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AN ALTERNATIVE AXIOMATIC PRESENTATION OF NELSON ALGEBRAS

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ABSTRACT. Nelson algebras were defined in 1967 by D. Brignole and A. Monteiro in terms of the language $\langle \wedge, \vee, \to, \sim, 1\rangle$. In 1962, D. Brignole solved the problem, proposed by A. Monteiro, of giving an axiomatization of Nelson algebras in terms of the connectives $\rightarrowtail$, $\wedge$ and the constant $0 = {\sim}1$, where the operation $\rightarrowtail$ is defined by $x \rightarrowtail y = (x \to y) \wedge ({\sim}y \to {\sim}x)$. In this work we present for the first time a complete proof of this fact, and also show the dependence and independence of some of the axioms proposed by Brignole.

1. PRELIMINARIES

Nelson algebras or $\mathscr{N}$-algebras were introduced by H. Rasiowa [Ras58] as an algebraic counterpart of Nelson's constructive logic with strong negation [Nel49]. Later, D. Brignole and A. Monteiro [BM67, Bri69] gave a characterization using identities, proving that Nelson algebras form a variety. This characterization was given in terms of the operations $\wedge$, $\vee$, $\to$, $\sim$ and the constant 1.

A different implication operation can be defined by

$$x \rightarrowtail y = (x \to y) \wedge ({\sim}y \to {\sim}x).$$

In 1962, Diana Brignole solved the problem proposed by A. Monteiro, of giving an axiomatization of Nelson algebras in terms of the connectives $\rightarrowtail$, $\wedge$ and the constant 0. This solution was presented at the annual meeting of the Unión Matemática Argentina, and a summary (containing some typos) was published in [Bri65]; however, to the best of our knowledge, the corresponding proof has not been published.

In [SV07], Spinks and Veroff used this axiomatization to prove that the variety of Nelson algebras is term equivalent to a variety of bounded 3-potent BCK-semilattices, and in [SV08a] and [SV08b] they proved that using the operation $\rightarrowtail$ Nelson algebras can be understood as residuated lattices, with the product given by the term

$$x * y = {\sim}(x \to {\sim}y) \vee {\sim}(y \to {\sim}x).$$

As a consequence, the corresponding logic, constructive logic with strong negation, can be seen as a substructural logic (see also [BC10]).

In this note, we give a complete proof of the axiomatization proposed by Brignole, prove the independence of some of the axioms from the rest and announce that two of the identities can be derived from the others, although we only have an automated proof of this fact.

Definition 1.1. A *Nelson algebra* is an algebra $\langle A, \wedge, \vee, \to, \sim, 1\rangle$ of type $(2,2,2,1,0)$ such that the following conditions are satisfied for all x, y, z in A:

(N1) $x \wedge (x \vee y) = x$,
(N2) $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$,
(N3) ${\sim}{\sim}x = x$,
(N4) ${\sim}(x \wedge y) = {\sim}x \vee {\sim}y$,

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(N5) $x \wedge \sim x = (x \wedge \sim x) \wedge (y \vee \sim y)$,
(N6) $x \to x = 1$,
(N7) $x \wedge (x \to y) = x \wedge (\sim x \vee y)$,
(N8) $(x \wedge y) \to z = x \to (y \to z)$.

We will denote by $\mathscr{N}$ the variety of Nelson algebras.

The axioms in this list form an independent set, see [MM96].

By axioms (N1) and (N2) we have that every Nelson algebra is a distributive lattice (see Sholander [Sho51]). Furthermore, if we define $0 = \sim 1$, we have that 0 and 1 are the bottom and top element of A, respectively.

In a Nelson algebra we can also define the following operations that will be used in this work:

- $\neg x := x \to (\sim 1)$,
- $x \rightarrowtail y := (x \to y) \wedge (\sim y \to \sim x)$.

Lemma 1.2. *Let $\langle A, \wedge, \vee, \to, \sim, 1 \rangle$ be a Nelson algebra. The following properties are satisfied in A:*

(a) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(b) $1 \to x = x$,
(c) $\sim x \leq \neg x$,
(d) $(x \to x) \wedge (\sim x \to \sim x) = 1$,
(e) $\sim y \leq y \to z$,
(f) $y \leq x \to y$,
(g) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(h) $x \to y = x \rightarrowtail (x \rightarrowtail y)$,
(i) $x \to (x \to y) = x \to y$.

Proof. The proofs of these items can be found in [Vig99]. □

The following definition is the set of equations given by Brignole in [Bri65], reordered and with some typos corrected.

Definition 1.3. A *Brignole algebra* is an algebra $\mathbf{A} = \langle A, \wedge_B, \rightarrowtail, 0 \rangle$ of type $(2,2,0)$ such that the following equations are satisfied for all $x, y, z \in A$:

(B1) $(x \rightarrowtail x) \rightarrowtail y = y$,
(B2) $x \wedge_B \sim_B (x \wedge_B \sim_B y) = x \wedge_B (x \rightarrowtail y)$,
(B3) $x \rightarrowtail (y \wedge_B z) = (x \rightarrowtail y) \wedge_B (x \rightarrowtail z)$,
(B4) $x \rightarrowtail y = \sim_B y \rightarrowtail \sim_B x$,
(B5) $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = (x \wedge_B y) \rightarrowtail ((x \wedge_B y) \rightarrowtail z)$,
(B6) $\sim_B (\sim_B x \wedge_B y) \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail y$,
(B7) $x \wedge_B (y \vee_B z) = (z \wedge_B x) \vee_B (y \wedge_B x)$,
(B8) $(x \wedge_B \sim_B x) \wedge_B (y \vee_B \sim_B y) = x \wedge_B \sim_B x$,
(B9) $(x \rightarrowtail y) \wedge_B y = y$,
(B10) $x \wedge_B (x \vee_B y) = x$,

where

- $\sim_B x := x \rightarrowtail 0$,
- $x \vee_B y := ((x \rightarrowtail 0) \wedge_B (y \rightarrowtail 0)) \rightarrowtail 0$.

We will denote by $\mathscr{B}$ the variety of Brignole algebras.

We are going to show that $\mathscr{B}$ and $\mathscr{N}$ are term equivalent.

2. TERM EQUIVALENCE BETWEEN $\mathscr{B}$ AND $\mathscr{N}$

Let us consider a Nelson algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \sim, 1\rangle$. Over $\mathbf{A}$ we can define the terms

- $x \rightarrowtail y := (x \rightarrow y) \wedge (\sim y \rightarrow \sim x)$,
- $x \wedge_{\mathrm{B}} y := x \wedge y$,
- $0 := \sim 1$,
- $\sim_{\mathrm{B}} x := x \rightarrowtail 0$,
- $x \vee_{\mathrm{B}} y := ((x \rightarrowtail 0) \wedge_{\mathrm{B}} (y \rightarrowtail 0)) \rightarrowtail 0$.

We are going to prove that $\langle A, \wedge_{\mathrm{B}}, \rightarrowtail, 0\rangle$ is a Brignole algebra. In order to see that, we need the following result:

Lemma 2.1. *In a Nelson algebra* $\mathbf{A}$ *the following identities hold for all* $x, y, z \in A$*:*

(a) $\sim_B x = \sim x$,
(b) $x \vee_B y = x \vee y$,
(c) $x \wedge (x \vee_B y) = x$, $x \wedge (y \vee_B z) = (z \wedge x) \vee_B (y \wedge x)$, $(x \wedge \sim_B x) \wedge (y \vee_B \sim_B y) = x \wedge \sim_B x$,
(d) $x \rightarrowtail x = 1$,
(e) $x = x \wedge (\sim x \rightarrow y)$,
(f) $1 \rightarrowtail x = x$,
(g) $(x \rightarrowtail x) \rightarrowtail y = y$,
(h) $(x \rightarrowtail y) \wedge y = y$,
(i) $x \wedge \sim(x \wedge \sim y) = x \wedge (x \rightarrowtail y)$,
(j) $x \rightarrowtail (y \wedge z) = (x \rightarrowtail y) \wedge (x \rightarrowtail z)$,
(k) $x \rightarrowtail y = \sim_B y \rightarrowtail \sim_B x$,
(l) $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = (x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail z)$,
(m) $\sim(\sim x \wedge y) \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail y$.

Proof. (a) $\sim_{\mathrm{B}} x = x \rightarrowtail 0 = x \rightarrowtail (\sim 1) = (x \rightarrow (\sim 1)) \wedge (\sim\sim 1 \rightarrow \sim x) \underset{(\mathrm{N}3)}{=} (x \rightarrow (\sim 1)) \wedge (1 \rightarrow (\sim x)) \underset{1.2\,(\mathrm{b})}{=} (x \rightarrow (\sim 1)) \wedge \sim x \underset{\mathrm{def.}\,\neg}{=} \neg x \wedge \sim x \underset{1.2(\mathrm{c})}{=} \sim x$.

(b) $x \vee_{\mathrm{B}} y = ((x \rightarrowtail 0) \wedge (y \rightarrowtail 0)) \rightarrowtail 0 \underset{\mathrm{def.}\,\sim}{=} \sim(\sim x \wedge \sim y) \underset{(\mathrm{N}4)}{=} \sim\sim x \vee \sim\sim y \underset{(\mathrm{N}3)}{=} x \vee y$.

(c) From item (b) and the definition of $\wedge_{\mathrm{B}}$ we have that this result is immediate from axioms (N1), (N2) and (N5).

From the first two conditions of this item we conclude that $\langle A; \wedge_{\mathrm{B}}, \vee_{\mathrm{B}}\rangle$ is a distributive lattice [Sho51]. Therefore, for the rest of the proof, we will use properties of distributive lattices without explicitly mentioning them.

(d) It follows immediately from Lemma 1.2 (d).

(e) It is a consequence of Lemma 1.2 (e) and (N3).

(f) $(1 \rightarrow y) \wedge (\sim y \rightarrow \sim 1) \underset{1.2(\mathrm{b})}{=} y \wedge (\sim y \rightarrow \sim 1) \underset{(\mathrm{e})}{=} y$.

(g) $(y \rightarrowtail y) \rightarrowtail x \underset{(\mathrm{d})}{=} 1 \rightarrowtail x \underset{(\mathrm{f})}{=} x$.

(h) $(x \rightarrowtail y) \wedge y = ((x \rightarrow y) \wedge (\sim y \rightarrow \sim x)) \wedge y = ((x \rightarrow y) \wedge y) \wedge (\sim y \rightarrow \sim x) \underset{1.2(\mathrm{f})}{=} y \wedge (\sim y \rightarrow \sim x) \underset{(\mathrm{e})}{=} y$.

(i) $x \wedge (x \rightarrowtail y) = x \wedge (x \rightarrow y) \wedge (\sim y \rightarrow \sim x) \underset{(\mathrm{N}7)}{=} x \wedge (\sim x \vee y) \wedge (\sim y \rightarrow \sim x) = ((x \wedge \sim x) \vee (x \wedge y)) \wedge (\sim y \rightarrow \sim x) = ((x \wedge \sim x) \wedge (\sim y \rightarrow \sim x)) \vee ((x \wedge y) \wedge (\sim y \rightarrow \sim x)) \underset{(\mathrm{e})}{=} (x \wedge \sim x) \vee ((x \wedge y) \wedge (\sim y \rightarrow \sim x)) = x \wedge (\sim x \vee (y \wedge (\sim y \rightarrow \sim x))) \underset{(\mathrm{e})}{=} x \wedge (\sim x \vee y) \underset{(\mathrm{N}3)}{=} x \wedge (\sim x \vee \sim\sim y) \underset{(\mathrm{N}4)}{=} x \wedge \sim(x \wedge \sim y)$.

(j) $x \rightarrowtail (y \wedge z) = (x \to (y \wedge z)) \wedge (\sim(y \wedge z) \to \sim x) \underset{1.2\,(a)}{=} (x \to y) \wedge (x \to z) \wedge (\sim(y \wedge z) \to \sim x) \underset{(N4)}{=} (x \to y) \wedge (x \to z) \wedge ((\sim y \vee \sim z) \to \sim x) \underset{1.2(g)}{=} (x \to y) \wedge (x \to z) \wedge (\sim y \to \sim x) \wedge (\sim z \to \sim x) = ((x \to y) \wedge (\sim y \to \sim x)) \wedge ((x \to z) \wedge (\sim z \to \sim x)) = (x \rightarrowtail y) \wedge (x \rightarrowtail z)$.

(k) It is an immediate consequence of the definition of $\rightarrowtail$ and (N3).

(l) $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) \underset{1.2(h)}{=} x \to (y \to z) \underset{(N8)}{=} (x \wedge y) \to z \underset{1.2\,(h)}{=} (x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail z)$.

(m) Observe that $\sim(\sim x \wedge y) \rightarrowtail (x \rightarrowtail y) \geq x \rightarrowtail y$ follows from (h). Let us see the other inequality. From the definition of $\rightarrowtail$, we can deduce the following: $u \rightarrowtail v \leq u \to v$.

Therefore, we have that $\sim(\sim x \wedge y) \rightarrowtail (x \rightarrowtail y) \leq \sim(\sim x \wedge y) \to (x \rightarrowtail y) \underset{\text{def.}}{=} (x \vee \sim y) \to ((x \to y) \wedge (\sim y \to \sim x)) \underset{1.2\,(a)}{=} ((x \vee \sim y) \to (x \to y)) \wedge ((x \vee \sim y) \to (\sim y \to \sim x))$.

Therefore,

$$\sim(\sim x \wedge y) \rightarrowtail (x \rightarrowtail y) \leq ((x \vee \sim y) \to (x \to y)) \wedge ((x \vee \sim y) \to (\sim y \to \sim x)). \tag{1}$$

Now, $(x \vee \sim y) \to (x \to y) \underset{1.2(g)}{=} (x \to (x \to y)) \wedge (\sim y \to (x \to y)) \underset{1.2(i)}{=} (x \to y) \wedge (\sim y \to (x \to y)) \leq x \to y$. Hence,

$$(x \vee \sim y) \to (x \to y) \leq x \to y. \tag{2}$$

On the other hand, $(x \vee \sim y) \to (\sim y \to \sim x) \underset{1.2(g)}{=} (x \to (\sim y \to \sim x)) \wedge (\sim y \to (\sim y \to \sim x)) \underset{1.2(i)}{=} (x \to (\sim y \to \sim x)) \wedge (\sim y \to \sim x) \underset{1.2(f)}{=} \sim y \to \sim x$. Then,

$$(x \vee \sim y) \to (\sim y \to \sim x) = \sim y \to \sim x. \tag{3}$$

From (1), (2) and (3) we conclude that

$$\sim(\sim x \wedge y) \rightarrowtail (x \rightarrowtail y) \leq (x \to y) \wedge (\sim y \to \sim x) = x \rightarrowtail y.$$

□

Theorem 2.2. *Let* $\mathbf{A} = \langle A, \wedge, \vee, \to, \sim, 1 \rangle$ *be a Nelson algebra. Then* $\langle A, \wedge_B, \rightarrowtail, 0 \rangle$ *is a Brignole algebra.*

Proof. From items (g), (h), (i), (j), (k), (l) and (m) of Lemma 2.1 it follows that **A** satisfies (B1), (B9), (B2), (B3), (B4), (B5) and (B6), respectively. The axioms (B10), (B7) and (B8) are verified considering Lemma 2.1 (c). □

Now, let us consider an algebra $\mathbf{A} = \langle A, \wedge_B, \rightarrowtail, 0 \rangle$ of type $(2,2,0)$ that satisfies equations (B1) to (B8). We define over **A** the following:

- $x \wedge y := x \wedge_B y$,
- $x \vee y := ((x \rightarrowtail 0) \wedge_B (y \rightarrowtail 0)) \rightarrowtail 0$,
- $x \to y := x \rightarrowtail (x \rightarrowtail y)$,
- $\sim x := x \rightarrowtail 0$,
- $1 := 0 \rightarrowtail 0$.

Our goal now is to prove that $\langle A, \wedge, \vee, \to, \sim, 1 \rangle$ is a Nelson algebra. Since we are going to prove later that the equations (B9) and (B10) can be proved from the other ones, we separate here the identities that can be proved without using those two equations.

Lemma 2.3. *In an algebra $\langle A, \wedge_B, \rightarrowtail, 0\rangle$ of type $(2,2,0)$ satisfying equations* (B1) *to* (B8)*, the following conditions are satisfied for all $x,y,z \in A$:*

(a) $\sim 1 = 0$,
(b) $1 = 1 \rightarrowtail 1$,
(c) $1 \rightarrowtail x = x$,
(d) $\sim\sim x = x$,
(e) $\sim(x \vee y) = \sim x \wedge \sim y$,
(f) $\sim(x \wedge y) = \sim x \vee \sim y$,
(g) $x \wedge (x \to y) = x \wedge (\sim x \vee y)$,
(h) $(x \wedge y) \to z = x \to (y \to z)$,
(i) $x \rightarrowtail x = 1$, *and in particular,* $x \rightarrowtail x = y \rightarrowtail y$.

Proof. (a) $\sim 1 = \sim(0 \rightarrowtail 0) \underset{\text{def.}}{=} (0 \rightarrowtail 0) \rightarrowtail 0 \underset{(B1)}{=} 0.$

(b) $1 = 0 \rightarrowtail 0 \underset{(B4)}{=} \sim 0 \rightarrowtail \sim 0 = (0 \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail 0) \underset{\text{def.}}{=} 1 \rightarrowtail 1.$

(c) $1 \rightarrowtail x \underset{(b)}{=} (1 \rightarrowtail 1) \rightarrowtail x \underset{(B1)}{=} x.$

(d) $x \underset{(B1)}{=} (y \rightarrowtail y) \rightarrowtail x \underset{(B4)}{=} (x \rightarrowtail 0) \rightarrowtail ((y \rightarrowtail y) \rightarrowtail 0) \underset{(B1)}{=} (x \rightarrowtail 0) \rightarrowtail 0 \underset{\text{def.}\sim}{=} \sim\sim x.$

(e) $\sim(x \vee y) = \sim(((x \rightarrowtail 0) \wedge (y \rightarrowtail 0)) \rightarrowtail 0) = \sim(\sim(\sim x \wedge \sim y)) \underset{(d)}{=} \sim x \wedge \sim y.$

(f) It follows from items (d) and (e).

(g) $x \wedge (x \to y) \underset{\text{def.}}{=} x \wedge (x \rightarrowtail (x \rightarrowtail y)) \underset{(B2)}{=} x \wedge \sim(x \wedge \sim(x \rightarrowtail y)) \underset{(d)\text{ and }(f)}{=} x \wedge (\sim x \vee (x \rightarrowtail y)) = (x \wedge \sim x) \vee (x \wedge (x \rightarrowtail y)) \underset{(B2)}{=} (x \wedge \sim x) \vee (x \wedge \sim(x \wedge \sim y)) = x \wedge (\sim x \vee \sim(x \wedge \sim y)) \underset{(f)}{=} x \wedge (\sim x \vee (\sim x \vee \sim\sim y)) \underset{(d)}{=} x \wedge (\sim x \vee \sim x \vee y) = x \wedge (\sim x \vee y).$

(h) By (B5), $(x \wedge y) \to z = (x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail z) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = x \to (y \to z)$.

(i) $x \rightarrowtail x \underset{(d)}{=} ((x \rightarrowtail x) \rightarrowtail 0) \rightarrowtail 0) \underset{(B1)}{=} 0 \rightarrowtail 0 \underset{\text{def.}}{=} 1.$ □

Lemma 2.4. *In a Brignole algebra* **A** *the following conditions are satisfied for all $x,y,z \in A$:*

(a) $x \wedge (x \vee y) = x$, $x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x)$ *and* $(x \wedge \sim x) \wedge (y \vee \sim y) = x \wedge \sim x$,
(b) $x \vee 1 = 1$,
(c) $x \rightarrowtail 1 = 1$,
(d) $x \to x = 1$.

Proof. (a) It is an immediate consequence of (B10), (B7) and (B8).

From now on, we will use the fact that the reduct $\langle A, \wedge, \vee\rangle$ is a distributive lattice [Sho51], with all its inherent properties.

(b) $x \vee 1 = x \vee \sim 0 \underset{\text{def.}}{=} \sim(\sim x \wedge \sim\sim 0) \underset{\text{Lemma 2.3(d)}}{=} \sim(\sim x \wedge 0) \underset{\text{def.}}{=} ((x \rightarrowtail 0) \wedge 0) \rightarrowtail 0 \underset{(B9)}{=} 0 \rightarrowtail 0 = 1.$

By this result, we can conclude that 1 is the top element of A.

(c) $1 \wedge (x \rightarrowtail 1) \underset{(B9)}{=} 1$, then $1 \leq x \rightarrowtail 1$. By (b), the equality follows.

(d) $x \to x = x \rightarrowtail (x \rightarrowtail x) \underset{\text{Lemma 2.3(i)}}{=} x \rightarrowtail 1 \underset{(c)}{=} 1.$ □

Theorem 2.5. *Let* $\langle A, \wedge_B, \rightarrowtail, 0\rangle$ *be a Brignole algebra. Then* $\mathbf{A} = \langle A, \wedge, \vee, \to, \sim, 1\rangle$ *is a Nelson algebra.*

Proof. By Lemma 2.4 (a), **A** satisfies (N1), (N2) and (N5). The items (d), (f), (g) and (h) from Lemma 2.3 prove the validity of (N3), (N4), (N7) and (N8), respectively, while Lemma 2.4 (d) proves (N6). □

Theorem 2.6. *The varieties of Nelson and Brignole algebras are term equivalent.*

Proof. If a Nelson algebra $\langle A, \wedge, \vee, \to, \sim, 1\rangle$ is obtained from a Brignole algebra $\langle A, \wedge, \rightarrowtail, 0\rangle$ as in Theorem 2.2, and if we define $x \Rightarrow y := (x \to y) \wedge (\sim y \to \sim x)$, we obtain $x \Rightarrow y = x \rightarrowtail y$:

$$\begin{aligned} x \Rightarrow y = (x \to y) \wedge (\sim y \to \sim x) &\underset{\text{def.}}{=} (x \rightarrowtail (x \rightarrowtail y)) \wedge (\sim y \rightarrowtail (\sim y \rightarrowtail \sim x)) \\ &\underset{(\text{B4})}{=} (x \rightarrowtail (x \rightarrowtail y)) \wedge (\sim y \rightarrowtail (x \rightarrowtail y)) \underset{(\text{B4})}{=} (\sim(x \rightarrowtail y) \rightarrowtail \sim x) \wedge (\sim(x \rightarrowtail y) \rightarrowtail y) \\ &\underset{(\text{B3})}{=} \sim(x \rightarrowtail y) \rightarrowtail (\sim x \wedge y) \underset{(\text{B4})}{=} \sim(\sim x \wedge y) \rightarrowtail (x \rightarrowtail y) \underset{(\text{B6})}{=} x \rightarrowtail y. \end{aligned}$$

If a Brignole algebra $\langle A, \wedge_B, \rightarrowtail, 0\rangle$ is obtained from a Nelson algebra $\langle A, \wedge, \vee, \to, \sim, 1\rangle$ as in Theorem 2.5, when we define $x \rightsquigarrow y := x \rightarrowtail (x \rightarrowtail y)$, we obtain that $x \rightsquigarrow y = x \to y$. This is a consequence of Lemma 1.2 (h). □

3. Independence of Brignole axioms

A natural question is which axioms of Definition 1.3 are independent. We have the following result:

Theorem 3.1. *In the variety* $\mathscr{B}$ *the axioms* (B1)*,* (B2)*,* (B4)*,* (B6) *and* (B7) *are independent.*

Proof. The examples in this section have been found by the programs Prover9 and Mace4 [McC10]. For each example, we indicate the elements for which the equation fails, while the rest of them have been checked to hold.

3.1. Independence of (B1).

$\rightarrowtail$	0	a	1
0	1	1	1
a	a	a	1
1	0	a	1

$\wedge_B$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

Axiom (B1) fails considering $x = a$ and $y = 0$. Indeed: $(a \rightarrowtail a) \rightarrowtail 0 = a \rightarrowtail 0 = a \neq 0$.

3.2. Independence of (B2).

$\rightarrowtail$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	1	1	1
1	0	a	b	1

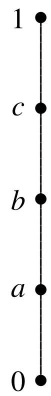
$\wedge_B$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Axiom (B2) fails considering $x = a$ and $y = b$.
Indeed: $a \wedge_{\mathrm{B}} ((a \wedge_{\mathrm{B}} (b \rightarrowtail 0)) \rightarrowtail 0) = a \wedge_{\mathrm{B}} ((a \wedge_{\mathrm{B}} 0) \rightarrowtail 0) = a \wedge_{\mathrm{B}} (0 \rightarrowtail 0) = a \wedge_{\mathrm{B}} 0 = 0$, and $a \wedge_{\mathrm{B}} (a \rightarrowtail b) = a \wedge_{\mathrm{B}} 1 = a$.

3.3. **Independence of (B4).**

$\rightarrowtail$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	1	1	1	1
c	a	a	b	1	1
1	0	a	b	c	1

$\wedge_{\mathrm{B}}$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

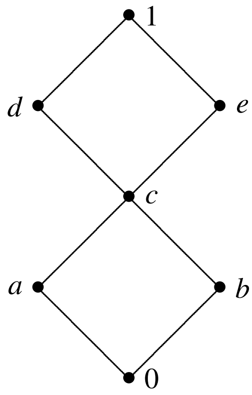


Axiom (B4) fails considering $x = c$ and $y = b$.
Indeed: $c \rightarrowtail b = b$, and $(b \rightarrowtail 0) \rightarrowtail (c \rightarrowtail 0) = b \rightarrowtail a = 1$.

3.4. **Independence of (B5).**

$\rightarrowtail$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	e	1	e	1	1	1	1
b	d	d	1	1	1	1	1
c	c	d	e	1	1	1	1
d	b	c	b	e	1	e	1
e	a	a	c	d	d	1	1
1	0	a	b	c	d	e	1

$\wedge_{\mathrm{B}}$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
c	0	a	b	c	c	c	c
d	0	a	b	c	d	c	d
e	0	a	b	c	c	e	e
1	0	a	b	c	d	e	1



Axiom (B5) fails considering $x = e$, $y = d$ and $z = 0$.

Indeed: $e \rightarrowtail (e \rightarrowtail (d \rightarrowtail (d \rightarrowtail 0))) = e \rightarrowtail (e \rightarrowtail (d \rightarrowtail b)) = e \rightarrowtail (e \rightarrowtail b) = e \rightarrowtail c = d$, and $(e \wedge_{\mathrm{B}} d) \rightarrowtail ((e \wedge_{\mathrm{B}} d) \rightarrowtail 0) = c \rightarrowtail (c \rightarrowtail 0) = c \rightarrowtail c = 1$.

3.5. **Independence of (B7).**

$\rightarrowtail$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	b	1	1	1
1	0	a	b	1

$\wedge_{\mathrm{B}}$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
1	0	a	b	1

Axiom (B7) fails considering $x = b$, $y = 0$ and $z = a$.

Indeed: $b \wedge_{\mathrm{B}} (0 \vee_{\mathrm{B}} a) = b \wedge_{\mathrm{B}} (((0 \rightarrowtail 0) \wedge_{\mathrm{B}} (a \rightarrowtail 0)) \rightarrowtail 0) = b \wedge_{\mathrm{B}} ((1 \wedge_{\mathrm{B}} a) \rightarrowtail 0) = b \wedge_{\mathrm{B}} (a \rightarrowtail 0) = b \wedge_{\mathrm{B}} a = b$, and $(a \wedge_{\mathrm{B}} b) \vee_{\mathrm{B}} (0 \wedge_{\mathrm{B}} b) = a \vee_{\mathrm{B}} 0 = ((a \rightarrowtail 0) \wedge_{\mathrm{B}} (0 \rightarrowtail 0)) \rightarrowtail 0 = (a \wedge_{\mathrm{B}} 1) \rightarrowtail 0 = a \rightarrowtail 0 = a$. □

4. Dependent axioms

In this section we will prove that axioms (B9) and (B10) can be derived from the other axioms for Brignole algebras.

Lemma 4.1. *Let* **A** *be an algebra* $\langle A, \wedge_B, \rightarrowtail, 0\rangle$ *satisfying the axioms* (B1) *to* (B8). *The following properties are satisfied for all* $x, y, z \in A$*:*

(a) $x = (x \rightarrowtail 0) \rightarrowtail (x \wedge_B 0)$,
(b) $0 \wedge_B 0 = 0$,
(c) $x \wedge_B y = y \wedge_B x$,
(d) $x \wedge_B x = x$,
(e) $0 \wedge_B 1 = 0$,
(f) $x \wedge_B 0 = 0$,
(g) $x \vee_B x = x$,
(h) $x \vee_B y = y \vee_B x$,
(i) $x \wedge_B (y \vee_B z) = (x \wedge_B y) \vee_B (x \wedge_B z)$,
(j) $x \vee_B 1 = 1$.

Proof.

(a) $x \underset{\text{Lemma 2.3 (d)}}{=} (x \rightarrowtail 0) \rightarrowtail 0 = {\sim}x \rightarrowtail 0 \underset{(\text{B6})}{=} {\sim}({\sim}{\sim}x \wedge_{\mathrm{B}} 0) \rightarrowtail ({\sim}x \rightarrowtail 0) \underset{\text{Lemma 2.3 (d) and def. } \sim}{=}$ ${\sim}(x \wedge_{\mathrm{B}} 0) \rightarrowtail {\sim}{\sim}x \underset{(\text{B4})}{=} {\sim}x \rightarrowtail (x \wedge_{\mathrm{B}} 0) = (x \rightarrowtail 0) \rightarrowtail (x \wedge_{\mathrm{B}} 0)$.

(b) Taking x to be 0 in (a), we have that $0 = (0 \rightarrowtail 0) \rightarrowtail (0 \wedge_B 0) \underset{(B1)}{=} 0 \wedge_B 0$.

(c) $x \wedge_B y \underset{\text{Lemma 2.3 (d)}}{=} ((x \wedge_B y) \rightarrowtail 0) \rightarrowtail 0 \underset{(b)}{=} ((x \wedge_B y) \rightarrowtail (0 \wedge_B 0)) \rightarrowtail 0 \underset{(B3)}{=} (((x \wedge_B y) \rightarrowtail 0) \wedge_B ((x \wedge_B y) \rightarrowtail 0)) \rightarrowtail 0 \underset{\text{def.}}{=} (x \wedge_B y) \vee_B (x \wedge_B y) \underset{(B7)}{=} y \wedge_B (x \vee_B x) \underset{\text{def.}}{=} y \wedge_B (((x \rightarrowtail 0) \wedge_B (x \rightarrowtail 0)) \rightarrowtail 0) \underset{(B3)}{=} y \wedge_B ((x \rightarrowtail (0 \wedge_B 0)) \rightarrowtail 0) \underset{(b)}{=} y \wedge_B ((x \rightarrowtail 0) \rightarrowtail 0) \underset{\text{Lemma 2.3 (d)}}{=} y \wedge_B x$.

(d) $x \wedge_B x \underset{\text{Lemma 2.3 (d)}}{=} \sim\sim x \wedge_B \sim\sim x \underset{\text{def.}}{=} ((x \rightarrowtail 0) \rightarrowtail 0) \wedge_B ((x \rightarrowtail 0) \rightarrowtail 0) \underset{(B3)}{=} (x \rightarrowtail 0) \rightarrowtail (0 \wedge_B 0) \underset{(b)}{=} (x \rightarrowtail 0) \rightarrowtail 0 \underset{\text{def.}}{=} \sim\sim x \underset{\text{Lemma 2.3 (d)}}{=} x$.

(e) We notice that $x \underset{(a)}{=} (x \rightarrowtail 0) \rightarrowtail (x \wedge_B 0) \underset{\text{(c), def.}}{=} \sim x \rightarrowtail (0 \wedge_B x)$.

Replacing x with $\sim y$, we obtain the equivalent

$$\sim y = y \rightarrowtail (0 \wedge_B \sim y). \tag{1}$$

Using (B2) and the definition of $\sim$, we have that $0 \wedge_B (0 \rightarrowtail x) = 0 \wedge_B ((0 \wedge_B (x \rightarrowtail 0)) \rightarrowtail 0)$. Then

$$(0 \wedge_B (x \rightarrowtail 0)) \rightarrowtail (0 \wedge_B (0 \rightarrowtail x)) = (0 \wedge_B (x \rightarrowtail 0)) \rightarrowtail (0 \wedge_B (0 \wedge_B (x \rightarrowtail 0)) \rightarrowtail 0). \tag{2}$$

The right side of (2) is of the same form as the right side of (1) taking y to be $0 \wedge_B (x \rightarrowtail 0)$, so we can rewrite (2) as

$$(0 \wedge_B (x \rightarrowtail 0)) \rightarrowtail (0 \wedge_B (0 \rightarrowtail x)) = (0 \wedge_B (x \rightarrowtail 0)) \rightarrowtail 0. \tag{3}$$

Replacing x by 0 in (3), we obtain

$$(0 \wedge_B (0 \rightarrowtail 0)) \rightarrowtail 0 = (0 \wedge_B (0 \rightarrowtail 0)) \rightarrowtail (0 \wedge_B (0 \rightarrowtail 0)) \underset{\text{Lemma 2.3 (i)}}{=} y \rightarrowtail y \underset{\text{Lemma 2.3 (i)}}{=} 0 \rightarrowtail 0,$$

that is,

$$(0 \wedge_B (0 \rightarrowtail 0)) \rightarrowtail 0 = 0 \rightarrowtail 0. \tag{4}$$

By Lemma 2.3 (d), (4) is equivalent to

$$0 \wedge_B (0 \rightarrowtail 0) = 0.$$

Therefore, $0 \wedge_B 1 = 0$.

(f) See the Appendix.

(g) $x \vee_B x \underset{\text{def.}}{=} ((x \rightarrowtail 0) \wedge_B (x \rightarrowtail 0)) \rightarrowtail 0 \underset{(d)}{=} (x \rightarrowtail 0) \rightarrowtail 0 \underset{\text{Lemma 2.3 (d)}}{=} x$.

(h) $x \vee_B y \underset{\text{def.}}{=} ((x \rightarrowtail 0) \wedge_B (y \rightarrowtail 0)) \rightarrowtail 0 \underset{(c)}{=} ((y \rightarrowtail 0) \wedge_B (x \rightarrowtail 0)) \rightarrowtail 0 \underset{\text{def.}}{=} y \vee_B x$.

(i) $x \wedge_B (y \vee_B z) \underset{(B7)}{=} (z \wedge_B x) \vee_B (y \wedge_B x) \underset{(c)}{=} (x \wedge_B z) \vee_B (x \wedge_B y) \underset{(h)}{=} (x \wedge_B y) \vee_B (x \wedge_B z)$.

(j) $x \vee_B 1 \underset{\text{Lemma 2.3 (d)}}{=} \sim\sim x \vee_B \sim\sim 1 \underset{\text{Lemma 2.3 (f)}}{=} \sim(\sim x \wedge_B \sim 1) \underset{\text{Lemma 2.3 (a)}}{=} \sim(\sim x \wedge_B 0) \underset{(f)}{=} \sim 0 \underset{\text{def.}}{=} 1$.

□

We are now in a position to show the following:

Theorem 4.2. *An algebra $\langle A, \wedge_B, \rightarrowtail, 0 \rangle$ of type $(2,2,0)$ is a Brignole algebra if and only if it satisfies the following equations for every $x, y, z \in A$:*

(B1) $(x \rightarrowtail x) \rightarrowtail y = y$,

(B2) $x \wedge_B \sim_B(x \wedge_B \sim_B y) = x \wedge_B (x \rightarrowtail y)$,

(B3) $x \rightarrowtail (y \wedge_B z) = (x \rightarrowtail y) \wedge_B (x \rightarrowtail z)$,

(B4) $x \rightarrowtail y = \sim_B y \rightarrowtail \sim_B x$,

(B5) $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = (x \wedge_B y) \rightarrowtail ((x \wedge_B y) \rightarrowtail z)$,
(B6) $\sim_B(\sim_B x \wedge_B y) \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail y$,
(B7) $x \wedge_B (y \vee_B z) = (z \wedge_B x) \vee_B (y \wedge_B x)$,
(B8) $(x \wedge_B \sim_B x) \wedge_B (y \vee_B \sim_B y) = x \wedge_B \sim_B x$,

where $\sim_B x := x \rightarrowtail 0$ *and* $x \vee_B y := ((x \rightarrowtail 0) \wedge_B (y \rightarrowtail 0)) \rightarrowtail 0$.

Proof. One of the implications is immediate. For the other one, let us prove (B9) first.

$$\begin{aligned}(x \rightarrowtail y) \wedge_B y &\underset{(B4)}{=} ((y \rightarrowtail 0) \rightarrowtail (x \rightarrowtail 0)) \wedge_B y \\ &\underset{\text{Lemma 2.3 (d)}}{=} ((y \rightarrowtail 0) \rightarrowtail (x \rightarrowtail 0)) \wedge_B (y \rightarrowtail 0) \rightarrowtail 0 \underset{(B3)}{=} (y \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail 0) \wedge_B 0) \\ &\underset{\text{Lemma 4.1 (f)}}{=} (y \rightarrowtail 0) \rightarrowtail 0 \underset{\text{Lemma 2.3 (d)}}{=} y.\end{aligned}$$

Hence, we have (B9).
Using Lemma 2.3 (f) and Lemma 4.1 (i), (h), and (c), we have

$$\sim(x \wedge_B \sim y) \wedge_B \sim(z \wedge_B \sim y) = \sim(\sim y \wedge_B (z \vee_B x)).$$

Taking $z = 1$ we obtain

$$\sim(x \wedge_B \sim y) \wedge_B \sim(1 \wedge_B \sim y) = \sim(\sim y \wedge_B (1 \vee_B x)). \tag{1}$$

Notice that

$$z \underset{(B9)}{=} z \wedge_B (z \rightarrowtail z) \underset{\text{Lemma 2.3 (i)}}{=} z \wedge_B 1 \underset{\text{Lemma 4.1 (c)}}{=} 1 \wedge_B z,$$

that is,

$$z = 1 \wedge_B z. \tag{2}$$

Taking $z = \sim y$ in (2), and replacing that in (1), we obtain

$$\begin{aligned}\sim(x \wedge_B \sim y) \wedge_B \sim\sim y = \sim(\sim y \wedge_B (1 \vee_B x)) &\underset{\text{Lemma 4.1 (h), (j)}}{=} \sim(\sim y \wedge_B 1) \\ \underset{\substack{\text{Lemma 4.1 (c)} \\ \text{and (2)}}}{=} \sim\sim y &\underset{\text{Lemma 2.3 (d)}}{=} y;\end{aligned}$$

by Lemma 2.3 (d) we have that

$$\sim(x \wedge_B \sim y) \wedge_B y = y,$$

and by Lemma 4.1 (i), Lemma 2.3 (d) and Lemma 4.1 (c) it follows that

$$y = y \wedge_B (\sim x \vee_B y). \tag{3}$$

If we change y and x in (3) by x and $\sim y$ respectively (and use Lemma 4.1 (h)), we obtain (B10). □

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APPENDIX

The following proof has been adapted from the output produced by the program Prover9 [McC10]. For simplicity we are going to replace $\wedge_B$ with $\wedge$.

1. $x \vee_B y = ((x \rightarrowtail 0) \wedge (y \rightarrowtail 0)) \rightarrowtail 0$ — definition of $\vee_B$
2. $\sim x = x \rightarrowtail 0$ — definition of $\sim$
3. $(x \rightarrowtail x) \rightarrowtail y = y$ — (B1)
4. $x \wedge \sim(x \wedge \sim y) = x \wedge (x \rightarrowtail y)$ — (B2)
5. $x \wedge ((x \wedge (y \rightarrowtail 0)) \rightarrowtail 0) = x \wedge (x \rightarrowtail y)$ — by (2) and (4)
6. $x \rightarrowtail (y \wedge z) = (x \rightarrowtail y) \wedge (x \rightarrowtail z)$ — (B3)
7. $(x \rightarrowtail y) \wedge (x \rightarrowtail z) = x \rightarrowtail (y \wedge z)$ — by (6)
8. $x \rightarrowtail y = \sim y \rightarrowtail \sim x$ — (B4)
9. $x \rightarrowtail y = (y \rightarrowtail 0) \rightarrowtail (x \rightarrowtail 0)$ — by (2) and (8)
10. $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = (x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail z)$ — (B5)
11. $\sim(\sim x \wedge y) \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail y$ — (B6)
12. $\sim((x \rightarrowtail 0) \wedge y) \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail y$ — by (2) and (11)
13. $(((x \rightarrowtail 0) \wedge y) \rightarrowtail 0) \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail y$ — by (2) and (12)
14. $x \wedge (y \vee_B z) = (z \wedge x) \vee_B (y \wedge x)$ — (B7)
15. $x \wedge (((y \rightarrowtail 0) \wedge (z \rightarrowtail 0)) \rightarrowtail 0) = (((z \wedge x) \rightarrowtail 0) \wedge ((y \wedge x) \rightarrowtail 0)) \rightarrowtail 0$ — by (1) and (14)
16. $x \rightarrowtail x = y \rightarrowtail y$ — by Lemma 2.4 (i)
17. $x \wedge y = y \wedge x$ — by Lemma 4.1 (c)
18. $x \wedge x = x$ — by Lemma 4.1 (d)
19. $0 = 0 \wedge (0 \rightarrowtail 0)$ — by Lemma 4.1 (e)

20. $x \rightarrowtail 0 = x \rightarrowtail (0 \wedge (x \rightarrowtail 0))$ by Lemma (4.1), (e), item (1)
21. $x \wedge ((x \wedge 0) \rightarrowtail 0) = x \wedge (x \rightarrowtail (y \rightarrowtail y))$ by (3) and (5)
22. $(x \rightarrowtail y) \wedge ((x \rightarrowtail (y \wedge 0)) \rightarrowtail 0) = (x \rightarrowtail y) \wedge ((x \rightarrowtail y) \rightarrowtail x)$ by (7) and (5)
23. $(x \rightarrowtail 0) \rightarrowtail ((y \rightarrowtail y) \rightarrowtail 0) = x$ by (9) and (3)
24. $(x \rightarrowtail 0) \rightarrowtail 0 = x$ by (3) and (23)
25. $0 \rightarrowtail (x \rightarrowtail 0) = x \rightarrowtail (y \rightarrowtail y)$ by (3) and (9)
26. $x \wedge (x \rightarrowtail y) = x \wedge ((y \rightarrowtail 0) \rightarrowtail (x \rightarrowtail 0))$ by (9)
27. $(x \rightarrowtail y) \wedge ((y \rightarrowtail 0) \rightarrowtail z) = (y \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail 0) \wedge z)$ by (9) and (7)
28. $((x \rightarrowtail 0) \rightarrowtail y) \wedge (z \rightarrowtail x) = (x \rightarrowtail 0) \rightarrowtail (y \wedge (z \rightarrowtail 0))$ by (9) and (7)
29. $(x \rightarrowtail (y \wedge z)) \rightarrowtail (((x \rightarrowtail y) \wedge (x \rightarrowtail z)) \rightarrowtail u) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail u)))$ by (7) and (10)
30. $(x \rightarrowtail (y \wedge z)) \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail u) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail u)))$ by (7) and (29)
31. $(x \wedge y) \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z)))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail ((x \wedge y) \rightarrowtail z))))$ by (10)
32. $((x \wedge y) \wedge (x \wedge y)) \rightarrowtail (((x \wedge y) \wedge (x \wedge y)) \rightarrowtail z) = (x \wedge y) \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail ((x \wedge y) \rightarrowtail z)))))$ by (10)
33. $(x \wedge y) \rightarrowtail (((x \wedge y) \wedge (x \wedge y)) \rightarrowtail z) = (x \wedge y) \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail ((x \wedge y) \rightarrowtail z)))))$ by (18) and (32)
34. $(x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail z) = (x \wedge y) \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail ((x \wedge y) \rightarrowtail z)))))$ by (18) and (33)
35. $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = (x \wedge y) \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail ((x \wedge y) \rightarrowtail z)))))$ by (10) and (34)
36. $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail ((x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail z)))))$ by (31) and (35)
37. $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail z)))))))$ by (10) and (36)
38. $((x \rightarrowtail y) \rightarrowtail 0) \rightarrowtail ((((x \rightarrowtail 0) \wedge y) \rightarrowtail 0) \rightarrowtail 0) = x \rightarrowtail y$ by (13) and (9)
39. $((x \rightarrowtail y) \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail 0) \wedge y) = x \rightarrowtail y$ by (24) and (38)
40. $x \vee_{\mathrm{B}} ((y \rightarrowtail 0) \wedge (z \rightarrowtail 0)) = ((((z \wedge (x \rightarrowtail 0)) \rightarrowtail 0) \wedge ((y \wedge (x \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0) \rightarrowtail 0$ by (15) and (1)
41. $((x \rightarrowtail 0) \wedge (((y \rightarrowtail 0) \wedge (z \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0 = ((((z \wedge (x \rightarrowtail 0)) \rightarrowtail 0) \wedge ((y \wedge (x \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0) \rightarrowtail 0$ by (1) and (40)
42. $((x \rightarrowtail 0) \wedge (((y \rightarrowtail 0) \wedge (z \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0 = ((z \wedge (x \rightarrowtail 0)) \rightarrowtail 0) \wedge ((y \wedge (x \rightarrowtail 0)) \rightarrowtail 0)$ by (24) and (41)
43. $((x \wedge (y \rightarrowtail 0)) \rightarrowtail 0) \wedge (((z \rightarrowtail 0) \wedge (x \rightarrowtail 0)) \rightarrowtail 0) = (((x \wedge (x \rightarrowtail y)) \rightarrowtail 0) \wedge ((z \wedge ((x \wedge (y \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (5) and (15)
44. $(x \wedge (((y \rightarrowtail 0) \wedge (z \rightarrowtail 0)) \rightarrowtail 0)) \wedge ((((z \wedge x) \rightarrowtail 0) \wedge ((y \wedge x) \rightarrowtail 0)) \rightarrowtail u) = (((z \wedge x) \rightarrowtail 0) \wedge ((y \wedge x) \rightarrowtail 0)) \rightarrowtail (0 \wedge u)$ by (15) and (7)
45. $(x \rightarrowtail x) \wedge (y \rightarrowtail z) = y \rightarrowtail (y \wedge z)$ by (16) and (7)
46. $(x \rightarrowtail y) \wedge (z \rightarrowtail z) = x \rightarrowtail (y \wedge x)$ by (16) and (7)
47. $(x \wedge y) \rightarrowtail (z \rightarrowtail z) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (x \wedge y))))$ by (16) and (10)
48. $0 \rightarrowtail ((x \wedge y) \rightarrowtail 0) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (x \wedge y))))$ by (25) and (47)
49. $(((x \rightarrowtail 0) \wedge x) \rightarrowtail 0) \rightarrowtail (y \rightarrowtail y) = x \rightarrowtail x$ by (16) and (13)
50. $((x \wedge (x \rightarrowtail 0)) \rightarrowtail 0) \rightarrowtail (y \rightarrowtail y) = x \rightarrowtail x$ by (17) and (49)
51. $0 \rightarrowtail (((x \wedge (x \rightarrowtail 0)) \rightarrowtail 0) \rightarrowtail 0) = x \rightarrowtail x$ by (25) and (50)
52. $0 \rightarrowtail (x \wedge (x \rightarrowtail 0)) = x \rightarrowtail x$ by (24) and (51)

53. $x \wedge (((y \rightarrowtail 0) \wedge x) \rightarrowtail 0) = x \wedge (x \rightarrowtail y)$ by (17) and (5)
54. $(x \rightarrowtail 0) \wedge ((x \rightarrowtail 0) \rightarrowtail 0) = (x \rightarrowtail 0) \wedge ((x \rightarrowtail 0) \rightarrowtail x)$ by (18) and (5)
55. $(x \rightarrowtail 0) \wedge x = (x \rightarrowtail 0) \wedge ((x \rightarrowtail 0) \rightarrowtail x)$ by (24) and (54)
56. $x \wedge (x \rightarrowtail 0) = (x \rightarrowtail 0) \wedge ((x \rightarrowtail 0) \rightarrowtail x)$ by (17) and (55)
57. $x \rightarrowtail ((x \wedge x) \rightarrowtail y) = x \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \rightarrowtail y)))$ by (18) and (10)
58. $x \rightarrowtail (x \rightarrowtail y) = x \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \rightarrowtail y)))$ by (18) and (57)
59. $((x \rightarrowtail 0) \rightarrowtail 0) \rightarrowtail (x \rightarrowtail (x \rightarrowtail 0)) = x \rightarrowtail (x \rightarrowtail 0)$ by (18) and (13)
60. $x \rightarrowtail (x \rightarrowtail (x \rightarrowtail 0)) = x \rightarrowtail (x \rightarrowtail 0)$ by (24) and (59)
61. $0 \wedge (0 \rightarrowtail 0) = 0 \wedge (0 \rightarrowtail (x \rightarrowtail x))$ by (18) and (21)
62. $x \wedge ((x \wedge y) \rightarrowtail 0) = x \wedge (x \rightarrowtail (y \rightarrowtail 0))$ by (24) and (5)
63. $x \wedge ((x \rightarrowtail 0) \rightarrowtail y) = (x \rightarrowtail 0) \rightarrowtail (0 \wedge y)$ by (24) and (7)
64. $((x \rightarrowtail 0) \rightarrowtail y) \wedge x = (x \rightarrowtail 0) \rightarrowtail (y \wedge 0)$ by (24) and (7)
65. $x \wedge ((x \rightarrowtail 0) \rightarrowtail y) = (x \rightarrowtail 0) \rightarrowtail (y \wedge 0)$ by (17) and (64)
66. $x \rightarrowtail (y \rightarrowtail 0) = y \rightarrowtail (x \rightarrowtail 0)$ by (24) and (9)
67. $(x \rightarrowtail 0) \rightarrowtail y = (y \rightarrowtail 0) \rightarrowtail x$ by (24) and (9)
68. $((((x \rightarrowtail 0) \rightarrowtail 0) \wedge 0) \rightarrowtail 0) \rightarrowtail x = (x \rightarrowtail 0) \rightarrowtail 0$ by (24) and (13)
69. $((x \wedge 0) \rightarrowtail 0) \rightarrowtail x = (x \rightarrowtail 0) \rightarrowtail 0$ by (24) and (68)
70. $(x \rightarrowtail 0) \rightarrowtail (x \wedge 0) = (x \rightarrowtail 0) \rightarrowtail 0$ by (67) and (69)
71. $(x \rightarrowtail 0) \rightarrowtail (x \wedge 0) = x$ by (24) and (70)
72. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge x) = x$ by (71) and (17)
73. $((x \rightarrowtail 0) \rightarrowtail x) \wedge ((x \rightarrowtail 0) \rightarrowtail 0) = x$ by (71) and (7)
74. $((x \rightarrowtail 0) \rightarrowtail x) \wedge x = x$ by (24) and (73)
75. $x \wedge ((x \rightarrowtail 0) \rightarrowtail x) = x$ by (17) and (74)
76. $((x \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \rightarrowtail (0 \wedge 0)) \rightarrowtail 0) = ((x \rightarrowtail 0) \rightarrowtail 0) \wedge (x \rightarrowtail (x \rightarrowtail 0))$ by (9) and (22)
77. $x \wedge (((x \rightarrowtail 0) \rightarrowtail (0 \wedge 0)) \rightarrowtail 0) = ((x \rightarrowtail 0) \rightarrowtail 0) \wedge (x \rightarrowtail (x \rightarrowtail 0))$ by (24) and (76)
78. $x \wedge (((x \rightarrowtail 0) \rightarrowtail 0) \rightarrowtail 0) = ((x \rightarrowtail 0) \rightarrowtail 0) \wedge (x \rightarrowtail (x \rightarrowtail 0))$ by (18) and (77)
79. $x \wedge (x \rightarrowtail 0) = ((x \rightarrowtail 0) \rightarrowtail 0) \wedge (x \rightarrowtail (x \rightarrowtail 0))$ by (24) and (78)
80. $x \wedge (x \rightarrowtail 0) = x \wedge (x \rightarrowtail (x \rightarrowtail 0))$ by (24) and (79)
81. $(x \rightarrowtail 0) \wedge (x \rightarrowtail y) = x \rightarrowtail ((0 \wedge (x \rightarrowtail 0)) \wedge y)$ by (20) and (7)
82. $x \rightarrowtail (0 \wedge y) = x \rightarrowtail ((0 \wedge (x \rightarrowtail 0)) \wedge y)$ by (7) and (81)
83. $(x \rightarrowtail x) \wedge (0 \rightarrowtail y) = 0 \rightarrowtail ((x \wedge (x \rightarrowtail 0)) \wedge y)$ by (52) and (7)
84. $(x \rightarrowtail x) \wedge y = y \wedge (z \rightarrowtail z)$ by (16) and (17)
85. $(x \rightarrowtail x) \wedge y = (y \rightarrowtail 0) \rightarrowtail ((y \rightarrowtail 0) \wedge 0)$ by (24) and (45)
86. $(x \rightarrowtail x) \wedge y = (y \rightarrowtail 0) \rightarrowtail (0 \wedge (y \rightarrowtail 0))$ by (17) and (85)
87. $(x \rightarrowtail x) \wedge (y \rightarrowtail (z \rightarrowtail z)) = 0 \rightarrowtail (0 \wedge (y \rightarrowtail 0))$ by (25) and (45)
88. $((x \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0) \rightarrowtail (0 \wedge ((x \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0)) = 0 \rightarrowtail (0 \wedge (x \rightarrowtail 0))$ by (86) and (87)
89. $(x \rightarrowtail x) \wedge ((y \rightarrowtail y) \rightarrowtail z) = (y \rightarrowtail y) \rightarrowtail (z \wedge (u \rightarrowtail u))$ by (84) and (45)
90. $(x \rightarrowtail x) \wedge z = (y \rightarrowtail y) \rightarrowtail (z \wedge (u \rightarrowtail u))$ by (3) and (89)
91. $(z \rightarrowtail 0) \rightarrowtail (0 \wedge (z \rightarrowtail 0)) = (y \rightarrowtail y) \rightarrowtail (z \wedge (u \rightarrowtail u))$ by (86) and (90)
92. $(z \rightarrowtail 0) \rightarrowtail (0 \wedge (z \rightarrowtail 0)) = z \wedge (u \rightarrowtail u)$ by (3) and (91)
93. $x \wedge (y \rightarrowtail y) = (x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0))$ by (92)
94. $(x \rightarrowtail x) \wedge (y \rightarrowtail (z \rightarrowtail z)) = y \rightarrowtail ((u \rightarrowtail u) \wedge y)$ by (84) and (45)
95. $((y \rightarrowtail (z \rightarrowtail z)) \rightarrowtail 0) \rightarrowtail (0 \wedge ((y \rightarrowtail (z \rightarrowtail z)) \rightarrowtail 0)) = y \rightarrowtail ((u \rightarrowtail u) \wedge y)$ by (86) and (94)
96. $0 \rightarrowtail (0 \wedge (y \rightarrowtail 0)) = y \rightarrowtail ((u \rightarrowtail u) \wedge y)$ by (88) and (95)
97. $0 \rightarrowtail (0 \wedge (y \rightarrowtail 0)) = y \rightarrowtail ((y \rightarrowtail 0) \rightarrowtail (0 \wedge (y \rightarrowtail 0)))$ by (86) and (96)
98. $0 \rightarrowtail ((x \wedge (x \rightarrowtail 0)) \wedge y) = ((0 \rightarrowtail y) \rightarrowtail 0) \rightarrowtail (0 \wedge ((0 \rightarrowtail y) \rightarrowtail 0))$ by (86) and (83)

99. $(x\wedge y)\wedge(x\rightarrowtail(x\rightarrowtail(y\rightarrowtail(y\rightarrowtail 0))))=(x\wedge y)\wedge((x\wedge y)\rightarrowtail 0)$ by (10) and (80)
100. $(x\rightarrowtail(x\rightarrowtail(y\rightarrowtail(y\rightarrowtail 0))))\wedge(x\wedge y)=(x\wedge y)\wedge((x\wedge y)\rightarrowtail 0)$ by (17) and (99)
101. $(x\rightarrowtail x)\wedge(((y\rightarrowtail 0)\wedge(x\rightarrowtail x))\rightarrowtail 0)=(x\rightarrowtail x)\wedge y$ by (3) and (53)
102. $(x\rightarrowtail x)\wedge((y\rightarrowtail(0\wedge y))\rightarrowtail 0)=(x\rightarrowtail x)\wedge y$ by (46) and (101)
103. $(((y\rightarrowtail(0\wedge y))\rightarrowtail 0)\rightarrowtail 0)\rightarrowtail(0\wedge(((y\rightarrowtail(0\wedge y))\rightarrowtail 0)\rightarrowtail 0))=(x\rightarrowtail x)\wedge y$ by (86) and (102)
104. $((0\rightarrowtail 0)\rightarrowtail(y\rightarrowtail(0\wedge y)))\rightarrowtail(0\wedge(((y\rightarrowtail(0\wedge y))\rightarrowtail 0)\rightarrowtail 0))=(x\rightarrowtail x)\wedge y$ by (67) and (103)
105. $(y\rightarrowtail(0\wedge y))\rightarrowtail(0\wedge(((y\rightarrowtail(0\wedge y))\rightarrowtail 0)\rightarrowtail 0))=(x\rightarrowtail x)\wedge y$ by (3) and (104)
106. $(y\rightarrowtail(0\wedge y))\rightarrowtail(0\wedge((0\rightarrowtail 0)\rightarrowtail(y\rightarrowtail(0\wedge y))))=(x\rightarrowtail x)\wedge y$ by (67) and (105)
107. $(y\rightarrowtail(0\wedge y))\rightarrowtail(0\wedge(y\rightarrowtail(0\wedge y)))=(x\rightarrowtail x)\wedge y$ by (3) and (106)
108. $(x\rightarrowtail(0\wedge x))\rightarrowtail(0\wedge(x\rightarrowtail(0\wedge x)))=(x\rightarrowtail 0)\rightarrowtail(0\wedge(x\rightarrowtail 0))$ by (86) and (107)
109. $x\wedge((y\wedge x)\rightarrowtail 0)=x\wedge(x\rightarrowtail(y\rightarrowtail 0))$ by (24) and (53)
110. $x\rightarrowtail(x\rightarrowtail(0\rightarrowtail(x\rightarrowtail 0)))=x\rightarrowtail(x\rightarrowtail x)$ by (25) and (58)
111. $(x\rightarrowtail(y\wedge z))\rightarrowtail(u\rightarrowtail u)=(x\rightarrowtail y)\rightarrowtail((x\rightarrowtail y)\rightarrowtail((x\rightarrowtail z)\rightarrowtail((x\rightarrowtail z)\rightarrowtail(x\rightarrowtail(y\wedge z)))))$ by (16) and (30)
112. $0\rightarrowtail((x\rightarrowtail(y\wedge z))\rightarrowtail 0)=(x\rightarrowtail y)\rightarrowtail((x\rightarrowtail y)\rightarrowtail((x\rightarrowtail z)\rightarrowtail((x\rightarrowtail z)\rightarrowtail(x\rightarrowtail(y\wedge z)))))$ by (25) and (111)
113. $(x\rightarrowtail(y\wedge z))\rightarrowtail((x\rightarrowtail y)\rightarrowtail((x\rightarrowtail y)\rightarrowtail((x\rightarrowtail z)\rightarrowtail((x\rightarrowtail z)\rightarrowtail u))))=(x\rightarrowtail y)\rightarrowtail((x\rightarrowtail y)\rightarrowtail((x\rightarrowtail z)\rightarrowtail((x\rightarrowtail z)\rightarrowtail((x\rightarrowtail(y\wedge z))\rightarrowtail u))))$ by (30) and (30)
114. $(0\wedge x)\wedge((0\wedge x)\wedge(((0\wedge x)\rightarrowtail 0)\rightarrowtail x))=0\wedge x$ by (63) and (75)
115. $(0\wedge x)\wedge((0\wedge x)\wedge((x\rightarrowtail 0)\rightarrowtail(0\wedge x)))=0\wedge x$ by (67) and (114)
116. $(0\wedge x)\wedge((0\wedge x)\wedge x)=0\wedge x$ by (72) and (115)
117. $(0\wedge x)\wedge(x\wedge(x\wedge 0))=0\wedge x$ by (17) and (116)
118. $x\wedge((x\rightarrowtail 0)\rightarrowtail(0\rightarrowtail(y\rightarrowtail y)))=(x\rightarrowtail 0)\rightarrowtail(0\wedge(0\rightarrowtail 0))$ by (61) and (63)
119. $(0\wedge x)\rightarrowtail(((0\wedge x)\wedge(x\wedge(x\wedge 0)))\rightarrowtail y)=(0\wedge x)\rightarrowtail((0\wedge x)\rightarrowtail((x\wedge(x\wedge 0))\rightarrowtail((x\wedge(x\wedge 0))\rightarrowtail y)))$ by (117) and (10)
120. $(0\wedge x)\rightarrowtail((0\wedge x)\rightarrowtail y)=(0\wedge x)\rightarrowtail((0\wedge x)\rightarrowtail((x\wedge(x\wedge 0))\rightarrowtail((x\wedge(x\wedge 0))\rightarrowtail y)))$ by (117) and (119)
121. $0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail y)))=(0\wedge x)\rightarrowtail((0\wedge x)\rightarrowtail((x\wedge(x\wedge 0))\rightarrowtail((x\wedge(x\wedge 0))\rightarrowtail y)))$ by (10) and (120)
122. $0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail y)))=(0\wedge x)\rightarrowtail((0\wedge x)\rightarrowtail(x\rightarrowtail(x\rightarrowtail((x\wedge 0)\rightarrowtail((x\wedge 0)\rightarrowtail y)))))$ by (10) and (121)
123. $0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail y)))=(0\wedge x)\rightarrowtail((0\wedge x)\rightarrowtail(x\rightarrowtail(x\rightarrowtail(x\rightarrowtail(x\rightarrowtail(0\rightarrowtail(0\rightarrowtail y)))))))$ by (10) and (122)
124. $0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail y)))=(0\wedge x)\rightarrowtail((0\wedge x)\rightarrowtail(x\rightarrowtail(x\rightarrowtail(0\rightarrowtail(0\rightarrowtail y)))))$ by (58) and (123)
125. $0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail y)))=0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail(x\rightarrowtail(x\rightarrowtail(0\rightarrowtail(0\rightarrowtail y)))))))$ by (10) and (124)
126. $0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail y)))=0\rightarrowtail(0\rightarrowtail(x\rightarrowtail(x\rightarrowtail(0\rightarrowtail(0\rightarrowtail y)))))$ by (58) and (125)
127. $(0\wedge x)\rightarrowtail((0\wedge((0\wedge x)\rightarrowtail 0))\wedge x)=y\rightarrowtail y$ by (16) and (82)
128. $(0\wedge x)\rightarrowtail((0\wedge(0\rightarrowtail(x\rightarrowtail 0)))\wedge x)=y\rightarrowtail y$ by (62) and (127)
129. $(0\wedge x)\rightarrowtail(x\wedge(0\wedge(0\rightarrowtail(x\rightarrowtail 0))))=y\rightarrowtail y$ by (17) and (128)

130. $x \wedge ((x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail (y \rightarrowtail y))) = (x \rightarrowtail 0) \rightarrowtail 0$ by (19) and (118)
131. $x \wedge ((x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail (y \rightarrowtail y))) = x$ by (24) and (130)
132. $0 \wedge (0 \rightarrowtail (x \rightarrowtail x)) = 0$ by (19) and (61)
133. $((x \rightarrowtail 0) \rightarrowtail 0) \wedge (0 \rightarrowtail x) = (x \rightarrowtail 0) \rightarrowtail 0$ by (19) and (28)
134. $x \wedge (0 \rightarrowtail x) = (x \rightarrowtail 0) \rightarrowtail 0$ by (24) and (133)
135. $x \wedge (0 \rightarrowtail x) = x$ by (24) and (134)
136. $((x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))) \rightarrowtail 0) \rightarrowtail (x \rightarrowtail 0) = x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (135) and (39)
137. $x \rightarrowtail (((x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))) \rightarrowtail 0) \rightarrowtail 0) = x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (66) and (136)
138. $x \rightarrowtail ((0 \rightarrowtail 0) \rightarrowtail (x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0)))) = x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (67) and (137)
139. $x \rightarrowtail (x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))) = x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (3) and (138)
140. $x \rightarrowtail (x \rightarrowtail x) = x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (110) and (139)
141. $(0 \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail x))) \wedge (0 \rightarrowtail (0 \rightarrowtail x)) = 0 \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail x))$ by (58) and (135)
142. $(0 \rightarrowtail (0 \rightarrowtail x)) \wedge (0 \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail x))) = 0 \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail x))$ by (17) and (141)
143. $0 \rightarrowtail (0 \rightarrowtail x) = 0 \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail x))$ by (135) and (142)
144. $x \wedge (((0 \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail 0)) = x$ by (9) and (131)
145. $x \wedge (((0 \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0) \rightarrowtail x) = x$ by (24) and (144)
146. $((x \rightarrowtail x) \rightarrowtail 0) \rightarrowtail (x \rightarrowtail 0) = x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (140) and (9)
147. $0 \rightarrowtail (x \rightarrowtail 0) = x \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (3) and (146)
148. $(x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail 0)) = (x \rightarrowtail 0) \rightarrowtail (x \rightarrowtail x)$ by (9) and (140)
149. $(x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail x) = (x \rightarrowtail 0) \rightarrowtail (x \rightarrowtail x)$ by (24) and (148)
150. $(x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail x) = 0 \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail 0)$ by (25) and (149)
151. $(x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail x) = 0 \rightarrowtail x$ by (24) and (150)
152. $(x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail ((x \wedge y) \rightarrowtail (x \wedge y))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (0 \rightarrowtail ((x \wedge y) \rightarrowtail 0)))))$ by (140) and (10)
153. $(x \wedge y) \rightarrowtail (0 \rightarrowtail ((x \wedge y) \rightarrowtail 0)) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (0 \rightarrowtail ((x \wedge y) \rightarrowtail 0)))))$ by (25) and (152)
154. $(x \wedge y) \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (x \wedge y))))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (0 \rightarrowtail ((x \wedge y) \rightarrowtail 0)))))$ by (48) and (153)
155. $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail ((x \wedge y) \rightarrowtail (x \wedge y))))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (0 \rightarrowtail ((x \wedge y) \rightarrowtail 0)))))$ by (31) and (154)
156. $x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (0 \rightarrowtail (y \rightarrowtail 0)))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (0 \rightarrowtail ((x \wedge y) \rightarrowtail 0)))))$ by (25) and (155)
157. $x \rightarrowtail (x \rightarrowtail (0 \rightarrowtail (y \rightarrowtail 0))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (0 \rightarrowtail ((x \wedge y) \rightarrowtail 0)))))$ by (147) and (156)
158. $x \rightarrowtail (x \rightarrowtail (0 \rightarrowtail (y \rightarrowtail 0))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (x \wedge y))))))))$ by (48) and (157)
159. $x \rightarrowtail (x \rightarrowtail (0 \rightarrowtail (y \rightarrowtail 0))) = x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail (x \wedge y))))$ by (37) and (158)
160. $(x \rightarrowtail (y \wedge z)) \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail (x \rightarrowtail (y \wedge z)))) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0)))))$ by (140) and (30)
161. $(x \rightarrowtail (y \wedge z)) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0)) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0)))))$ by (25) and (160)
162. $(x \rightarrowtail (y \wedge z)) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (x \rightarrowtail (y \wedge z)))))) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0)))))$ by (112) and (161)
163. $(x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail (x \rightarrowtail (y \wedge z)))))) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0)))))$ by (113) and (162)

164. $(x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail z) \rightarrowtail 0)))) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0)))))$ by (25) and (163)
165. $(x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail z) \rightarrowtail 0))) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0)))))$ by (147) and (164)
166. $(x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail z) \rightarrowtail 0))) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (x \rightarrowtail (y \wedge z)))))))))$ by (112) and (165)
167. $(x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail z) \rightarrowtail 0))) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail ((x \rightarrowtail z) \rightarrowtail (x \rightarrowtail (y \wedge z)))))$ by (37) and (166)
168. $0 \rightarrowtail ((x \wedge y) \rightarrowtail 0) = x \rightarrowtail (x \rightarrowtail (0 \rightarrowtail (y \rightarrowtail 0)))$ by (159) and (48)
169. $0 \rightarrowtail ((x \rightarrowtail (y \wedge z)) \rightarrowtail 0) = (x \rightarrowtail y) \rightarrowtail ((x \rightarrowtail y) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail z) \rightarrowtail 0)))$ by (167) and (112)
170. $(0 \rightarrowtail x) \wedge (((0 \rightarrowtail x) \rightarrowtail 0) \rightarrowtail y) = ((0 \rightarrowtail x) \rightarrowtail 0) \rightarrowtail (((x \rightarrowtail 0) \rightarrowtail 0) \wedge y)$ by (151) and (27)
171. $(0 \rightarrowtail x) \wedge (((0 \rightarrowtail x) \rightarrowtail 0) \rightarrowtail y) = ((0 \rightarrowtail x) \rightarrowtail 0) \rightarrowtail (x \wedge y)$ by (24) and (170)
172. $(0 \rightarrowtail (0 \rightarrowtail x)) \wedge (0 \rightarrowtail y) = 0 \rightarrowtail ((0 \rightarrowtail (0 \rightarrowtail x)) \wedge y)$ by (143) and (7)
173. $0 \rightarrowtail ((0 \rightarrowtail x) \wedge y) = 0 \rightarrowtail ((0 \rightarrowtail (0 \rightarrowtail x)) \wedge y)$ by (7) and (172)
174. $(x \wedge 0) \wedge ((0 \rightarrowtail (y \rightarrowtail y)) \wedge (((0 \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0) \rightarrowtail x)) = x \wedge 0$ by (65) and (145)
175. $(x \wedge 0) \wedge (((0 \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0) \rightarrowtail ((y \rightarrowtail y) \wedge x)) = x \wedge 0$ by (171) and (174)
176. $(x \wedge 0) \wedge (((0 \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)))) = x \wedge 0$ by (86) and (175)
177. $((0 \wedge (0 \rightarrowtail 0)) \rightarrowtail 0) \wedge ((x \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (56) and (42)
178. $(0 \rightarrowtail 0) \wedge ((x \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (135) and (177)
179. $(0 \rightarrowtail 0) \wedge ((x \wedge 0) \rightarrowtail 0) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (178)
180. $(((x \wedge 0) \rightarrowtail 0) \rightarrowtail 0) \rightarrowtail (0 \wedge (((x \wedge 0) \rightarrowtail 0) \rightarrowtail 0)) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (86) and (179)
181. $((0 \rightarrowtail 0) \rightarrowtail (x \wedge 0)) \rightarrowtail (0 \wedge (((x \wedge 0) \rightarrowtail 0) \rightarrowtail 0)) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (67) and (180)
182. $(x \wedge 0) \rightarrowtail (0 \wedge (((x \wedge 0) \rightarrowtail 0) \rightarrowtail 0)) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (181)
183. $(x \wedge 0) \rightarrowtail (0 \wedge ((0 \rightarrowtail 0) \rightarrowtail (x \wedge 0))) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (67) and (182)
184. $(x \wedge 0) \rightarrowtail (0 \wedge (x \wedge 0)) = (((0 \rightarrowtail 0) \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (183)
185. $(x \wedge 0) \rightarrowtail (0 \wedge (x \wedge 0)) = (0 \wedge (((x \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (184)
186. $(x \wedge 0) \rightarrowtail (0 \wedge (x \wedge 0)) = (0 \wedge (((x \rightarrowtail 0) \wedge 0) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (185)
187. $(x \wedge 0) \rightarrowtail (0 \wedge (x \wedge 0)) = (0 \wedge ((0 \wedge (x \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (17) and (186)
188. $(x \wedge 0) \rightarrowtail (0 \wedge (x \wedge 0)) = (0 \wedge (0 \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail 0))) \rightarrowtail 0$ by (62) and (187)
189. $(x \wedge 0) \rightarrowtail (0 \wedge (x \wedge 0)) = (0 \wedge (0 \rightarrowtail x)) \rightarrowtail 0$ by (24) and (188)
190. $(0 \wedge x) \rightarrowtail (x \wedge 0) = y \rightarrowtail y$ by (16) and (17)
191. $(0 \wedge x) \rightarrowtail (y \rightarrowtail y) = 0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \wedge 0))))$ by (190) and (10)
192. $0 \rightarrowtail ((0 \wedge x) \rightarrowtail 0) = 0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \wedge 0))))$ by (25) and (191)
193. $0 \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))) = 0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \wedge 0))))$ by (168) and (192)

194. $0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0)) = 0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \wedge 0))))$ by (143) and (193)
195. $(0 \wedge x) \wedge (((x \wedge 0) \rightarrowtail 0) \rightarrowtail ((0 \wedge x) \rightarrowtail 0)) = (0 \wedge x) \wedge (y \rightarrowtail y)$ by (190) and (26)
196. $((x \rightarrowtail x) \rightarrowtail 0) \rightarrowtail (((0 \wedge y) \rightarrowtail 0) \wedge (y \wedge 0)) = (0 \wedge y) \rightarrowtail (y \wedge 0)$ by (190) and (39)
197. $0 \rightarrowtail (((0 \wedge x) \rightarrowtail 0) \wedge (x \wedge 0)) = (0 \wedge x) \rightarrowtail (x \wedge 0)$ by (3) and (196)
198. $(0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (y \rightarrowtail y))) = (0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (x \wedge 0))$ by (190) and (58)
199. $(0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (0 \rightarrowtail ((0 \wedge x) \rightarrowtail 0))) = (0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (x \wedge 0))$ by (25) and (198)
200. $(0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))))) = (0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (x \wedge 0))$ by (168) and (199)
201. $(0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0)))) = (0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (x \wedge 0))$ by (143) and (200)
202. $0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail (0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0)))))) = (0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (x \wedge 0))$ by (10) and (201)
203. $0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \rightarrowtail 0)))) = (0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (x \wedge 0))$ by (126) and (202)
204. $0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail 0))) = (0 \wedge x) \rightarrowtail ((0 \wedge x) \rightarrowtail (x \wedge 0))$ by (60) and (203)
205. $0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail 0))) = 0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail (x \wedge 0))))$ by (10) and (204)
206. $0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail 0))) = 0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))$ by (194) and (205)
207. $((x \wedge y) \rightarrowtail 0) \wedge (y \wedge x) = (x \wedge y) \wedge ((x \wedge y) \rightarrowtail 0)$ by (17) and (17)
208. $((x \wedge y) \rightarrowtail 0) \wedge (y \wedge x) = (x \rightarrowtail (x \rightarrowtail (y \rightarrowtail (y \rightarrowtail 0)))) \wedge (x \wedge y)$ by (100) and (207)
209. $0 \rightarrowtail ((0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail (x \rightarrowtail 0)))) \wedge (0 \wedge x)) = (0 \wedge x) \rightarrowtail (x \wedge 0)$ by (208) and (197)
210. $0 \rightarrowtail ((0 \rightarrowtail (0 \rightarrowtail (x \rightarrowtail 0))) \wedge (0 \wedge x)) = (0 \wedge x) \rightarrowtail (x \wedge 0)$ by (206) and (209)
211. $0 \rightarrowtail ((0 \rightarrowtail (x \rightarrowtail 0)) \wedge (0 \wedge x)) = (0 \wedge x) \rightarrowtail (x \wedge 0)$ by (173) and (210)
212. $(0 \rightarrowtail 0) \wedge (((x \rightarrowtail 0) \wedge (0 \rightarrowtail 0)) \rightarrowtail 0) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (75) and (43)
213. $(0 \rightarrowtail 0) \wedge ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (46) and (212)
214. $(((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0) \rightarrowtail 0) \rightarrowtail (0 \wedge (((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0) \rightarrowtail 0)) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (86) and (213)
215. $((0 \rightarrowtail 0) \rightarrowtail (x \rightarrowtail (0 \wedge x))) \rightarrowtail (0 \wedge (((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0) \rightarrowtail 0)) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (67) and (214)
216. $(x \rightarrowtail (0 \wedge x)) \rightarrowtail (0 \wedge (((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0) \rightarrowtail 0)) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (215)
217. $(x \rightarrowtail (0 \wedge x)) \rightarrowtail (0 \wedge ((0 \rightarrowtail 0) \rightarrowtail (x \rightarrowtail (0 \wedge x)))) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (67) and (216)
218. $(x \rightarrowtail (0 \wedge x)) \rightarrowtail (0 \wedge (x \rightarrowtail (0 \wedge x))) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (217)
219. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = (((0 \wedge (0 \rightarrowtail (0 \rightarrowtail 0))) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (108) and (218)
220. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = (((0 \wedge (0 \rightarrowtail 0)) \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (80) and (219)
221. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = ((0 \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge ((0 \rightarrowtail 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (135) and (220)
222. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = ((0 \rightarrowtail 0) \wedge ((x \wedge ((0 \wedge 0) \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (221)
223. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = ((0 \rightarrowtail 0) \wedge ((x \wedge (0 \rightarrowtail 0)) \rightarrowtail 0)) \rightarrowtail 0$ by (18) and (222)
224. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = ((0 \rightarrowtail 0) \wedge ((0 \rightarrowtail 0) \rightarrowtail (x \rightarrowtail 0))) \rightarrowtail 0$ by (109) and (223)
225. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = ((0 \rightarrowtail 0) \wedge (x \rightarrowtail 0)) \rightarrowtail 0$ by (3) and (224)

226. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = (((x \rightarrowtail 0) \rightarrowtail 0) \rightarrowtail (0 \wedge ((x \rightarrowtail 0) \rightarrowtail 0))) \rightarrowtail 0$ by (86) and (225)
227. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = (x \rightarrowtail (0 \wedge ((x \rightarrowtail 0) \rightarrowtail 0))) \rightarrowtail 0$ by (24) and (226)
228. $(x \rightarrowtail 0) \rightarrowtail (0 \wedge (x \rightarrowtail 0)) = (x \rightarrowtail (0 \wedge x)) \rightarrowtail 0$ by (24) and (227)
229. $(x \wedge 0) \wedge (((0 \rightarrowtail (y \rightarrowtail y)) \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0)) = x \wedge 0$ by (228) and (176)
230. $0 \rightarrowtail ((x \wedge (x \rightarrowtail 0)) \wedge y) = ((0 \rightarrowtail y) \rightarrowtail (0 \wedge (0 \rightarrowtail y))) \rightarrowtail 0$ by (228) and (98)
231. $x \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0) = 0 \rightarrowtail (0 \wedge (x \rightarrowtail 0))$ by (228) and (97)
232. $x \wedge (y \rightarrowtail y) = (x \rightarrowtail (0 \wedge x)) \rightarrowtail 0$ by (228) and (93)
233. $x \wedge (y \rightarrowtail y) = (x \rightarrowtail (x \wedge 0)) \rightarrowtail 0$ by (17) and (232)
234. $x \wedge (((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0) \rightarrowtail 0) = x \wedge (x \rightarrowtail ((y \rightarrowtail y) \rightarrowtail 0))$ by (232) and (62)
235. $x \wedge ((0 \rightarrowtail 0) \rightarrowtail (x \rightarrowtail (0 \wedge x))) = x \wedge (x \rightarrowtail ((y \rightarrowtail y) \rightarrowtail 0))$ by (67) and (234)
236. $x \wedge (x \rightarrowtail (0 \wedge x)) = x \wedge (x \rightarrowtail ((y \rightarrowtail y) \rightarrowtail 0))$ by (3) and (235)
237. $x \wedge (x \rightarrowtail (0 \wedge x)) = x \wedge (x \rightarrowtail 0)$ by (3) and (236)
238. $(0 \wedge x) \wedge (((x \wedge 0) \rightarrowtail 0) \rightarrowtail ((0 \wedge x) \rightarrowtail 0)) = ((0 \wedge x) \rightarrowtail ((0 \wedge x) \wedge 0)) \rightarrowtail 0$ by (233) and (195)
239. $(0 \wedge x) \wedge (((x \wedge 0) \rightarrowtail 0) \rightarrowtail ((0 \wedge x) \rightarrowtail 0)) = ((0 \wedge x) \rightarrowtail (0 \wedge (0 \wedge x))) \rightarrowtail 0$ by (17) and (238)
240. $(((x \rightarrowtail x) \rightarrowtail 0) \rightarrowtail (x \rightarrowtail x)) \rightarrowtail (0 \wedge (((x \rightarrowtail x) \rightarrowtail 0) \rightarrowtail (x \rightarrowtail x)))) \rightarrowtail 0 = x \rightarrowtail x$ by (232) and (75)
241. $((0 \rightarrowtail (x \rightarrowtail x)) \rightarrowtail (0 \wedge (((x \rightarrowtail x) \rightarrowtail 0) \rightarrowtail (x \rightarrowtail x)))) \rightarrowtail 0 = x \rightarrowtail x$ by (3) and (240)
242. $((0 \rightarrowtail (x \rightarrowtail x)) \rightarrowtail (0 \wedge (0 \rightarrowtail (x \rightarrowtail x)))) \rightarrowtail 0 = x \rightarrowtail x$ by (3) and (241)
243. $((0 \rightarrowtail (x \rightarrowtail x)) \rightarrowtail 0) \rightarrowtail 0 = x \rightarrowtail x$ by (132) and (242)
244. $0 \rightarrowtail (x \rightarrowtail x) = x \rightarrowtail x$ by (24) and (243)
245. $((0 \wedge x) \rightarrowtail (x \wedge 0)) \wedge y = (y \rightarrowtail (0 \wedge y)) \rightarrowtail 0$ by (190) and (232)
246. $(x \wedge 0) \wedge (((y \rightarrowtail y) \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0)) = x \wedge 0$ by (244) and (229)
247. $(x \wedge 0) \wedge (0 \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0)) = x \wedge 0$ by (3) and (246)
248. $(x \wedge 0) \wedge ((x \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail x) \rightarrowtail 0)))) = x \wedge 0$ by (169) and (247)
249. $(x \wedge 0) \wedge ((x \rightarrowtail 0) \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail 0))) = x \wedge 0$ by (3) and (248)
250. $(x \wedge 0) \wedge ((x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail 0))) = x \wedge 0$ by (25) and (249)
251. $(x \wedge 0) \wedge ((x \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail x)) = x \wedge 0$ by (24) and (250)
252. $(x \wedge 0) \wedge (0 \rightarrowtail x) = x \wedge 0$ by (151) and (251)
253. $(0 \rightarrowtail x) \wedge (x \wedge 0) = x \wedge 0$ by (17) and (252)
254. $(x \rightarrowtail x) \wedge (0 \rightarrowtail y) = 0 \rightarrowtail ((x \rightarrowtail x) \wedge y)$ by (244) and (7)
255. $((0 \rightarrowtail y) \rightarrowtail (0 \wedge (0 \rightarrowtail y))) \rightarrowtail 0 = 0 \rightarrowtail ((x \rightarrowtail x) \wedge y)$ by (232) and (254)
256. $((0 \rightarrowtail y) \rightarrowtail (0 \wedge (0 \rightarrowtail y))) \rightarrowtail 0 = 0 \rightarrowtail ((y \rightarrowtail (0 \wedge y)) \rightarrowtail 0)$ by (232) and (255)
257. $((0 \rightarrowtail y) \rightarrowtail (0 \wedge (0 \rightarrowtail y))) \rightarrowtail 0 = (y \rightarrowtail 0) \rightarrowtail ((y \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail ((y \rightarrowtail y) \rightarrowtail 0)))$ by (169) and (256)
258. $((0 \rightarrowtail y) \rightarrowtail (0 \wedge (0 \rightarrowtail y))) \rightarrowtail 0 = (y \rightarrowtail 0) \rightarrowtail ((y \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail 0))$ by (3) and (257)
259. $((0 \rightarrowtail y) \rightarrowtail (0 \wedge (0 \rightarrowtail y))) \rightarrowtail 0 = (y \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail ((y \rightarrowtail 0) \rightarrowtail 0))$ by (25) and (258)
260. $((0 \rightarrowtail y) \rightarrowtail (0 \wedge (0 \rightarrowtail y))) \rightarrowtail 0 = (y \rightarrowtail 0) \rightarrowtail (0 \rightarrowtail y)$ by (24) and (259)
261. $((0 \rightarrowtail x) \rightarrowtail (0 \wedge (0 \rightarrowtail x))) \rightarrowtail 0 = 0 \rightarrowtail x$ by (151) and (260)
262. $0 \rightarrowtail ((x \wedge (x \rightarrowtail 0)) \wedge y) = 0 \rightarrowtail y$ by (261) and (230)
263. $x \wedge ((x \wedge (x \rightarrowtail 0)) \rightarrowtail 0) = x \wedge (x \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0))$ by (237) and (62)
264. $x \wedge (x \rightarrowtail ((x \rightarrowtail 0) \rightarrowtail 0)) = x \wedge (x \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0))$ by (62) and (263)
265. $x \wedge (x \rightarrowtail x) = x \wedge (x \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0))$ by (24) and (264)
266. $(x \rightarrowtail (x \wedge 0)) \rightarrowtail 0 = x \wedge (x \rightarrowtail ((x \rightarrowtail (0 \wedge x)) \rightarrowtail 0))$ by (233) and (265)
267. $(x \wedge ((y \rightarrowtail (0 \wedge 0)) \rightarrowtail 0)) \wedge ((((y \wedge x) \rightarrowtail 0) \wedge ((y \wedge x) \rightarrowtail 0)) \rightarrowtail z) = (((y \wedge x) \rightarrowtail 0) \wedge ((y \wedge x) \rightarrowtail 0)) \rightarrowtail (0 \wedge z)$ by (7) and (44)

268. $(x\wedge((y\rightarrowtail 0)\rightarrowtail 0))\wedge((((y\wedge x)\rightarrowtail 0)\wedge((y\wedge x)\rightarrowtail 0))\rightarrowtail z)=(((y\wedge x)\rightarrowtail 0)\wedge((y\wedge x)\rightarrowtail 0))\rightarrowtail(0\wedge z)$ by (18) and (267)
269. $(x\wedge y)\wedge((((y\wedge x)\rightarrowtail 0)\wedge((y\wedge x)\rightarrowtail 0))\rightarrowtail z)=(((y\wedge x)\rightarrowtail 0)\wedge((y\wedge x)\rightarrowtail 0))\rightarrowtail(0\wedge z)$ by (24) and (268)
270. $(x\wedge y)\wedge(((y\wedge x)\rightarrowtail 0)\rightarrowtail z)=(((y\wedge x)\rightarrowtail 0)\wedge((y\wedge x)\rightarrowtail 0))\rightarrowtail(0\wedge z)$ by (18) and (269)
271. $(x\wedge y)\wedge(((y\wedge x)\rightarrowtail 0)\rightarrowtail z)=((y\wedge x)\rightarrowtail 0)\rightarrowtail(0\wedge z)$ by (18) and (270)
272. $((x\wedge 0)\rightarrowtail 0)\rightarrowtail(0\wedge((0\wedge x)\rightarrowtail 0))=((0\wedge x)\rightarrowtail(0\wedge(0\wedge x)))\rightarrowtail 0$ by (271) and (239)
273. $((x\wedge 0)\rightarrowtail 0)\rightarrowtail(0\wedge(0\rightarrowtail(x\rightarrowtail 0)))=((0\wedge x)\rightarrowtail(0\wedge(0\wedge x)))\rightarrowtail 0$ by (62) and (272)
274. $0\rightarrowtail(((x\rightarrowtail x)\wedge 0)\wedge y)=0\rightarrowtail y$ by (3) and (262)
275. $0\rightarrowtail((0\wedge(x\rightarrowtail x))\wedge y)=0\rightarrowtail y$ by (17) and (274)
276. $0\rightarrowtail(((0\rightarrowtail(0\wedge 0))\rightarrowtail 0)\wedge y)=0\rightarrowtail y$ by (233) and (275)
277. $0\rightarrowtail(((0\rightarrowtail 0)\rightarrowtail 0)\wedge y)=0\rightarrowtail y$ by (18) and (276)
278. $0\rightarrowtail(0\wedge x)=0\rightarrowtail x$ by (3) and (277)
279. $0\rightarrowtail(x\rightarrowtail 0)=x\rightarrowtail((x\rightarrowtail(0\wedge x))\rightarrowtail 0)$ by (278) and (231)
280. $x\wedge(0\rightarrowtail(x\rightarrowtail 0))=(x\rightarrowtail(x\wedge 0))\rightarrowtail 0$ by (279) and (266)
281. $(0\rightarrowtail x)\wedge(0\rightarrowtail y)=0\rightarrowtail(x\wedge(0\wedge y))$ by (278) and (7)
282. $0\rightarrowtail(x\wedge y)=0\rightarrowtail(x\wedge(0\wedge y))$ by (7) and (281)
283. $0\rightarrowtail((0\rightarrowtail(x\rightarrowtail 0))\wedge x)=(0\wedge x)\rightarrowtail(x\wedge 0)$ by (282) and (211)
284. $0\rightarrowtail(x\wedge(0\rightarrowtail(x\rightarrowtail 0)))=(0\wedge x)\rightarrowtail(x\wedge 0)$ by (17) and (283)
285. $0\rightarrowtail((x\rightarrowtail(x\wedge 0))\rightarrowtail 0)=(0\wedge x)\rightarrowtail(x\wedge 0)$ by (280) and (284)
286. $(x\rightarrowtail x)\rightarrowtail((x\rightarrowtail x)\rightarrowtail(0\rightarrowtail((x\rightarrowtail 0)\rightarrowtail 0)))=(0\wedge x)\rightarrowtail(x\wedge 0)$ by (169) and (285)
287. $(x\rightarrowtail x)\rightarrowtail((x\rightarrowtail x)\rightarrowtail(0\rightarrowtail x))=(0\wedge x)\rightarrowtail(x\wedge 0)$ by (24) and (286)
288. $(x\rightarrowtail x)\rightarrowtail(0\rightarrowtail x)=(0\wedge x)\rightarrowtail(x\wedge 0)$ by (3) and (287)
289. $0\rightarrowtail x=(0\wedge x)\rightarrowtail(x\wedge 0)$ by (3) and (288)
290. $(0\rightarrowtail x)\wedge y=(y\rightarrowtail(0\wedge y))\rightarrowtail 0$ by (289) and (245)
291. $((x\wedge 0)\rightarrowtail(0\wedge(x\wedge 0)))\rightarrowtail 0=x\wedge 0$ by (290) and (253)
292. $((0\wedge(0\rightarrowtail x))\rightarrowtail 0)\rightarrowtail 0=x\wedge 0$ by (189) and (291)
293. $(((0\rightarrowtail(0\wedge 0))\rightarrowtail 0)\rightarrowtail 0)\rightarrowtail 0=x\wedge 0$ by (290) and (292)
294. $(((0\rightarrowtail 0)\rightarrowtail 0)\rightarrowtail 0)\rightarrowtail 0=x\wedge 0$ by (18) and (293)
295. $(0\rightarrowtail 0)\rightarrowtail 0=x\wedge 0$ by (3) and (294)
296. $0=x\wedge 0$ by (3) and (295)

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