Charged meson masses under strong magnetic fields: Gauge invariance and Schwinger phases

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We study the role of the Schwinger phase (SP) that appears in the propagator of a charged particle in the presence of a static and uniform magnetic field $\vec{B}$. We first note that this phase cannot be removed by a gauge transformation; far from this, we show that it plays an important role in the restoration of the symmetries of the system. Next, we analyze the effect of SPs in the one-loop corrections to charged pion and rho meson self-energies. To carry out this analysis we consider first a simple form for the meson-quark interactions, and then we study the π^+ and ρ^+ propagators within the Nambu-Jona-Lasinio model, performing a numerical analysis of the B dependence of meson lowest energy states. For both π^+ and ρ^+ mesons, we compare the numerical results arising from the full calculation—in which SPs are included in the propagators, and meson wave functions correspond to states of definite Landau quantum number—and those obtained within alternative schemes in which SPs are neglected (or somehow eliminated) and meson states are described by plane waves of definite four-momentum.

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I. INTRODUCTION

The study of the behavior of charged particles in the presence of an intense magnetic field within the framework of relativistic quantum field theory has a long history (see, e.g., Ref. [1] and references therein). In recent years, the interest in this topic has been renewed in the context of the physics of strong interactions [2–4]. The motivation arises mostly from the realization that intense magnetic fields might play an important role in the study of the early Universe [5], in the analysis of high energy noncentral heavy ion collisions [6,7], and in the description of compact stellar objects like the magnetars [8,9]. It is well known that magnetic fields also induce interesting phenomena like the chiral magnetic effect [10–12], the enhancement of the QCD quark-antiquark condensate ("magnetic catalysis") [13], and the decrease of critical temperatures for chiral restoration and deconfinement QCD transitions [14].

In the above context, the study of the properties of magnetized light hadrons shows up as a very relevant task. In fact, this subject has been addressed by several works in the framework of various approaches to nonperturbative QCD. These include, e.g., Nambu-Jona-Lasinio (NJL)-like models [15–29,29–37], quark-meson models [38,39], chiral perturbation theory [40–42], hidden local symmetry [43], path integral Hamiltonians [44,45], and QCD sum rules [46]. In addition, results for the π and ρ meson spectra in the presence of background magnetic fields have been obtained from lattice QCD (LQCD) calculations [14,47–51].

In models with explicit quark degrees of freedom, like, e.g., the NJL model or the meson-quark model, the determination of meson properties demands the evaluation of quark loops. In the presence of a magnetic field $\vec{B}$, the calculation of these loops requires some care due to the appearance of Schwinger phases (SPs) [52] associated with quark propagators. These phases are not invariant under either translational or gauge transformations. When all external legs in the quark loop correspond to neutral particles, SPs cancel out, and one can take the usual momentum basis to diagonalize the corresponding loop correction; this is the case, for example, of one-loop corrections to neutral meson self-energies. In contrast, when some of the external legs correspond to charged particles—as in the case of the one-loop correction to a charged meson mass—Schwinger phases do not cancel, leading to a breakdown of translational invariance that prevents proceeding as in the neutral case. In this situation, some existing calculations within the NJL model [15,18,22,23,26,30,34] just neglect Schwinger phases; if this is done, one can set meson transverse momenta to zero, considering only the translational invariant part of the quark propagators to determine charged meson masses. In fact, it has even been argued [53] that this way to proceed would be consistent with gauge invariance. On the other hand, a method that fully takes into account the translational-breaking effects introduced by SPs has been presented in Ref. [27] for the calculation of charged pion masses, and then it has been subsequently extended for the determination of the charged pion masses at finite temperature [28], for the analysis of other charged pion properties [31], for the determination of charged kaon [35] and rho meson masses [37], and for the study of diquark and nucleon masses [54]. This method, based on the use of the eigenfunctions associated to magnetized relativistic particles, allows one to diagonalize the charged meson polarization functions in order to obtain the corresponding meson masses.

The main objective of this paper is to clarify the role played by Schwinger phases in the calculation of quark loops associated to the determination of charged meson properties in the presence of an external magnetic field. One important point to be shown is that these phases cannot be "gauged away": if a SP does not vanish in a given gauge, it cannot be removed by any gauge transformation. In fact, the assumption of neglecting SPs might be considered at best as some kind of approximation in which the polarization functions are forced to be gauge invariant, instead of *gauge covariant*, as they should be. To be fully consistent and self-contained we devote the first few sections of this paper to reviewing some properties of the SP as well as to providing the explicit form of quantum fields and propagators for particles with spin 0, 1/2, and 1. Then, we dedicate one section to the determination of one-loop corrections to the charged pion and rho meson self-energies in the context of the quark-meson model, and another section is devoted to the calculation of charged pion and rho meson masses in the framework of the NJL model. Throughout these calculations we focus on the role of SPs and the preservation of gauge properties of the involved quantities. In this way we uncover the issues that appear when the SPs are neglected, providing further support to the method introduced in Ref. [31]. We also show that the assumption of neglecting SPs may have a significant qualitative impact on the theoretical predictions for the behavior of meson masses under a strong magnetic field.

This work is organized as it follows. In Sec. II we review the definition of the SP and state its explicit form in commonly used gauges. Then, we show how the SP plays an important role in the preservation of the expected symmetries of the system—although it is not itself translational and gauge invariant—and we discuss the related constraints on the form of the invariant part of charged particle propagators. In Sec. III we present the explicit form of charged particle quantum fields in the presence of an external magnetic field. The corresponding expressions are given in a quite general form, in terms of eigenfunctions associated to the more commonly used gauges. In Sec. IV we provide the explicit form of the charged particle propagators; this is done in terms of both the field eigenfunctions and the product of a SP and a gauge invariant function obtained using the Schwinger proper time method. Next, in Sec. V we determine the leading order correction to the charged pion and to the charged rho meson self-energies for some typical quark-meson interaction Lagrangian. In particular, we show that these corrections are diagonal in the basis of the corresponding meson eigenfunctions. We also show that this implies taking into account some transverse momentum fluctuations, which would have been neglected by disregarding the SP (and considering plane wave meson wave functions). In Sec. VI we extend the analysis to the calculation of the charged pion and rho meson masses in the framework of the NJL model. To give an idea of the importance of properly taking into account the SP, we perform a numerical analysis of the effect of the magnetic field on these masses, comparing the results obtained from the expressions that include/neglect the SP. Finally, in Sec. VII we provide a summary of our work, together with our main conclusions. We also include Appendixes A–C, and D to provide some formulas related with the formalism used throughout our work.

II. SCHWINGER PHASE AND CHARGED PARTICLE PROPAGATORS

A. Gauge transformations and gauge fixing for a constant magnetic field

We start by considering the electromagnetic field strength $F^{\mu\nu}$ associated with a general electromagnetic field $\mathcal{A}^\mu(x)$,

$$F^{\mu\nu} = \partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu. \tag{2.1}$$

Throughout this work we use the Minkowski metric $g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, while for a space-time coordinate four-vector x^μ we adopt the notation $x^\mu = (t, \vec{x})$, with $\vec{x} = (x^1, x^2, x^3)$. We also consider the covariant derivative $\mathcal{D}^\mu$ that appears in the field equations associated with an electrically charged particle,

$$\mathcal{D}^\mu = \partial^\mu + iQ\mathcal{A}^\mu, \tag{2.2}$$

where Q is the particle electric charge. Now, under a gauge transformation $\Lambda(x)$ the electromagnetic field transforms as

$$\mathcal{A}^\mu \to \tilde{\mathcal{A}}^\mu = \mathcal{A}^\mu + \partial^\mu \Lambda. \tag{2.3}$$

While the electromagnetic field strength $F^{\mu\nu}$ is invariant under this transformation, the operator $\mathcal{D}^\mu$ transforms in a covariant way, namely

$$\mathcal{D}^\mu \to \tilde{\mathcal{D}}^\mu = e^{-iQ\Lambda(x)} \mathcal{D}^\mu e^{iQ\Lambda(x)}. \tag{2.4}$$

So far we have considered a general external electromagnetic field $\mathcal{A}^\mu(x)$. In what follows we concentrate on the case associated with a static and uniform magnetic field $\vec{B}$. The tensor $F^{\mu\nu}$ is given in this case by

$$F^{ij} = F_{ij} = -\epsilon_{ijk} B^k, \qquad F^{0j} = 0, \tag{2.5}$$

with $i,\, j = 1,\, 2,\, 3$, whereas the corresponding electromagnetic field can be written as

$$\mathcal{A}^\mu(x) = \frac{1}{2} x_\nu F^{\nu\mu} + \partial^\mu \Psi(x), \tag{2.6}$$

where $\Psi(x)$ is, in principle, an arbitrary function. For any form of this function one obtains a particular gauge. Without losing generality, one can now choose the axis 3 to be parallel (or antiparallel) to the magnetic field, writing $\vec{B} = (0, 0, B)$. In addition, one can take $\Psi(x)$ to be only a function of the spatial coordinates that are perpendicular to $\vec{B}$, i.e., x^1 and x^2. The reason is that only the components F^{12} and F^{21} of the field strength tensor are different from zero, which implies that only $\partial^1 \mathcal{A}^2$ and $\partial^2 \mathcal{A}^1$ are relevant. In what follows we adopt this coordinate choice.

For the considered situation, some commonly used gauges are

Symmetric gauge (SG),

$$\Psi(x) = 0, \qquad \mathcal{A}^\mu(x) = \left(0, -\frac{B}{2} x^2, \frac{B}{2} x^1, 0\right); \tag{2.7}$$

Landau gauge 1 (LG1),

$$\Psi(x) = \frac{B}{2} x^1 x^2, \qquad \mathcal{A}^\mu(x) = (0, -B x^2, 0, 0); \tag{2.8}$$

Landau gauge 2 (LG2),

$$\Psi(x) = -\frac{B}{2} x^1 x^2, \qquad \mathcal{A}^\mu(x) = (0, 0, B x^1, 0). \tag{2.9}$$

In what follows, we refer to them as "standard gauges."

According to the above introduced coordinate choice, given a four-vector V^μ we find it convenient to distinguish between "parallel" components, V^0 and V^3, and "perpendicular" components, V^1 and V^2. Thus, we introduce the definitions

$$V^\mu_\parallel \equiv (V^0, 0, 0, V^3), \qquad V^\mu_\perp \equiv (0, V^1, V^2, 0). \tag{2.10}$$

In addition, we define the metric tensors

$$g^{\mu\nu}_\parallel = \mathrm{diag}(1, 0, 0, -1), \quad g^{\mu\nu}_\perp = \mathrm{diag}(0, -1, -1, 0). \tag{2.11}$$

The scalar products of parallel and perpendicular vectors are thus given by

$$\begin{aligned} &V^\mu_\parallel W_{\parallel\mu} = V_\parallel \cdot W_\parallel = V^0 W^0 - V^3 W^3, \\ &V^\mu_\perp W_{\perp\mu} = -\vec{V}_\perp \cdot \vec{W}_\perp = -(V^1 W^1 + V^2 W^2), \\ &V^\mu_\parallel W_{\perp\mu} = 0. \end{aligned} \tag{2.12}$$

B. Schwinger phase

We introduce here the so-called Schwinger phase, which will be a relevant quantity throughout this work. Given a particle P with electric charge Q_P, we denote the associated SP by $\Phi_\mathrm{P}(x, y)$; its explicit form is [52]

$$\Phi_\mathrm{P}(x, y) = Q_\mathrm{P} \int_x^y d\xi_\mu \left[\mathcal{A}^\mu(\xi) + \frac{1}{2} F^{\mu\nu}(\xi_\nu - y_\nu)\right], \tag{2.13}$$

where $F^{\mu\nu}$ is assumed to be constant, and the integration is performed along an arbitrary path that connects x with y. In general, the SP is found to be not invariant under either translations or gauge transformations. On the other hand, the integral in Eq. (2.13) is shown to be path independent; thus, it can be evaluated using a straight line path. In this way, using Eq. (2.6) one can obtain a closed expression for the SP associated to a static and uniform magnetic field in an arbitrary gauge. It reads as

$$\Phi_\mathrm{P}(x, y) = \frac{Q_\mathrm{P}}{2} x_\mu F^{\mu\nu} y_\nu - Q_\mathrm{P}[\Psi(x) - \Psi(y)]. \tag{2.14}$$

From Eqs. (2.3) and (2.6), it is seen that under a gauge transformation the SP transforms as

$$\Phi_\mathrm{P}(x, y) \to \tilde{\Phi}_\mathrm{P}(x, y) = \Phi_\mathrm{P}(x, y) - Q_\mathrm{P}[\Lambda(x) - \Lambda(y)]. \tag{2.15}$$

The expressions for the SP in the particular gauges introduced above can now be readily obtained from Eq. (2.14). We have

$$\mathrm{SG:}\ \Phi_\mathrm{P}(x, y) = -\frac{Q_\mathrm{P} B}{2}(x^1 y^2 - y^1 x^2); \tag{2.16}$$

$$\mathrm{LG1:}\ \Phi_\mathrm{P}(x, y) = -\frac{Q_\mathrm{P} B}{2}(x^2 + y^2)(x^1 - y^1); \tag{2.17}$$

$$\mathrm{LG2:}\ \Phi_\mathrm{P}(x, y) = \frac{Q_\mathrm{P} B}{2}(x^1 + y^1)(x^2 - y^2). \tag{2.18}$$

It is worth noticing that in all cases the SP includes products that mix the coordinates of the points x^μ and y^μ. Clearly, there is no way in which these combinations could be expressed in terms of the difference between a scalar function evaluated at x^μ and the same function evaluated at y^μ. Therefore, it follows from Eq. (2.15) that if the SP does not vanish in a given gauge it will be nonvanishing *in any gauge*. This means that the SP cannot be "gauged away."

C. Charged particle propagators in a static and uniform magnetic field

Let us now study the propagators of charged particles. We start by considering the propagator of a spin zero meson, e.g., a charged pion, which we denote as $\Delta_{\pi^\mathcal{Q}}(x,y)$ with $\mathcal{Q}=\pm 1$. The particle charge is then given by $Q_\pi = \mathcal{Q}e$, where e denotes the proton charge.

The equation that defines the meson propagator is

$$(\mathcal{D}^\mu \mathcal{D}_\mu + m_\pi^2)\Delta_{\pi^\mathcal{Q}}(x,y) = -\delta^{(4)}(x-y). \qquad (2.19)$$

If we now perform a gauge transformation, using Eq. (2.4) we get

$$(\tilde{\mathcal{D}}^\mu \tilde{\mathcal{D}}_\mu + m_\pi^2)e^{-iQ_\pi\Lambda(x)}\Delta_{\pi^\mathcal{Q}}(x,y) = -e^{-iQ_\pi\Lambda(x)}\delta^{(4)}(x-y); \qquad (2.20)$$

hence, the propagator has to transform according to

$$\Delta_{\pi^\mathcal{Q}}(x,y) \to \tilde{\Delta}_{\pi^\mathcal{Q}}(x,y) = e^{-iQ_\pi\Lambda(x)}\Delta_{\pi^\mathcal{Q}}(x,y)e^{iQ_\pi\Lambda(y)}, \qquad (2.21)$$

which is the natural extension of a gauge covariant transformation for the case of a bilocal object. It is seen that the phase difference appearing in the transformed propagator is just the same quantity that appears in Eq. (2.15) for the gauge-transformed SP. Therefore, one can always write the meson propagator as

$$\Delta_{\pi^\mathcal{Q}}(x,y) = e^{i\Phi_{\pi^\mathcal{Q}}(x,y)}\bar{\Delta}_{\pi^\mathcal{Q}}(x,y), \qquad (2.22)$$

where $\bar{\Delta}_{\pi^\mathcal{Q}}(x,y)$ is a gauge invariant function; the gauge dependence of the propagator is carried by the SP, which has a well defined expression.

Since we are dealing with a system subject to a static and uniform magnetic field, the invariance under translations in time and space, under rotations around any axis parallel to the magnetic field, and under boosts in directions parallel to the magnetic field is expected to be preserved. Translations in time, as well as translations and boosts in the direction of $\vec{B}$, can be treated in the same way as in the case of a free particle, since they do not involve the axes 1 or 2. Thus, let us focus on the translations in the plane perpendicular to $\vec{B}$ and in the rotations around the $\vec{B}$ direction. Noticeably, the expected invariance seems to be at odds with the fact that the charged pion propagator is known to be not invariant under these transformations. The aim of the following discussion is to clarify this point and see how the invariance implies further constraints on the form of the propagator.

Let us first consider space translations in the perpendicular plane, i.e., a general transformation of the form $x^\mu \to x'^\mu = x^\mu + b_\perp^\mu$. From Eq. (2.6), under this transformation one has

$$\begin{aligned}\mathcal{A}^\mu(x) &\to \mathcal{A}_t^\mu(x)\\ &\equiv \mathcal{A}^\mu(x')\\ &= \mathcal{A}^\mu(x) - \frac{1}{2}F^{\mu\nu}b_{\perp\nu} + \partial^\mu\Psi(x') - \partial^\mu\Psi(x). \end{aligned}\qquad (2.23)$$

It is rather easy to see that this is fully equivalent to a gauge transformation

$$\mathcal{A}_\mu(x) \to \tilde{\mathcal{A}}^\mu(x) = \mathcal{A}^\mu(x) + \partial^\mu\Lambda_t(x;b_\perp), \qquad (2.24)$$

with

$$\Lambda_t(x;b_\perp) = \Psi(x') - \Psi(x) - \frac{1}{2}x_\mu F^{\mu\nu}b_{\perp\nu}. \qquad (2.25)$$

From Eq. (2.25) we can readily get the expressions of $\Lambda_t(x;b_\perp)$ in the particular gauges introduced in the previous subsections. We have

$$\mathrm{SG}:\ \Lambda_t(x;b_\perp) = -\frac{B}{2}(x^1b^2 - x^2b^1); \qquad (2.26)$$

$$\mathrm{LG1}:\ \Lambda_t(x;b_\perp) = \frac{B}{2}b^2(2x^1 + b^1); \qquad (2.27)$$

$$\mathrm{LG2}:\ \Lambda_t(x;b_\perp) = -\frac{B}{2}b^1(2x^2 + b^2). \qquad (2.28)$$

A similar relation between the translation $x^\mu \to x'^\mu = x^\mu + b_\perp^\mu$ and the gauge transformation in Eqs. (2.24) and (2.25) can be obtained for the Schwinger phase. Under the translation, the SP transforms as

$$\begin{aligned}\Phi_{\pi^\mathcal{Q}}(x,y) &\to \Phi_{\pi^\mathcal{Q},t}(x,y)\\ &\equiv \Phi_{\pi^\mathcal{Q}}(x',y')\\ &= \frac{Q_\pi}{2}x'_\mu F^{\mu\nu}y'_\nu - Q_\pi[\Psi(x') - \Psi(y')], \end{aligned}\qquad (2.29)$$

whereas performing the corresponding gauge transformation one gets

$$\begin{aligned}\Phi_{\pi^\mathcal{Q}}(x,y) &\to \tilde{\Phi}_{\pi^\mathcal{Q}}(x,y)\\ &= \Phi_{\pi^\mathcal{Q}}(x,y) - Q_\pi[\Lambda_t(x;b_\perp) - \Lambda_t(y;b_\perp)]. \end{aligned}\qquad (2.30)$$

Taking into account the form of $\Lambda_t(x;b_\perp)$ in Eq. (2.25) we observe that $\Phi_{\pi^\mathcal{Q},t} = \tilde{\Phi}_{\pi^\mathcal{Q}}(x,y)$. Now we can turn back to Eq. (2.19), writing the propagator as in Eq. (2.22). From the above equations it is seen that under the considered translation the operator $(\mathcal{D}^\mu\mathcal{D}_\mu + m_\pi^2)$ and the factor $\exp[i\Phi_{\pi^\mathcal{Q}}(x,y)]$ transform in the same way as under the gauge transformation $\Lambda_t(x,b_\perp)$. Together with the requirement that Eq. (2.19) be translational invariant, this implies that the gauge invariant factor $\bar{\Delta}_{\pi^{\mathcal{Q}}}(x,y)$ has to also be translational invariant. Thus, we can write

$$\bar{\Delta}_{\pi^{\mathcal{Q}}}(x,y) = \bar{\Delta}_{\pi^{\mathcal{Q}}}(x-y), \tag{2.31}$$

and it is possible to obtain a Fourier transform $\bar{\Delta}_{\pi^{\mathcal{Q}}}(v_\parallel, v_\perp)$ that satisfies

$$\bar{\Delta}_{\pi^{\mathcal{Q}}}(x-y) = \int \frac{d^4v}{(2\pi)^4} e^{-iv(x-y)} \bar{\Delta}_{\pi^{\mathcal{Q}}}(v_\parallel, v_\perp). \tag{2.32}$$

Notice that in this expression we have made it explicit that one gets in general different dependences on parallel and perpendicular momenta.

We consider now a rotation of arbitrary angle α around an axis parallel to the magnetic field $\vec{B}$. The effect of such a rotation on an arbitrary vector v^μ acts only on the perpendicular component $v_\perp^\mu$. Choosing the axis 3 in the direction of $\vec{B}$, for a rotation matrix $R_{\hat{3}}(\alpha)$ we have $x'^\mu = \mathcal{R}^\mu_\nu x^\nu$, with

$$\mathcal{R}^\mu_\nu \equiv R_{\hat{3}}(\alpha)^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.33}$$

At this point it is important to specify if we adopt an active or a passive point of view for the rotation; here we adopt the passive point of view and define $\bar{x}'^\mu = \bar{\mathcal{R}}^\mu{}_\nu x^\nu$, with $\bar{\mathcal{R}} \equiv R_{\hat{3}}(-\alpha)$. From Eq. (2.6), under the rotation $R_{\hat{3}}(\alpha)$ the electromagnetic field transforms as

$$\begin{aligned} \mathcal{A}^\mu(x) &\to \mathcal{A}^\mu_r(x) \\ &= \mathcal{R}^\mu{}_\nu \mathcal{A}^\nu(\bar{x}') \\ &= -\frac{1}{2}\mathcal{R}^\mu{}_\tau F^{\tau\delta}\bar{\mathcal{R}}_\delta{}^\nu x_\nu + \mathcal{R}^\mu{}_\tau \frac{\partial\Psi(z)}{\partial z_\tau}\bigg|_{z=\bar{x}'}. \end{aligned} \tag{2.34}$$

Noting that $\mathcal{R}^\mu{}_\tau F^{\tau\delta}\bar{\mathcal{R}}_\delta{}^\nu = F^{\mu\nu}$, the result in Eq. (2.34) can be reinterpreted as a gauge transformation

$$\mathcal{A}^\mu(x) \to \tilde{\mathcal{A}}^\mu(x) = \mathcal{A}^\mu(x) + \partial^\mu \Lambda_r(x;\alpha), \tag{2.35}$$

where

$$\Lambda_r(x,\alpha) = \Psi(\bar{x}') - \Psi(x). \tag{2.36}$$

As in the case of the translations, the equivalence between the rotation $R_{\hat{3}}(\alpha)$ and the gauge transformation $\Lambda_r(x;\alpha)$ is also obtained for the SP, i.e., one gets $\Phi_{\pi^{\mathcal{Q}},r}(x,y) = \tilde{\Phi}_{\pi^{\mathcal{Q}}}(x,y)$. The explicit expressions for the function $\Lambda_r(x;\alpha)$ in the standard gauges read as

$$\text{SG:} \qquad \Lambda_r(x;\alpha) = 0; \tag{2.37}$$

$$\text{LG1:} \qquad \Lambda_r(x;\alpha) = -\frac{B}{2}\sin\alpha\Big[2x^1x^2\sin\alpha + ((x^1)^2 - (x^2)^2)\cos\alpha\Big]; \tag{2.38}$$

$$\text{LG2:} \qquad \Lambda_r(x;\alpha) = \frac{B}{2}\sin\alpha\Big[2x^1x^2\sin\alpha + ((x^1)^2 - (x^2)^2)\cos\alpha\Big]. \tag{2.39}$$

Turning back once again to Eq. (2.19), and writing the propagator as in Eq. (2.22), we observe that under a rotation $R_{\hat{3}}(\alpha)$ the operator $(\mathcal{D}^\mu\mathcal{D}_\mu + m_\pi^2)$ and the factor $e^{i\Phi_{\pi^{\mathcal{Q}}}(x,y)}$ transform in the same way as under the gauge transformation $\Lambda_r(x;\alpha)$. This equivalence, together with the requirement that Eq. (2.19) be invariant under $R_{\hat{3}}(\alpha)$ rotations, implies that $\bar{\Delta}_{\pi^{\mathcal{Q}}}(x-y)$ has to be invariant under these transformations. Since, in addition, the propagator has to be invariant under boosts along the axis 3, one can conclude that the Fourier transform $\bar{\Delta}_{\pi^{\mathcal{Q}}}(v_\parallel, v_\perp)$ defined in Eq. (2.32) can depend only on the quantities $v_\parallel^2$ and $v_\perp^2$.

An entirely similar analysis can be performed for the case of spin 1/2 and spin 1 particles. Thus, the spin 1/2 fermion propagator $S_f(x,y)$ can be written as

$$S_f(x,y) = e^{i\Phi_f(x,y)}\bar{S}_f(x-y), \tag{2.40}$$

with

$$\bar{S}_f(x-y) = \int \frac{d^4v}{(2\pi)^4} e^{-iv(x-y)} \bar{S}_f(v_\parallel, v_\perp). \tag{2.41}$$

Here the propagator is a matrix in Dirac space that involves products of the γ^μ Dirac matrices. Owing to the invariance under rotations around the axis 3 (i.e., the $\vec{B}$ axis) and under boosts in that direction, it is easy to see that $\bar{S}_f(v_\parallel, v_\perp)$ has to be a function of $v_\parallel^2$, $v_\perp^2$, $\gamma_\parallel \cdot v_\parallel$ and $\vec{\gamma}_\perp \cdot \vec{v}_\perp$.

In the case of a charged vector meson propagator, for instance, a ρ meson propagator $D^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x,y)$, we can write

$$D^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x,y) = e^{i\Phi_{\rho^{\mathcal{Q}}}(x,y)}\bar{D}^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x-y), \tag{2.42}$$

with

$$\bar{D}^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x-y) = \int \frac{d^4v}{(2\pi)^4} e^{-iv(x-y)}\bar{D}^{\nu\gamma}_{\rho^{\mathcal{Q}}}(v_\parallel, v_\perp). \tag{2.43}$$

Similarly to the previous cases, invariance under rotations around the axis 3 and under boosts in that direction implies that $\bar{D}^{\nu\gamma}_{\rho^{\mathcal{Q}}}(v_\parallel, v_\perp)$ will be given by a linear combination of tensors of order 2 built from the tensors $g^{\mu\nu}_\parallel$, $g^{\mu\nu}_\perp$, $F^{\mu\nu}$, and the vectors $v^\mu_\parallel$, $v^\mu_\perp$, with coefficients given by functions that depend only on $v^2_\parallel$ and $v^2_\perp$.

Obviously, the above statements can be corroborated by carrying out detailed calculations of the propagators. This is sketched in Sec. IV, where the explicit forms of $\bar{\Delta}_{\pi^{\mathcal{Q}}}(v_\parallel, v_\perp)$, $\bar{S}_f(v_\parallel, v_\perp)$, and $\bar{D}^{\nu\gamma}_{\rho^{\mathcal{Q}}}(v_\parallel, v_\perp)$ are given.

In conclusion, we have seen that the Schwinger phase carries all gauge, translation, and rotation noninvariance that is present in particle propagators. In fact, this should not be surprising, since the breakdown of translational and rotational symmetry is precisely produced by the gauge choice. When calculating a physical quantity we specify a particular gauge and use propagators that, in general, break both translational and rotational invariance; however, all these symmetries are simultaneously recovered in the final result.

III. QUANTUM FIELDS OF CHARGED PARTICLES IN A MAGNETIC FIELD

A. A set of basic functions

Let us consider a charged scalar particle in a static and homogeneous magnetic field. We introduce the scalar functions $\mathcal{F}_Q(x,\bar{q})$, solutions of the eigenvalue equation

$$\mathcal{D}^\mu \mathcal{D}_\mu \mathcal{F}_Q(x,\bar{q}) = f_{\bar{q}}\mathcal{F}_Q(x,\bar{q}), \tag{3.1}$$

where Q is the particle electric charge, $\mathcal{D}^\mu$ is the covariant derivative defined in Eq. (2.2), and the corresponding electromagnetic field $\mathcal{A}^\mu(x)$ is of the form given by Eq. (2.6). In Eq. (3.1), $\bar{q}$ stands for a set of four labels that are needed to completely specify each eigenfunction. One can be more explicit and write the eigenvalue equation in the form

$$\left[\partial^\mu\partial_\mu - Q\vec{B}\cdot\vec{L} + \frac{Q^2}{4}(\vec{x}\times\vec{B})^2\right]e^{iQ\Psi(x)}\mathcal{F}_Q(x,\bar{q}) = f_{\bar{q}}e^{iQ\Psi(x)}\mathcal{F}_Q(x,\bar{q}), \tag{3.2}$$

where $L^k = i\epsilon_{klm}x_l\partial_m$. From this equation it is seen that while the eigenfunctions $\mathcal{F}_Q(x,\bar{q})$ are gauge dependent, the eigenvalues $f_{\bar{q}}$ are not. As discussed in the previous section, the magnetic field can always be taken to lie along the axis 3, and then $\Psi(x)$ can be assumed to depend only on the two spatial coordinates perpendicular to $\vec{B}$, x^1, and x^2. Consequently, as in the case of a free particle, the eigenvalues of the components of the four-momentum along the time direction, q^0, and the magnetic field direction, q^3, can be taken as two of the labels required to specify $\mathcal{F}_Q(x,\bar{q})$. On the other hand, as is well known, the eigenvalues of Eq. (3.1) are given by

$$f_{\bar{q}} = -[(q^0)^2 - (2k+1)B_Q - (q^3)^2], \tag{3.3}$$

where $B_Q \equiv |QB|$, and k is a non-negative integer, to be related with the so-called Landau level. This means that the eigenvalues depend only on three of the labels included in $\bar{q}$. There is a degeneracy, which arises, of course, as a consequence of gauge invariance; to fully specify the eigenfunctions, a fourth quantum number χ is required, i.e., one has $\bar{q} = (q^0, k, \chi, q^3)$.

Although it is not strictly necessary, the quantum number χ can be conveniently chosen according to the gauge in which the eigenvalue problem is analyzed [55]. In particular, since for the standard gauges SG, LG1, and LG2 one has unbroken continuous symmetries, in those cases it is natural to consider quantum numbers χ associated with the corresponding group generators. Usual choices are

$$\text{SG:} \quad \chi = \imath, \quad \text{non-negative integer, associated to } L^3 (\text{eigenvalue of } L^3: \ m = \text{sign}(QB)(\imath - k)); \tag{3.4}$$

$$\text{LG1:} \quad \chi = q^1, \quad \text{real number, eigenvalue of } -i\frac{\partial}{\partial x^1}; \tag{3.5}$$

$$\text{LG2:} \quad \chi = q^2, \quad \text{real number, eigenvalue of } -i\frac{\partial}{\partial x^2}. \tag{3.6}$$

The explicit forms of the functions $\mathcal{F}_Q(x,\bar{q})$ for the above standard gauges and quantum numbers χ are given in Appendix A. They are shown to satisfy the completeness and orthogonality relations

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \mathcal{F}_Q(x,\bar{q})^* \mathcal{F}_Q(y,\bar{q}) = \delta^{(4)}(x-y), \tag{3.7}$$

$$\int d^4x\, \mathcal{F}_Q(x,\bar{q}')^* \mathcal{F}_Q(x,\bar{q}) = \hat{\delta}_{\bar{q}\bar{q}'}. \tag{3.8}$$

Here we have introduced some shorthand notation whose explicit form depends on the chosen gauge. For SG we have

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \equiv \frac{1}{(2\pi)^2}\sum_{k,\imath=0}^{\infty}\int \frac{dq^0}{2\pi}\frac{dq^3}{2\pi},$$
$$\hat{\delta}_{\bar{q}\bar{q}'} \equiv (2\pi)^4 \delta_{kk'}\delta_{\imath\imath'}\delta(q^0-q'^0)\delta(q^3-q'^3), \tag{3.9}$$

while for LG1 (LG2) we have

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \equiv \frac{1}{2\pi}\sum_{k=0}^{\infty}\int \frac{dq^0}{2\pi}\frac{dq^i}{2\pi}\frac{dq^3}{2\pi},$$
$$\hat{\delta}_{\bar{q}\bar{q}'} \equiv (2\pi)^4 \delta_{kk'}\delta(q^0-q'^0)\delta(q^i-q'^i)\delta(q^3-q'^3), \tag{3.10}$$

where $i=1$ $(i=2)$. For later use we also find it convenient to define $\breve{q}=(k,\chi,q^3)$, $s=\mathrm{sign}(QB)$, and

$$\sum\!\!\!\!\!\!\int_{\{\bar{q}_E\}} = \sum\!\!\!\!\!\!\int_{\bar{q}} 2\pi\delta(q^0-E). \tag{3.11}$$

It can be shown that the functions $\mathcal{F}_Q(x,\bar{q})$ satisfy the useful relations [56]

$$\begin{aligned}
&\mathcal{D}^0\mathcal{F}_Q(x,\bar{q}) = -iq^0\mathcal{F}_Q(x,\bar{q}),\\
&(\mathcal{D}^1 \pm i\mathcal{D}^2)\mathcal{F}_Q(x,\bar{q})\\
&\quad = (\mp s)[(2k+1\mp s)B_Q]^{1/2}\mathcal{F}_Q(x,\bar{q}_{k\mp s}),\\
&\mathcal{D}^3\mathcal{F}_Q(x,\bar{q}) = -iq^3\mathcal{F}_Q(x,\bar{q}),
\end{aligned} \tag{3.12}$$

where $\bar{q}_{k\pm s}=(q^0,k\pm s,\chi,q^3)$. We also notice that under a gauge transformation $\Lambda(x)$ the functions $\mathcal{F}_Q(x,\bar{q})$ [with $\bar{q}=(q^0,k,\chi,q^3)$] transform as

$$\mathcal{F}_Q(x,\bar{q}) \to \tilde{\mathcal{F}}_Q(x,\bar{q}) = e^{-iQ\Lambda(x)}\mathcal{F}_Q(x,\bar{q}). \tag{3.13}$$

B. Spin 0 charged particles: The charged pions

Let us start by considering the gauged Klein-Gordon action for a pointlike charged pion in the presence of a static and homogeneous magnetic field. We have

$$\mathcal{S}_{\mathrm{KG}} = -\int d^4x\, \pi^{\mathcal{Q}}(x)^*(\mathcal{D}^\mu\mathcal{D}_\mu + m_\pi^2)\pi^{\mathcal{Q}}(x), \tag{3.14}$$

where, as in Sec. II C, we have denoted the pion charge by $Q_\pi = \mathcal{Q}e$, with $\mathcal{Q}=\pm1$. From the action in Eq. (3.14) one gets the associated gauged Klein-Gordon equation, namely

$$(\mathcal{D}^\mu\mathcal{D}_\mu + m_\pi^2)\pi^{\mathcal{Q}}(x) = 0. \tag{3.15}$$

We notice that, taking into account Eq. (2.4), the gauge invariance of the gauged Klein-Gordon action requires that under a gauge transformation $\Lambda(x)$ the $\pi^{\mathcal{Q}}(x)$ field transform as

$$\pi^{\mathcal{Q}}(x) \to \tilde{\pi}^{\mathcal{Q}}(x) = e^{-iQ_\pi\Lambda(x)}\pi^{\mathcal{Q}}(x). \tag{3.16}$$

Using the notation introduced in the previous subsection, the quantized charged pion field can be written as

$$\begin{aligned}
\pi^{\mathcal{Q}}(x) &= \pi^{-\mathcal{Q}}(x)^\dagger\\
&= \sum\!\!\!\!\!\!\int_{\{\bar{q}_{E_\pi}\}} \frac{1}{2E_\pi}\left\{a_\pi^{\mathcal{Q}}(\breve{q})\mathbb{F}^{\mathcal{Q}}(x,\bar{q}) + a_\pi^{-\mathcal{Q}}(\breve{q})^\dagger\mathbb{F}^{-\mathcal{Q}}(x,\bar{q})^*\right\}.
\end{aligned} \tag{3.17}$$

Here the pion energy is given by $E_\pi = \sqrt{m_\pi^2+(2k+1)B_\pi+(q^3)^2}$, with $k\geq 0$, while the functions $\mathbb{F}^{\mathcal{Q}}(x,\bar{q})$ are given by

$$\mathbb{F}^{\mathcal{Q}}(x,\bar{q}) = \mathcal{F}_{Q_\pi}(x,\bar{q}). \tag{3.18}$$

According to Eqs. (3.7) and (3.8), they satisfy the relations

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^{\mathcal{Q}}(x,\bar{q})^*\mathbb{F}^{\mathcal{Q}}(y,\bar{q}) = \delta^{(4)}(x-y), \tag{3.19}$$

$$\int d^4x \mathbb{F}^{\mathcal{Q}}(x,\bar{q})^*\mathbb{F}^{\mathcal{Q}}(x,\bar{q}') = \hat{\delta}_{\bar{q}\bar{q}'}. \tag{3.20}$$

On the other hand, the creation and annihilation operators in Eq. (3.17) satisfy the commutation relations

$$\begin{aligned}
[a_\pi^{\mathcal{Q}}(\breve{q}), a_\pi^{\pm\mathcal{Q}}(\breve{q}')] &= [a_\pi^{\mathcal{Q}}(\breve{q})^\dagger, a_\pi^{\pm\mathcal{Q}}(\breve{q}')^\dagger]\\
&= [a_\pi^{\mathcal{Q}}(\breve{q}), a_\pi^{-\mathcal{Q}}(\breve{q}')^\dagger] = 0,\\
[a_\pi^{\mathcal{Q}}(\breve{q}), a_\pi^{\mathcal{Q}}(\breve{q}')^\dagger] &= 2E_\pi(2\pi)^3\delta_{kk'}\delta_{\chi\chi'}\delta(q^3-q'^3).
\end{aligned} \tag{3.21}$$

Note that according to the above definitions the operators $a_\pi^{\mathcal{Q}}(\breve{q})$ and $a_\pi^{-\mathcal{Q}}(\breve{q})$ turn out to have different dimensions from the creation and annihilation operators that are usually defined in absence of the external magnetic field.

C. Spin 1/2 charged particles: The quarks

Let us consider the gauged Dirac action for a pointlike quark of flavor f in the presence of a static and homogeneous magnetic field. We express the quark charge as $Q_f = \mathcal{Q}_f e$, with $\mathcal{Q}_u = 2/3$, $\mathcal{Q}_d = -1/3$ for $f = u, d$. The gauged action is given by

$$\mathcal{S}_D = \int d^4x\, \bar{\psi}_f(x)(i\not{D} - m_f)\psi_f(x), \tag{3.22}$$

where, as usual, $\bar{\psi}_f = \psi_f^\dagger\gamma^0$ and $\not{D} = \gamma_\mu \mathcal{D}^\mu$; the associated gauged Dirac equation reads as

$$(i\not{D} - m_f)\psi_f(x) = 0. \tag{3.23}$$

In a similar way as in the case of the charged pion, gauge invariance of the gauged Dirac action requires that under a gauge transformation $\Lambda(x)$ the field $\psi_f(x)$ transforms as

$$\psi_f(x) \to \tilde{\psi}_f(x) = e^{-iQ_f\Lambda(x)}\psi_f(x). \tag{3.24}$$

The quantized quark fields are given by

$$\psi_f(x) = \sum\!\!\!\!\!\!\int_{\{\bar{q}_{E_f}\}} \sum_{a=1,2} \frac{1}{2E_f}\{b_f(\check{q},a)U_f(x,\bar{q},a) + d_f(\check{q},a)^\dagger V_f(x,\bar{q},a)\}, \tag{3.25}$$

where the quark energy is given by $E_f = \sqrt{m_f^2 + 2kB_f + (q^3)^2}$, with $k \geq 0$; for $k = 0$, only the value $a = 1$ in the sum over a is allowed. Once again we use here the definitions $\bar{q} = (q^0, k, \chi, q^3)$, $\check{q} = (k, \chi, q^3)$, $B_f = |Q_f B|$, and $s = \mathrm{sign}(Q_f B)$. The spinors U_f and V_f in Eq. (3.25) can be written as

$$\begin{aligned} U_f(x,\bar{q},a) &= \mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})u_{\mathcal{Q}_f}(k,q^3,a), \\ V_f(x,\bar{q},a) &= \tilde{\mathbb{E}}^{-\mathcal{Q}_f}(x,\bar{q})v_{-\mathcal{Q}_f}(k,q^3,a), \end{aligned} \tag{3.26}$$

where $\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})$ and $\tilde{\mathbb{E}}^{-\mathcal{Q}_f}(x,\bar{q})$ are Ritus functions [57]. Their explicit forms are

$$\begin{aligned} \mathbb{E}^{\mathcal{Q}}(x,\bar{q}) &= \sum_{\lambda=\pm}\Gamma^\lambda \mathcal{F}_Q(x,\bar{q}_\lambda), \\ \tilde{\mathbb{E}}^{-\mathcal{Q}}(x,\bar{q}) &= \sum_{\lambda=\pm}\Gamma^\lambda \mathcal{F}_{-Q}(x,\bar{q}_{-\lambda})^*, \end{aligned} \tag{3.27}$$

with $\Gamma^\pm = (1 \pm S^3)/2$, $S^3 = i\gamma^1\gamma^2$ being the three component of the spin operator in the spin one half representation. We have also used the definition $\bar{q}_\lambda = (q^0, k_{s\lambda}, \chi, q^3)$, with $k_{s\pm} = k - (1 \mp s)/2$. The explicit form of the spinors $u_{\mathcal{Q}_f}(k,q^3,a)$ and $v_{-\mathcal{Q}_f}(k,q^3,a)$ in Eq. (3.26), as well as the anticommutation relations between the fermion creation and annihilation operators and some properties of the functions $\mathbb{E}^{\mathcal{Q}}(x,\bar{q})$ are given in Appendix B. Using these properties it is easy to show that the spinors U_f and V_f satisfy orthogonality and completeness relations, namely [56]

$$\begin{aligned} &\int d^4x\, \bar{U}_f(x,\bar{q},a)U_f(x,\bar{q}',a') = 2m_f\hat{\delta}_{\bar{q}\bar{q}'}\delta_{aa'}, \\ &\int d^4x\, \bar{V}_f(x,\bar{q},a)V_f(x,\bar{q}',a') = -2m_f\hat{\delta}_{\bar{q}\bar{q}'}\delta_{aa'}, \\ &\int d^4x\, \bar{V}_f(-x,\bar{q},a)U_f(x,\bar{q}',a') = \int d^4x\, \bar{U}_f(x,\bar{q},a)V_f(-x,\bar{q}',a') = 0 \end{aligned} \tag{3.28}$$

and

$$\frac{1}{2m_f}\sum\!\!\!\!\!\!\int_{\bar{q}}\sum_a [U_f(x,\bar{q},a)\bar{U}(x',\bar{q},a) - V_f(-x,\bar{q},a)\bar{V}_f(-x',\bar{q},a)] = \delta^{(4)}(x-x'). \tag{3.29}$$

On the right-hand side of this last equation, an identity in Dirac space is understood.

D. Spin 1 charged particles: The charged rho mesons

We consider here the gauged Proca action for a charged pointlike rho meson in the presence of a static and homogeneous magnetic field. Expressing the rho charge as $Q_\rho = \mathcal{Q}e$, with $\mathcal{Q} = \pm 1$, we have

$$\mathcal{S}_P = \int d^4x\left\{-\frac{1}{2}\rho^{\mathcal{Q},\mu\nu}(x)^\dagger\rho^{\mathcal{Q}}_{\mu\nu}(x) + m_\rho^2\rho^{\mathcal{Q},\mu}(x)^\dagger\rho^{\mathcal{Q}}_\mu(x) + \frac{i}{2}Q_\rho F^{\mu\nu}[\rho^{\mathcal{Q}}_\mu(x)^\dagger\rho^{\mathcal{Q}}_\nu(x) - \rho^{\mathcal{Q}}_\nu(x)^\dagger\rho^{\mathcal{Q}}_\mu(x)]\right\}, \tag{3.30}$$

where $\rho^{\mathcal{Q}}_{\mu\nu} = \mathcal{D}_\mu \rho^{\mathcal{Q}}_\nu - \mathcal{D}_\nu \rho^{\mathcal{Q}}_\mu$. The associated gauged Proca equation reads as

$$\mathcal{D}^\mu \mathcal{D}_\mu \rho^{\mathcal{Q}}_\nu(x) + m_\rho^2 \rho^{\mathcal{Q}}_\nu(x) - 2iQ_\rho F_\nu{}^\alpha \rho^{\mathcal{Q}}_\alpha(x) = 0, \tag{3.31}$$

with

$$\mathcal{D}^\mu \rho^{\mathcal{Q}}_\mu(x) = 0. \tag{3.32}$$

In the same way as in the previous cases, gauge invariance of the gauged Proca action requires that under a gauge transformation $\Lambda(x)$ the rho field transforms as

$$\rho^{\mathcal{Q}}_\mu(x) \to \tilde{\rho}^{\mathcal{Q}}_\mu(x) = e^{-iQ_\rho \Lambda(x)} \rho^{\mathcal{Q}}_\mu(x). \tag{3.33}$$

The quantized charged rho field can be written as

$$\rho^{\mathcal{Q},\mu}(x) = \sum\!\!\!\!\!\!\int_{\{\bar{q}_{E_\rho}\}} \sum_c \frac{1}{2E_\rho} \Big[a^{\mathcal{Q}}_\rho(\breve{q}, c) W^\mu_{\mathcal{Q}}(x, \bar{q}, c) + a^{-\mathcal{Q}}_\rho(\breve{q}, c)^\dagger W^\mu_{-\mathcal{Q}}(x, \bar{q}, c)^* \Big], \tag{3.34}$$

where the rho energy is given by $E_\rho = \sqrt{m_\rho^2 + (2k+1)B_\rho + (q^3)^2}$, and we have used the definitions of $\bar{q}$ and $\breve{q}$ introduced in the previous subsections, together with $B_\rho = |Q_\rho B|$ and $s = \text{sign}(Q_\rho B)$. It is important to point out that in this case the sum over the integer index k starts at $k = -1$, instead of zero.

The functions $W^\mu_{\mathcal{Q}}(x, \bar{q}, c)$ in Eq. (3.34) are given by

$$W^\mu_{\mathcal{Q}}(x, \bar{q}, c) = \mathbb{R}^{\mathcal{Q},\mu\nu}(x, \bar{q}) \epsilon_{\mathcal{Q},\nu}(k, q^3, c), \tag{3.35}$$

where, as in the case of spin $1/2$ fields [see Eqs. (3.26)], we have separated the wave function into a function $\mathbb{R}^{\mathcal{Q},\mu\nu}$ that depends on x and $\bar{q}$ and a polarization vector $\epsilon_{\mathcal{Q},\nu}(k, q^3, c)$, the index c denoting the polarization state. Explicit expressions for these vectors—dictated by the orthogonality relation Eq. (3.32)—are given in Appendix C. In fact, we note that for $k = -1$ there is only one possible polarization vector; this means that the index c can only take the value $c = 1$ in this case. For $k = 0$ two polarization vectors can be constructed; thus in that case c can take values of 1 and 2, while for $k \geq 1$ the sum over c in Eq. (3.34) runs over the full set of values $c = 1, 2, 3$.

The functions $\mathbb{R}^{\mathcal{Q},\mu\nu}$ are given by

$$\mathbb{R}^{\mathcal{Q},\mu\nu}(x, \bar{q}) = \sum_{\lambda=-1,0,1} \mathcal{F}_{Q_\rho}(x, \bar{q}_\lambda) \Upsilon^{\mu\nu}_\lambda, \tag{3.36}$$

where $\bar{q}_\lambda = (q^0, k - s\lambda, \chi, q^3)$ (notice that this definition of $\bar{q}_\lambda$ is different from the one used in the case of charged fermions). There are various possible choices for the tensors $\Upsilon^{\mu\nu}_\lambda$; here we use

$$\Upsilon^{\mu\nu}_0 = g^{\mu\nu}_\parallel, \qquad \Upsilon^{\mu\nu}_{\pm 1} = \frac{1}{2}(g^{\mu\nu}_\perp \mp S^{\mu\nu}_3), \tag{3.37}$$

where $S^{\mu\nu}_3 = i(\delta^\mu{}_1 \delta^\nu{}_2 - \delta^\mu{}_2 \delta^\nu{}_1)$ is the three component of the spin operator in the spin one representation. Orthogonality and completeness relations for the functions $\mathbb{R}^{\mathcal{Q},\mu\nu}$, as well as other useful relations involving these functions and the $\Upsilon^{\mu\nu}_\lambda$ tensors, are given in Appendix C. In that Appendix we also quote the commutation relations between the creation and annihilation operators for the charged rho fields.

As discussed in Appendix C, for $k \geq 0$ an additional vector, orthogonal to the physical polarization vectors, can be introduced [see Eq. (C18)]. We keep for this new vector the notation $\epsilon^\mu_{\mathcal{Q}}(k, q^3, c)$, taking for the polarization index the value $c = 0$, and we refer to the associated polarization as "longitudinal." If we extend the set of charged rho meson wave functions $W^\mu_{\mathcal{Q}}(x, \bar{q}, c)$ by including the corresponding "longitudinal" wave function $W^\mu_{\mathcal{Q}}(x, \bar{q}, 0) \equiv \mathbb{R}^{\mathcal{Q},\mu\nu}(x, \bar{q}) \epsilon_{\mathcal{Q},\nu}(k, q^3, 0)$, we get for these functions orthogonality and completeness relations, namely

$$\int d^4x W^\mu_{\mathcal{Q}}(x, \bar{q}', c')^* W_{\mathcal{Q},\mu}(x, \bar{q}, c) = -\zeta_c \hat{\delta}_{\bar{q}\bar{q}'} \delta_{cc'} \tag{3.38}$$

and

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \sum_{c=c_{\min}}^{c_{\max}} \zeta_c W^\mu_{\mathcal{Q}}(x, \bar{q}, c)^* W^\nu_{\mathcal{Q}}(x', \bar{q}, c) = -g^{\mu\nu} \delta^{(4)}(x - x'). \tag{3.39}$$

In these equations the coefficients ζ_c are defined as $\zeta_0 = -1$, $\zeta_1 = \zeta_2 = \zeta_3 = 1$, while $c_{\min}$ and $c_{\max}$ are given by

$$c_{\min} = \begin{cases} 1 & \text{if } k = -1 \\ 0 & \text{if } k \geq 0 \end{cases}, \qquad c_{\max} = \begin{cases} 1 & \text{if } k = -1 \\ 2 & \text{if } k = 0 \\ 3 & \text{if } k \geq 1 \end{cases}. \tag{3.40}$$

IV. EXPLICIT FORM OF THE CHARGED PARTICLE PROPAGATORS

A. The spin 0 charged particle propagator

As discussed above, the charged pion propagator $\Delta_{\pi^{\mathcal{Q}}}(x, y)$ satisfies Eq. (2.19), and its behavior under a gauge transformation is given by Eq. (2.21). Using the functions $\mathbb{F}^{\mathcal{Q}}(x, \bar{q})$ defined in Eq. (3.18) and the properties of the functions $\mathcal{F}_Q(x, \bar{q})$ discussed in Sec. III A, it can be easily seen that $\Delta_{\pi^{\mathcal{Q}}}(x, y)$ can be expressed as

$$\Delta_{\pi^{\mathcal{Q}}}(x, y) = \sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^{\mathcal{Q}}(x, \bar{q}) \hat{\Delta}_{\pi^{\mathcal{Q}}}(k, q_\parallel) \mathbb{F}^{\mathcal{Q}}(y, \bar{q})^*, \tag{4.1}$$

with

$$\hat{\Delta}_{\pi^{\mathcal{Q}}}(k, q_\parallel) = \frac{1}{q_\parallel^2 - m_\pi^2 - (2k+1)B_\pi + i\epsilon}. \tag{4.2}$$

In fact, from this expression of the propagator, taking into account the gauge transformation properties of the functions $\mathcal{F}_Q(x,\bar{q})$ in Eq. (3.13), it is immediately seen that it transforms in the covariant way given by Eq. (2.21).

In addition, as is well known, an alternative form for the charged pion propagator can be obtained using the Schwinger proper time method. If $\Delta_{\pi^Q}(x,y)$ is written as in Eqs. (2.22) and (2.32), i.e.,

$$\Delta_{\pi^Q}(x,y) = e^{i\Phi_{\pi^Q}(x,y)} \int \frac{d^4v}{(2\pi)^4} e^{-iv(x-y)} \bar{\Delta}_{\pi^Q}(v_\parallel, v_\perp), \quad (4.3)$$

one gets

$$\bar{\Delta}_{\pi^Q}(v_\parallel, v_\perp) = -i\int_0^\infty d\sigma \frac{1}{\cos(\sigma B_\pi)} \exp\left[-i\sigma\left(m_\pi^2 - v_\parallel^2 + \vec{v}_\perp^2 \frac{\tan(\sigma B_\pi)}{\sigma B_\pi} - i\epsilon\right)\right]. \quad (4.4)$$

This expression can also be obtained starting from Eq. (4.1), as shown, e.g., in Appendix D of Ref. [3]. Notice that, as expected from the discussion in Sec. II C, $\bar{\Delta}_{\pi^Q}(v_\parallel, v_\perp)$ depends only on $v_\parallel^2$ and $v_\perp^2$.

B. The spin 1/2 charged particle propagator

The spin $1/2$ charged particle propagator $S_f(x,y)$ satisfies the equation

$$(i\not{D} - m_f)S_f(x,y) = \delta^{(4)}(x-y). \quad (4.5)$$

In terms of the Ritus functions in Eq. (3.27), it can be expressed as

$$S_f(x,y) = \sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathbb{E}^{Q_f}(x,\bar{q}) \hat{S}_f(k,q_\parallel) \bar{\mathbb{E}}^{Q_f}(y,\bar{q}), \quad (4.6)$$

where

$$\hat{S}_f(k,q_\parallel) = \frac{\hat{\not{\Pi}}_s + m_f}{q_\parallel^2 - m_f^2 - 2kB_f + i\epsilon}, \quad (4.7)$$

with the definitions $\hat{\Pi}_s^\mu = (q^0, 0, -s\sqrt{2kB_f}, q^3)$ and $\bar{\mathbb{E}}^{Q_f}(y,\bar{q}) = \gamma^0 \mathbb{E}^{Q_f}(y,\bar{q})^\dagger \gamma^0$. The above expression can be obtained using the relations in Appendix B. Moreover, taking into account the gauge transformation properties of these functions [see Eq. (3.13)], from Eq. (4.6) it is easy to see that under a gauge transformation $\Lambda(x)$ the propagator transforms, as it should, in the covariant way

$$S_f(x,y) \to \tilde{S}_f(x,y) = e^{-iQ_f\Lambda(x)} S_f(x,y) e^{iQ_f\Lambda(y)}. \quad (4.8)$$

As in the case of spin 0 particles, an alternative form of this propagator can be obtained using the Schwinger proper time method. If $S_f(x,y)$ is written as in Eqs. (2.40) and (2.41), i.e.,

$$S_f(x,y) = e^{i\Phi_f(x,y)} \int \frac{d^4v}{(2\pi)^4} e^{-iv(x-y)} \bar{S}_f(v_\parallel, v_\perp), \quad (4.9)$$

one gets

$$\bar{S}_f(v_\parallel, v_\perp) = -i\int_0^\infty d\sigma \exp\left[-i\sigma\left(m_f^2 - v_\parallel^2 + \vec{v}_\perp^2 \frac{\tan(\sigma B_f)}{\sigma B_f} - i\epsilon\right)\right] \times \left[(v_\parallel \cdot \gamma_\parallel + m_f)(1 - s\gamma^1\gamma^2 \tan(\sigma B_f)) - \frac{\vec{v}_\perp \cdot \vec{\gamma}_\perp}{\cos^2(\sigma B_f)}\right]. \quad (4.10)$$

The derivation of this expression starting from Eq. (4.6) can be found, e.g., in Ref. [58]. We note that $\bar{S}_f(p_\parallel, p_\perp)$ satisfies the constraints imposed by the invariance under rotations around the axis 3 (i.e., the $\vec{B}$ axis) and under boosts in that direction discussed in Sec. II C.

C. The spin 1 charged particle propagator

The spin 1 charged particle propagator $D^{\nu\gamma}_{\rho^Q}(x,y)$ satisfies the equation

$$[(\mathcal{D}^\alpha\mathcal{D}_\alpha + m_\rho^2)g_{\mu\nu} - \mathcal{D}_\mu\mathcal{D}_\nu + 2iQ_\rho F_{\mu\nu}]D^{\nu\gamma}_{\rho^Q}(x,y) = \delta_\mu{}^\gamma \delta^{(4)}(x-y); \quad (4.11)$$

it can be expressed as

$$D^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x,y) = \sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{R}^{\mathcal{Q},\nu\alpha}(x,\bar{q})\hat{D}_{\rho^{\mathcal{Q}},\alpha\beta}(k,q_\parallel)\mathbb{R}^{\mathcal{Q},\gamma\beta}(y,\bar{q})^*, \tag{4.12}$$

with

$$\hat{D}_{\rho^{\mathcal{Q}},\alpha\beta}(k,q_\parallel)) = \frac{-g_{\alpha\beta}+(1-\delta_{k,-1})\Pi_\alpha(k,q_\parallel)\Pi_\beta(k,q_\parallel)^*/m_\rho^2}{q_\parallel^2 - m^2 - (2k+1)B_\rho + i\epsilon}. \tag{4.13}$$

Here we have introduced the four-vector $\Pi^\mu(k,q_\parallel)$, given by

$$\Pi^\mu(k,q_\parallel) = \left(q^0, i\sqrt{B_\rho/2}\left(\sqrt{k+1}-\sqrt{k}\right), -s\sqrt{B_\rho/2}\left(\sqrt{k+1}+\sqrt{k}\right), q^3\right). \tag{4.14}$$

This vector, which is defined only for $k \geq 0$, plays in some cases a role equivalent to the one played by the four-momentum vector for $B=0$. It is easy to see that

$$\Pi_\mu(k,q_\parallel)^*\Pi^\mu(k,q_\parallel) = (q^0)^2 - (2k+1)B_Q - (q^3)^2, \tag{4.15}$$

which is equal to m_ρ^2 for $q^0 = E_\rho$.

Taking into account the properties of $\mathbb{R}^{\mathcal{Q},\mu\nu}$ functions quoted in Appendix C, it can be shown that $D^{\nu\gamma}_{\rho^{\mathcal{Q}}}(x,y)$, expressed as in Eq. (4.12), satisfies Eq. (4.11). Moreover, using the gauge transformation properties of the functions $\mathcal{F}_Q(x,\bar{q})$ [see Eq. (3.13)] it is easy to see that, as in the case of spin 0 and spin 1/2 particles, the propagator transforms in a covariant way under a gauge transformation.

As in the previous cases, an alternative form for the charged ρ meson propagator can be obtained using the Schwinger proper time method. If $D^{\mu\nu}_{\rho^{\mathcal{Q}}}(x,y)$ is written as in Eqs. (2.42) and (2.43), i.e.,

$$D^{\mu\nu}_{\rho^{\mathcal{Q}}}(x,y) = e^{i\Phi_{\rho^{\mathcal{Q}}}(x,y)}\int \frac{d^4v}{(2\pi)^4} e^{-iv(x-y)}\bar{D}^{\mu\nu}_{\rho^{\mathcal{Q}}}(v_\parallel, v_\perp), \tag{4.16}$$

one gets

$$\begin{aligned}\bar{D}^{\mu\nu}_{\rho^{\mathcal{Q}}}(v_\parallel,v_\perp) = i\int_0^\infty \frac{d\sigma}{\cos(\sigma B_\rho)}\exp\left[-i\sigma\left(m_\rho^2 - v_\parallel^2 + \vec{v}_\perp^{\,2}\frac{\tan(\sigma B_\rho)}{\sigma B_\rho} - i\epsilon\right)\right]\\ \times\left\{\mathbb{O}_1^{\mu\nu}(v) - \left[2\sin^2(\sigma B_\rho) - 1 + \frac{\vec{v}_\perp^{\,2}}{m_\rho^2}\tan^2(\sigma B_\rho) + i\frac{B_\rho}{2m_\rho^2}\tan(\sigma B_\rho)\right]\mathbb{O}_2^{\mu\nu}(v)\right.\\ -\frac{1}{m_\rho^2}\left[\mathbb{O}_3^{\mu\nu}(v) + [1+\tan^2(\sigma B_\rho)]\mathbb{O}_4^{\mu\nu}(v) + \mathbb{O}_5^{\mu\nu}(v)\right]\\ \left.+ i\left[\sin(2\sigma B_\rho) + \frac{\vec{v}_\perp^{\,2}}{m_\rho^2}\tan(\sigma B_\rho) + \frac{iB_\rho}{2m_\rho^2}\right]\mathbb{O}_6^{\mu\nu}(v) + \frac{i\tan(\sigma B_\rho)}{m_\rho^2}\mathbb{O}_7^{\mu\nu}(v)\right\},\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}&\mathbb{O}_1^{\mu\nu}(v) = g_\parallel^{\mu\nu}, \qquad \mathbb{O}_2^{\mu\nu}(v) = g_\perp^{\mu\nu}, \qquad \mathbb{O}_3^{\mu\nu}(v) = v_\parallel^\mu v_\parallel^{\nu*},\\ &\mathbb{O}_4^{\mu\nu}(v) = v_\perp^\mu v_\perp^{\nu}{}^*, \qquad \mathbb{O}_5^{\mu\nu}(v) = v_\perp^\mu v_\parallel^{\nu*} + v_\parallel^\mu v_\perp^{\nu}{}^*, \qquad \mathbb{O}_6^{\mu\nu}(v) = -iQ_\rho F^{\mu\nu}/B_\rho,\\ &\mathbb{O}_7^{\mu\nu}(v) = iQ_\rho[F^{\mu\alpha}v_{\perp\alpha}v_\parallel^{\nu*} + v_\parallel^\mu v_{\perp\alpha}{}^* F^{\alpha\nu}]/B_\rho.\end{aligned} \tag{4.18}$$

This expression can be obtained from Eq. (4.11) using the same methods as in the previous cases. In fact, an equivalent result has been obtained for the W boson propagator in Ref. [59]. Once again, it is found that $\bar{D}^{\mu\nu}_{\rho^{\mathcal{Q}}}(p_\parallel, p_\perp)$ satisfies the constraints imposed by the invariance under rotations around the axis 3 (i.e., the $\vec{B}$ axis) and under boosts in that direction discussed in Sec. II C.

V. MESON-QUARK INTERACTIONS AND ONE-LOOP CORRECTIONS TO CHARGED MESON TWO-POINT CORRELATORS

In the previous section we have considered charged noninteracting boson and fermion fields in the presence of an external magnetic field; let us now analyze the situation in which these particles interact with each other. The type of interactions to be considered here are quite generic.

In fact, they can be found in several effective approaches for low energy strong interactions, such as, e.g., meson-quark models in hadronic physics and meson-nucleon models in nuclear physics. As simple but relevant issues, in this section we discuss the one-loop corrections to the charged pion and rho meson two-point correlators. It is worth mentioning that in the presence of an external magnetic field these mesons turn out to get mixed. Since in this paper we are mainly concerned about the role played by the Schwinger phase in this type of calculation, these mixing terms will be neglected; i.e., the corrections to spin 0 and spin 1 meson self-energies will be treated separately.

A. Pion-quark interactions and one-loop correction to the charged pion two-point correlator

Let us consider the quark-pion interaction Lagrangian

$$\mathcal{L}_{\rm int}^{(\pi q)} = g_s \bar\psi(x) i\gamma_5 \vec\tau \psi(x)\vec\pi(x). \tag{5.1}$$

Here, $\psi(x)$ stands for a fermion field doublet; for definiteness, we take it to be

$$\psi(x) = \begin{pmatrix}\psi_u(x)\\ \psi_d(x)\end{pmatrix}, \tag{5.2}$$

where the fields $\psi_f(x)$, with $f = u, d$, are associated to u and d quarks. Using the same notation as in previous sections, we have $Q_u = 2e/3$, $Q_d = -e/3$, e being the proton charge. As usual, τ_i are the Pauli matrices, and pion charge and isospin states are related by $\pi^\pm = (\pi_1 \mp i\pi_2)/\sqrt{2}$, $\pi^0 = \pi_3$. The gauge transformation properties for charged fields given in the previous section guarantee that the interaction Lagrangian in Eq. (5.1) is gauge invariant.

We analyze now the leading order correction (LOC) to the two-point π^+ correlator. One has

$$i\Delta_{\pi^+}^{(\rm LOC)}(y,y') = \frac{i^2}{2}\int d^4x\, d^4x' \langle 0|T[\pi^+(y)\pi^+(y')^\dagger \mathcal{L}_{\rm int}^{(\pi q)}(x)\mathcal{L}_{\rm int}^{(\pi q)}(x')]|0\rangle, \tag{5.3}$$

where the contributions that lead to vacuum-vacuum subdiagrams have been omitted [60]. Considering the relevant terms in $\mathcal{L}_{\rm int}^{(\pi q)}$ we have

$$\Delta_{\pi^+}^{(\rm LOC)}(y,y') = -ig_s^2\int d^4x\, d^4x'\,\Delta_{\pi^+}(y,x)J_{\pi^+}(x,x')\Delta_{\pi^+}(x',y'), \tag{5.4}$$

where $J_{\pi^+}(x,x')$ is the polarization function in coordinate space,

$$J_{\pi^+}(x,x') = -2N_c {\rm tr}_D[iS_u(x,x')i\gamma_5 iS_d(x',x)i\gamma_5], \tag{5.5}$$

${\rm tr}_D$ denoting trace in Dirac space.

We also introduce the π^+ polarization function in $\bar q$ space (or Ritus space), $J_{\pi^+}(\bar q,\bar q')$, defined by

$$J_{\pi^+}(\bar q,\bar q') = \int d^4x\, d^4x'\,\mathbb{F}^+(x,\bar q)^* J_{\pi^+}(x,x')\mathbb{F}^+(x',\bar q'). \tag{5.6}$$

This equation can be inverted using the completeness relation for the functions $\mathcal{F}_Q(x,\bar q)$ [Eq. (3.19)], obtaining

$$J_{\pi^+}(x,x') = \sum\!\!\!\!\!\!\!\int_{\bar q,\bar q'} \mathbb{F}^+(x,\bar q) J_{\pi^+}(\bar q,\bar q')\mathbb{F}^+(x',\bar q')^*. \tag{5.7}$$

From this last relation, together with Eq. (4.1) and the orthogonality relation Eq. (3.20), the leading order correction to the π^+ propagator can be written as

$$\Delta_{\pi^+}^{(\rm LOC)}(y,y') = -ig_s^2 \sum\!\!\!\!\!\!\!\int_{\bar q,\bar q'} \mathbb{F}^+(y,\bar q)\hat\Delta_{\pi^+}(k,q_\parallel)J_{\pi^+}(\bar q,\bar q') \times \hat\Delta_{\pi^+}(k',q'_\parallel)\mathbb{F}^+(y',\bar q')^*, \tag{5.8}$$

where $\hat\Delta_{\pi^+}(k,q_\parallel)$ is given by Eq. (4.2).

Note that none of the functions appearing in the rhs of Eq. (5.6) is by itself an invariant quantity. However, being $\mathcal{L}_{\rm int}^{(\pi q)}(x)$ gauge invariant, so must be $J_{\pi^+}(\bar q,\bar q')$. In fact, on the basis of gauge, translational, and rotational symmetries, we expect $J_{\pi^+}(\bar q,\bar q')$ to be of the form $J_{\pi^+}(\bar q,\bar q') = \hat\delta_{\bar q\bar q'}\hat J_{\pi^+}(k,q_\parallel)$. In the following we will see how this comes out by explicit calculation.

We start by considering Eq. (5.5), writing the quark propagators in the form given by Eqs. (2.40) and (2.41). In this way we get

$$J_{\pi^+}(x,x') = e^{i\Phi_{\pi^+}(x,x')}\bar J_{\pi^+}(x-x'), \tag{5.9}$$

where

$$\bar J_{\pi^+}(x-x') = \int \frac{d^4v}{(2\pi)^4} e^{-iv(x-x')}\bar J_{\pi^+}(v_\parallel,v_\perp), \tag{5.10}$$

with

$$\bar J_{\pi^+}(v_\parallel,v_\perp) = -2N_c\int\frac{d^4p}{(2\pi)^4}{\rm tr}_D[i\bar S^u(p_\parallel^+,p_\perp^+)i\gamma_5 i\bar S^d(p_\parallel^-,p_\perp^-)i\gamma_5]. \tag{5.11}$$

Here we have used the definition $p_\mu^\pm = p_\mu \pm v_\mu/2$. In addition, in Eq. (5.9) we have made use of the relation

$$\Phi^{Q_u}(x,x') + \Phi^{Q_d}(x',x) = \Phi^{Q_u - Q_d}(x,x') = \Phi^{Q_{\pi^+}}(x,x'). \tag{5.12}$$

We see from the above equations that $J_{\pi^+}(x,x')$ can be written as the product of a SP and a function $\bar{J}_{\pi^+}(x-x')$, which is both gauge and translational invariant. Thus, under a gauge transformation the polarization function transforms in the same way as the SP. On the other hand, as in the case of a bare charged pion propagator, invariance under rotations around the axis 3 (i.e., the $\vec{B}$ axis) and under boosts in that direction imply that $\bar{J}_{\pi^+}(v_\parallel, v_\perp)$ has to be a function of $v_\parallel^2$ and $v_\perp^2$; this is indeed corroborated by the explicit form given below.

Replacing Eq. (5.9) in Eq. (5.6), and taking into account Eq. (3.18), we get

$$J_{\pi^+}(\bar{q},\bar{q}') = \int \frac{d^4 v}{(2\pi)^4} h_{\pi^+}(\bar{q},\bar{q}',v_\parallel,v_\perp)\bar{J}_{\pi^+}(v_\parallel,v_\perp), \tag{5.13}$$

where

$$h_{\rm P}(\bar{q},\bar{q}',v_\parallel,v_\perp) = \int d^4x d^4x' \mathcal{F}_{Q_{\rm P}}(x,\bar{q})^* \mathcal{F}_{Q_{\rm P}}(x',\bar{q}') e^{i\Phi_{\rm P}(x,x')} e^{-iv(x-x')}. \tag{5.14}$$

It is easy to see that $h_{\rm P}(\bar{q},\bar{q}',v_\parallel,v_\perp)$ is gauge invariant, given the gauge transformation properties of the SP and the functions $\mathcal{F}_{Q_{\rm P}}(x,\bar{q})$. One can carry out its explicit calculation in any of the standard gauges, SG, LG1, and LG2, obtaining

$$h_{\rm P}(\bar{q},\bar{q}',v_\parallel,v_\perp) = \delta_{\chi\chi'} \check{h}_{\rm P}(k,q_\parallel,k',q'_\parallel,v_\parallel,v_\perp). \tag{5.15}$$

Here $\delta_{\chi\chi'}$ stands for $\delta_{ll'}$, $\delta(q^1 - q'^1)$ and $\delta(q^2 - q'^2)$ for SG, LG1, and LG2, respectively, while the function $\check{h}_{\rm P}(k,q_\parallel,k',q'_\parallel,v_\parallel,v_\perp)$ is given by

$$\check{h}_{\rm P}(k,q_\parallel,k',q'_\parallel,v_\parallel,v_\perp) = (2\pi)^4 \delta^{(2)}(q_\parallel - q'_\parallel)(2\pi)^2 \delta^{(2)}(q_\parallel - v_\parallel) f_{kk'}(v_\perp), \tag{5.16}$$

with

$$f_{kk'}(v_\perp) = \frac{4\pi(-i)^{k+k'}}{B_P}\sqrt{\frac{k!}{k'!}}\left(\frac{2\vec{v}_\perp^2}{B_P}\right)^{\frac{k'-k}{2}} L_k^{k'-k}\left(\frac{2\vec{v}_\perp^2}{B_P}\right) \times e^{-\vec{v}_\perp^2/B_P} e^{is(k-k')\phi_\perp}. \tag{5.17}$$

We have used here the definition $B_P = |BQ_{\rm P}|$ and introduced the angle $\phi_\perp$, given by $\vec{v}_\perp = |\vec{v}_\perp|(\cos\phi_\perp, \sin\phi_\perp)$. Note that in the present case $B_P = B_\pi = e|B|$ and $s = {\rm sign}(B)$.

As stated, $\bar{J}_{\pi^+}(v_\parallel,v_\perp)$ is found to be a function of $\vec{v}_\perp^2$ [see Eq. (5.24) below]. Performing the integral over $\phi_\perp$, one arrives at the expected form

$$J_{\pi^+}(\bar{q},\bar{q}') = \hat{\delta}_{\bar{q}\bar{q}'} \hat{J}_{\pi^+}(k,q_\parallel), \tag{5.18}$$

where

$$\hat{J}_{\pi^+}(k,q_\parallel) = \int_0^\infty d|\vec{v}_\perp|^2 \bar{J}_{\pi^+}(q_\parallel,v_\perp)\rho_k(\vec{v}_\perp^2), \tag{5.19}$$

with

$$\rho_k(\vec{v}_\perp^2) = \frac{(-1)^k}{B_\pi} e^{-\vec{v}_\perp^2/B_\pi} L_k\left(\frac{2\vec{v}_\perp^2}{B_\pi}\right). \tag{5.20}$$

It is worth recalling that, due to the presence of the nonvanishing Schwinger phase $\Phi^{Q_{\pi^+}}(x,x')$, the polarization function $J_{\pi^+}(x,x')$ is not just a function of $x-x'$, and therefore it cannot be diagonalized by a Fourier transform into momentum space. Instead, one can obtain a diagonal, gauge invariant function $J_{\pi^+}(\bar{q},\bar{q}') = \hat{\delta}_{\bar{q}\bar{q}'}\hat{J}_{\pi^+}(k,q_\parallel)$ through the above described transformation into $\bar{q}$ space. As shown by Eq. (5.19), the Fourier transform $\bar{J}_{\pi^+}(q_\parallel,v_\perp)$ of the translational invariant part of $J_{\pi^+}(x,x')$ does not coincide with $\hat{J}_{\pi^+}(k,q_\parallel)$; in fact, the latter can be obtained from the integration of $\bar{J}_{\pi^+}(q_\parallel,v_\perp)$ over the perpendicular momentum $v_\perp$, weighted by the function $\rho_k(\vec{v}_\perp^2)$ given by Eq. (5.20). On the other hand, in the absence of the SP in Eq. (5.14) one could replace the functions $\mathcal{F}_{Q_{\rm P}}(x,\bar{q})$ by plane waves, and the function h_{π^+} in Eq. (5.13) would be simply given by $(2\pi)^8\delta^{(4)}(q-q')\delta^{(4)}(q-v)$; this is, indeed, what is done in the case of neutral mesons.

Given that $J_{\pi^+}(\bar{q},\bar{q}')$ is diagonal in $\bar{q}$ space, from Eq. (5.4) the LOC to the propagator can be written as

$$\Delta_{\pi^+}^{(\rm LOC)}(y,y') = \sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathbb{F}^+(y,\bar{q})\hat{\Delta}_{\pi^+}^{(\rm LOC)}(k,q_\parallel)\mathbb{F}^+(y',\bar{q})^*, \tag{5.21}$$

where

$$\hat{\Delta}_{\pi^+}^{(\rm LOC)}(k,q_\parallel) = \hat{\Delta}_{\pi^+}(k,q_\parallel)\hat{\Sigma}_{\pi^+}(k,q_\parallel)\hat{\Delta}_{\pi^+}(k,q_\parallel), \tag{5.22}$$

with

$$\hat{\Sigma}_{\pi^+}(k,q_\parallel) = -ig_s^2 \hat{J}_{\pi^+}(k,q_\parallel). \tag{5.23}$$

To get the final form of $\hat{J}_{\pi^+}(k,q_\parallel)$ from Eq. (5.19) we need the explicit expression of $\bar{J}_{\pi^+}(v_\parallel,v_\perp)$. The latter can be readily obtained from Eq. (5.11), taking into account the invariant part of the quark propagators given in Eq. (4.10). One has

$$\bar{J}_{\pi^+}(v_\parallel, v_\perp) = -\frac{iN_c}{4\pi^2}\int_{-1}^{1} dx \int_0^\infty \frac{dz}{t_+} e^{-z\phi(x,v_\parallel^2)} e^{-(t_+^2-t_-^2)\vec{v}_\perp^2/(4t_+)} \times \left\{ \left[m_u m_d + \frac{1}{z} + (1-x^2)\frac{v_\parallel^2}{4} \right](1-t_u t_d) + \left[\frac{1}{t_+} - \left(1 - \frac{t_-^2}{t_+^2}\right)\frac{\vec{v}_\perp^2}{4} \right](1-t_u^2)(1-t_d^2) \right\}, \tag{5.24}$$

where we have used the definition

$$\phi(x, v_\parallel^2) = (m_u^2 + m_d^2)/2 - x(m_u^2 - m_d^2)/2 - (1-x^2)v_\parallel^2/4, \tag{5.25}$$

as well as

$$\begin{aligned} t_u &= \tanh\left[(1-x)zB_u/2\right], \\ t_d &= \tanh\left[(1+x)zB_d/2\right], \\ t_\pm &= t_u/B_u \pm t_d/B_d. \end{aligned} \tag{5.26}$$

Replacing this expression in Eq. (5.19) and performing the integral over $v_\perp$ we finally obtain

$$\hat{J}_{\pi^+}(k, q_\parallel) = -\frac{iN_c}{4\pi^2}\int_{-1}^{1} dx \int_0^\infty dz\, e^{-z\phi(x,q_\parallel^2)} \frac{1}{\alpha_+}\left(\frac{\alpha_-}{\alpha_+}\right)^k \times \left\{ \left[m_u m_d + \frac{1}{z} + (1-x^2)\frac{q_\parallel^2}{4} \right](1-t_u t_d) + \frac{\alpha_- + k(\alpha_- - \alpha_+)}{\alpha_+\alpha_-}(1-t_u^2)(1-t_d^2) \right\}, \tag{5.27}$$

where we have defined $\alpha_\pm$ as

$$\alpha_\pm = \frac{t_u}{B_u} + \frac{t_d}{B_d} \pm B_\pi \frac{t_u}{B_u}\frac{t_d}{B_d} = t_+ \pm B_\pi \frac{t_u}{B_u}\frac{t_d}{B_d}. \tag{5.28}$$

The integral on the rhs of Eq. (5.27) is divergent, so it has to be regularized. This can be done, e.g., by subtracting the $B = 0$ contribution, leaving a finite B-dependent piece. In addition, an analytical extension of the function $\hat{J}_{\pi^+}(k, q_\parallel)$ can be performed for large positive values of $q_\parallel^2$.

We end this subsection by noting that $\Delta_{\pi^+}^{(\mathrm{LOC})}(y, y')$ can be expressed in an alternative way. By looking at Eqs. (2.22) and (2.31) for the bare propagator together with Eq. (5.9) for the polarization function, one can foresee that the translational noninvariance of the dressed propagator will be carried by Schwinger phases at any order of the calculation. On this basis, we explicitly separate the corresponding SP in the LOC to the propagator, writing

$$\Delta_{\pi^+}^{(\mathrm{LOC})}(y, y') = e^{i\Phi_{\pi^+}(y,y')}\bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y, y'). \tag{5.29}$$

Then we can use Eqs. (2.22), (2.31), (5.4), and (5.9) to obtain

$$\bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y, y') = -ig_s^2 \int d^4x\, d^4x'\, e^{i\varphi(y-x', x-y')} \times \bar{\Delta}_{\pi^+}(y-x)\bar{J}_{\pi^+}(x-x')\bar{\Delta}_{\pi^+}(x'-y'), \tag{5.30}$$

where

$$\begin{aligned} &\varphi(y-x', x-y') \\ &\quad = \Phi_{\pi^+}(y,x) + \Phi_{\pi^+}(x,x') + \Phi_{\pi^+}(x',y') + \Phi_{\pi^+}(y',y) \\ &\quad = \frac{Q_\pi}{2}(y_\mu - x'_\mu)F^{\mu\nu}(x_\nu - y'_\nu). \end{aligned} \tag{5.31}$$

It is worth noticing that the phase $\varphi(y-x', x-y')$ is in general nonvanishing for nonzero B. Moreover, it is invariant under gauge transformations, translations, rotations around the direction of $\vec{B}$, and boosts in that direction. This implies that once the SP has been extracted, the remaining factor $\bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y, y')$ should have all the associated invariance properties; in particular, one should be able to write $\bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y, y') = \bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y-y')$. Indeed, using the Fourier transforms defined in Eqs. (2.32) and (5.10), and changing variables x and x' to $z = y - x'$ and $z' = x - y'$ we obtain

$$\bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y, y') = -ig_s^2 \int \frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\frac{d^4t}{(2\pi)^4}\bar{\Delta}_{\pi^+}(r_\parallel, r_\perp)\bar{J}_{\pi^+}(s_\parallel, s_\perp)\bar{\Delta}_{\pi^+}(t_\parallel, t_\perp)e^{-i(r-s+t)(y-y')}\int d^4z\, d^4z'\, e^{i\varphi(z,z')}e^{i(r-s)z'}e^{i(t-s)z}; \tag{5.32}$$

thus, we can write

$$\bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y, y') = \bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(y-y') = \int \frac{d^4v}{(2\pi)^4} e^{-iv(y-y')}\bar{\Delta}_{\pi^+}^{(\mathrm{LOC})}(v_\parallel, v_\perp). \tag{5.33}$$

Now, noting that $\varphi(z, z')$ depends only on the perpendicular components of z and z', we get

$$\bar{\Delta}^{(\mathrm{LOC})}_{\pi^+}(v_\parallel, v_\perp) = -ig_s^2 \int \frac{d^2 r_\perp}{(2\pi)^2}\frac{d^2 t_\perp}{(2\pi)^2} \bar{\Delta}_{\pi^+}(v_\parallel, r_\perp)\bar{J}_{\pi^+}(v_\parallel, r_\perp + t_\perp - v_\perp)\bar{\Delta}_{\pi^+}(v_\parallel, t_\perp) \times \int d^2 z_\perp\, d^2 z'_\perp e^{i\varphi(z_\perp, z'_\perp)} e^{-i(\vec{v}_\perp - \vec{t}_\perp)\cdot \vec{z}'_\perp} e^{-i(\vec{v}_\perp - \vec{r}_\perp)\cdot \vec{z}_\perp}, \tag{5.34}$$

and finally we can perform the integrals over $z_\perp$ and $z'_\perp$ and make the change of variables

$$r_\perp = v_\perp - \sqrt{\frac{B_\pi}{2}} u_\perp, \qquad t_\perp = v_\perp - \sqrt{\frac{B_\pi}{2}} u'_\perp, \tag{5.35}$$

obtaining

$$\bar{\Delta}^{(\mathrm{LOC})}_{\pi^+}(v_\parallel, v_\perp) = -i\frac{g_s^2}{(2\pi)^2} \int d^2 u_\perp\, d^2 u'_\perp e^{i\varphi\left(\sqrt{\frac{2}{B_\pi}} u'_\perp, \sqrt{\frac{2}{B_\pi}} u_\perp\right)} \times \bar{\Delta}_{\pi^+}\left(v_\parallel, v_\perp - \sqrt{\frac{B_\pi}{2}} u_\perp\right) \bar{J}_{\pi^+}\left(v_\parallel, v_\perp - \sqrt{\frac{B_\pi}{2}}(u'_\perp + u_\perp)\right) \bar{\Delta}_{\pi^+}\left(v_\parallel, v_\perp - \sqrt{\frac{B_\pi}{2}} u'_\perp\right). \tag{5.36}$$

At this point we can remark on the important role played by the Schwinger phase, which is responsible for the existence of the phase $\varphi(\sqrt{\frac{2}{B_\pi}} u'_\perp, \sqrt{\frac{2}{B_\pi}} u_\perp)$ [see Eq. (5.31)]. As shown in Eq. (5.36), the latter drives the fluctuation of transverse momenta suffered by the charged particles when they propagate in the presence of the magnetic field. Were the phase $\varphi(z_\perp, z'_\perp)$ omitted in Eq. (5.34), one would directly obtain

$$\bar{\Delta}^{(\mathrm{LOC})}_{\pi^+}(v_\parallel, v_\perp) = -ig_s^2 \bar{\Delta}_{\pi^+}(v_\parallel, v_\perp)\bar{J}_{\pi^+}(v_\parallel, v_\perp)\bar{\Delta}_{\pi^+}(v_\parallel, v_\perp), \tag{5.37}$$

losing any transverse momentum fluctuation.

A final comment on the $B \to 0$ limit of Eq. (5.36) is pertinent. It is easy to see that in this limit the integral over $u_\perp$ and $u'_\perp$ only affects the phase $\varphi(\sqrt{\frac{2}{B_\pi}} u'_\perp, \sqrt{\frac{2}{B_\pi}} u_\perp)$. To regulate the oscillatory integrals one can introduce factors $e^{-\epsilon|u^i|}$, $e^{-\epsilon|u'^i|}$, with $i = 1, 2$, and then take $\epsilon \to 0^+$, obtaining

$$\lim_{B\to 0} \bar{\Delta}^{(\mathrm{LOC})}_{\pi^+}(v_\parallel, v_\perp) = -ig_s^2 \left[\lim_{B\to 0} \bar{\Delta}_{\pi^+}(v_\parallel, v_\perp)\right]\left[\lim_{B\to 0} \bar{J}_{\pi^+}(v_\parallel, v_\perp)\right]\left[\lim_{B\to 0} \bar{\Delta}_{\pi^+}(v_\parallel, v_\perp)\right]. \tag{5.38}$$

This is the expected result. If there is no magnetic field, there is no fluctuation of transverse momenta; this is a consequence of translation invariance, which implies the conservation of the four components of the momentum. On the contrary, in the presence of a static and uniform magnetic field the situation is different. As we have seen in Sec. II, in that case translation invariance in the transverse direction is realized in a nontrivial way, being related to gauge transformations. In addition, in Sec. III we have seen that the wave functions associated to charged particles depend on the chosen gauge, and cannot be written in terms of definite four-momentum states. In fact, this leads to a fluctuation in the transverse spatial directions that is translated into a fluctuation in the transverse directions of the momentum. Our result in Eq. (5.36) shows how these fluctuations affect the evaluation of the LOC to the pion propagator, in particular, the part of the propagator that is invariant under gauge transformations, translations, rotations around the $\vec{B}$ axis, and boosts in the spatial direction parallel to the magnetic field.

On the basis of charge conservation, it is not difficult to realize that the appearance of a SP as in Eq. (5.29) will be valid at any order of correction, and, therefore, it also applies to the full propagator.

B. Rho meson-quark interactions and one-loop correction to the charged rho meson two-point correlator

Let us consider the rho meson-quark interaction Lagrangian

$$\mathcal{L}_{\rm int}^{(\rho q)} = g_v \vec{\rho}_\mu(x)\bar{\psi}(x)\gamma^\mu\vec{\tau}\psi(x). \tag{5.39}$$

As usual, ρ charge states are related to isospin states by $\rho^\pm_\mu = (\rho_{1,\mu} \mp i\rho_{2,\mu})/\sqrt{2}$ and $\rho^0_\mu = \rho_{3,\mu}$.

The leading order correction to the two-point ρ^+ correlator is given by

$$iD^{(\rm LOC)}_{\rho^+,\mu\nu}(y,y') = \frac{i^2}{2}\int d^4x\, d^4x' \langle 0|T[\rho^+_\mu(y)\rho^+_\nu(y')^\dagger \mathcal{L}_{\rm int}^{(\rho q)}(x)\mathcal{L}_{\rm int}^{(\rho q)}(x')]|0\rangle. \tag{5.40}$$

Considering the relevant terms in $\mathcal{L}_{\rm int}^{(\rho q)}$ we have

$$D^{(\rm LOC)}_{\rho^+,\mu\nu}(y,y') = -ig_v^2\int d^4x\, d^4x' D_{\rho^+,\mu\alpha}(y,x)J^{\alpha\beta}_{\rho^+}(x,x')D_{\rho^+,\beta\nu}(x',y'), \tag{5.41}$$

where $J^{\alpha\beta}_{\rho^+}(x,x')$ is the polarization function in coordinate space,

$$J^{\alpha\beta}_{\rho^+}(x,x') = -2N_c {\rm tr}_D[iS_u(x,x')\gamma^\beta iS_d(x',x)\gamma^\alpha]. \tag{5.42}$$

As in the charged pion case, we introduce the polarization function in $\bar{q}$ space (or Ritus space), $J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}')$, given by

$$J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}') = \int d^4x\, d^4x' \mathbb{R}^{+,\mu\alpha}(x,\bar{q})^* J_{\rho^+,\mu\nu}(x,x')\mathbb{R}^{+,\nu\alpha'}(x',\bar{q}'). \tag{5.43}$$

Using the completeness relation, Eq. (C5), one gets

$$J^{\mu\nu}_{\rho^+}(x,x') = \sum\!\!\!\!\!\!\!\int_{\bar{q},\bar{q}'} \mathbb{R}^{+,\mu\alpha}(x,\bar{q})J_{\rho^+,\alpha\alpha'}(\bar{q},\bar{q}')\mathbb{R}^{+,\nu\alpha'}(x',\bar{q}')^*. \tag{5.44}$$

Then, from Eq. (4.12) and the orthogonality relation [Eq. (C4)], the LOC to the propagator can be written as

$$D^{(\rm LOC)}_{\rho^+,\mu\nu}(y,y') = -ig_v^2 \sum\!\!\!\!\!\!\!\int_{\bar{q},\bar{q}'} \mathbb{R}^+_{\mu\alpha}(y,\bar{q})\hat{D}^{\alpha\alpha'}_{\rho^+}(k,q_\parallel)J_{\rho^+,\alpha'\beta'}(\bar{q},\bar{q}')\hat{D}^{\beta'\beta}_{\rho^+}(k',q_\parallel)\mathbb{R}^+_{\nu\beta}(y',\bar{q}')^*. \tag{5.45}$$

One can also take into account the explicit form of the functions $\mathbb{R}^+_{\mu\nu}$ in Eq. (3.36) to write $J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}')$ as

$$J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}') = \sum_{\lambda,\lambda'=-1}^{+1}(\Upsilon^{\mu\alpha}_\lambda)^*\Upsilon^{\nu\alpha'}_{\lambda'}\int d^4x d^4x' \mathcal{F}_{Q_{\rho^+}}(x,\bar{q}_\lambda)^*\mathcal{F}_{Q_{\rho^+}}(x',\bar{q}'_{\lambda'})J_{\rho^+,\mu\nu}(x,x'), \tag{5.46}$$

where $\bar{q}_\lambda = (q^0, k_\lambda, \chi, q^3)$, $k_\lambda = k - s\lambda$, $s = {\rm sign}(Q_{\rho^+}B) = {\rm sign}(B)$.

Proceeding as in the π^+ case, we go back to Eq. (5.42) and write the quark propagators in the form given by Eqs. (2.40) and (2.41). This leads to

$$J^{\mu\nu}_{\rho^+}(x,x') = e^{i\Phi_{\rho^+}(x,x')}\int\frac{d^4v}{(2\pi)^4}e^{-iv(x-x')}\bar{J}^{\mu\nu}_{\rho^+}(v_\parallel,v_\perp), \tag{5.47}$$

where

$$\bar{J}^{\mu\nu}_{\rho^+}(v_\parallel,v_\perp) = -2N_c\int\frac{d^4p}{(2\pi)^4}{\rm tr}_D\left[i\bar{S}^u(p^+_\parallel,p^+_\perp)\gamma^\nu i\bar{S}^d(p^-_\parallel,p^-_\perp)\gamma^\mu\right], \tag{5.48}$$

with $p^{\pm}_{\mu} = p^{\pm}_{\mu} + v_{\mu}/2$. Replacing these equations into Eq. (5.46) we get

$$J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}') = \int \frac{d^4 v}{(2\pi)^4} \sum_{\lambda,\lambda'=-1}^{+1} (\Upsilon^{\mu\alpha}_{\lambda})^* \Upsilon^{\nu\alpha'}_{\lambda'} \bar{J}_{\rho^+,\mu\nu}(v_\parallel, v_\perp) h_{\rho^+}(\bar{q}_\lambda, \bar{q}'_{\lambda'}, v_\parallel, v_\perp), \tag{5.49}$$

where the function h_{P} is given by Eq. (5.14). As in the case of the charged pion, in the standard gauges one can carry out explicit calculations that lead to

$$J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}') = (2\pi)^4 \delta^{(2)}(q_\parallel - q'_\parallel)\delta_{\chi\chi'} \int \frac{d^2 v_\perp}{(2\pi)^2} \sum_{\lambda,\lambda'} (\Upsilon^{\mu\alpha}_{\lambda})^* \Upsilon^{\nu\alpha'}_{\lambda'} \bar{J}_{\rho^+,\mu\nu}(q_\parallel, v_\perp) f_{k_\lambda k'_{\lambda'}}(v_\perp), \tag{5.50}$$

where $f_{kk'}(v_\perp)$ is given by Eq. (5.17) and $\delta_{\chi\chi'}$ stands for $\delta_{\iota\iota'}$, $\delta(q^1 - q'^1)$ and $\delta(q^2 - q'^2)$ for SG, LG1, and LG2, respectively.

To proceed further, one can carry out the calculation of $\bar{J}^{\mu\nu}_{\rho^+}(v_\parallel, v_\perp)$ from Eq. (5.48). As expected from symmetry arguments, the explicit calculation shows that one can write

$$\bar{J}^{\mu\nu}_{\rho^+}(v_\parallel, v_\perp) = \sum_{i=1}^{7} c_i(v_\parallel, v_\perp) \mathbb{O}^{\mu\nu}_i(v), \tag{5.51}$$

where $\mathbb{O}^{\mu\nu}_i(v)$ are the operators defined in Eq. (4.18) and $c_i(v_\parallel, v_\perp)$ are scalar functions that depend on $v_\parallel^2$ and $v_\perp^2$. Then one has

$$J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}') = (2\pi)^4 \delta^{(2)}(q_\parallel - q'_\parallel)\delta_{\chi\chi'} \sum_{i=1}^{7} \int \frac{d|\vec{v}_\perp|^2}{8\pi^2} c_i(q_\parallel, v_\perp) Z^{\alpha\alpha'}_i(k, k', v_\perp^2), \tag{5.52}$$

where

$$Z^{\alpha\alpha'}_i(k, k', v_\perp^2) = \int_0^{2\pi} d\phi_\perp \sum_{\lambda,\lambda'} (\Upsilon^{\mu\alpha'}_{\lambda})^* \Upsilon^{\nu\alpha}_{\lambda'} \mathbb{O}_{i,\mu\nu}(v) f_{k_\lambda k'_{\lambda'}}(v_\perp). \tag{5.53}$$

By performing the above integral for each one of the operators $\mathbb{O}_{i,\mu\nu}(v)$, it is seen that $Z^{\alpha\alpha'}_i(k, k', v_\perp^2) \propto \delta_{kk'}$, and consequently $J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}')$ can be written as

$$J^{\alpha\alpha'}_{\rho^+}(\bar{q},\bar{q}') = \hat{\delta}_{\bar{q}\bar{q}'} \hat{J}^{\alpha\alpha'}_{\rho^+}(k, q_\parallel). \tag{5.54}$$

The expression for $\hat{J}^{\alpha\alpha'}_{\rho^+}(k, q_\parallel)$ can be obtained taking into account the Schwinger form of the translational invariant part of quark propagators; see Eq. (4.10). In this way, for $k \geq 0$ we obtain

$$\hat{J}^{\alpha\alpha'}_{\rho^+}(k, q_\parallel) = \sum_{i=1}^{7} d_i(k, q_\parallel) \mathbb{O}^{\alpha\alpha'}_i(\Pi), \tag{5.55}$$

where $\Pi^\mu(k, q_\parallel)$ is the four-vector defined in Eq. (4.14). The explicit expressions of the functions $d_i(k, q_\parallel)$ are given in Appendix D. As one can see from Eqs. (D2), for the particular case $k = -1$ (where Π^μ is not defined), we get $d_2(-1, q_\parallel) = d_6(-1, q_\parallel)$, while the remaining coefficients are zero. In this case one has $\hat{J}^{\alpha\alpha'}_{\rho^+}(-1, q_\parallel) \propto \mathbb{O}^{\alpha\alpha'}_2 + \mathbb{O}^{\alpha\alpha'}_6 = 2\Upsilon^{\alpha\alpha'}_{-s}$.

From Eq. (5.45), we see now that the one-loop correction to the charged rho meson two-point correlator can be expressed as

$$D^{(\mathrm{LOC})}_{\rho^+,\mu\nu}(y, y') = \sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathbb{R}^+_{\mu\alpha}(y, \bar{q}) \hat{D}^{(\mathrm{LOC})\alpha\alpha'}_{\rho^+}(k, q_\parallel) \mathbb{R}^+_{\nu\alpha'}(y', \bar{q}')^*, \tag{5.56}$$

with

$$\hat{D}^{(\mathrm{LOC})\alpha\alpha'}_{\rho^+}(k, q_\parallel) = \hat{D}^{\alpha\beta}_{\rho^+}(k, q_\parallel) \hat{\Sigma}_{\rho^+,\beta\beta'}(k, q_\parallel) \hat{D}^{\beta'\alpha}_{\rho^+}(k, q_\parallel), \tag{5.57}$$

$\hat{\Sigma}_{\rho^+}(k, q_\parallel)$ being the one-loop ρ^+ meson self-energy, related to the polarization function $\hat{J}_{\rho^+}(k, q_\parallel)$ by

$$\hat{\Sigma}^{\alpha\alpha'}_{\rho^+}(k, q_\parallel) = -i g_v^2 \hat{J}^{\alpha\alpha'}_{\rho^+}(k, q_\parallel). \tag{5.58}$$

For the description of physical ρ^+ meson states, it is also useful to project $\hat{J}_{\rho^+}(k, q_\parallel)$ on the polarization state basis.

In this way, one can define a matrix $\mathbf{J}^{cc'}_{\rho^+}(k, q_\parallel)$ given by

$$\mathbf{J}^{cc'}_{\rho^+}(k, q_\parallel) = \epsilon_{+,\alpha}(k, q^3, c)^* \hat{J}^{\alpha\alpha'}_{\rho^+}(k, q_\parallel)\epsilon_{+,\alpha'}(k, q^3, c'), \tag{5.59}$$

where $\epsilon^+_\alpha(k, q^3, c)$ are the polarization vectors introduced in Eq. (3.35). In the case $k = -1$, i.e., the lowest Landau level, only $c = 1$ is allowed. One has

$$\hat{\mathbf{J}}^{11}_{\rho^+}(-1, q_\parallel) = -i\frac{N_c}{4\pi^2}\int_{-1}^{1} dx \int_0^\infty dz\, e^{-z\phi(x,q^2_\parallel)} \times \frac{(1+t_u)(1+t_d)}{\alpha_+}\left[m_u m_d + \frac{1}{z} + \frac{1-x^2}{4}q^2_\parallel\right], \tag{5.60}$$

where $\phi(x, q^2_\parallel)$, t_f, and α^+ have been defined in Eqs. (5.25), (5.26), and (5.28), respectively. As in the case of the charged pion [see Eq. (5.27) for $\hat{J}_{\pi^+}(k, q_\parallel)$], this expression is divergent and has to be regularized. Again, this can be done by subtracting the $B = 0$ contribution, leaving a well-defined B-dependent piece. In addition, the function $\hat{J}_{\rho^+}(k, q_\parallel)$ can be analytically extended for large positive values of $q^2_\parallel$.

VI. MAGNETIZED CHARGED PION AND RHO MASSES IN THE NAMBU-JONA-LASINIO MODEL

In this section we consider an extended NJL model in the presence of an external magnetic field. The corresponding Lagrangian reads as

$$\mathcal{L} = \bar\psi(i\not{D} - m_0)\psi + G_s\left[(\bar\psi\psi)^2 + (\bar\psi i\gamma_5\vec\tau\psi)^2\right] - G_v(\bar\psi\gamma_\mu\vec\tau\psi)^2, \tag{6.1}$$

where $\psi(x)$ is the $u - d$ quark doublet defined in Eq. (5.2) and $\mathcal{D}^\mu$ is the covariant derivative in Eq. (2.2). Models like the one described by Eq. (6.1) have often been used to study the influence of an external magnetic field on meson masses. In fact, the NJL model was introduced more than 60 years ago for the description of spontaneous chiral symmetry breaking and dynamical mass generation [61,62]; then, during the late 80s and earlier 90s, the approach was reinterpreted as an effective model for low energy QCD [63–65]. For a large enough value of the coupling constant G_s, it is seen that the model describes adequately the breakdown of chiral symmetry, and leads to a phenomenologically reasonable value for the chiral quark-antiquark condensate at the mean field level. In turn, this implies that the quarks acquire an effective dynamical mass $M_f \approx 300$–400 MeV $\gg m_0$. In the simple model given by Eq. (6.1), it turns out that $M_u = M_d$ even in the presence of an external magnetic field; however, the magnetic field can break this degeneracy if more general flavor mixing interactions are included (for details see, e.g., Ref. [36]).

In the above framework, mesons can be described as quantum fluctuations in the large N_c approximation (which, in this context, is equivalent to the well known random phase approximation); i.e., they can be introduced via the summation of an infinite number of quark loops. Here we are particularly interested in the masses of the charged pion (lightest charged meson in the absence of the external magnetic field) and the charged rho meson. Concerning the latter, we recall that there has been some discussion about the possibility that the presence of a strong magnetic field may induce $\rho^\pm$ condensation. Our interest here is not to perform a detailed analysis of meson masses in the presence of the magnetic field but to study the effect of Schwinger phases, showing how the results get modified if SPs are neglected. Therefore, as done in Sec. V, we study here $\pi^\pm$ and $\rho^\pm$ masses separately. A full analysis, in which $\pi^+ - \rho^+$ mixing is explicitly considered, can be found in Ref. [37].

Let us first take $G_v = 0$ in Eq. (6.1) and concentrate just on the charged pion mass. Following Ref. [66] we introduce the charged pseudoscalar currents

$$j_+(x) = \sqrt{2}\bar\psi_u(x)i\gamma_5\psi_d(x), \quad j_-(x) = \sqrt{2}\bar\psi_d(x)i\gamma_5\psi_u(x). \tag{6.2}$$

Next, we define the two-point function $\Pi_{P^+}(x, x')$ as the two-point correlator between these two currents. To zeroth order in G_s we have

$$\Pi^{(0)}_{P^+}(x, x') = \langle 0|T[j_-(x)j_+(x')]|0\rangle = J_{\pi^+}(x, x'), \tag{6.3}$$

where $J_{\pi^+}(x, x')$ is given by Eq. (5.5). The full two-point function in the large N_c approximation is obtained as

$$\begin{aligned}\Pi_{P^+}(x, x') &= J_{\pi^+}(x, x') + 2iG_s\int d^4z J_{\pi^+}(x, z)J_{\pi^+}(z, x') \\ &\quad + (2iG_s)^2\int d^4z d^4z' J_{\pi^+}(x, z)J_{\pi^+}(z, z')J_{\pi^+}(z', x') + \ldots\end{aligned} \tag{6.4}$$

Then from Eqs. (5.7) and (3.20) one readily gets

$$\begin{aligned}\Pi_{P^+}(x, x') &= \sum\!\!\!\!\!\!\int_{\bar q} \mathbb{F}^+(x, \bar q)\hat{J}_{\pi^+}(k, q_\parallel) \\ &\quad \times\Big\{1 + 2iG\hat{J}_{\pi^+}(k, q_\parallel) \\ &\quad + [2iG\hat{J}_{\pi^+}(k, q_\parallel)]^2 + \ldots\Big\}\mathbb{F}^+(x', \bar q)^* \\ &= \sum\!\!\!\!\!\!\int_{\bar q} \mathbb{F}^+(x, \bar q)\hat{J}_{\pi^+}(k, q_\parallel)K^{-1}_{\pi^+}\mathbb{F}^+(x', \bar q)^*,\end{aligned} \tag{6.5}$$

where we have defined

$$K_{\pi^+}(k, q_\parallel) = 1 - 2iG\hat{J}_{\pi^+}(k, q_\parallel), \tag{6.6}$$

and $\hat{J}_{\pi^+}(k, q_\parallel)$ is the function given by Eq. (5.27), in which we have replaced the quark masses m_f by the dynamical masses M_f. Thus, one can obtain the π^+ pole mass for each Landau level k by solving the equation

$$K_{\pi^+}|_{q_\parallel^2 = m_{\pi^+}^2} = 0. \tag{6.7}$$

Charged pion masses were determined for the first time in this way in Refs. [27,31]. For the lowest Landau level $k = 0$ one obtains

$$\hat{J}_{\pi^+}(0, q_\parallel) = -\frac{iN_c}{4\pi^2}\int_{-1}^{1} dx \int_0^\infty dz \frac{1}{\alpha_+} e^{-z\phi(x,q_\parallel^2)} \left\{ \left[M_u M_d + \frac{1}{z} + (1-x^2)\frac{q_\parallel^2}{4} \right](1 - t_u t_d) + \frac{1}{\alpha_+}(1-t_u^2)(1-t_d^2) \right\}. \tag{6.8}$$

In the derivation of Eq. (6.5), it is worth paying attention to Eqs. (5.7) and (5.18), which show that $J_{\pi^+}(x, x')$ is diagonal in the basis of eigenstates of the Klein-Gordon operator in Eq. (3.15). In usual quantum field theory, particle states are given at zero order in perturbation theory by plane waves (i.e., they have definite four-momentum). In contrast, in our case there is an external static and uniform magnetic field that plays the role of a background; consequently, as discussed in Sec. III, the zero order charged particle states correspond to wave functions expressed in terms of the functions $\mathcal{F}_Q(x, \bar{q})$.

Tracing back the derivation of Eq. (6.5), we can see what happens if the SP is neglected. The diagonal condition in Eq. (5.18) arises in fact from Eqs. (5.13) and (5.14); if one intends to make an approximation in which the SP in Eq. (5.14) is removed, one should also replace the wave functions by plane waves, $\mathcal{F}_{Q_{\rm P}}(x, \bar{q}) \to \exp(-iqx)$, in order to guarantee translational invariance. Thus, we denote this procedure as "plane wave approximation" (PWA). Within this approximation, the two-interacting quark state—or the pion, in the context discussed in Sec. V—is no longer specified by the set of quantum numbers $\bar{q} = (q^0, k, \chi, q^3)$ but by the four-momentum $q^\mu = (q^0, q^1, q^2, q^3)$. In this way one obtains

$$h_{\rm P}^{\rm PWA}(q, q', v) = (2\pi)^8 \delta^{(4)}(q - q')\delta^{(4)}(q - v), \tag{6.9}$$

losing the effect of the magnetic field in this part of the calculation. After a trivial integration over v, according to Eq. (5.13) one gets

$$J_{\pi^+}^{\rm PWA}(q, q') = (2\pi)^4 \delta^{(4)}(q - q')\hat{J}_{\pi^+}^{\rm PWA}(q_\parallel, q_\perp), \tag{6.10}$$

where

$$\hat{J}_{\pi^+}^{\rm PWA}(q_\parallel, q_\perp) = \bar{J}_{\pi^+}(q_\parallel, q_\perp), \tag{6.11}$$

with $\bar{J}_{\pi^+}(q_\parallel, q_\perp)$ given by Eq. (5.24) [notice that the calculation of $\bar{J}_{\pi^+}(q_\parallel, q_\perp)$ only involves the translational invariant part of quark propagators; hence it is not affected by Schwinger phases].

We notice that, in some sense, the result in Eq. (6.10) can be misleading. Given that the magnetic field is assumed to be uniform, one would expect the system to be invariant under translations in space-time, and this seems to be confirmed by the conservation of four-momentum arising from Eq. (6.10). Nevertheless, in the presence of the magnetic field it is found that translational invariance (in the plane perpendicular to $\vec{B}$) is realized in a nontrivial way, related to gauge transformations. We refer here to Sec. II, in which this issue has been discussed in detail.

From the above results it is easy to see that within the PWA one can define a π^+ pole mass (taking $q_\perp = 0$) as the solution of the equation

$$K_{\pi^+}^{\rm PWA}|_{q_\parallel^2 = m_{\pi^+}^2} = 1 - 2iG\hat{J}_{\pi^+}^{\rm PWA}(q_\parallel, 0)|_{q_\parallel^2 = m_{\pi^+}^2} = 0, \tag{6.12}$$

where, according to Eq. (5.24),

$$\hat{J}_{\pi^+}^{\rm PWA}(q_\parallel, 0) = -\frac{iN_c}{4\pi^2}\int_{-1}^{1} dx \int_0^\infty dz \frac{1}{t_+} e^{-z\phi(x,q_\parallel^2)} \left\{ \left[M_u M_d + \frac{1}{z} + (1-x^2)\frac{q_\parallel^2}{4} \right](1 - t_u t_d) + \frac{1}{t_+}(1-t_u^2)(1-t_d^2) \right\}. \tag{6.13}$$

Comparing Eq. (6.13) with our result in Eq. (6.8), it is seen that the PWA expression can be obtained from the full result by the replacement $\alpha_+ \to t_+$ in the integrand. We also note that Eq. (6.13) is consistent with Eq. (80) of Ref. [53], where an alternative method has been used to evaluate the effects of the magnetic field on charged pion masses. The difference between Eqs. (6.13) and (6.8) shows that the approach in Ref. [53] does not fully take into account the effects arising from the presence of the Schwinger phase.

An important point to be stressed is the fact that within the PWA the two-quark state has zero total transverse momentum. One can see, however, that this cannot be possible: the two-quark state, as a whole, has to behave as a charged bound system immersed in a magnetic field, whose quantum ground state—which must have some nonvanishing zero-point energy—cannot be described by a particle at rest. In fact, the charged meson state cannot have any definite momentum in the plane perpendicular to the magnetic field. The situation can be better understood by looking at Eq. (5.19), which shows that our result for $\hat{J}_{\pi^+}(k, q_\parallel)$ arises from the convolution of $\bar{J}_{\pi^+}(q_\parallel, v_\perp)$ with the function $\rho_k(\vec{v}_\perp^2)$ given in Eq. (5.20). In fact, this function describes the total transverse momentum distribution due to the vibration of the two-quark quantum state in the presence of the external magnetic field. Notice that, expressed in this way, the plane wave approximation would correspond to a distribution $\rho_k^{\rm PWA}(\vec{v}_\perp^2) = \delta(|\vec{v}_\perp|^2 - |\vec{q}_\perp|^2)$.

Let us consider now the rho meson sector. As mentioned above, for simplicity we analyze the situation in which the $\rho^+ - \pi^+$ mixing is neglected. The study of the ρ^+ meson in this simplified scenario can be performed by eliminating the pseudoscalar-pseudoscalar coupling $(\bar{\psi} i\gamma_5 \vec{\tau}\psi)^2$ in Eq. (6.1). To proceed we introduce the charged vector currents

$$j_+^\mu(x) = \sqrt{2}\bar{\psi}_u(x)\gamma^\mu\psi_d(x), \quad j_-^\mu(x) = \sqrt{2}\bar{\psi}_d(x)\gamma^\mu\psi_u(x), \tag{6.14}$$

and define the two-point function $\Pi_{V^+}^{\mu\nu}(x,x')$ as the two-point correlator between both currents. To zero order in G_v we have

$$\Pi_{V^+}^{(0)\mu\nu}(x,x') = \langle 0|T[j_-^\mu(x) j_+^\nu(x')]|0\rangle = J_{\rho^+}^{\mu\nu}(x,x'), \tag{6.15}$$

where $J_{\rho^+}^{\mu\nu}(x,x')$ is given by Eq. (5.42). Now, as in the case of the π^+, we can evaluate the full vector two-point function in the large N_c approximation,

$$\begin{aligned}\Pi_{V^+}^{\mu\nu}(x,x') = {}& J_{\rho^+}^{\mu\nu}(x,x') + (-2iG_V)\int d^4z\, J_{\rho^+}^{\mu\alpha}(x,z) g_{\alpha\beta} J_{\rho^+}^{\beta\nu}(z,x') \\ &+ (-2iG_V)^2 \int d^4z\, d^4z'\, J_{\rho^+}^{\mu\alpha}(x,z) g_{\alpha\alpha'} J_{\rho^+}^{\alpha'\beta'}(z,z') g_{\beta'\beta} J_{\rho^+}^{\beta\nu}(z',x') + \ldots\end{aligned} \tag{6.16}$$

Using Eq. (5.44) together with Eqs. (C4) and (5.54), and resumming the loop contributions, we obtain

$$\begin{aligned}\Pi_{V^+}^{\mu\nu}(x,x') = {}& \sum\!\!\!\!\!\!\!\int_{\bar{q},\bar{q}'} \mathbb{R}^{+,\mu\alpha}(x,\bar{q}) \hat{J}_{\rho^+,\alpha\alpha'}(k,q_\parallel)(K_{\rho^+}^{-1})^{\alpha'}_{\beta} \\ &\times \mathbb{R}^{+,\nu\beta}(x',\bar{q})^*,\end{aligned} \tag{6.17}$$

where

$$K_{\rho^+}^{\alpha\beta}(k,q_\parallel) = g^{\alpha\beta} + 2iG_V \hat{J}_{\rho^+}^{\alpha\beta}(k,q_\parallel). \tag{6.18}$$

In this way, taking $q^3 = 0$, the charged rho pole masses m_ρ can be obtained for each Landau level by solving the equation

$$\det K_{\rho^+} = 0 \tag{6.19}$$

for $q_\parallel^\mu = (E_\rho, 0, 0, 0)$, $E_\rho^2 = m_\rho^2 + (2k+1)B_\rho$.

In the limit $B \to 0$, it can be seen from the coefficients $d_i(k, q_\parallel)$ in Appendix D that $\hat{J}_{\rho^+}^{\alpha\beta}(k,q_\parallel)$ can be written in terms of $\mathbb{O}_1^{\alpha\beta} + \mathbb{O}_2^{\alpha\beta} = g^{\alpha\beta}$ and $\mathbb{O}_3^{\alpha\beta}(\Pi) + \mathbb{O}_4^{\alpha\beta}(\Pi) + \mathbb{O}_5^{\alpha\beta}(\Pi) = \Pi^\alpha\Pi^{\beta *}$, where $\Pi^\alpha \to (q^0, 0, 0, q^3)$. In this limit the ρ^+ meson can be taken to be at rest, and one gets three degenerate masses that correspond to the ρ^+ polarization states. On the other hand, for nonzero B the mass states depend on the value of k. For the lowest Landau level $k = -1$, which corresponds to the lightest charged ρ state, from the results in Appendix D it is seen that $d_2(-1, q_\parallel) = d_6(-1, q_\parallel)$, while the remaining coefficients $d_i(-1, q_\parallel)$ are zero. In addition, according to the definitions in Appendix C, one has

$$\begin{aligned}-\frac{1}{2}(\mathbb{O}_2^{\alpha\beta} + \mathbb{O}_6^{\alpha\beta}) &= -\Upsilon_{-s}^{\alpha\beta} \\ &= \epsilon_+^\alpha(-1, q^3, 1)\epsilon_+^\beta(-1, q^3, 1)^*,\end{aligned} \tag{6.20}$$

so we can write

$$\hat{J}_{\rho^+}^{\alpha\beta}(-1, q_\parallel) = -2d_2(-1, q_\parallel)\epsilon_+^\alpha(-1, q^3, 1)\epsilon_+^\beta(-1, q^3, 1)^*. \tag{6.21}$$

Hence, it is found that for the lowest Landau level there is only one mass eigenstate, which corresponds to $c = 1$ in the polarization state basis. From the expressions in Appendix D the coefficient on the right-hand side of Eq. (6.21) is given by

$$-2d_2(-1,q_\parallel) = -i\frac{N_c}{4\pi^2}\int_{-1}^{1}dx\int_0^\infty dz\, e^{-z\phi(x,q_\parallel^2)}\frac{(1+t_u)(1+t_d)}{\alpha_+}\left[M_uM_d + \frac{1}{z} + \frac{1-x^2}{4}q_\parallel^2\right] = \hat{\mathbf{J}}^{11}_{\rho^+}(-1,q_\parallel), \tag{6.22}$$

consistently with the result found in Sec. V B; see Eq. (5.60) (here the quark masses have been replaced by the effective masses M_u and M_d). We see that the ρ^+ state is in this case also an eigenstate of the spin operator S_3, with eigenvalue $s_3 = s = \mathrm{sg}(Q_\rho B)$. From Eq. (6.19), the mass of this state can be obtained as the solution of

$$1 - 2iG_V\hat{\mathbf{J}}^{11}_{\rho^+}(-1,q_\parallel) = 0. \tag{6.23}$$

As in the charged pion case, it is interesting to see how the results get modified if the plane wave approximation is used. As stated, in the PWA the Schwinger phase $\Phi_{\rho^+}(x,x')$ is neglected, and one should replace $\mathcal{F}_{Q_P}(x,\bar{q}) \to \exp(-iqx)$, which leads to $\mathbb{R}^{Q,\mu\nu} = \exp(-iqx)\sum_\lambda \Upsilon^{\mu\nu}_\lambda = \exp(-iqx)g^{\mu\nu}$. In this way, from Eqs. (5.44) and (5.47) one gets

$$J^{\mathrm{PWA},\mu\nu}_{\rho^+}(x,x') = \int\frac{d^4q}{(2\pi)^4}\frac{d^4q'}{(2\pi)^4}e^{-iqx}g^{\mu\alpha}J^{\mathrm{PWA}}_{\rho^+,\alpha\alpha'}(q,q')e^{iq'x'}g^{\nu\alpha'} = \int\frac{d^4v}{(2\pi)^4}e^{-iv(x-x')}\bar{J}^{\mu\nu}_{\rho^+}(v_\parallel,v_\perp), \tag{6.24}$$

and consequently

$$J^{\mathrm{PWA},\alpha\alpha'}_{\rho^+}(q,q') = (2\pi)^4\delta^{(4)}(q-q')\bar{J}^{\alpha\alpha'}_{\rho^+}(q_\parallel,q_\perp), \tag{6.25}$$

where $\bar{J}^{\alpha\alpha'}_{\rho^+}(q_\parallel,q_\perp)$ is the quark loop function given by Eq. (5.48). Notice that Eq. (6.25) can also be obtained from Eq. (5.49), taking into account the PWA result in Eq. (6.9).

Within the PWA approximation the lowest energy ρ^+ states correspond to the situation in which the meson is at rest. It is easy to see that the pole masses for the different polarization states can be obtained as the solutions of

$$\det K^{\mathrm{PWA}}_{\rho^+} = 0, \tag{6.26}$$

where

$$K^{\mathrm{PWA},\alpha\beta}_{\rho^+} = g^{\alpha\beta} + 2iG_V\bar{J}^{\alpha\beta}_{\rho^+}(q_\parallel,0), \tag{6.27}$$

with $q^3 = 0$. We can compare the PWA result with the full result for the lowest Landau level $k = -1$ by taking the projection of $\bar{J}^{\alpha\beta}_{\rho^+}(q_\parallel,0)$ onto the polarization state $s_3 = s$, i.e., taking the piece of $\bar{J}^{\alpha\beta}_{\rho^+}(q_\parallel,0)$ proportional to $-\Upsilon^{\alpha\beta}_{-s}$. The explicit calculation of the quark loop leads to the equation

$$1 - 2iG_V\hat{\mathbf{J}}^{11,\mathrm{PWA}}_{\rho^+}(q_\parallel,0) = 0, \tag{6.28}$$

where

$$\hat{\mathbf{J}}^{11,\mathrm{PWA}}_{\rho^+}(q_\parallel,0) = -i\frac{N_c}{4\pi^2}\int_{-1}^{1}dx\int_0^\infty dz\, e^{-z\phi(x,q_\parallel^2)}\frac{(1+t_u)(1+t_d)}{t_+} \times\left[M_uM_d + \frac{1}{z} + \frac{1-x^2}{4}q_\parallel^2\right]. \tag{6.29}$$

Hence, in the same way as in the case of the charged pion, the mass of the lowest ρ^+ state within the PWA can be obtained from the full result in Eq. (6.22) by replacing the factor $1/\alpha_+$ by $1/t_+$ in the integrand. It can be seen that the expressions in Eqs. (6.28) and (6.29) are consistent with Eq. (24) of Ref. [30], which shows that the method used in that reference turns out to be equivalent to the PWA.

To complete this section, we find it worth estimating the importance of taking into account Schwinger phases in the calculation of charged meson properties as functions of the magnetic field. Therefore, in what follows we analyze the B dependence of π^+ and ρ^+ masses, comparing the results obtained from Eqs. (6.7) and (6.23) with those found within the plane wave approximation, i.e., those obtained from Eqs. (6.12) and (6.28).

We recall that the above expressions for the quark loop integrals are divergent and have to be regularized. Here, as done, e.g., in Refs. [30,37,53], we use the so-called magnetic field independent regularization, in which we subtract from the integrals the corresponding expressions in the $B \to 0$ limit, and then we add them in a regularized form. In fact, as noticed in Ref. [30], to properly regularize the function $\hat{\mathbf{J}}^{11,\mathrm{PWA}}_{\rho^+}(q_\parallel,0)$ in Eq. (6.29) it is necessary to introduce a modification of the method, considering not only the $B \to 0$ limit but also a linear term in B. To perform the numerical calculations, for definiteness we choose here the same set of model parameters as in Ref. [37], viz. $m_0 = 5.833$ MeV, $\Lambda = 587.9$ MeV, and $G_s\Lambda^2 = 2.44$, where Λ is a 3D cutoff parameter that is introduced to regularize the ultraviolet divergent quark loops in the $B = 0$ limit. For vanishing external field, this parametrization leads to an effective quark mass $M_0 = 400$ MeV and a quark-antiquark condensate $\langle\psi_f\bar{\psi}_f\rangle_0 = -(241\ \text{MeV})^3$; in addition, one obtains the empirical values of the pion mass and decay constant in vacuum, namely $m_{\pi,0} = 138$ MeV and $f_{\pi,0} = 92.4$ MeV. Regarding the vector couplings, we take $G_v\Lambda^2 = 2.651$, which leads to $m_{\rho,0} = 770$ MeV for $B = 0$. It is worth mentioning that we have checked that our results remain basically unchanged if one uses other standard parameters, like, e.g., those considered in Refs. [30,53].

In Figs. 1 and 2 we display our numerical results for the charged pion and charged rho mesons, respectively. The curves show the values of the ratio $E_P/m_{P,0}$, where $P = \pi^+, \rho^+$, as functions of eB. Here $m_{P,0}$ is the particle mass at $B = 0$, while E_P stands for the energy of the P meson in its lowest state, i.e., $E_{\pi^+} = \sqrt{m_{\pi^+}^2 + B_\pi}$ and $E_{\rho^+} = \sqrt{m_{\rho^+}^2 - B_\rho}$, where m_P is the meson mass for nonzero B. We stress that to determine the mass of the lowest energy state from our full calculation one has to take $q^3 = 0$ and $k = 0$ ($k = -1$) for the pion (rho meson), whereas within the PWA one has to take $\vec{q} = 0$.

From Fig. 1 it is seen that, for the whole considered range of values of eB, the PWA leads to values of the ratio $E_{\pi^+}/m_{\pi,0}$ that are larger than those obtained from the full calculation, in which the SP is properly taken into account. In turn, the latter are larger than those obtained within the "pointlike approximation" (PLA), in which the meson is considered as a particle with no internal structure (in this pointlike limit, one has $E_{\pi^+}^{\text{PLA}} = \sqrt{m_{\pi,0}^2 + eB}$). On the other hand, the results can be compared with the values arising from LQCD calculations [50,51]. These are found to be close to or even lower than those corresponding to the PLA, which implies that the proper treatment of Schwinger phases improves the agreement between LQCD results and NJL model predictions for the dependence of E_{π^+} with the magnetic field. It is also worth mentioning that, as shown in Ref. [37], $\rho^+ - \pi^+$ mixing effects (which have been neglected in the calculations shown in Fig. 1) tend to bring NJL results even closer to LQCD values.

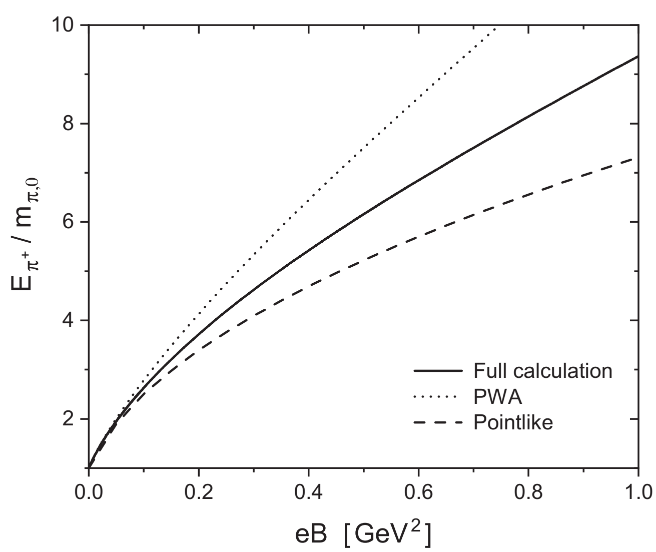

FIG. 1. Ratio $E_{\pi^+}/m_{\pi,0}$ as a function of eB. Here E_{π^+} stands for the energy of the lowest π^+ state (corresponding to the Landau level $k = 0$), while $m_{\pi,0}$ is the charged pion mass at vanishing external magnetic field.

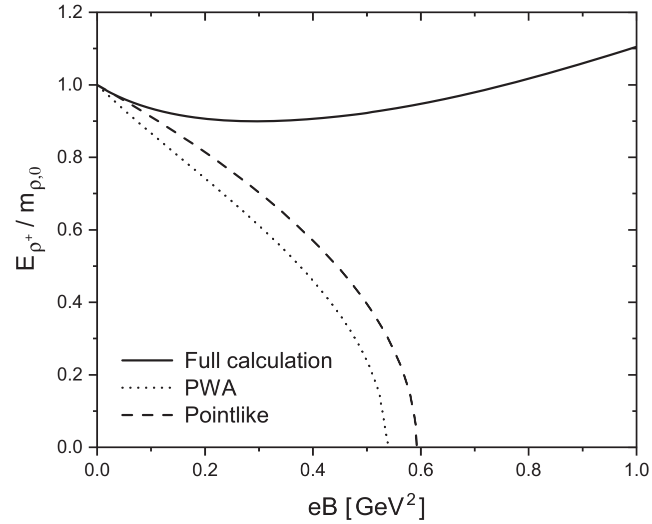

FIG. 2. Ratio $E_{\rho^+}/m_{\rho,0}$ as a function of eB. Here E_{ρ^+} stands for the energy of the lowest ρ^+ state (corresponding to the Landau level $k = -1$), while $m_{\rho,0}$ is the charged rho mass at vanishing external magnetic field.

Now, as can be seen from Fig. 2, the effect of taking into account the SP is even more striking in the case of the charged rho meson energy. Indeed, the results from PWA and PLA approximations (dotted and dashed lines in the figure) seem to indicate that E_{ρ^+} vanishes at some critical magnetic field—driving in this way a possible ρ^+ meson condensation—while this is not what comes out from the full calculation, in which the SP is properly included (full line in Fig. 2). Regarding LQCD calculations, in this case it is found [45,50,67] that the ratio $E_{\rho^+}/m_{\rho,0}$ shows some decrease for low values of eB, while for $eB \gtrsim 0.7\ \text{GeV}^2$ it tends to stabilize at a value of about 0.7; hence, no ρ^+ meson condensation is expected from these results. In fact, values consistent with the behavior predicted by LQCD can be obtained within the NJL model—taking into account the effect of the SP—by considering B-dependent couplings [37]. We recall that these results correspond to the Landau level $k = -1$, for which there is no mixing between the ρ^+ and the charged pion.

VII. CONCLUSIONS

In this paper we have studied the role of the Schwinger phases appearing in the propagators of charged particles in the presence of a static and uniform magnetic field $\vec{B}$. These propagators are not gauge invariant objects; if one performs a gauge transformation, they transform in a well defined covariant way. In fact, it is seen that the noninvariance can be isolated in a Schwinger phase Φ_P, in such a way that the propagator can be written as $\exp(i\Phi_P)$ times a gauge invariant function. As a first result we have shown that the SP cannot be removed by a gauge transformation; far from this, we have seen that it plays an important role in the restoration of the symmetries of the system.

The presence of a static and uniform magnetic field does not alter the homogeneity of space-time, although it does break space isotropy. Still, isotropy is preserved in transverse directions, taking the direction of $\vec{B}$ as a symmetry axis. Therefore, the studied system has to be invariant under translations, under rotations around the direction of $\overrightarrow{B}$, and under boost transformations in this direction. As a consequence of the existence of this set of symmetries, for any Lorentz tensor one should distinguish between "longitudinal" components (time components and spatial components in the direction of $\vec{B}$) and "perpendicular components" (spatial components in the direction perpendicular to $\vec{B}$), which in general will show different behaviors.

The equations that describe the dynamics of a charged particle in a static and uniform magnetic field involve the electromagnetic field A^μ. Even if one assumes that the physical system has the above stated symmetries, it has to be taken into account that both the homogeneity and the transverse isotropy of space become broken when one chooses a specific gauge to set A^μ. Looking at the propagators of charged particles, we have shown that this breakdown manifests itself in the SP, whereas the part of the propagators that is invariant under gauge transformations is found to be also invariant under translations and rotations around the direction of $\vec{B}$. Additionally, we have seen that a translation in a direction perpendicular to $\vec{B}$, as well as a rotation around the direction of $\vec{B}$, are equivalent to gauge transformations. Explicit expressions have been given for some gauges that are usually considered in the literature, namely the symmetric gauge and the Landau gauges 1 and 2.

As an application to particular physical quantities, we have analyzed the effect of the SP in the one-loop corrections to charged pion and rho meson self-energies. To carry out this analysis we have firstly considered standard meson-quark interactions, and then we have studied the π^+ and ρ^+ propagators within the Nambu-Jona-Lasinio model, performing a numerical analysis of the B dependence of meson lowest energy states. For both π^+ and ρ^+ mesons (for simplicity, $\pi^+ - \rho^+$ mixing has not been considered), we have compared the numerical results arising from the full calculation—in which SPs are included in the propagators, and meson wave functions correspond to states of definite Landau quantum number—and those obtained within a plane wave approximation—in which SPs are neglected (or simply eliminated) and meson states are described by plane waves of definite four-momentum.

In the case of the pion, from our analysis it is seen that the polarization function is diagonal in the basis of π^+ eigenfunctions $\mathcal{F}_+(x,\bar{q})$, and can be written as a convolution of a gauge invariant function $\bar{J}_{\pi^+}(v_\parallel, v_\perp)$—calculated from the gauge invariant part of the polarization function, after a transformation to momentum space—with a function $h_{\pi^+}(\bar{q},\bar{q}',v_\parallel,v_\perp)$ given by a projection of these eigenfunctions onto plane waves, modulated by the SP [see Eqs. (5.13) and (5.14)]. Moreover, after some integration we have found that the polarization function can be written as an integral of the function $\bar{J}_{\pi^+}(q_\parallel, v_\perp)$ over the perpendicular momentum $v_\perp$, weighted by a given function $\rho_k(\vec{v}_\perp^2)$ [see Eq. (5.19)]; i.e., the polarization function depends on definite values of the energy q^0 and the parallel momentum q^3 of the two-quark system as a whole, while the perpendicular momentum has no definite value but some distribution. This is what one would expect for a charged particle, which must have some zero-point energy when it is submerged in a magnetic field. In contrast, within the PWA the polarization function can be transformed to four-momentum space as $\bar{J}_{\pi^+}(q_\parallel, q_\perp)$, where $q_\perp$ would be the perpendicular momentum of the two-quark system (the pion, in the case of the NJL model). Formally this would correspond to take $\rho_k(\vec{v}_\perp^2) = \delta(|\vec{v}_\perp|^2 - |\vec{q}_\perp|^2)$, although the perpendicular momentum $\vec{q}_\perp$ is not a well defined quantity for a charged particle in a magnetic field. Alternatively, the effect of the SP on the one-loop correction to the π^+ propagator can be seen from Eq. (5.36), where once again our result shows the fluctuations of the perpendicular momenta of the coupled two-quark system. These fluctuations are due to the presence of a gauge invariant phase φ, which arises from a combination of Schwinger phases along a closed path [see Eq. (5.31)]; if this phase is eliminated, the effect of the fluctuations gets lost, as shown in Eq. (5.37).

It is worth emphasizing that even though the function $h_{\pi^+}(\bar{q},\bar{q}',v_\parallel,v_\perp)$ involves several gauge dependent quantities, its explicit expression, given by Eqs. (5.15)–(5.17), is itself gauge independent. This has been checked by performing the corresponding calculations in the three standard gauges mentioned above. In this way, it has been shown that the inclusion of the SP allows us to carry out a full calculation of the polarization function, using the proper wave functions of charged particles and preserving both the invariance under gauge transformations and the symmetries under translations and rotations around the direction of $\vec{B}$.

The above qualitative discussion applies also to the case of the ρ^+ meson propagator, although the explicit expressions are more involved due to the more complex Lorentz structure. It is worth mentioning that we have introduced in this case a description of spin one charged fields in the presence of the magnetic field. As proposed by Ritus for the case of spin $1/2$ fermion fields [57], in our formalism we have separated the meson wave functions as a product of a tensor that carries the spatial coordinates and a polarization vector. Then the explicit expression for the meson propagator and the corresponding one-loop correction have been obtained.

Finally, as mentioned above, we have carried out a numerical analysis of the B dependence of π^+ and ρ^+ meson masses (and lowest state energies) within the NJL model. Using a three-momentum cutoff and a so-called magnetic field independent regularization [30,37,53], we have found that our full calculation leads to a B dependence of the charged pion mass that clearly improves the agreement with LQCD results, in comparison with the one obtained using the PWA. Moreover, there is still room for further improvement, e.g., by considering $\rho^+ - \pi^+$ mixing as done in Ref. [37]. Concerning the charged rho meson, we have found a qualitative difference between our results and those obtained within the PWA. Indeed, our calculations show that if the presence of the SP is properly taken into account, the ρ^+ mass does not vanish for any considered value of the magnetic field, a fact that can be relevant in connection with the occurrence of ρ meson condensation for strong magnetic fields. Our results are in the same line as those obtained by LQCD analyses [45,50,67], which indicate that the value of the energy of the lowest ρ^+ state tends to stabilize at $E_{\rho^+}/m_{\rho,0} \sim 0.7$ for $eB > 0.8$ GeV^2. Let us recall that this state corresponds to the Landau level $k = -1$, which does not mix with the pion. We have also checked that our numerical results do not suffer significant changes if one uses other standard model parameters, like, e.g., those considered in Refs. [30,53].

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APPENDIX A: THE FUNCTIONS $\mathcal{F}_Q(x,\bar{q})$ IN THE STANDARD GAUGES

We give here the expressions for the functions $\mathcal{F}_Q(x,\bar{q})$, eigenfunctions of the operator $\mathcal{D}^\mu\mathcal{D}_\mu$ [see Eq. (3.1)], in the standard gauges SG, LG1, and LG2. As in the main text, we choose the axis 3 in the direction of the magnetic field, and use the notation $B_Q = |QB|$ and $s = \mathrm{sign}(QB)$.

It is worth pointing out that the functions $\mathcal{F}_Q(x,\bar{q})$ can be determined up to a global phase, which in general can depend on k. In the following expressions for SG, LG1, and LG2 the corresponding phases have been fixed by requiring $\mathcal{F}_Q(x,\bar{q})$ to satisfy Eqs. (5.14)–(5.16), with $f_{kk'}(v_\perp)$ given by Eq. (5.17).

1. Symmetric gauge

In the SG we take $\chi = \imath$, where $\imath$ is a nonnegative integer. Thus, the set of quantum numbers used to characterize a given particle state is $\bar{q} = (q^0, k, \imath, q^3)$. In addition, we introduce polar coordinates r, ϕ to denote the vector $\vec{x}_\perp = (x^1, x^2)$ that lies in the plane perpendicular to the magnetic field. The functions $\mathcal{F}_Q(x,\bar{q})$ in this gauge are given by

$$\mathcal{F}_Q(x,\bar{q})^{(\mathrm{SG})} = \sqrt{2\pi} e^{-i(q^0x^0 - q^3x^3)} e^{-is(k-\imath)\phi} R_{k,\imath}(r), \qquad (A1)$$

where

$$R_{k,\imath}(r) = N_{k,\imath}\xi^{(k-\imath)/2} e^{-\xi/2} L_\imath^{k-\imath}(\xi), \qquad (A2)$$

with $\xi = B_Q r^2/2$. Here we have used the definition $N_{k,\imath} = (B_Q \imath!/k!)^{1/2}$, while $L_j^m(x)$ are generalized Laguerre polynomials.

2. Landau gauges LG1 and LG2

For the gauges LG1 and LG2 we take $\chi = q^j$ with $j = 1$ and $j = 2$, respectively. Thus, we have $\bar{q} = (q^0, k, q^j, q^3)$. The corresponding functions $\mathcal{F}_Q(x,\bar{q})$ are given by

$$\mathcal{F}_Q(x,\bar{q})^{(\mathrm{LG1})} = (-is)^k N_k e^{-i(q^0x^0 - q^1x^1 - q^3x^3)} D_k(\rho_s^{(1)}), \qquad (A3)$$

$$\mathcal{F}_Q(x,\bar{q})^{(\mathrm{LG2})} = N_k e^{-i(q^0x^0 - q^2x^2 - q^3x^3)} D_k(\rho_s^{(2)}), \qquad (A4)$$

where $\rho_s^{(1)} = \sqrt{2B_Q}(x^2 + sq^1/B_Q)$, $\rho_s^{(2)} = \sqrt{2B_Q}(x^1 - sq^2/B_Q)$ and $N_k = (4\pi B_Q)^{1/4}/\sqrt{k!}$. The cylindrical parabolic functions $D_k(x)$ in the above equations are defined as

$$D_k(x) = 2^{-k/2} e^{-x^2/4} H_k(x/\sqrt{2}), \qquad (A5)$$

where $H_k(x)$ are Hermite polynomials, with the standard convention $H_{-1}(x) = 0$.

APPENDIX B: WAVE FUNCTION PROPERTIES AND ANTICOMMUTATION RELATIONS FOR SPIN 1/2 CHARGED PARTICLES IN A UNIFORM MAGNETIC FIELD

As stated in Eqs. (3.26), the fermion wave functions can be written as

$$\begin{aligned} U_f(x,\bar{q},a) &= \mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})u_{\mathcal{Q}_f}(k,q^3,a), \\ V_f(x,\bar{q},a) &= \tilde{\mathbb{E}}^{-\mathcal{Q}_f}(x,\bar{q})v_{-\mathcal{Q}_f}(k,q^3,a), \end{aligned} \tag{B1}$$

where the functions $\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})$ and $\tilde{\mathbb{E}}^{-\mathcal{Q}_f}(x,\bar{q})$ are defined by Eqs. (3.27). It is easy to see that the matrices $\Gamma^{\pm}$ appearing in these definitions satisfy

$$\begin{aligned} \Gamma^\lambda\Gamma^\lambda &= \Gamma^\lambda, \qquad \Gamma^\lambda\Gamma^{-\lambda} = 0, \\ \Gamma^\lambda\gamma^\mu_\parallel &= \gamma^\mu_\parallel\Gamma^\lambda, \qquad \Gamma^\lambda\gamma^\mu_\perp = \gamma^\mu_\perp\Gamma^{-\lambda}. \end{aligned} \tag{B2}$$

It can be shown that the functions $\mathbb{E}_{\bar{p}}(x)$ satisfy orthogonality and completeness relations, namely

$$\int d^4x \bar{\mathbb{E}}^{\mathcal{Q}_f}(x,\bar{q})\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q}') = \hat{\delta}_{\bar{q}\bar{q}'}[\mathcal{I} + \delta_{k0}(\Gamma^s - \mathcal{I})] \tag{B3}$$

and

$$\sum\!\!\!\!\!\!\!\int_{\bar{q}} \mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})\bar{\mathbb{E}}^{\mathcal{Q}_f}(x',\bar{q}) = \delta^{(4)}(x-x')\mathcal{I}, \tag{B4}$$

where $\mathcal{I}$ stands for the identity in Dirac space, and, as in the main text, we have used the definitions $\mathcal{Q}_f = Q_f/e$, $s = \mathrm{sign}(Q_f B)$, and $\bar{\mathbb{E}}^{\mathcal{Q}_f}(x,\bar{q}) = \gamma^0\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})^\dagger\gamma^0$. In addition, they satisfy the useful relation

$$i\not{D}\mathbb{E}^{\mathcal{Q}_f}(x,\bar{q}) = \mathbb{E}^{\mathcal{Q}_f}(x,\bar{q})\hat{\not{\Pi}}_s(q^0,k,q^3), \tag{B5}$$

where $\hat{\Pi}^\mu_s(q^0,k,q^3) = (q^0, 0, -s\sqrt{2k|Q_f B|}, q^3)$.

On the other hand, the spinors $u_{\mathcal{Q}_f}(k,q^3,a)$ and $v_{-\mathcal{Q}_f}(k,q^3,a)$, $a = 1, 2$, in Eqs. (B1) are given by

$$u_{\mathcal{Q}_f}(k,q^3,a) = \frac{1}{\sqrt{2(E_f+m_f)}}[\hat{\not{\Pi}}_s(E_f,k,q^3) + m_f\mathcal{I}]\begin{pmatrix}\phi^{(a)} \\ \phi^{(a)}\end{pmatrix}, \tag{B6}$$

$$v_{-\mathcal{Q}_f}(k,q^3,a) = \frac{1}{\sqrt{2(E_f+m_f)}}[-\hat{\not{\Pi}}_{-s}(E_f,k,q^3) + m_f\mathcal{I}]\begin{pmatrix}\tilde{\phi}^{(a)} \\ -\tilde{\phi}^{(a)}\end{pmatrix}, \tag{B7}$$

where $\phi^{(1)\dagger} = -\tilde{\phi}^{(2)\dagger} = (1,0)$ and $\phi^{(2)\dagger} = \tilde{\phi}^{(1)\dagger} = (0,1)$. We use here the Weyl representation for Dirac matrices, namely

$$\gamma_0 = \begin{pmatrix} 0 & \mathcal{I} \\ \mathcal{I} & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \tag{B8}$$

where σ_i, with $i = 1, 2, 3$, are the Pauli matrices. It can be shown that the spinors satisfy the relations

$$\begin{aligned} \sum_{a=1,2} u_{\mathcal{Q}_f}(k,q^3,a)\bar{u}_{\mathcal{Q}_f}(k,q^3,a) &= \hat{\not{\Pi}}_s(E_f,k,q^3) + m_f\mathcal{I}, \\ \sum_{a=1,2} v_{-\mathcal{Q}_f}(k,q^3,a)\bar{v}_{-\mathcal{Q}_f}(k,q^3,a) &= \hat{\not{\Pi}}_{-s}(E_f,k,q^3) - m_f\mathcal{I}. \end{aligned} \tag{B9}$$

We finally quote the anticommutation relations between creation and annihilation operators in Eq. (3.25). They read as

$$\begin{aligned} \{b_f(\breve{q},a), b_f(\breve{q}',a')\} &= \{d_f(\breve{q},a), d_f(\breve{q}',a')\} = 0, \\ \{b_f(\breve{q},a), d_f(\breve{q}',a')\} &= \{b_f(\breve{q},a), d_f(\breve{q}',a')^\dagger\} = 0, \\ \{b_f(\breve{q},a), b_f(\breve{q}',a')^\dagger\} &= \{d_f(\breve{q},a), d_f(\breve{q}',a')^\dagger\} \\ &= 2E_f\delta_{aa'}(2\pi)^3\delta_{kk'}\delta_{\chi\chi'}\delta(q^3-q'^3). \end{aligned} \tag{B10}$$

APPENDIX C: WAVE FUNCTION PROPERTIES AND COMMUTATION RELATIONS FOR MASSIVE SPIN 1 CHARGED PARTICLES IN A UNIFORM MAGNETIC FIELD

According to Eq. (3.35), the wave functions $W^\mu_{\mathcal{Q}}(x,\bar{q},c)$ are given by

$$W^\mu_{\mathcal{Q}}(x,\bar{q},c) = \mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q})\epsilon_{\mathcal{Q},\nu}(k,q^3,c), \tag{C1}$$

where $\mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q})$ is given by Eqs. (3.36) and (3.37), while $\epsilon_{\mathcal{Q},\nu}(k,q^3,c)$ are the charged rho meson polarization vectors. As in the main text, we define $\mathcal{Q} = \mathrm{sign}(Q_\rho)$.

The tensors $\mathbb{R}^{\mathcal{Q},\mu\nu}$ involve the functions $\mathcal{F}_{\mathcal{Q}}(x,\bar{q})$ and the tensors $\Upsilon_\lambda^{\mu\nu}$, defined by Eq. (3.37). The latter obey some useful relations, namely

$$\Upsilon_\lambda^{\mu\nu} = (\Upsilon_\lambda^{\nu\mu})^* = \Upsilon_{-\lambda}^{\nu\mu}, \qquad \Upsilon_\lambda^{\mu\alpha}\Upsilon_{\lambda',\alpha\nu} = \delta_{\lambda\lambda'}\Upsilon_\lambda{}^\mu{}_\nu, \quad \sum_{\lambda=-1,0,1}\Upsilon_\lambda^{\mu\nu} = g^{\mu\nu}. \tag{C2}$$

It is also useful to introduce the projector $(\mathcal{P}_{k,s})^{\mu\nu}$, defined by

$$\begin{aligned}(\mathcal{P}_{k,s})^{\mu\nu} &= g^{\mu\nu} - \delta_{k,-1}\Upsilon_0^{\mu\nu} - (\delta_{k,-1}+\delta_{k,0})\Upsilon_s^{\mu\nu}\\ &= \Upsilon_{-s}^{\mu\nu} + (1-\delta_{k,-1})\Upsilon_0^{\mu\nu} + (1-\delta_{k,-1}-\delta_{k,0})\Upsilon_s^{\mu\nu},\end{aligned} \tag{C3}$$

which satisfies $(\mathcal{P}_{k,s})^{\mu\alpha}(\mathcal{P}_{k,s})_{\alpha\nu} = (\mathcal{P}_{k,s})^\mu{}_\nu$. Here $s = \mathrm{sign}(Q_\rho B) = \pm 1$.

The functions $\mathbb{R}^{\mathcal{Q},\mu\nu}$ are shown to satisfy orthogonality and completeness relations, viz.

$$\int d^4x \mathbb{R}^{\mathcal{Q},\mu\alpha}(x,\bar{q})\mathbb{R}^{\mathcal{Q}}_{\nu\alpha}(x,\bar{q}')^* = \hat{\delta}_{\bar{q}\bar{q}'}(\mathcal{P}_{k,s})^\mu{}_\nu \tag{C4}$$

and

$$\sum\!\!\!\!\!\!\int_{\bar{q}} \mathbb{R}^{\mathcal{Q},\mu\alpha}(x,\bar{q})\mathbb{R}^{\mathcal{Q}}_{\nu\alpha}(x',\bar{q})^* = \delta^{(4)}(x-x')\delta^\mu{}_\nu. \tag{C5}$$

It can also be seen that

$$(\mathcal{P}_{k,s})^\mu{}_\alpha \mathbb{R}^{\mathcal{Q},\alpha\nu}(x,\bar{q}) = \mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q}). \tag{C6}$$

For $k \geq 0$, one can find some other useful relations that involve the four-vector $\Pi^\mu(k,q_\parallel)$ defined in Eq. (4.14). These are

$$\mathcal{D}_\mu \mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q}) = -i\mathcal{F}_{\mathcal{Q}}(x,\bar{q})\Pi^\nu(k,q_\parallel)^*, \tag{C7}$$

$$\mathcal{D}_\mu \mathcal{F}_{\mathcal{Q}}(x,\bar{q}) = -i\mathbb{R}^{\mathcal{Q}}_{\mu\nu}(x,\bar{q})\Pi^\nu(k,q_\parallel), \tag{C8}$$

$$(\mathcal{P}_{k,s})^\mu{}_\alpha \Pi^\alpha(k,q_\parallel) = \Pi^\mu(k,q_\parallel). \tag{C9}$$

Let us consider now the polarization vectors $\epsilon_\nu(k,q^3,c)$. Their form is dictated by the transversality condition $\mathcal{D}^\mu \rho_\mu^{\mathcal{Q}}(x) = 0$ in Eq. (3.32), which implies that for $q^0 = E_\rho$ one must have

$$\mathcal{D}_\mu W^\mu_{\mathcal{Q}}(x,\bar{q},c) = \mathcal{D}_\mu \mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q})\epsilon_{\mathcal{Q},\nu}(k,q^3,c) = 0. \tag{C10}$$

Taking into account Eq. (C7), it is seen that the transversality is trivially satisfied for $k=-1$, since in that case $\mathcal{F}_{\mathcal{Q}}(x,\bar{q})$ is zero. For $k \geq 0$, according to Eq. (C7) the condition (C10) can be expressed as

$$\Pi^\mu(k,q_\parallel)^*|_{q^0=E_\rho}\epsilon_{\mathcal{Q},\mu}(k,q^3,c) = 0. \tag{C11}$$

For $k \geq 1$ there are three linearly independent vectors that satisfy Eq. (C11). A convenient choice is

$$\begin{aligned}\epsilon^\mu_{\mathcal{Q}}(k,q^3,1) &= \frac{1}{\sqrt{2}}\frac{1}{m_\perp m_{2\perp}}[\Pi_+(E_\rho,0,0,q^3) + m_\perp^2(0,1,is,0)],\\ \epsilon^\mu_{\mathcal{Q}}(k,q^3,2) &= \frac{1}{m_\perp}(q^3,0,0,E_\rho),\\ \epsilon^\mu_{\mathcal{Q}}(k,q^3,3) &= \frac{1}{\sqrt{2}}\frac{1}{m_\rho m_{2\perp}}\left[\Pi_-\left(E_\rho,\frac{\Pi_+^*}{2},is\frac{\Pi_+^*}{2},q^3\right) + m_{2\perp}^2(0,1,-is,0)\right],\end{aligned} \tag{C12}$$

where we have used the definitions

$$\begin{aligned}m_\perp &= \sqrt{m_\rho^2 + (2k+1)B_\rho},\\ m_{2\perp} &= \sqrt{m_\rho^2 + kB_\rho},\\ \Pi_+ &= -\Pi^1(k,q_\parallel) + is\,\Pi^2(k,q_\parallel) = -i\sqrt{2(k+1)B_\rho},\\ \Pi_- &= -\Pi^1(k,q_\parallel) - is\,\Pi^2(k,q_\parallel) = i\sqrt{2kB_\rho},\end{aligned} \tag{C13}$$

with $B_\rho = |Q_\rho B|$. Using these polarization vectors one recovers the known expressions for a vector boson in a constant magnetic field; see, e.g., Ref. [68].

For $k=0$ two independent nontrivial transverse polarization vectors can be constructed. A suitable choice is

$$\begin{aligned}\epsilon^\mu_{\mathcal{Q}}(0,q^3,1) &= \frac{1}{\sqrt{2}}\frac{1}{m_\perp m_{2\perp}}(E_\rho\Pi_+, m_\perp^2, is\,m_\perp^2, q^3\Pi_+),\\ \epsilon^\mu_{\mathcal{Q}}(0,q^3,2) &= \frac{1}{m_\perp}(q^3,0,0,E_\rho),\end{aligned} \tag{C14}$$

where $m_\perp$, $m_{2\perp}$, Π_+, and E_ρ are understood to be evaluated at $k=0$. It can be seen that $\epsilon^\mu_{\mathcal{Q}}(0,q^3,2)$ satisfies

$$S_3^{\mu\nu}\epsilon_{\mathcal{Q},\nu}(0,q^3,2) = 0, \tag{C15}$$

while $\epsilon^\mu(0,q^3,1)$ is not an eigenvector of S_3.

For $k = -1$, one has $\mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q}) \propto \Upsilon^{\mu\nu}_{-s}$. This leaves only one nontrivial polarization vector, which can be conveniently written as

$$\epsilon^{\mu}_{\mathcal{Q}}(-1,q^3,1) = \frac{1}{\sqrt{2}}(0,1,is,0). \tag{C16}$$

As in the case of $\epsilon_{\mathcal{Q},\nu}(0,q^3,2)$, it is easy to see that this vector has a definite spin projection in the direction of the magnetic field. Indeed, one has

$$S_3^{\mu\nu}\epsilon_{\mathcal{Q},\nu}(-1,q^3,1) = s\epsilon^{\mu}_{\mathcal{Q}}(-1,q^3,1). \tag{C17}$$

Finally, for $k \geq 0$ one can also define an additional, “longitudinal,” polarization vector. We keep for this vector the notation $\epsilon^{\mu}_{\mathcal{Q}}(k,q^3,c)$, taking for the polarization index the value $c = 0$. It is given by

$$\epsilon^{\mu}_{\mathcal{Q}}(k,q^3,0) = \frac{1}{m_\rho}\Pi^{\mu}(k,q_\parallel)\big|_{q^0=E_\rho}, \tag{C18}$$

where, as stated, $k \geq 0$. For $k = -1$ no longitudinal vector is introduced.

It is worth noticing that the full set of four polarization vectors satisfies orthogonality and completeness relations, namely

$$\epsilon^{\mu}_{\mathcal{Q}}(k,q^3,c)^*\epsilon_{\mathcal{Q},\mu}(k,q^3,c') = -\zeta_c\delta_{cc'} \tag{C19}$$

and

$$\sum_{c=c_{\min}}^{c_{\max}}\zeta_c\epsilon^{\mu}_{\mathcal{Q}}(k,q^3,c)\epsilon^{\nu}_{\mathcal{Q}}(k,q^3,c)^* = -(\mathcal{P}_{k,s})^{\mu\nu}, \tag{C20}$$

where $\zeta_0 = -1$ and $\zeta_1 = \zeta_2 = \zeta_3 = 1$, while $c_{\min}$ and $c_{\max}$ are given by

$$c_{\min} = \begin{cases} 1 & \text{if } k=-1 \\ 0 & \text{if } k\geq 0 \end{cases}, \quad c_{\max} = \begin{cases} 1 & \text{if } k=-1 \\ 2 & \text{if } k=0 \\ 3 & \text{if } k\geq 1 \end{cases}. \tag{C21}$$

For $k \geq 1$, from Eqs. (C20), (C18), and (C2) it is seen that the sum over the physical polarizations $c = 1, 2, 3$ satisfies

$$\sum_{c=1}^{3}\epsilon^{\mu}_{\mathcal{Q}}(k,q^3,c)\epsilon^{\nu}_{\mathcal{Q}}(k,q^3,c)^* = -\left[g^{\mu\nu} - \frac{\Pi^{\mu}(k,q_\parallel)\Pi^{\nu}(k,q_\parallel)^*}{m_\rho^2}\right], \tag{C22}$$

where the vectors $\Pi^{\mu}(k,q_\parallel)$ are assumed to be “on shell,” i.e., one has to take $q^0 = E_\rho$.

As stated in the main text, one can also extend the set of charged rho meson wave functions $W^{\mu}_{\mathcal{Q}}(x,\bar{q},c)$ including a “longitudinal” wave function $W^{\mu}_{\mathcal{Q}}(x,\bar{q},0) \equiv \mathbb{R}^{\mathcal{Q},\mu\nu}(x,\bar{q})\epsilon_{\mathcal{Q},\nu}(k,q^3,0)$. In this way one gets for these functions the orthogonality and completeness relations in Eqs. (3.38) and (3.39).

We conclude this Appendix by quoting the commutation relations for the creation and annihilation operators in Eq. (3.34). One has

$$\begin{aligned}
[a^{\mathcal{Q}}_{\rho}(\breve{q},c), a^{\pm\mathcal{Q}}_{\rho}(\breve{q}',c')] &= [a^{\mathcal{Q}}_{\rho}(\breve{q},c)^\dagger, a^{\pm\mathcal{Q}}_{\rho}(\breve{q}',c')^\dagger] \\
&= [a^{\mathcal{Q}}_{\rho}(\breve{q},c), a^{-\mathcal{Q}}_{\rho}(\breve{q}',c')^\dagger] = 0, \\
[a^{\mathcal{Q}}_{\rho}(\breve{q},c), a^{\mathcal{Q}}_{\rho}(\breve{q}',c')^\dagger] &= [a^{-\mathcal{Q}}_{\rho}(\breve{q},c), a^{-\mathcal{Q}}_{\rho}(\breve{q}',c')^\dagger] \\
&= 2E_\rho(2\pi)^3\delta_{cc'}\delta_{kk'}\delta_{\chi\chi'}\delta(q^3-q'^3).
\end{aligned} \tag{C23}$$

APPENDIX D: EXPLICIT FORM OF THE COEFFICIENTS OF THE OPERATORS FOR THE ONE-LOOP ρ^+ MESON POLARIZATION FUNCTION

As stated in Sec. V B, the one-loop correction to the ρ^+ propagator in $\bar{q}$ space can be written in terms of a set of tensors $\mathbb{O}_i^{\alpha\alpha'}(\Pi)$, with $i = 1, \ldots 7$. We give here the explicit expressions for the corresponding coefficients $d_i(k,q_\parallel)$, introduced in Eq. (5.55). The latter have been obtained taking into account the Schwinger form of quark propagators in Eq. (4.10). In general they can be written in the form

$$d_i(k,q_\parallel) = -i\frac{N_c}{4\pi^2}\int_{-1}^{1}dx\int_0^{\infty}dz\frac{e^{-z\phi(x,q_\parallel^2)}}{\alpha_+}\left(\frac{\alpha_-}{\alpha_+}\right)^k f^{(i)}_{k,q_\parallel}(x,z), \tag{D1}$$

where $\phi(x,q_\parallel^2)$ and $\alpha^{\pm}$ are defined in Eqs. (5.25) and (5.28), respectively. After some calculation, the functions $f^{(i)}_{k,q_\parallel}(x,z)$ are found to be given by

$$f^{(1)}_{k,q_\parallel}(x,z) = -(1-t_u t_d)\left[m_u m_d + (1-x^2)\frac{q_\parallel^2}{4}\right] - \frac{\alpha_- + k(\alpha_- - \alpha_+)}{\alpha_+\alpha_-}(1-t_u^2)(1-t_d^2), \qquad k\geq 0$$
$$f^{(2)}_{k,q_\parallel}(x,z) = f^{(2a)}_{k,q_\parallel}(x,z) + f^{(2b)}_{k,q_\parallel}(x,z) + (2k+1) f^{(2c)}_{k,q_\parallel}(x,z)$$
$$f^{(3)}_{k,q_\parallel}(x,z) = \frac{1}{2}(1-x^2)(1-t_u t_d), \qquad k\geq 0$$
$$f^{(4)}_{k,q_\parallel}(x,z) = \frac{1}{B_\rho}\frac{\alpha_+ - \alpha_-}{\alpha_+\alpha_-}(1-t_u^2)(1-t_d^2), \qquad k\geq 1$$
$$f^{(5)}_{k,q_\parallel}(x,z) = f^{(5a)}_{k,q_\parallel}(x,z) + f^{(5b)}_{k,q_\parallel}(x,z),$$
$$f^{(6)}_{k,q_\parallel}(x,z) = f^{(2a)}_{k,q_\parallel}(x,z) - f^{(2b)}_{k,q_\parallel}(x,z) + f^{(2c)}_{k,q_\parallel}(x,z),$$
$$f^{(7)}_{k,q_\parallel}(x,z) = -f^{(5a)}_{k,q_\parallel}(x,z) + f^{(5b)}_{k,q_\parallel}(x,z), \tag{D2}$$

where

$$f^{(2a)}_{k,q_\parallel}(x,z) = -\frac{1}{2}\frac{\alpha_-}{\alpha_+}(1+t_u)(1+t_d)\left[m_u m_d + \frac{1}{z} + (1-x^2)\frac{q_\parallel^2}{4}\right], \qquad k\geq -1,$$
$$f^{(2b)}_{k,q_\parallel}(x,z) = -\frac{1}{2}\frac{\alpha_+}{\alpha_-}(1-t_u)(1-t_d)\left[m_u m_d + \frac{1}{z} + (1-x^2)\frac{q_\parallel^2}{4}\right], \qquad k\geq 1,$$
$$f^{(2c)}_{k,q_\parallel}(x,z) = \frac{\alpha_+ - \alpha_-}{2\alpha_+\alpha_-}(1-t_u^2)(1-t_d^2), \qquad k\geq 1,$$
$$f^{(5a)}_{k,q_\parallel}(x,z) = \frac{1}{4\alpha_+}\left[(1+x)\frac{t_u(1+t_u)(1-t_d^2)}{B_u} + (1-x)\frac{t_d(1+t_d)(1-t_u^2)}{B_d}\right], \qquad k\geq 0,$$
$$f^{(5b)}_{k,q_\parallel}(x,z) = \frac{1}{4\alpha_-}\left[(1+x)\frac{t_u(1-t_u)(1-t_d^2)}{B_u} + (1-x)\frac{t_d(1-t_d)(1-t_u^2)}{B_d}\right], \qquad k\geq 1.$$

Here, as in the main text, we have used the definitions $t_u = \tanh[(1-x)zB_u/2]$, $t_d = \tanh[(1+x)zB_d/2]$, with $B_f = |Q_f B|$ for $f = u, d$. For $k = 0$ and $k = -1$ some of the above functions vanish; therefore, for each expression we have explicitly indicated the range of values of k to be taken into account.

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