|^s &\le \frac{1}{p}\|Ax\|^{sp} + \frac{1}{q}\|B^*x\|^{sq} \\ &= \frac{1}{p}\|Ax\|^{2\frac{sp}{2}} + \frac{1}{q}\|B^*x\|^{2\frac{sq}{2}} \\ &= \frac{1}{p}\langle |A|^2 x, x\rangle^{\frac{sp}{2}} + \frac{1}{q}\langle |B^*|^2 x, x\rangle^{\frac{sq}{2}} \\ &\le \frac{1}{p}\langle |A|^{sp} x, x\rangle + \frac{1}{q}\langle |B^*|^{sq} x, x\rangle,\end{aligned}$$

for $x \in \mathcal{H}$, $\|x\| = 1$. Therefore, we obtain

$$\|Ax\|^s \|B^*x\|^s \le \left\langle \left(\frac{1}{p}|A|^{sp} + \frac{1}{q}|B^*|^{sq}\right) x, x\right\rangle \tag{22}$$

for $x \in \mathcal{H}$, $\|x\| = 1$.

By making use of (21) and (22), we obtain

$$\left|\left\langle B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right) Ax, x\right\rangle\right|^s \le \frac{1}{2^s}\left\langle \left(\frac{1}{p}|A|^{sp} + \frac{1}{q}|B^*|^{sq}\right) x, x\right\rangle,$$

for $x \in \mathcal{H}$, $\|x\| = 1$, and by taking the supremum over all x of norm one, we obtain (18). □

Corollary 3. *If $r \ge 1$ and $A \in \mathcal{B}(\mathcal{H})$, then, for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\omega(AS_{\mathcal{Z}}A) \leq \frac{1}{2^{\frac{1}{r}}}\left(\omega^r\left(A^2\right) + \left\|\frac{1}{p}|A|^{rp} + \frac{1}{q}|A^*|^{rq}\right\|\right)^{\frac{1}{r}},$$

provided that $rp \geq 2$, $rq \geq 2$; *and for* $s > 0$,

$$\omega\left(A\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right) \leq \frac{1}{2}\left\|\frac{1}{p}|A|^{sp} + \frac{1}{q}|A^*|^{sq}\right\|^{\frac{1}{s}},$$

provided that $sp \geq 2$, $sq \geq 2$.

Remark 2. *If we take* $p = q = 2$ *in (17) and (18), we obtain*

$$\omega(BS_{\mathcal{Z}}A) \leq \frac{1}{2^{\frac{1}{r}}}\left(\omega^r(BA) + \frac{1}{2}\left\||A|^{2r} + |B^*|^{2r}\right\|\right)^{\frac{1}{r}},$$

for $r \geq 1$, *and for* $s \geq 1$,

$$\omega\left(B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right) \leq \frac{1}{2^{1+1/s}}\left\||A|^{2s} + |B^*|^{2s}\right\|^{\frac{1}{s}}.$$

In these inequalities, when we take $B = A$, *we obtain*

$$\omega(AS_{\mathcal{Z}}A) \leq \frac{1}{2^{\frac{1}{r}}}\left(\omega^r\left(A^2\right) + \frac{1}{2}\left\||A|^{2r} + |A^*|^{2r}\right\|\right)^{\frac{1}{r}},$$

for $r \geq 1$, *and*

$$\omega\left(A\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right) \leq \frac{1}{2^{1+1/s}}\left\||A|^{2s} + |A^*|^{2s}\right\|^{\frac{1}{s}}$$

for $s \geq 1$.

Further, if we take $r = 2$ *in (17), we obtain*

$$\omega(BS_{\mathcal{Z}}A) \leq \frac{\sqrt{2}}{2}\left(\omega^2(BA) + \left\|\frac{1}{p}|A|^{2p} + \frac{1}{q}|B^*|^{2q}\right\|\right)^{\frac{1}{2}},$$

for $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *The case* $p = q = 2$ *also offers*

$$\omega(BS_{\mathcal{Z}}A) \leq \frac{\sqrt{2}}{2}\left(\omega^2(BA) + \frac{1}{2}\left\||A|^4 + |B^*|^4\right\|\right)^{\frac{1}{2}}.$$

Moreover, if we take $B = A$ *in these inequalities, we have*

$$\omega(AS_{\mathcal{Z}}A) \leq \frac{\sqrt{2}}{2}\left(\omega^2\left(A^2\right) + \left\|\frac{1}{p}|A|^{2p} + \frac{1}{q}|A^*|^{2q}\right\|\right)^{\frac{1}{2}},$$

and

$$\omega(AS_{\mathcal{Z}}A) \leq \frac{\sqrt{2}}{2}\left(\omega^2\left(A^2\right) + \frac{1}{2}\left\||A|^4 + |A^*|^4\right\|\right)^{\frac{1}{2}}.$$

Furthermore, for $s = 2$, *we also have*

$$\omega\left(B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A\right) \leq \frac{1}{2}\left\|\frac{1}{p}|A|^{2p} + \frac{1}{q}|B^*|^{2q}\right\|^{\frac{1}{2}},$$

for $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *The case* $p = q = 2$ *also offers*

$$\omega\left(B\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)A\right)\le\frac{\sqrt{2}}{4}\left\||A|^4+|B^*|^4\right\|^{\frac{1}{2}}.$$

If we consider $B=A$, we obtain

$$\omega\left(A\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)A\right)\le\frac{1}{2}\left\|\frac{1}{p}|A|^{2p}+\frac{1}{q}|A^*|^{2q}\right\|^{\frac{1}{2}},$$

and

$$\omega\left(A\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)A\right)\le\frac{\sqrt{2}}{4}\left\||A|^4+|A^*|^4\right\|^{\frac{1}{2}}.$$

In the subsequent theorem, we establish an upper bound for the numerical radius by utilizing a convex combination of $|A|$ and $|B^*|$.

Theorem 3. *We assume that $S_{\mathcal{Z}}$ is the Selberg operator defined above and $A,B\in\mathcal{B}(\mathcal{H})$; then, for $\alpha\in[0,1]$,*

$$\omega^2(BS_{\mathcal{Z}}A)\le\frac{1}{2}\left(\omega^2(BA)+\left\|(1-\alpha)|A|^2+\alpha|B^*|^2\right\|\|A\|^{2\alpha}\|B\|^{2(1-\alpha)}\right)\tag{23}$$

and

$$\omega^2\left(B\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)A\right)\le\frac{1}{4}\left\|(1-\alpha)|A|^2+\alpha|B^*|^2\right\|\|A\|^{2\alpha}\|B\|^{2(1-\alpha)}.\tag{24}$$

Furthermore, in specific instances, we obtain

$$\omega^2(BS_{\mathcal{Z}}A)\le\frac{1}{2}\left(\omega^2(BA)+\frac{1}{2}\left\||A|^2+|B^*|^2\right\|\|A\|\|B\|\right)$$

and

$$\omega^2\left(B\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)A\right)\le\frac{1}{8}\left\||A|^2+|B^*|^2\right\|\|A\|\|B\|.$$

Proof. From (19), for $r=2$, we also have

$$\begin{aligned}
&\left|\langle BS_{\mathcal{Z}}Ax,x\rangle\right|^2\\
&\le\frac{1}{2}\left(\left|\langle BAx,x\rangle\right|^2+\|Ax\|^2\|B^*x\|^2\right)\\
&=\frac{1}{2}\left(\left|\langle BAx,x\rangle\right|^2+\langle|A|^2x,x\rangle\langle|B^*|^2x,x\rangle\right)\\
&=\frac{1}{2}\left(\left|\langle BAx,x\rangle\right|+\langle|A|^2x,x\rangle^{1-\alpha}\langle|B^*|^2x,x\rangle^{\alpha}\langle|A|^2x,x\rangle^{\alpha}\langle|B^*|^2x,x\rangle^{1-\alpha}\right)\\
&\le\frac{1}{2}\left(\left|\langle BAx,x\rangle\right|+\left((1-\alpha)\langle|A|^2x,x\rangle+\alpha\langle|B^*|^2x,x\rangle\right)\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)}\right)\\
&=\frac{1}{2}\left(\left|\langle BAx,x\rangle\right|+\left\langle\left[(1-\alpha)|A|^2+\alpha|B^*|^2\right]x,x\right\rangle\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)}\right),
\end{aligned}$$

for all $x\in\mathcal{H}$.

If we take the supremum over all x of norm one, we obtain

$$\begin{aligned}
&\omega^2(BS_{\mathcal{Z}}A)\\
&=\sup_{\|x\|=1}\left|\langle BS_{\mathcal{Z}}Ax,x\rangle\right|^2\\
&\le\frac{1}{2}\sup_{\|x\|=1}\left(\left|\langle BAx,x\rangle\right|^2+\left\langle\left[(1-\alpha)|A|^2+\alpha|B^*|^2\right]x,x\right\rangle\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)}\right).
\end{aligned}$$

Hence,

$$\omega^2(BS_{\mathcal{Z}}A) \le \frac{1}{2}\sup_{\|x\|=1}|\langle BAx,x\rangle|^2 + \frac{1}{2}\sup_{\|x\|=1}\left(\left\langle\left[(1-\alpha)|A|^2+\alpha|B^*|^2\right]x,x\right\rangle\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)}\right) \tag{25}$$

and since

$$\begin{aligned}&\sup_{\|x\|=1}\left(\left\langle\left[(1-\alpha)|A|^2+\alpha|B^*|^2\right]x,x\right\rangle\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)}\right)\\ &\le \sup_{\|x\|=1}\left\langle\left[(1-\alpha)|A|^2+\alpha|B^*|^2\right]x,x\right\rangle \sup_{\|x\|=1}\|Ax\|^{2\alpha}\sup_{\|x\|=1}\|B^*x\|^{2(1-\alpha)}\\ &= \left\|(1-\alpha)|A|^2+\alpha|B^*|^2\right\|\|A\|^{2\alpha}\|B\|^{2(1-\alpha)},\end{aligned}$$

by (25), we obtain the desired result (23).

By (21), we obtain for $s=2$ that

$$\begin{aligned}\left|\left\langle B\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)Ax,x\right\rangle\right|^2 &\le \frac{1}{4}\|Ax\|^2\|B^*x\|^2 = \frac{1}{4}\langle|A|^2x,x\rangle\langle|B^*|^2x,x\rangle\\ &\le \frac{1}{4}\left\langle\left[(1-\alpha)|A|^2+\alpha|B^*|^2\right]x,x\right\rangle\|Ax\|^{2\alpha}\|B^*x\|^{2(1-\alpha)}\end{aligned}$$

and by taking the supremum over all x of norm one, we obtain (24). □

Corollary 4. *If $A\in\mathcal{B}(\mathcal{H})$, then, for $\alpha\in[0,1]$,*

$$\omega^2(AS_{\mathcal{Z}}A) \le \frac{1}{2}\left(\omega^2\left(A^2\right)+\left\|(1-\alpha)|A|^2+\alpha|A^*|^2\right\|\|A\|^2\right)$$

and

$$\omega^2\left(A\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)A\right) \le \frac{1}{4}\left\|(1-\alpha)|A|^2+\alpha|A^*|^2\right\|\|A\|^2.$$

In particular, we have

$$\omega^2(AS_{\mathcal{Z}}A) \le \frac{1}{2}\left(\omega^2\left(A^2\right)+\frac{1}{2}\left\||A|^2+|A^*|^2\right\|\|A\|^2\right)$$

and

$$\omega^2\left(A\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)A\right) \le \frac{1}{8}\left\||A|^2+|A^*|^2\right\|\|A\|^2.$$

It is a well-known fact that every two-sided ideal of $\mathcal{B}(\mathcal{H})$ includes $\mathcal{K}_0(\mathcal{H})$, the ideal comprising finite rank operators. Consequently, we have $S_{\mathcal{Z}}\in\mathcal{K}_0(\mathcal{H})\subseteq C_{|||\cdot|||}$, where $C_{|||\cdot|||}$ represents the ideal defined by a specific unitarily invariant norm $|||\cdot|||$. We conclude this section by deriving the following inequalities applicable to such norms.

Theorem 4. *We asume that $S_{\mathcal{Z}}$ is the Selberg operator defined above, $A,B\in\mathcal{B}(\mathcal{H})$ and $|||\cdot|||$ is a unitarily invariant norm; then, we have norm inequalities*

$$|||AS_{\mathcal{Z}}B||| \le \frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2+\frac{\|A\|}{\|B\|}|B^*|^2\right\||||S_{\mathcal{Z}}|||.$$

In particular, we conclude that

$$\|AS_{\mathcal{Z}}B\| \leq \frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|\|S_{\mathcal{Z}}\|. \tag{26}$$

In addition, if A and B belong to the ideal associated to $|||\cdot|||$, *then*

$$|||AS_{\mathcal{Z}}B||| \leq \min\{\mu(A, S_{\mathcal{Z}}, B), \nu(A, S_{\mathcal{Z}}, B)\}, \tag{27}$$

where

$$\mu(A, S_{\mathcal{Z}}, B) = \frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\||||S_{\mathcal{Z}}|||$$

and

$$\nu(A, S_{\mathcal{Z}}, B) = \frac{1}{2}\|S_{\mathcal{Z}}\|\left|\left|\left|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right|\right|\right|.$$

Proof. As $S_{\mathcal{Z}} \in \mathcal{K}_0(\mathcal{H})$ in consequence $AS_{\mathcal{Z}}B \in \mathcal{K}_0(\mathcal{H})$, then, as $S_{\mathcal{Z}}$ is a positive operator, we have, by Lemma 2.1 in [14],

$$\begin{aligned} s_j(AS_{\mathcal{Z}}B) &\leq \frac{1}{2}s_j((A^*A + BB^*)^{\frac{1}{2}}S_{\mathcal{Z}}(A^*A + BB^*)^{\frac{1}{2}}) \\ &\leq \frac{1}{2}\left\||A|^2 + |B^*|^2\right\|s_j(S_{\mathcal{Z}}). \end{aligned} \tag{28}$$

Replacing A by $\frac{A}{\|A\|}$ and B by $\frac{B}{\|B\|}$ in (28), respectively, we obtain

$$s_j(AS_{\mathcal{Z}}B) \leq \frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|s_j(S_{\mathcal{Z}})$$

for any $j = 1, 2, \ldots$. Thus, for any $k \in \mathbb{N}$, we obtain

$$\frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|\sum_{j=1}^{k} s_j(S_{\mathcal{Z}}) \geq \sum_{j=1}^{k} s_j(AS_{\mathcal{Z}}B),$$

or, equivalently,

$$\frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|\|S_{\mathcal{Z}}\|_{(k)} \geq \|AS_{\mathcal{Z}}B\|_{(k)}$$

for any $k \in \mathbb{N}$. Then, by the Ky Fan dominance Theorem, we conclude that

$$|||AS_{\mathcal{Z}}B||| \leq \frac{1}{2}\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\||||S_{\mathcal{Z}}||| \tag{29}$$

for any unitarily invariant norm $|||\cdot|||$.

On the other hand, if we assume that A and B belong to the ideal associated to $|||\cdot|||$, then, by Theorem 2.4 in [15], we obtain

$$s_j(AS_{\mathcal{Z}}B) \leq \frac{1}{2}s_j\left(\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right)\|S_{\mathcal{Z}}\|.$$

Then,

$$\begin{aligned} \|AS_{\mathcal{Z}}B\|_{(k)} = \sum_{j=1}^{k} s_j(AS_{\mathcal{Z}}B) &\leq \frac{1}{2}\|S_{\mathcal{Z}}\|\sum_{j=1}^{k} s_j\left(\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right) \\ &= \frac{1}{2}\|S_{\mathcal{Z}}\|\left\|\frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2\right\|_{(k)}, \end{aligned}$$

or, equivalently, by the Ky Fan dominance Theorem, we achieve

$$|||AS_{\mathcal{Z}}B||| \leq \frac{1}{2}\|S_{\mathcal{Z}}\| \left|\left|\left| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right|\right|\right|. \tag{30}$$

From (29) and (30), we derive the inequality (27). □

From Inequality (26), we obtain a new refinement of Lemma 1.2 in [11] for the Selberg operator, since

$$\omega(BS_{\mathcal{Z}}A) \leq \|AS_{\mathcal{Z}}B\| \leq \frac{1}{2}\left\| \frac{\|B\|}{\|A\|}|A|^2 + \frac{\|A\|}{\|B\|}|B^*|^2 \right\| \|S_{\mathcal{Z}}\|.$$

3. Inequalities for Summations with the Selberg Operator

In this section, we delve into studying bounds related to the summation of operators, placing special focus on the Selberg operator.

The opening proposition in this section provides a broadened perspective of the inequality established in Theorem 1. The inequality reads as follows:

$$\left\| BS_{\mathcal{Z}}A - \frac{1}{2}BA \right\|^2 = \left\| B\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)A \right\|^2 \leq \frac{1}{4}\|A\|^2\|B\|^2 = \frac{1}{4}\left\| |A|^2 \right\| \left\| |B^*|^2 \right\|,$$

and this inequality holds for any $A, B \in \mathcal{B}(\mathcal{H})$.

Theorem 5. *We assume that $S_{\mathcal{Z}}$ is the Selberg operator defined above, $A_i, B_i \in \mathcal{B}(\mathcal{H})$, $i \in \{1, \ldots, m\}$ and $p_i \geq 0$, $i \in \{1, \ldots, m\}$ with $\sum_{i=1}^m p_i = 1$. Then, we have norm inequality*

$$\left\| \sum_{i=1}^m p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^m p_i B_i A_i \right\|^2 \leq \frac{1}{4}\left\| \sum_{i=1}^m p_i |A_i|^2 \right\| \left\| \sum_{i=1}^m p_i |B_i^*|^2 \right\| \tag{31}$$

and numerical radius inequality

$$\omega\left(\sum_{i=1}^m p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^m p_i B_i A_i \right) \leq \left\| \sum_{i=1}^m p_i \frac{|A_i|^2 + |B_i^*|^2}{4} \right\|. \tag{32}$$

Proof. From (8), we obtain

$$\left| \left\langle \left(B_i S_{\mathcal{Z}} A_i - \frac{1}{2}B_i A_i \right) x, y \right\rangle \right| \leq \frac{1}{2}\|A_i x\| \|B_i^* y\|$$

for all $i \in \{1, \ldots, m\}$ and $x, y \in \mathcal{H}$.

If we multiply by $p_i \geq 0$, $i \in \{1, \ldots, m\}$ and sum, we obtain

$$\sum_{i=1}^m p_i \left| \left\langle \left(B_i S_{\mathcal{Z}} A_i - \frac{1}{2}B_i A_i \right) x, y \right\rangle \right| \leq \frac{1}{2}\sum_{i=1}^m p_i \|A_i x\| \|B_i^* y\|. \tag{33}$$

By the generalized triangle inequality, we have

$$\begin{aligned} \sum_{i=1}^m p_i \left| \left\langle \left(B_i S_{\mathcal{Z}} A_i - \frac{1}{2}B_i A_i \right) x, y \right\rangle \right| &\geq \left| \sum_{i=1}^m p_i \left\langle \left(B_i S_{\mathcal{Z}} A_i - \frac{1}{2}B_i A_i \right) x, y \right\rangle \right| \\ &= \left| \left\langle \left(\sum_{i=1}^m p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^m p_i B_i A_i \right) x, y \right\rangle \right| \end{aligned}$$

for $x, y \in \mathcal{H}$.

By the Cauchy–Bunyakowsky–Schwarz inequality, we have

$$\begin{aligned}\sum_{i=1}^{m} p_i \|A_i x\| \|B_i^* y\| &\leq \left(\sum_{i=1}^{m} p_i \|A_i x\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} p_i \|B_i^* y\|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{m} p_i \langle |A_i|^2 x, x\rangle\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} p_i \langle |B_i^*|^2 y, y\rangle\right)^{\frac{1}{2}}\end{aligned}$$

for $x, y \in \mathcal{H}$. This implies that

$$\sum_{i=1}^{m} p_i \|A_i x\| \|B_i^* y\| \leq \left\langle \sum_{i=1}^{m} p_i |A_i|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{m} p_i |B_i^*|^2 y, y\right\rangle^{\frac{1}{2}} \tag{34}$$

for all $x, y \in \mathcal{H}$. By making use of (33) and (34), we obtain

$$\left|\left\langle \left(\sum_{i=1}^{m} p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^{m} p_i B_i A_i\right) x, y\right\rangle\right| \leq \frac{1}{2}\left\langle \sum_{i=1}^{m} p_i |A_i|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{m} p_i |B_i^*|^2 y, y\right\rangle^{\frac{1}{2}} \tag{35}$$

for $x, y \in \mathcal{H}$.

If we take the supremum over all x and y of norm one, we obtain

$$\begin{aligned}\left\|\sum_{i=1}^{m} p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^{m} p_i B_i A_i\right\| &= \sup_{\|x\|=\|y\|=1}\left|\left\langle \left(\sum_{i=1}^{m} p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^{m} p_i B_i A_i\right) x, y\right\rangle\right| \\ &\leq \frac{1}{2}\sup_{\|x\|=\|y\|=1}\left(\left\langle \sum_{i=1}^{m} p_i |A_i|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{m} p_i |B_i^*|^2 y, y\right\rangle^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\sup_{\|x\|=1}\left\langle \sum_{i=1}^{m} p_i |A_i|^2 x, x\right\rangle^{\frac{1}{2}} \sup_{\|y\|=1}\left\langle \sum_{i=1}^{m} p_i |B_i^*|^2 y, y\right\rangle^{\frac{1}{2}} \\ &= \frac{1}{2}\left\|\sum_{i=1}^{m} p_i |A_i|^2\right\|^{\frac{1}{2}} \left\|\sum_{i=1}^{m} p_i |B_i^*|^2\right\|^{\frac{1}{2}},\end{aligned}$$

which proves (31).

From (35), we derive

$$\begin{aligned}&\left|\left\langle \left(\sum_{i=1}^{m} p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^{m} p_i B_i A_i\right) x, x\right\rangle\right| \\ &\leq \frac{1}{2}\left\langle \sum_{i=1}^{m} p_i |A_i|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{m} p_i |B_i^*|^2 x, x\right\rangle^{\frac{1}{2}} \\ &\leq \frac{1}{4}\left(\left\langle \sum_{i=1}^{m} p_i |A_i|^2 x, x\right\rangle + \left\langle \sum_{i=1}^{m} p_i |B_i^*|^2 x, x\right\rangle\right) \\ &= \frac{1}{4}\left\langle \left(\sum_{i=1}^{m} p_i |A_i|^2 + \sum_{i=1}^{m} p_i |B_i^*|^2\right) x, x\right\rangle = \left\langle \left(\sum_{i=1}^{m} p_i \frac{|A_i|^2 + |B_i^*|^2}{4}\right) x, x\right\rangle,\end{aligned}$$

and by taking the supremum over all x of norm one, we obtain (32). □

Theorem 6. *With the assumptions of Theorem 5, we have the following numerical radius inequalities:*

$$\omega^2\left(\sum_{i=1}^{m} p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^{m} p_i B_i A_i\right) \leq \frac{1}{2}\left\|\frac{1}{p}\left(\sum_{i=1}^{m} p_i |A_i|^2\right)^p + \frac{1}{q}\left(\sum_{i=1}^{m} p_i |B_i^*|^2\right)^q\right\| \tag{36}$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\omega^2\left(\sum_{i=1}^m p_iB_iS_{\mathcal{Z}}A_i-\frac{1}{2}\sum_{i=1}^m p_iB_iA_i\right) \le \frac{1}{4}\left(\omega\left(\sum_{i=1}^m p_i|B_i^*|^2\sum_{i=1}^m p_i|A_i|^2\right)+\left\|\sum_{i=1}^m p_i|A_i|^2\right\|\left\|\sum_{i=1}^m p_i|B_i^*|^2\right\|\right). \tag{37}$$

Proof. By (35) and by taking the square and $y=x$, we determine that

$$\left|\left\langle\left(\sum_{i=1}^m p_iB_iS_{\mathcal{Z}}A_i-\frac{1}{2}\sum_{i=1}^m p_iB_iA_i\right)x,x\right\rangle\right|^2 \le \frac{1}{2}\left\langle\sum_{i=1}^m p_i|A_i|^2x,x\right\rangle\left\langle\sum_{i=1}^m p_i|B_i^*|^2x,x\right\rangle \tag{38}$$

for $x\in\mathcal{H}$.

By Young's inequality, we have

$$\left\langle\sum_{i=1}^m p_i|A_i|^2x,x\right\rangle\left\langle\sum_{i=1}^m p_i|B_i^*|^2x,x\right\rangle \le \frac{1}{p}\left\langle\sum_{i=1}^m p_i|A_i|^2x,x\right\rangle^p+\frac{1}{q}\left\langle\sum_{i=1}^m p_i|B_i^*|^2x,x\right\rangle^q$$

for $x\in\mathcal{H}$ and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

By the McCarthy inequality, we also have

$$\frac{1}{p}\left\langle\sum_{i=1}^m p_i|A_i|^2x,x\right\rangle^p+\frac{1}{q}\left\langle\sum_{i=1}^m p_i|B_i^*|^2x,x\right\rangle^q \le \frac{1}{p}\left\langle\left(\sum_{i=1}^m p_i|A_i|^2\right)^p x,x\right\rangle+\frac{1}{q}\left\langle\left(\sum_{i=1}^m p_i|B_i^*|^2\right)^q x,x\right\rangle$$

for $x\in\mathcal{H}$ with $\|x\|=1$. This yields that

$$\frac{1}{p}\left\langle\sum_{i=1}^m p_i|A_i|^2x,x\right\rangle^p+\frac{1}{q}\left\langle\sum_{i=1}^m p_i|B_i^*|^2x,x\right\rangle^q \le \left\langle\left[\frac{1}{p}\left(\sum_{i=1}^m p_i|A_i|^2\right)^p+\frac{1}{q}\left(\sum_{i=1}^m p_i|B_i^*|^2\right)^q\right]x,x\right\rangle, \tag{39}$$

for $x\in\mathcal{H}$ with $\|x\|=1$.

Therefore, by (38) and (39), we obtain

$$\left|\left\langle\left(\sum_{i=1}^m p_iB_iS_{\mathcal{Z}}A_i-\frac{1}{2}\sum_{i=1}^m p_iB_iA_i\right)x,x\right\rangle\right|^2 \le \frac{1}{2}\left\langle\left[\frac{1}{p}\left(\sum_{i=1}^m p_i|A_i|^2\right)^p+\frac{1}{q}\left(\sum_{i=1}^m p_i|B_i^*|^2\right)^q\right]x,x\right\rangle$$

for $x\in\mathcal{H}$ with $\|x\|=1$.

Finally, if we take the supremum over all x of norm one, we deduce the desired result (36).

If we use Buzano's inequality

$$|\langle u,e\rangle\langle e,v\rangle| \le \frac{1}{2}(|\langle u,v\rangle|+\|u\|\|v\|),$$

where $\|e\|=1$, we obtain

$$\begin{aligned}&\langle \sum_{i=1}^{m} p_i|A_i|^2 x, x\rangle \langle x, \sum_{i=1}^{m} p_i|B_i^*|^2 x\rangle \\ &\leq \frac{1}{2}\left(\left|\langle \sum_{i=1}^{m} p_i|A_i|^2 x, \sum_{i=1}^{m} p_i|B_i^*|^2 x\rangle\right| + \left\|\sum_{i=1}^{m} p_i|A_i|^2 x\right\| \left\|\sum_{i=1}^{m} p_i|B_i^*|^2 x\right\| \right) \\ &= \frac{1}{2}\left(\left|\langle \sum_{i=1}^{m} p_i|B_i^*|^2 \sum_{i=1}^{m} p_i|A_i|^2 x, x\rangle\right| + \left\|\sum_{i=1}^{m} p_i|A_i|^2 x\right\| \left\|\sum_{i=1}^{m} p_i|B_i^*|^2 x\right\| \right)\end{aligned}$$

for $x \in \mathcal{H}$ with $\|x\| = 1$.

By (38), we obtain

$$\begin{aligned}&\left|\langle \left(\sum_{i=1}^{m} p_i B_i S_{\mathcal{Z}} A_i - \frac{1}{2}\sum_{i=1}^{m} p_i B_i A_i \right) x, x\rangle\right|^2 \\ &\leq \frac{1}{4}\left(\left|\langle \sum_{i=1}^{m} p_i|B_i^*|^2 \sum_{i=1}^{m} p_i|A_i|^2 x, x\rangle\right| + \left\|\sum_{i=1}^{m} p_i|A_i|^2 x\right\| \left\|\sum_{i=1}^{m} p_i|B_i^*|^2 x\right\| \right)\end{aligned}$$

for $x \in \mathcal{H}$ with $\|x\| = 1$.

By taking the supremum over all x of norm one, we obtain the desired result (37). □

4. Conclusions

In conclusion, this paper delves into the study of the Selberg operator, exploring its properties and relationships with other bounded operators. By establishing bounds for the norm and numerical radius of the product of three operators, with one of them being a Selberg operator, valuable insights are gained into the behavior of operator compositions involving the Selberg operator. Additionally, the paper presents various bounds for the summation of operators, particularly the Selberg operator. In this study, it is important to note that we employ a unitarily invariant norm, denoted as $|||\cdot|||$, throughout our analysis.

This work serves as a starting point for future research in the field and lays the foundation for exploring more complex aspects of the Selberg operator and its implications in different areas of math. By emphasizing the importance of symmetry in math, this study opens up opportunities for further investigation and potential applications in related studies.

A fascinating open problem is to find the best possible limits for the size of the Selberg operator in various situations. It would be valuable to explore whether we can discover tighter bounds or more general limits that apply to a wider range of operator combinations. By determining the optimal norm bounds, we can gain a deeper understanding of how the Selberg operator behaves and its limitations in different contexts.

Additionally, another interesting area for further exploration is studying the Selberg operator within the framework of operator algebras. This involves examining its properties and behavior in relation to mathematical structures called C^*-algebras or von Neumann algebras. By investigating the Selberg operator in these algebraic settings, we can uncover deeper insights into its structural properties and its connections with other operators. This line of research can provide a more comprehensive understanding of the Selberg operator and its role in the broader context of operator theory.

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