Reflections on the q-Fourier transform and the q-Gaussian function

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October 15, 2013

Abstract

The standard q-Fourier Transform (qFT) of a constant diverges, which begs for a better treatment. In addition, Hilhorst has conclusively proved that the ordinary qFT is not of a one-to-one character for an infinite set of functions [J. Stat. Mech. P10023 (2010)]. Generalizing the ordinary qFT analyzed in [Milan J. Math. **76** (2008) 307],

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we appeal here to a complex q-Fourier transform, and show that the problems above mentioned are overcome.

Keywords: q-Fourier transform, tempered ultradistributions, complex-plane generalization, one-to-one character.

1   Introduction

Nonextensive statistical mechanics (NEXT) [1, 2, 3], a well known generalization of the Boltzmann-Gibbs (BG) one, is used in many scientific and technological endeavors. NEXT central concept is that of a nonadditive (though extensive [4]) entropic information measure characterized by the real index q (with q = 1 recovering the standard BG entropy). Applications include cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spin glasses [8], turbulence in the heliosphere [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], and Fokker-Planck equation's studies [16], etc. NEXT can be advantageously expressed via q-generalizations of standard mathematical concepts [17]. One can mention, for instance, the logarithm and exponential functions, addition and multiplication, Fourier transform (FT), the Central Limit Theorem [18], plane waves, and the representation of the Dirac delta into plane waves [20, 21, 22, 23].

Until recently, a generic analytical expression for the inverse q-FT for arbitrary functions and any value of q did not exist [24]. This situation was adequately remedied in [25], whose authors, by using tempered ultra-

distributions [19, 30], introduced a complex q-Fourier transform $F(k, q)$ which exhibits nice properties and is one-to-one. In turn, this overcame a serious flaw of the original F_q-definition, i.e., not being of the essential one-to-one nature [27]. Investigations of this kind and related questions are relevant for field theory and condensed matter physics, engineering (e.g., image and signal processing), and mathematical areas for which the standard FT and its inverse play important roles.

In this work we focus attention on q-Gaussians, an essential tool of q-statistics [26], that was not discussed in [25]. q-Gaussian behavior is often encountered in quite distinct settings [26]. In particular, one has to mention experimental scenarios in which data are gathered using a set-up that performs a normalization preprocessing. The ensuing normalized input, as recorded by the measurement device, will always be q-Gaussian distributed, if the incoming data exhibit elliptical symmetry, a rather common feature [26]. The q-Fourier transform of the q-Gaussian was discussed in [18], but the corresponding treatment also presented the flaws above alluded to, a situation deserving further discussion that we tackle below.

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2 Preliminaries

It is necessary, before proceeding, to review materials developed in [25] (more details in the Appendix). So-called q-exponentials

$$e_q(x) = [1 + (1 - q)x]_+^{1/(1-q)}, \qquad (2.1)$$

are the hallmark of Tsallis's statistics [1], being generalizations of the ordinary exponential functions and coinciding with them for $q \to 1$. Here we will deal with *complex* q-exponentials, i.e., $e_q(ikx)$ for $1 \leq q < 2$ with k a real number (see ([23])

$$e_q(ikx) = [1 + i(1 - q)kx]^{\frac{1}{1-q}}. \qquad (2.2)$$

Our central tools are *distributions*, that is, linear functionals that map a set of conventional and well-behaved functions, called test functions, onto the set of real (complex) numbers. In this sense, (2.2) is to be regarded as a distribution. *Tempered distributions* constitute a subset of the distributions-set for which the test functions are members of a special space called Schwartz' one $\mathcal{S}$, a function-space in which its members possess derivatives that are rapidly decreasing. $\mathcal{S}$ exhibits a notable property: *the Fourier transform is an automorphism on $\mathcal{S}$*, a property that allows, by duality, to define the

Fourier transform for elements in the dual space of $\mathcal{S}$. This dual is the space of tempered distributions. In physics it is not uncommon to face functions that grow exponentially in space or time. In such circumstances Schwartz' space of tempered distributions is too restrictive. Instead, ultra-distributions satisfy that need [19], being continuous linear functionals defined on the space of entire functions rapidly decreasing on straight lines parallel to the real axis [19].

An important fact about ultra-distributions is the following: *a tempered distribution is the cut of a tempered ultra-distribution.* We are not speaking here of the "cut" of an analytic function (see Refs. [30, 31]). Accordingly, $e_q(ikx)$ is the cut along the real k-axis of a tempered ultra-distribution [30, 31], an essential fact for our present endeavor,

$$E_q(ikx) = \{H(x)H[\Im(k)] - H(-x)H[-\Im(k)]\}\left[1 + i(1-q)kx\right]^{\frac{1}{1-q}}, \quad (2.3)$$

with $H(x)$ the Heaviside's step function and $\Im(k)$ the imaginary part of the complex number k. The relationship between $e_q(ikx)$ and $E_q(ikx)$ becomes more clear noting that, if $f(k)$ is a tempered distribution and $F(k)$ is the corresponding tempered ultra-distribution, then ([25, 30, 31])

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$$f(k) = F(k + i0) - F(k - i0). \tag{2.4}$$

Thus, (2.4) leads to

$$e_q(ikx) = E_q[i(k + i0)x] - E_q[i(k - i0)x] \tag{2.5}$$

At this stage we introduce the set $\Lambda_{[1,2),\infty}$, defined as

$$\Lambda_{[1,2),\infty} = \{f(x)/f(x) \in \Lambda^+_{[1,2),\infty} \wedge f(x) \in \Lambda^-_{[1,2),\infty}\}, \tag{2.6}$$

where

$$\Lambda^+_{[1,2),\infty} = \Big\{ f(x)/f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} \in \mathcal{L}^1[\mathbb{R}^+] \wedge$$

$$[f(x) \geq 0; 1 \leq q < 2]\} \tag{2.7}$$

and

$$\Lambda^-_{[1,2),\infty} = \Big\{ f(x)/f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}} \in \mathcal{L}^1[\mathbb{R}^-] \wedge$$

$$[f(x) \geq 0; 1 \leq q < 2]\} \tag{2.8}$$

With the help of Λ and by recourse to (2.3) together with the fact that for a

given $F(k, q)$

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} dq\, \delta(q - 1 - \epsilon)\, F(k, q) = F(k) \tag{2.9}$$

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so that

$$f(x) = \frac{1}{2\pi} \oint_\Gamma dk\, F(k)e^{-ikx} \tag{2.10}$$

we can define a complex Umarov-Tsallis-Steinberg (UTS) q-Fourier trans-

form (of $f(x) \in \Lambda_{[1,2),\infty}$) in the following way

$$F(k, q) = [H(q-1) - H(q-2)] \times$$

$$\left\{ H[\Im(k)] \int_0^\infty f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}},\ dx - \right.$$

$$\left. H[-\Im(k)] \int_{-\infty}^0 f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}\ dx \right\} \tag{2.11}$$

Here q is a real variable such that $1 \le q < 2$. *The cut along the real axis*

of this transform is the real UTS q-Fourier transform given in [18], [23] (see

[25] for a simple application of this transform). Taking into account that for

$q = 1$ the q-Fourier transform is the usual Fourier transform and using the

formula for the inversion of the complex Fourier transform straightforwardly

leads to the inversion formula for (2.11).

Consider

$$F(k) = \lim_{\epsilon \to 0+} \int_1^2 \delta(q - 1 - \epsilon)F(k, q)dq,$$

together with

$$f(x) = \frac{1}{2\pi} \int_\Gamma dk\, F(k) e^{-ikx}.$$

Since for $q = 1$ our equation (2.11) is the complex Fourier transform

$$F(k) = H[\Im(k)] \int_0^\infty dx\, f(x) e^{ikx} - H[-\Im(k)] \int_{-\infty}^0 dx\, f(x) e^{ikx},$$

from (2.11) we find

$$f(x) = \frac{1}{2\pi} \oint_\Gamma \left[\lim_{\epsilon \to 0^+} \int_1^2 F(k,q)\delta(q-1-\epsilon)\, dq \right] e^{-ikx}\, dk. \qquad (2.12)$$

Eqs. (2.11) and (2.12) solve the problem of inversion of the q-Fourier trans-

form, which is of a one-to-one nature (see [27] for fixed q). Clearly, from (2.4)

and (2.5), on the real axis, one gets for (2.11) and (2.12)

$$F(k,q) = [H(q-1) - H(q-2)] \times$$

$$\int_{-\infty}^\infty f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}\, dx, \qquad (2.13)$$

for the real transform, and

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\lim_{\epsilon \to 0^+} \int_1^2 F(k,q)\delta(q-1-\epsilon)\, dq \right] e^{-ikx}\, dk, \qquad (2.14)$$

for its inverse.

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3 Series expansion of the q-Fourier transform

Consider now the function

$$e_q[ikx\, f(x)^{q-1}] \equiv h(x,k,q) = \{1+(1-q)ikx[f(x)]^{q-1}\}^{\frac{1}{1-q}}, \qquad (3.1)$$

with $f(x) \in \Lambda_{[1,2),\infty}$, that constitutes a generalization of $e_q[ikx]$, whose treatment was discussed above. Using the series expansions of the logarithm and the exponential function, we write

$$e_q[ikx\, f(x)^{q-1}] \equiv \{1+(1-q)ikx[f(x)]^{q-1}\}^{\frac{1}{1-q}} = e^{\frac{1}{1-q}\ln\{1+(1-q)ikx[f(x)]^{q-1}\}} =$$

$$e^{\frac{1}{1-q}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(1-q)^n(ikx)^n[f(x)]^{n(q-1)}} =$$

$$e^{\frac{1}{1-q}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(1-q)^n(ikx)^n e^{n(q-1)\ln f(x)}} =$$

$$e^{\frac{1}{1-q}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(1-q)^n(ikx)^n \sum_{m=0}^{\infty}\frac{n^m}{m!}(q-1)^m[\ln f(x)]^m} =$$

$$e^{\left[\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{n^{m-1}}{m!}(ikx)^n \ln^m[f(x)](q-1)^{n+m-1}\right]} =$$

Performing the change of variables $n' = m+n$, $m' = m$ and then making $n' = n$ and $m' = m$ we obtain:

$$e^{\left\{\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\frac{(n+1-m)^{m-1}}{m!}(ikx)^{n-m+1}\ln^m[f(x)]\right](q-1)^n\right\}}. \qquad (3.2)$$

Let $g(x, k, n)$ be given by

$$g(x, k, n) = \sum_{m=0}^{n} \frac{(n - m + 1)^{m-1}}{m!} (ikx)^{n-m+1} \ln^m[f(x)]. \qquad (3.3)$$

Then,

$$e_q[ikx\, f(x)^{q-1}] \equiv \{1 + (1 - q)ikx[f(x)]^{q-1}\} = h(x, k, q) = e^{\sum_{n=0}^{\infty} g(x,k,n)(q-1)^n} \qquad (3.4)$$

or

$$e_q[ikx\, f(x)^{q-1}] \equiv e^{ikx} e^{\sum_{n=1}^{\infty} g(x,k,n)(q-1)^n}. \qquad (3.5)$$

According to the exponential function's expansion we have

$$e^{\sum_{n=1}^{\infty} g(x,k,n)(q-1)^n} = \sum_{p=0}^{\infty} \frac{\left(\sum_{n=1}^{\infty} g(x, k, n)(q - 1)^n \right)^p}{p!}, \qquad (3.6)$$

or:

$$e^{\sum_{n=1}^{\infty} g(x,k,n)(q-1)^n} = 1 + \sum_{s=1}^{\infty} g(k, x, s)(q - 1)^S +$$

$$\frac{1}{2!} \sum_{s_1=1}^{\infty} \sum_{s_2=1}^{\infty} g(k, x, s_1)g(k, x, s_2)(q - 1)^{s_1+s_2} + \cdots +$$

This sum can be rearranged. Let $l(x, k, n)$ be given by:

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$$l(x, k, n) = \frac{1}{n!} \sum_{s=n}^{\infty} \sum_{s_1=1}^{s-n+1} \sum_{s_2=1}^{s-s_1-n+2} \cdots \sum_{s_{n-1}=1}^{s-s_1-s_2-\cdots-s_{n-2}-1}$$

$$g(x, k, s_1)g(x, k, s_2)\ldots g(x, k, s_{n-1}) \cdots$$

$$g(x, k, s - s_1 - s_2 - \cdots - s_{n-1})(q-1)^s \tag{3.7}$$

Then we have:

$$e_q[ikx\, f(x)^{q-1}] \equiv h(x, k, q) = e^{ikx}\left[1 + \sum_{n=1}^{\infty} l(x, k, n)\right], \tag{3.8}$$

Finally (see Section 2) we can write the q-Fourier transform of $e_q[ikx\, f(x)^{q-1}]$ in the fashion

$$F(k, q) = [H(q - 1) - H(q - 2)]\times$$

$$\left\{ H[\Im(k)] \int_0^{\infty} f(x)h(x, k, q)\, dx - H[-\Im(k)] \int_{-\infty}^{0} f(x)h(x, k, q)\, dx \right\}. \tag{3.9}$$

4 The q-Fourier transform of the q-Gaussian

As stated in the Introduction, our purpose is to calculate the q-Fourier transform of the q-Gaussian. The calculation is too involved if one wishes to consider the pertinent expansions up to arbitrary order on $q-1$. We will content ourselves here with a first order approach. Accordingly,

$$h(x, k, q) = e^{ikx}[1 + g(x, k, 1)(q - 1)], \tag{4.1}$$

with

$$g(x, k, 1) = \frac{(ikx)^2}{2} + ikx \ln[f(x)], \tag{4.2}$$

so that, up to first order, we have for the q-Fourier transform

$$F(k, q) = [H(q - 1) - H(q - 2)] \times$$

$$\left\{ H[\Im(k)] \int_0^\infty \left\{ 1 + \left\{ \frac{(ikx)^2}{2} + ikx \ln[f(x)] \right\} (q - 1) \right\} f(x) e^{ikx} \, dx - \right.$$

$$\left. H[-\Im(k)] \int_{-\infty}^0 \left\{ 1 + \left\{ \frac{(ikx)^2}{2} + ikx \ln[f(x)] \right\} (q - 1) \right\} f(x) e^{ikx} \, dx \right\} \tag{4.3}$$

Let $G(k)$ and $G(k, \beta)$ be given by:

$$G(k) = \left\{ H[\Im(k)] \int_0^\infty f(x) e^{ikx} \, dx - H[-\Im(k)] \int_{-\infty}^0 f(x) e^{ikx} \, dx \right\} \tag{4.4}$$

$$G(k, \beta) = \left\{ H[\Im(k)] \int_0^\infty [f(x)]^\beta e^{ikx} \, dx - H[-\Im(k)] \int_{-\infty}^0 [f(x)]^\beta e^{ikx} \, dx \right\}, \tag{4.5}$$

enabling us to write

$$F(k, q) = [H(q - 1) - H(q - 2)] \times$$

$$G(k) + \left[\frac{k^2}{2} \frac{\partial^2}{\partial k^2} G(k) + k \frac{\partial}{\partial k} \frac{\partial}{\partial \beta} G(k, \beta) \right]_{\beta=1} (q - 1). \tag{4.6}$$

Let $f(x)$ be the q-Gaussian

$$f(x) = C_{q'}[1 + (q' - 1)\alpha x^2]^{\frac{1}{1-q'}}, \tag{4.7}$$

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where:

$$C_{q'} = \frac{\sqrt{(q'-1)\alpha}}{B\left(\frac{1}{2}, \frac{1}{q'-1}\frac{1}{2}\right)} \quad q' \neq 1, \tag{4.8}$$

$$C_1 = \sqrt{\frac{\alpha}{\pi}}. \tag{4.9}$$

Thus, using [28] we obtain

$$G(k, q') = H[\Im(k)]C_{q'}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{2-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{1}{1-q'}}}\left[\frac{2}{(1-q')i\alpha k}\right]^{\frac{2-q'}{1-q'}-\frac{1}{2}} \times$$

$$\left\{\mathbf{H}_{\frac{2-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(1-q')\alpha}\right) - \mathbf{N}_{\frac{2-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(1-q')\alpha}\right)\right\} -$$

$$H[-\Im(k)]C_{q'}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{2-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{1}{1-q'}}}\left[\frac{2}{(q'-1)i\alpha k}\right]^{\frac{2-q'}{1-q'}-\frac{1}{2}} \times$$

$$\left\{\mathbf{H}_{\frac{2-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(q'-1)\alpha}\right) - \mathbf{N}_{\frac{2-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(q'-1)\alpha}\right)\right\}, \tag{4.10}$$

$$G(k, q', \beta) = H[\Im(k)]C_{q'}^{\beta}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{\beta+1-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{\beta}{1-q'}}}\left[\frac{2}{(1-q')i\alpha k}\right]^{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \times$$

$$\left\{\mathbf{H}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(1-q')\alpha}\right) - \mathbf{N}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(1-q')\alpha}\right)\right\} -$$

$$H[-\Im(k)]C_{q'}^{\beta}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{\beta+1-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{\beta}{1-q'}}}\left[\frac{2}{(q'-1)i\alpha k}\right]^{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}} \times$$

$$\left\{\mathbf{H}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(q'-1)\alpha}\right) - \mathbf{N}_{\frac{\beta+1-q'}{1-q'}-\frac{1}{2}}\left(\frac{ik}{(q'-1)\alpha}\right)\right\}, \tag{4.11}$$

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where $\mathbf{H}$ and $\mathbf{N}$ are the Struve and Neumann functions, respectively. The q-Fourier transform of the q-Gaussian is now

$$F(k, q, q') = [H(q-1) - H(q-2)] \times$$

$$G(k, q') + \left[\frac{k^2}{2} \frac{\partial^2}{\partial k^2} G(k, q') + k \frac{\partial}{\partial k} \frac{\partial}{\partial \beta} G(k, q', \beta) \right]_{\beta=1} (q-1). \qquad (4.12)$$

The cuts on the real axis of $G(k, q')$ and $G(k, q', \beta)$ are

$$G(k, q') = C_{q'} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{2-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{1}{1-q'}}} \left[\frac{2}{(1-q')i\alpha(k+i0)} \right]^{\frac{2-q'}{1-q'} - \frac{1}{2}} \times$$

$$\left\{ \mathbf{H}_{\frac{2-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k+i0)}{(1-q')\alpha} \right) - \mathbf{N}_{\frac{2-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k+i0)}{(1-q')\alpha} \right) \right\} +$$

$$C_{q'} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{2-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{1}{1-q'}}} \left[\frac{2}{(q'-1)i\alpha(k-i0)} \right]^{\frac{2-q'}{1-q'} - \frac{1}{2}} \times$$

$$\left\{ \mathbf{H}_{\frac{2-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k-i0)}{(q'-1)\alpha} \right) - \mathbf{N}_{\frac{2-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k-i9)}{(q'-1)\alpha} \right) \right\}, \qquad (4.13)$$

$$G(k, q', \beta) = C_{q'}^{\beta} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\beta+1-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{\beta}{1-q'}}} \left[\frac{2}{(1-q')i\alpha(k+i0)} \right]^{\frac{\beta+1-q'}{1-q'} - \frac{1}{2}} \times$$

$$\left\{ \mathbf{H}_{\frac{\beta+1-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k+i0)}{(1-q')\alpha} \right) - \mathbf{N}_{\frac{\beta+1-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k+i0)}{(1-q')\alpha} \right) \right\} +$$

$$C_{q'}^{\beta} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\beta+1-q'}{1-q'}\right)}{[(q'-1)\alpha]^{\frac{\beta}{1-q'}}} \left[\frac{2}{(q'-1)i\alpha(k-i0)} \right]^{\frac{\beta+1-q'}{1-q'} - \frac{1}{2}} \times$$

$$\left\{ \mathbf{H}_{\frac{\beta+1-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k-i0)}{(q'-1)\alpha} \right) - \mathbf{N}_{\frac{\beta+1-q'}{1-q'} - \frac{1}{2}} \left(\frac{i(k-i0)}{(q'-1)\alpha} \right) \right\}. \qquad (4.14)$$

Now, the real q-Fourier transform takes the form

$$F(k, q, q') = [H(q-1) - H(q-2)] \times$$

$$G(k, q') + \left[\frac{k^2}{2} \frac{\partial^2}{\partial k^2} G(k, q') + k \frac{\partial}{\partial k} \frac{\partial}{\partial \beta} G(k, q', \beta) \right]_{\beta=1} (q-1). \qquad (4.15)$$

5   The q-Fourier transform of the q-Gaussian

for fixed q

In this section we will provide an alternative path to the computation of the q-Fourier transform, for a q-Gaussian, in the case of a fixed q-value. Note that previously, an interesting calculation of this transformation, for the real axis, has been presented in Ref. [33].

We start with

$$F(k, q) = H[\Im(k)] \int_0^\infty C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \times$$

$$\left\{ 1 + (1-q)ikx \left\{ C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \right\} (q-1) \right\}^{\frac{1}{1-q}} dx -$$

$$H[-\Im(k)] \int_{-\infty}^0 C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \times$$

$$\left\{ 1 + (1-q)ikx \left\{ C_q [1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \right\} (q-1) \right\}^{\frac{1}{1-q}} dx, \qquad (5.1)$$

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$1 \leq q < 2$. Simplifying terms we obtain

$$F(k, q) = H[\Im(k)] \int_0^\infty C_q[(q-1)\alpha x^2 + e^{-\frac{i\pi}{2}}(q-1)C_q^{q-1}kx + 1]^{\frac{1}{1-q}} \, dx -$$

$$H[-\Im(k)] \int_{-\infty}^0 C_q[(q-1)\alpha x^2 + e^{\frac{i\pi}{2}}(q-1)C_q^{q-1}kx + 1]^{\frac{1}{1-q}} \, dx. \qquad (5.2)$$

Effecting the change of variables $\sqrt{(q-1)\alpha} \, x = y$ the q-Fourier transform

adopts the appearance

$$F(k, q) = \frac{H[\Im(k)]}{\sqrt{(q-1)\alpha}} \int_0^\infty C_q \left[y^2 + e^{-\frac{i\pi}{2}} C_q^{q-1} \sqrt{\frac{q-1}{\alpha}} ky + 1 \right]^{\frac{1}{1-q}} \, dy -$$

$$\frac{H[-\Im(k)]}{\sqrt{(q-1)\alpha}} \int_0^\infty C_q \left[y^2 + e^{\frac{i\pi}{2}} C_q^{q-1} \sqrt{\frac{q-1}{\alpha}} ky + 1 \right]^{\frac{1}{1-q}} \, dy. \qquad (5.3)$$

Using now the result given in [34] ($\mathbf{P}_\nu^\mu$ is the associated Legendre function)

$$\mathbf{P}_\nu^\mu(z) = \frac{2^\mu \Gamma(1 - 2\mu)(z^2 - 1)^{\frac{\mu}{2}}}{\Gamma(1 - \mu)\Gamma(-\mu - \nu)\Gamma(\nu - \mu + 1)} \times$$

$$\int_0^\infty \left(1 + 2tz + t^2\right)^{\mu - \frac{1}{2}} t^{-1-\nu-\mu} \, dt, \qquad (5.4)$$

and we can write:

$$\int_0^\infty \left(1 + 2tz + t^2\right)^{\mu - \frac{1}{2}} \, dt = \Gamma(-\mu)2^{-\mu-1}(z^2 - 1)^{\frac{\mu}{2}} \mathbf{P}_{-\mu-1}^\mu(z), \qquad (5.5)$$

where:

$$\gamma = \frac{C_q^{q-1}}{2} \sqrt{\frac{q-1}{\alpha}} \quad \mu = \frac{1}{1-q} + \frac{1}{2}, \qquad (5.6)$$

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so that

$$F(k, q) = C_q \frac{\Gamma(-\mu)}{\sqrt{(q-1)\alpha}} 2^{-\mu-1} e^{-\frac{i\pi\mu}{2}} (\gamma^2 k^2 + 1)^{\frac{\mu}{2}} \times$$

$$\left\{ H[\Im(k)] \mathbf{P}^\mu_{-1-\mu}(e^{-\frac{i\pi}{2}}\gamma k) - H[-\Im(k)] e^{i\pi\mu} \mathbf{P}^\mu_{-1-\mu}(e^{\frac{i\pi}{2}}\gamma k) \right\}, \qquad (5.7)$$

which is the q-Fourier transform of the q-Gaussian on the complex plane for fixed q.

Our next step is to evaluate the q-Fourier transform of the q-Gaussian on the real axis. This needs calculating the cut of (5.7) along the real axis. Using [35] we reach the equality

$$(\gamma^2 k^2 + 1)^{\frac{\mu}{2}} \left\{ H[\Im(k)] \mathbf{P}^\mu_{-1-\mu}(e^{-\frac{i\pi}{2}}\gamma k) - H[-\Im(k)] e^{i\pi\mu} \mathbf{P}^\mu_{-1-\mu}(e^{\frac{i\pi}{2}}\gamma k) \right\} =$$

$$H[\Im(k)] \frac{(\gamma k + i)^\mu}{\Gamma(1-\mu)} F\left(-\mu, 1+\mu, ; 1-\mu; \frac{1+i\gamma k}{2}\right) -$$

$$H[-\Im(k)] e^{i\pi\mu} \frac{(\gamma k - i)^\mu}{\Gamma(1-\mu)} F\left(-\mu, 1+\mu, ; 1-\mu; \frac{1-i\gamma k}{2}\right) = C(k, q), \quad (5.8)$$

where F is the hypergeometric function.

The cut $c(k, q)$ along the real axis of $C(k, q)$ is

$$c(k, q) = \frac{(\gamma k + i)^\mu}{\Gamma(1-\mu)} F\left(-\mu, 1+\mu, ; 1-\mu; \frac{1+i\gamma k}{2}\right) +$$

$$e^{i\pi\mu} \frac{(\gamma k - i)^\mu}{\Gamma(1-\mu)} F\left(-\mu, 1+\mu, ; 1-\mu; \frac{1-i\gamma k}{2}\right). \qquad (5.9)$$

Now, according to [36]

$$F(-\mu, 1+\mu; 1-\mu; z) = (1-z)^{-\mu} F(1, -2\mu; 1-\mu; z), \qquad (5.10)$$

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and then

$$c(k, q) = \frac{2^\mu e^{\frac{i\pi\mu}{2}}}{\Gamma(1-\mu)} \left[F\left(1, -2\mu; 1-\mu, \frac{1+i\gamma k}{2}\right) + \right.$$
$$\left. F\left(1, -2\mu; 1-\mu, \frac{1-i\gamma k}{2}\right) \right], \qquad (5.11)$$

so that via the result [37] we have

$$F\left(1, -2\mu; 1-\mu; \frac{1+i\gamma k}{2}\right) = -F\left(1, -2\mu; 1-\mu; \frac{1-i\gamma k}{2}\right) +$$
$$\frac{2\Gamma(1-\mu)\sqrt{\pi}}{\Gamma(\frac{1}{2}-\mu)}(1+\gamma^2 k^2)^\mu. \qquad (5.12)$$

Eqs. (5.11) and (5.12) lead to the cut $f(k, q)$ of $F(k, q)$ in (5.7)

$$f(k, q) = \left(1 + \frac{C_q^{2(q-1)}(q-1)k^2}{4\alpha}\right)^{\frac{1}{1-q}+\frac{1}{2}}, \qquad (5.13)$$

which is the q-Fourier transform of the q-Gaussian on the real axis for fixed

q. From (5.13) we see that the q-Fourier transform of a q-Gaussian is another

q'-Gaussian with

$$q' = 1 - \frac{2(1-q)}{3+q}$$

This result can also be obtained using the real q-Fourier transform

$$f(k, q) = \int\limits_{-\infty}^{\infty} C_q[1 + (q-1)\alpha x^2]^{\frac{1}{1-q}} \times$$

$$\left\{1 + (1-q)ikx \left\{C_q[1 + (q-1)\alpha x^2]^{\frac{1}{1-q}}\right\}^{q-1}\right\}^{\frac{1}{1-q}} dx, \qquad (5.14)$$

$1 \le q < 2$.

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Conclusions

By recourse to tempered ultra-distributions we have evaluated a complex q-Fourier transform $F(k, q)$ for the q-Gaussian that is one-to-one, solving thus a flaw of the original F_q-definition, i.e., not being of the essential one-to-one nature, as illustrated, for instance, in [25]. The computation has been done both for a floating q and for a fixed one. An essential piece of the Tsallis' machinery has thus been rebuilt.

6   Appendix: Tempered Ultradistributions and Distributions of Exponential Type

For the benefit of the reader we give a brief summary of the main properties of distributions of exponential type and tempered ultra-distributions.

Notations. The notations are almost textually taken from Ref. [31]. Let $\mathbb{R}^n$ (res. $\mathbb{C}^n$) be the real (resp. complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, ..., x_n)$ (resp $z = (z_1, z_2, ..., z_n)$). We shall use the notations:

(a) $x + y = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$; $\alpha x = (\alpha x_1, \alpha x_2, ..., \alpha x_n)$

(b) $x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, ..., x_n \geqq 0$

(c)$x \cdot y = \sum\limits_{j=1}^{n} x_j y_j$

(d)$| x | = \sum\limits_{j=1}^{n} | x_j |$

Let $\mathbb{N}^{n}$ be the set of n-tuples of natural numbers. If $p \in \mathbb{N}^{n}$, then $p = (p_1, p_2, ..., p_n)$, and p_j is a natural number, $1 \leqq j \leqq n$. $p + q$ stands for $(p_1 + q_1, p_2 + q_2, ..., p_n + q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, ..., p_n \geqq q_n$. x^p entails $x_1^{p_1} x_2^{p_2} ... x_n^{p_n}$. We shall denote by $| p | = \sum\limits_{j=1}^{n} p_j$ and call D^p the differential operator $\partial^{p_1 + p_2 + ... + p_n} / \partial x_1^{p_1} \partial x_2^{p_2} ... \partial x_n^{p_n}$

For any natural k we define $x^k = x_1^k x_2^k ... x_n^k$ and $\partial^k / \partial x^k = \partial^{nk} / \partial x_1^k \partial x_2^k ... \partial x_n^k$

The space $\mathcal{H}$ of test functions such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any p and q, being defined [see Ref. ([31])] by means of the countably set of norms

$$\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p, \, x} e^{p|x|} \left| D^q \hat{\phi}(x) \right| \quad , \quad p = 0, 1, 2, ... \tag{6.1}$$

The space of continuous linear functionals defined on $\mathcal{H}$ is the space Λ_∞ of the distributions of the exponential type given by (ref.[31]).

$$T = \frac{\partial^k}{\partial x^k} \left[e^{k|x|} f(x) \right] \tag{6.2}$$

where k is an integer such that $k \geqq 0$ and $f(x)$ is a bounded continuous function. In addition we have $\mathcal{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \Lambda_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (ref[32]).

The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \overline{\hat{\phi}(x)} \, e^{iz\cdot x} \, dx \tag{6.3}$$

According to ref.[31], $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call $\mathcal{H}$ the set of all such functions.

$$\mathcal{H} = \mathcal{F}\{\mathcal{H}\} \tag{6.4}$$

The topology in $\mathcal{H}$ is defined by the countable set of semi-norms:

$$\|\phi\|_k = \sup_{z \in V_k} |z|^k |\phi(z)|, \tag{6.5}$$

where $V_k = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n :\mid Im z_j \mid \leqq k, 1 \leqq j \leqq n\}$

The dual of $\mathcal{H}$ is the space $\mathcal{U}$ of tempered ultra-distributions [see Ref. ([31])]. In other words, a tempered ultra-distribution is a continuous linear functional defined on the space $\mathcal{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis. Moreover, we have $\mathcal{H} \subset \mathcal{S} \subset \mathcal{S}' \subset \mathcal{U}$.

$\mathcal{U}$ can also be characterized in the following way [see Ref. ([31])]: let $\mathcal{A}_\omega$ be the space of all functions $F(z)$ such that:

A)- $F(z)$ is analytic for $\{z \in \mathbb{C}^n : |Im(z_1)| > p, |Im(z_2)| > p, ..., |Im(z_n)| > p\}$.

B)- $F(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, ..., |Im(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $F(z)$.

Let $\boldsymbol{\Pi}$ be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$. Then $\boldsymbol{\mathcal{U}}$ is the quotient space

$\boldsymbol{C}$)- $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{A}_\omega}/\boldsymbol{\Pi}$

By a pseudo-polynomial we understand a function of z of the form

$\sum_s z_j^s G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n)$ with $G(z_1, ..., z_{j-1}, z_{j+1}, ..., z_n) \in \boldsymbol{\mathcal{A}_\omega}$

Due to these properties it is possible to represent any ultra-distribution as [see Ref. ([31])]

$$F(\phi) = < F(z), \phi(z) > = \oint_\Gamma F(z)\phi(z) \, dz \tag{6.6}$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup ... \Gamma_n$, where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$).

Eq. (6.6) will be our fundamental representation for a tempered ultra-distribution. Use is also made of the "Dirac formula" for ultra-distributions [see Ref. ([30])]

$$F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)} \, dt \tag{6.7}$$

where the "density" $f(t)$ is such that

$$\oint_\Gamma F(z)\phi(z) \, dz = \int_{-\infty}^{\infty} f(t)\phi(t) \, dt. \tag{6.8}$$

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While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (6.8) should be interpreted in the sense of distribution theory. Another important property of the analytic representation is the fact that on Γ, $F(z)$ is bounded by a power of z [31]

$$|F(z)| \leq C|z|^p, \tag{6.9}$$

where C and p depend on F.

The representation (6.6) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ does not alter the ultra-distribution:

$$\oint_\Gamma \{F(z) + P(z)\}\phi(z) \, dz = \oint_\Gamma F(z)\phi(z) \, dz + \oint_\Gamma P(z)\phi(z) \, dz$$

However,

$$\oint_\Gamma P(z)\phi(z) \, dz = 0.$$

As $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing), we obtain:

$$\oint_\Gamma \{F(z) + P(z)\}\phi(z) \, dz = \oint_\Gamma F(z)\phi(z) \, dz. \tag{6.10}$$

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