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L^p-boundedness properties of variation operators in the Schrödinger setting

J.J. Betancor · J.C. Fariña · E. Harboure · L. Rodríguez-Mesa

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Abstract In this paper we establish the L^p-boundedness properties of the variation operators associated with the heat semigroup, Riesz transforms and commutator between Riesz transforms and multiplication by $BMO(\mathbb{R}^n)$-functions in the Schrödinger setting.

Keywords Variation operator · Commutator operator · BMO · Schrödinger

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1 Introduction and main results

We consider the time independent Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$, defined by

$$\mathcal{L} = -\Delta + V,$$

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J.J. Betancor · J.C. Fariña (✉) · L. Rodríguez-Mesa
Departamento de Análisis Matemático, Universidad de la Laguna, Campus de Anchieta, Avda. Astrofísico Francisco Sánchez, s/n, 38271 La Laguna (Sta. Cruz de Tenerife), Spain
e-mail: jcfarina@ull.es

J.J. Betancor
e-mail: jbetanco@ull.es

L. Rodríguez-Mesa
e-mail: lrguez@ull.es

E. Harboure
Instituto de Matemática Aplicada del Litoral, IMAL, Universidad Nacional del Litoral, C/Gëmes, Santa Fe, Argentina
e-mail: harboure@santafe-conicet.gov.ar

where the potential V is nonzero, nonnegative and belongs, for some $q \geq n/2$, to the reverse Hölder class B_q, that is, there exists $C > 0$ such that

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{1/q} \leq \frac{C}{|B|}\int_B V(x)dx,$$

for every ball B in $\mathbb{R}^n$. Since any nonnegative polynomial belongs to B_q for every $1 < q < \infty$, the Hermite operator $-\Delta + |x|^2$ falls under our considerations.

Harmonic analysis associated with the operator $\mathcal{L}$ has been studied by several authors in the last decade. Most of them had, as starting point, the paper of Shen [25]. This author investigated L^p-boundedness properties of the Riesz transforms formally defined in the $\mathcal{L}$-setting by

$$R^{\mathcal{L}} = \nabla \mathcal{L}^{-1/2}, \tag{1}$$

where as usual ∇ denotes the gradient operator and the negative square root $\mathcal{L}^{-1/2}$ of $\mathcal{L}$ is given by the functional calculus as follows

$$\mathcal{L}^{-1/2} f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy, \tag{2}$$

being

$$K(x, y) = -\frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{-1/2} \Gamma(x, y, \tau) d\tau.$$

Here, for every $\tau \in \mathbb{R}$, $\Gamma(x, y, \tau)$, $x, y \in \mathbb{R}^n$, represents the fundamental solution for the operator $\mathcal{L} + i\tau$.

Bongioanni, Harboure and Salinas [3] studied the behavior in L^p spaces of the commutator operator $[R^{\mathcal{L}}, b]$ defined by

$$\left[R^{\mathcal{L}}, b\right] f = bR^{\mathcal{L}}(f) - R^{\mathcal{L}}(bf),$$

where b is in an appropriate class containing the space *BMO* of bounded mean oscillation functions.

The heat semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ generated by the operator $-\mathcal{L}$ can be written on $L^2(\mathbb{R}^n)$ as the following integral operator

$$W_t^{\mathcal{L}}(f)(x) = \int_{\mathbb{R}^n} W_t^{\mathcal{L}}(x, y) f(y) dy, \quad f \in L^2\left(\mathbb{R}^n\right).$$

The semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ is C_0 in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, but it is not Markovian. The main properties of the kernel $W_t^{\mathcal{L}}(x, y)$, $x, y \in \mathbb{R}^n$, $t > 0$, can be encountered in [12].

Other operators associated with the Schrödinger operator $\mathcal{L}$ have been studied on $L^p(\mathbb{R}^n)$ and on other kind of function spaces also in [1, 2, 11, 12, 15, 28], and [30].

The variation operators were considered in ergodic theory ([5] and [19]) in order to measure the speed of convergence. A precedent of the Bourgain's result ([5]) is the celebrated Lèpingle's ρ-variation inequality in martingale theory ([21], see also [22]). Suppose that $\{T_t\}_{t>0}$ is an uniparametric system of linear operators bounded in $L^p(\mathbb{R}^n)$, for some $1 \leq p < \infty$, such that $\lim_{t\to 0^+} T_t f$ exists in $L^p(\mathbb{R}^n)$.

If $\rho > 2$, the variation operator $V_\rho(T_t)$ is given by

$$V_\rho(T_t)(f)(x) = \sup_{\{t_j\}_{j\in\mathbb{N}}\searrow 0}\left(\sum_{j=0}^{\infty}\left|T_{t_j}f(x) - T_{t_{j+1}}f(x)\right|^\rho\right)^{1/\rho}, \quad f \in L^p\left(\mathbb{R}^n\right).$$

Here the supremum is taken over all the real decreasing sequences $\{t_j\}_{j\in\mathbb{N}}$ that converge to zero. By E_ρ we denote the space that includes all the functions $w : (0,\infty) \longrightarrow \mathbb{R}$, such that

$$\|w\|_{E_\rho} = \sup_{\{t_j\}_{j\in\mathbb{N}}\searrow 0}\left(\sum_{j=0}^{\infty}\left|w(t_j) - w(t_{j+1})\right|^\rho\right)^{1/\rho} < \infty.$$

$\|\cdot\|_{E_\rho}$ is a seminorm on E_ρ. It can be written

$$V_\rho(T_t)(f) = \|T_t f\|_{E_\rho}.$$

Variation operators for C_0-semigroups of operators and singular integrals have been analyzed on L^p-spaces by Campbell et al. ([6, 7] and [19]). More recently, in [9] and [17] the authors have studied variation operators on L^p-spaces for semigroups and Riesz transforms in the Ornstein-Uhlenbeck and Hermite settings.

As it was mentioned, for every $1 \le p < \infty$, the semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ is C_0 in $L^p(\mathbb{R}^n)$, that is, for every $f \in L^p(\mathbb{R}^n)$, $W_t^{\mathcal{L}}(f) \to f$, as $t \to 0^+$, in $L^p(\mathbb{R}^n)$. The L^p-boundedness properties of the oscillation and variation operators for $\{W_t^{\mathcal{L}}\}_{t>0}$ are established in the following.

Theorem 1.1 *Let $\rho > 2$. Then, the variation operator $V_\rho(W_t^{\mathcal{L}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

Note that, since $\{W_t^{\mathcal{L}}\}_{t>0}$ is not Markovian, none part of Theorem 1.1 can be deduced from [18, Theorem 3.3]. However, the referee has pointed out us that recently Le Merdy and Xu ([20]) have extended [18, Theorem 3.3] to all the contractive and strongly continuous semigroups. Then, from [20, Corollary 3.4] we deduce that the variation operator $V_\rho(W_t^{\mathcal{L}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$. The behavior of $V_\rho(W_t^{\mathcal{L}})$ in $L^1(\mathbb{R}^n)$ established in Theorem 1.1 does not follows from the results in [20].

According to standard ideas, Shen in [25] actually defined (although he did not write it in this way), for every $\ell = 1, \dots, n$, the ℓ-th Riesz transform in the $\mathcal{L}$-context by

$$R_\ell^{\mathcal{L}}(f)(x) = \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} R_\ell^{\mathcal{L}}(x,y)f(y)dy, \quad \text{a.e. } x \in \mathbb{R}^n, \tag{3}$$

provided that $f \in L^p(\mathbb{R}^n)$ and either

(i) $1 \le p < \infty$ and $V \in B_n$; or
(ii) $1 \le p < p_0$, $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, and $V \in B_q$, $n/2 \le q < n$.

Here, for every $x, y \in \mathbb{R}^n$, $x \neq y$,

$$R_\ell^{\mathcal{L}}(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{-1/2}\frac{\partial}{\partial x_\ell}\Gamma(x,y,\tau)d\tau.$$

In the sequel we complete Shen's result proving that actually the limit in (3) exists and the Riesz transform $R^{\mathcal{L}} = (R_1^{\mathcal{L}}, \dots, R_n^{\mathcal{L}})$ can be represented by (1) on $C_c^\infty(\mathbb{R}^n)$, the space of C^∞-functions in $\mathbb{R}^n$ that have compact support.

Proposition 1.1 *Let $\ell = 1, \dots, n$. Suppose that one of the following two conditions holds*:

(i) $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, *and* $V \in B_n$;
(ii) $f \in L^p(\mathbb{R}^n)$, $1 \leq p < p_0$, *where* $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, *and* $V \in B_q$, $n/2 \leq q < n$.

Then, there exists the following limit

$$\lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_\ell^{\mathcal{L}}(x,y) f(y) dy, \quad \text{a.e. } x \in \mathbb{R}^n.$$

Moreover, if $f \in C_c^\infty(\mathbb{R}^n)$, $\mathcal{L}^{-1/2} f$, as defined in (2), admits partial derivative with respect to x_ℓ almost everywhere in $\mathbb{R}^n$ and

$$\frac{\partial}{\partial x_\ell}\mathcal{L}^{-1/2} f(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R_\ell^{\mathcal{L}}(x,y) f(y) dy, \quad \text{a.e. } x \in \mathbb{R}^n. \tag{4}$$

For every $\varepsilon > 0$, the ε-truncation $R_\ell^{\mathcal{L},\varepsilon}$ of $R_\ell^{\mathcal{L}}$ is defined as usual by

$$R_\ell^{\mathcal{L},\varepsilon}(f)(x) = \int_{|x-y|>\varepsilon} R_\ell^{\mathcal{L}}(x,y) f(y) dy, \quad \ell = 1, \dots, n.$$

The behavior on L^p spaces of the variation operators associated with the family of truncations $\{R_\ell^{\mathcal{L},\varepsilon}\}_{\varepsilon>0}$, $\ell = 1, \dots, n$, is contained in the following result.

Theorem 1.2 *Let $\ell = 1, \dots, n$. Assume that $\rho > 2$ and that $V \in B_q$, with $q \geq n/2$, and $1 < p < p_0$, where $\frac{1}{p_0} = (\frac{1}{q} - \frac{1}{n})_+$. Then, the variation operator $V_\rho(R_\ell^{\mathcal{L},\varepsilon})$ is bounded from $L^p(\mathbb{R}^n)$ into itself. Moreover, $V_\rho(R_\ell^{\mathcal{L},\varepsilon})$ is bounded from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

By $BMO(\mathbb{R}^n)$ we denote the usual John-Nirenberg space. A locally integrable function b on $\mathbb{R}^n$ is in $BMO(\mathbb{R}^n)$ if and only if there exists $C > 0$ such that

$$\frac{1}{|B|}\int_B |b(x) - b_B| dx \leq C,$$

for every ball B in $\mathbb{R}^n$. Here $b_B = \frac{1}{|B|}\int_B b(x)dx$, where B is a ball in $\mathbb{R}^n$. For $f \in BMO(\mathbb{R}^n)$ we define

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |b(x) - b_B| dx,$$

where the supremum is taken over all the balls B in $\mathbb{R}^n$.

For every $V \in B_{n/2}$, we consider the function γ defined by

$$\gamma(x) = \sup\left\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1\right\}, \quad x \in \mathbb{R}^n.$$

Under our assumptions it is not hard to see that $0 < \gamma(x) < \infty$, for all $x \in \mathbb{R}^n$. This function γ was introduced in [24] when the potential V satisfies that

$$\max_{x \in B} V(x) \leq C \frac{1}{|B|} \int_B V(y)dy,$$

for every ball B in $\mathbb{R}^n$, to study the Neumann problem for the operator $\mathcal{L}$ in the region above a Lipschitz graph. The main properties of γ were showed in [25, Sect. 1] (see also [24]). Here, the function γ plays a crucial role.

In [3] the space $BMO_\theta(\gamma)$, $\theta \geq 0$, was defined as follows. Let $\theta \geq 0$. A locally integrable function b in $\mathbb{R}^n$ is in $BMO_\theta(\gamma)$ provided that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y) - b_{B(x,r)}\right| dy \leq C\left(1 + \frac{r}{\gamma(x)}\right)^\theta,$$

for all $x \in \mathbb{R}^n$ and $r > 0$. We denote for $b \in BMO_\theta(\gamma)$

$$\|b\|_{BMO_\theta(\gamma)} = \sup_{x \in \mathbb{R}^n,\, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \left|b(y) - b_{B(x,r)}\right| dy \left(1 + \frac{r}{\gamma(x)}\right)^{-\theta}.$$

Note that $BMO(\mathbb{R}^n) = BMO_0(\gamma) \subset BMO_\theta(\gamma) \subset BMO_{\theta'}(\gamma)$, when $0 \leq \theta \leq \theta'$. We set $BMO_\infty(\gamma) = \bigcup_{\theta>0} BMO_\theta(\gamma)$. As it is pointed out in [3], $BMO_\infty(\gamma)$ is in general larger than $BMO(\mathbb{R}^n)$.

For $b \in BMO_\infty(\gamma)$ and $\ell = 1, \ldots, n$, the commutator operator $C^{\mathcal{L}}_{b,\ell}$ is defined by

$$C^{\mathcal{L}}_{b,\ell}(f) = bR^{\mathcal{L}}_\ell(f) - R^{\mathcal{L}}_\ell(bf), \quad f \in C^\infty_c\big(\mathbb{R}^n\big).$$

Note that $bf \in L^1(\mathbb{R}^n)$, for every $f \in C^\infty_c(\mathbb{R}^n)$ and $b \in BMO_\infty(\gamma)$.

In [3, Theorem 1] it was shown that, for every $b \in BMO_\infty(\gamma)$ and $\ell = 1, \ldots, n$, the operator $C^{\mathcal{L}}_{b,\ell}$ is bounded from $L^p(\mathbb{R}^n)$ into itself, provided that $1 < p < p_0$, where $\frac{1}{p_0} = (\frac{1}{q} - \frac{1}{n})_+$ and $V \in B_q$, $q \geq n/2$.

In the next result we obtain a pointwise representation of the commutator operator by a principal value integral.

Proposition 1.2 *Let $\ell = 1, \ldots, n$. If $b \in BMO_\infty(\gamma)$, $V \in B_q$, with $q \geq n/2$, and $f \in L^p(\mathbb{R}^n)$, where $1 < p < p_0$ and $\frac{1}{p_0} = (\frac{1}{q} - \frac{1}{n})_+$, then*

$$C^{\mathcal{L}}_{b,\ell}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \big(b(x) - b(y)\big) R^{\mathcal{L}}_\ell(x,y) f(y)dy, \quad \text{a.e. } x \in \mathbb{R}^n.$$

For every $b \in BMO_\infty(\gamma)$, $\varepsilon > 0$, and $\ell = 1, \dots, n$, we define the ε-truncation $C_{b,\ell}^{\mathcal{L},\varepsilon}$ of $C_{b,\ell}^{\mathcal{L}}$ by

$$C_{b,\ell}^{\mathcal{L},\varepsilon}(f)(x) = \int_{|x-y|>\varepsilon} \big(b(x) - b(y)\big) R_\ell^{\mathcal{L}}(x,y) f(y) dy, \quad x \in \mathbb{R}^n.$$

The L^p-boundedness properties of the variation operators associated with the family of truncations $\{C_{b,\ell}^{\mathcal{L},\varepsilon}\}_{\varepsilon>0}$ are contained in the following.

Theorem 1.3 *Let $\ell = 1, \dots, n$ and $b \in BMO_\infty(\gamma)$. Assume that $V \in B_q$, with $q \geq n/2$, and $1 < p < p_0$, where $\frac{1}{p_0} = (\frac{1}{q} - \frac{1}{n})_+$. Then, if $\rho > 2$, the variation operator $V_\rho(C_{b,\ell}^{\mathcal{L},\varepsilon})$ is bounded from $L^p(\mathbb{R}^n)$ into itself.*

In [25, p. 516] it was proved that if V is a nonnegative polynomial, then $V \in B_q$, for every $1 < q < \infty$. Then, as special cases of our results appear the corresponding ones to the Hermite operator $H = -\Delta + |x|^2$ ([9] and [10]).

This paper is organized as follows. In Sect. 2 we describe a general procedure that we shall use to prove our main results and we present the L^p-properties of the variation operators associated with the classical ($V \equiv 0$) heat semigroup $\{W_t\}_{t>0}$, Riesz transforms R_ℓ and their commutators $C_{b,\ell}$, $\ell = 1, \dots, n$, that will be very useful to our purposes. The proof of Theorem 1.1 is carried out in Sect. 3. We present proofs of Proposition 1.1 and Theorem 1.2 in Sect. 4. Finally, in Sect. 5 we give proofs for Proposition 1.2 and Theorem 1.3.

Throughout this paper by c and C we will always denote positive constants that can change in each occurrence. If $1 < p < \infty$, by p' we represent the exponent conjugated of p, that is, $p' = \frac{p}{p-1}$.

2 Procedure and auxiliary results

In order to establish boundedness properties for harmonic analysis operators (semigroup, maximal operators, Riesz transforms, Littlewood-Paley functions, ...) in the Schrödinger setting it is usual to exploit that $\mathcal{L}$ is actually a nice perturbation of the Laplacian operator $-\Delta$. We now describe a general procedure to analyze harmonic operators associated with the Schrödinger operator. Suppose that T is a $\mathcal{L}$-harmonic analysis operator and that $\mathcal{T}$ is the corresponding Δ-harmonic operator. According to the function γ described above, we split $\mathbb{R}^n \times \mathbb{R}^n$ in two parts as follows

$$A = \big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| < \gamma(x)\big\},$$

and

$$B = \big(\mathbb{R}^n \times \mathbb{R}^n\big) \setminus A.$$

The sets A and B are usually called local and global region associated with $\mathcal{L}$, respectively. The local part of the operator T is defined by

$$T_{\mathrm{loc}}(f)(x) = T(f\chi_{B(x,\gamma(x))})(x), \quad x \in \mathbb{R}^n.$$

In a similar way we consider the operator

$$\mathcal{T}_{\rm loc}(f)(x) = \mathcal{T}(f\chi_{B(x,\gamma(x))})(x), \quad x \in \mathbb{R}^n.$$

Then, we decompose the operator T through

$$T = (T_{\rm loc} - \mathcal{T}_{\rm loc}) + \mathcal{T}_{\rm loc} + (T - T_{\rm loc}).$$

It is clear that $(T - T_{\rm loc})(f)(x) = T(f\chi_{\mathbb{R}^n \setminus B(x,\gamma(x))})(x)$. Since the set $\{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}^n \setminus B(x, \gamma(x))\}$ is sufficiently far away from the diagonal (usual line of singularities) $\{(x, x) : x \in \mathbb{R}^n\}$, the operator $T - T_{\rm loc}$ will be controlled by a positive and L^p-bounded operator. We said that $\mathcal{L}$ is actually a nice perturbation of the Laplacian operator $-\Delta$. That niceness leads to the operators T and $\mathcal{T}$ to have the same singularity in the local region. Then, cancelation of singularities in $T_{\rm loc} - \mathcal{T}_{\rm loc}$ takes place and $T_{\rm loc} - \mathcal{T}_{\rm loc}$ is controlled by a positive and L^p-bounded operator for the given range of p. In this way L^p-boundedness of T is reduced to the corresponding property for the operator $\mathcal{T}_{\rm loc}$. Finally, L^p-boundedness properties of the operator $\mathcal{T}_{\rm loc}$ rely on well known properties for the classical harmonic operator $\mathcal{T}$.

This procedure has been used in [25] to establish L^p-boundedness properties for $\mathcal{L}$-Riesz transform.

We will employ this comparative method to describe the behavior in L^p-spaces of the variation operators for the heat semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ generated by $-\mathcal{L}$, Riesz transforms and commutators of Riesz transforms with the multiplication by $BMO_\infty(\gamma)$-functions in the Schrödinger setting. Following this pattern we will need to know L^p-boundedness properties of the variation operators associated with the classical heat semigroup, Riesz transforms and commutators between Riesz transforms and multiplication by $BMO(\mathbb{R}^n)$-functions.

In [18, Theorem 3.3] it was established that that if $\{T_t\}_{t>0}$ is a symmetric diffusion semigroup (in the sense of [26, p. 65]) then the variation operator $V_\rho(T_t)$, with $\rho > 2$, is bounded from $L^p(\mathbb{R}^n)$ into itself for every $1 < p < \infty$. This result applies to the symmetric diffusion semigroup $\{W_t\}_{t>0}$ generated by the Euclidean Laplacian Δ. Recently, Crescimbeni, Macías, Menárguez, Torrea and Viviani ([9]), by using vector valued Calderón-Zygmund theory, have proved that the operators $V_\rho(W_t)$ map $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, for each $\rho > 2$. These results are contained in the following.

Theorem 2.1 ([18, Theorem 3.3] and [9, Theorem 1.1]) *Let $\rho > 2$. Then, the variation operator $V_\rho(W_t)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

For every $\ell = 1, \ldots, n$, the ℓ-th Riesz transform R_ℓ is defined by

$$R_\ell f(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{x_\ell - y_\ell}{|x - y|^{n+1}} f(y)dy, \quad \text{a.e. } x \in \mathbb{R}^n,$$

for each $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. For every $\varepsilon > 0$ and $\ell = 1, \ldots, n$, the ε-truncation of R_ℓ is given by

$$R_\ell^\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} \frac{x_\ell - y_\ell}{|x - y|^{n+1}} f(y)dy, \quad x \in \mathbb{R}^n.$$

We denote the kernel function of R_ℓ by $R_\ell(z) = \frac{z_\ell}{|z|^{n+1}}$, $z = (z_1, \dots, z_n) \in \mathbb{R}^n \setminus \{0\}$. The variation operators for R_ℓ^ε, $\ell = 1, \dots, n$, were investigated in [6] and [7] where the following results were proved.

Theorem 2.2 ([6, Theorems 1.1 and 1.2] and [7, Corollary 1.4]) *Let $\ell = 1, \dots, n$. Assume that $\rho > 2$. Then, the variation operator $V_\rho(R_\ell^\varepsilon)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

Let us mention that by using transference methods Gillespie and Torrea ([14, Theorem B]) have proved dimension free $L^p(\mathbb{R}^n, |x|^\alpha dx)$ norm inequalities, for every $1 < p < \infty$ and $-1 < \alpha < p - 1$, for variation operators of the Riesz transform R_ℓ, $\ell = 1, \dots, n$.

Next let $b \in BMO(\mathbb{R}^n)$. It is well known that, for every $\ell = 1, \dots, n$, the commutator operator $C_{b,\ell}$ defined by

$$C_{b,\ell}(f) = bR_\ell(f) - R_\ell(bf),$$

is bounded from $L^p(\mathbb{R}^n)$ into itself, and for each $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$C_{b,\ell} f(x) = \lim_{\varepsilon \to 0^+} C_{b,\ell}^\varepsilon(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

where

$$C_{b,\ell}^\varepsilon(f)(x) = \int_{|x-y|>\varepsilon} \big(b(x) - b(y)\big) R_\ell(x-y) f(y) dy, \quad x \in \mathbb{R}^n$$

([8, Theorem I]).

L^p-boundedness properties for the variation operators associated with $C_{b,\ell}$, $\ell = 1, \dots, n$, are stated in the following. To our knowledge the result is new, so we provide a proof.

Theorem 2.3 *Let $b \in BMO(\mathbb{R}^n)$ and $\ell = 1, 2, \dots, n$. Assume that $\rho > 2$. Then, the variation operator $V_\rho(C_{b,\ell}^\varepsilon)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$.*

Proof Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Inspired in the ideas developed in [14] our goal is to estimate the sharp maximal function

$$\big(V_\rho\big(C_{b,\ell}^\varepsilon\big)(f)\big)^\#(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |V_\rho\big(C_{b,\ell}^\varepsilon\big)(f)(y) - c_{B(x,r)}| dy, \quad x \in \mathbb{R}^n,$$

where, for every $x \in \mathbb{R}^n$ and $r > 0$, $c_{B(x,r)}$ is a constant that will be specified later.

Let $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ and denote by $B = B(x_0, r_0)$. We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{4B}$ and $f_2 = f\chi_{(4B)^c}$, and we write

$$\begin{aligned} C_{b,\ell}^\varepsilon f &= (b - b_B) R_\ell^\varepsilon(f) - R_\ell^\varepsilon\big((b-b_B) f_1\big) - R_\ell^\varepsilon\big((b-b_B) f_2\big) \\ &= A_1^\varepsilon(f) + A_2^\varepsilon(f) + A_3^\varepsilon(f), \quad \varepsilon > 0. \end{aligned}$$

We have that

$$\begin{aligned}
&\int_{(4B)^c} \frac{|b(y)-b_B|}{|x-y|^n}|f(y)|dy \\
&\quad=\sum_{k=2}^{\infty}\int_{2^k r_0<|y-x_0|\leq 2^{k+1}r_0}\frac{|b(y)-b_B|}{|x-y|^n}|f(y)|dy \\
&\quad\leq \sum_{k=2}^{\infty}\left(\int_{2^k r_0<|y-x_0|\leq 2^{k+1}r_0}\frac{|b(y)-b_B|^{p'}}{|x-y|^n}dy\right)^{1/p'} \\
&\qquad\times\left(\int_{2^k r_0<|y-x_0|\leq 2^{k+1}r_0}\frac{|f(y)|^p}{|x-y|^n}dy\right)^{1/p} \\
&\quad\leq C\sum_{k=2}^{\infty}\left(\frac{1}{(2^{k+1}r_0)^n}\int_{|y-x_0|\leq 2^{k+1}r_0}\big|b(y)-b_B\big|^{p'}dy\right)^{1/p'}\frac{1}{(2^k r_0)^{n/p}}\|f\|_{L^p(\mathbb{R}^n)} \\
&\quad\leq C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\sum_{k=2}^{\infty}\frac{k}{(2^k r_0)^{n/p}},\quad x\in B.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
&V_\rho\big(R_\ell^\varepsilon\big)\big((b-b_B)f_2\big)(x) \\
&\quad=\sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\searrow 0}\left(\sum_{j=0}^{\infty}\left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j}R_\ell(x-y)\big(b(y)-b_B\big)f(y)\chi_{(4B)^c}(y)dy\right|^\rho\right)^{1/\rho} \\
&\quad\leq\sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\searrow 0}\sum_{j=0}^{\infty}\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j}\frac{|b(y)-b_B\|f(y)|}{|x-y|^n}\chi_{(4B)^c}(y)dy \\
&\quad\leq\int_{(4B)^c}\frac{|b(y)-b_B\|f(y)|}{|x-y|^n}dy\leq Cr_0^{-n/p}\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)},\quad x\in B.
\end{aligned}$$

We denote $c_B=-V_\rho(R_\ell^\varepsilon)((b-b_B)f_2)(x_0)$. It is clear that $c_B=V_\rho(A_3^\varepsilon)(f)(x_0)$. We have

$$\begin{aligned}
&\frac{1}{|B|}\int_B\big|V_\rho\big(C_{b,\ell}^\varepsilon\big)(f)(x)-c_B\big|dx \\
&\quad=\frac{1}{|B|}\int_B\big|\,\big\|A_1^\varepsilon(f)(x)+A_2^\varepsilon(f)(x)+A_3^\varepsilon(f)(x)\big\|_{E_\rho}-\big\|A_3^\varepsilon(f)(x_0)\big\|_{E_\rho}\big|dx \\
&\quad\leq\frac{1}{|B|}\int_B\big\|A_1^\varepsilon(f)(x)+A_2^\varepsilon(f)(x)+A_3^\varepsilon(f)(x)-A_3^\varepsilon(f)(x_0)\big\|_{E_\rho}dx \\
&\quad\leq\frac{1}{|B|}\int_B\big\|A_1^\varepsilon(f)(x)\big\|_{E_\rho}dx+\frac{1}{|B|}\int_B\big\|A_2^\varepsilon(f)(x)\big\|_{E_\rho}dx \\
&\qquad+\frac{1}{|B|}\int_B\big\|A_3^\varepsilon(f)(x)-A_3^\varepsilon(f)(x_0)\big\|_{E_\rho}dx. \qquad (5)
\end{aligned}$$

We analyze each term. Firstly we obtain

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left\|A_1^\varepsilon(f)(x)\right\|_{E_\rho}dx\\
&\quad=\frac{1}{|B|}\int_B \left|b(x)-b_B\right|V_\rho\big(R_\ell^\varepsilon\big)(f)(x)dx\\
&\quad\le\left(\frac{1}{|B|}\int_B \left|b(x)-b_B\right|^{r'}dx\right)^{1/r'}\left(\frac{1}{|B|}\int_B\big(V_\rho\big(R_\ell^\varepsilon\big)(f)(x)\big)^r dx\right)^{1/r}\\
&\quad\le C\|b\|_{BMO(\mathbb{R}^n)}\mathcal{M}_r\big(V_\rho\big(R_\ell^\varepsilon\big)(f)\big)(z),\quad z\in B.
\end{aligned}\tag{6}$$

Here $1\le r<\infty$ and $\mathcal{M}_r$ is the r-maximal function defined by

$$\mathcal{M}_r(g)(x)=\sup_{x\in B}\left(\frac{1}{|B|}\int_B\left|g(y)\right|^r dy\right)^{1/r},\quad x\in\mathbb{R}^n,$$

for every measurable function g on $\mathbb{R}^n$.

On the other hand, according to [7, Theorem A], we have that

$$\begin{aligned}
&\frac{1}{|B|}\int_B \left\|A_2^\varepsilon(f)(x)\right\|_{E_\rho}dx\\
&\quad\le\left(\frac{1}{|B|}\int_B\left|V_\rho\big(R_\ell^\varepsilon\big)\big((b-b_B)f_1\big)(x)\right|^\beta dx\right)^{1/\beta}\\
&\quad\le C\left(\frac{1}{|B|}\int_{4B}\left|b(x)-b_B\right|^\beta\left|f(x)\right|^\beta dx\right)^{1/\beta}\\
&\quad\le C\left(\frac{1}{|B|}\int_{4B}\left|b(x)-b_B\right|^{s'\beta}dx\right)^{1/(s'\beta)}\left(\frac{1}{|B|}\int_{4B}\left|f(x)\right|^{s\beta}dx\right)^{1/(s\beta)}\\
&\quad\le C\|b\|_{BMO(\mathbb{R}^n)}\left(\frac{1}{|B|}\int_{4B}\left|f(x)\right|^{s\beta}dx\right)^{1/(s\beta)},
\end{aligned}$$

where $1<s,\beta<\infty$. Then, for every $1<r<\infty$, we have

$$\frac{1}{|B|}\int_B\left\|A_2^\varepsilon(f)(x)\right\|_{E_\rho}dx\le C\|b\|_{BMO(\mathbb{R}^n)}\mathcal{M}_r(f)(z),\quad z\in B.\tag{7}$$

In order to study $\|A_3^\varepsilon(f)(x)-A_3^\varepsilon(f)(x_0)\|_{E_\rho}$ we use a procedure developed in [14]. We have that

$$\left\|R_\ell^\varepsilon\big((b-b_B)f_2\big)(x)-R_\ell^\varepsilon\big((b-b_B)f_2\big)(x_0)\right\|_{E_\rho}\le H_1(x)+H_2(x),\quad x\in B,\tag{8}$$

where, for every $x\in B$,

$$H_1(x)=\left\|\int_{|x-y|>\varepsilon}\big(R_\ell(x-y)-R_\ell(x_0-y)\big)\big(b(y)-b_B\big)f_2(y)dy\right\|_{E_\rho}$$

and

$$H_2(x)=\left\|\int_{\mathbb{R}^n}\big(\chi_{\{|x-y|>\varepsilon\}}(y)-\chi_{\{|x_0-y|>\varepsilon\}}(y)\big)R_\ell(x_0-y)\big(b(y)-b_B\big)f_2(y)dy\right\|_{E_\rho}.$$

By using Minkowski inequality and well known properties of the function R_ℓ we get

$$\begin{aligned}H_1(x)&\leq\int_{\mathbb{R}^n}\big|R_\ell(x-y)-R_\ell(x_0-y)\big|\big|b(y)-b_B\big|\big|f(y)\big|\chi_{(4B)^c}(y)dy\\&\leq C\sum_{k=2}^{\infty}\int_{2^kr_0\leq|x_0-y|\leq2^{k+1}r_0}\frac{|x-x_0|}{|x_0-y|^{n+1}}\big|b(y)-b_B\big|\big|f(y)\big|dy\\&\leq C\sum_{k=2}^{\infty}\frac{1}{2^k}\frac{1}{(2^kr_0)^n}\int_{2^{k+1}B}\big|b(y)-b_B\big|\big|f(y)\big|dy,\quad x\in B.\end{aligned}\tag{9}$$

To analyze H_2 we split the integral appearing in the norm in four terms as follows. Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a real decreasing sequence that converges to zero. We have

$$\begin{aligned}&\int_{\mathbb{R}^n}\big|\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)-\chi_{\{\varepsilon_{j+1}<|x_0-y|<\varepsilon_j\}}(y)\big|\big|R_\ell(x_0-y)\big|\big|b(y)-b_B\big|\big|f_2(y)\big|dy\\&\leq C\bigg(\int_{\mathbb{R}^n}\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_{j+1}+r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\\&\quad\times\frac{1}{|x_0-y|^n}\big|b(y)-b_B\big|\big|f_2(y)\big|dy\\&\quad+\int_{\mathbb{R}^n}\chi_{\{\varepsilon_j<|x_0-y|<\varepsilon_j+r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\\&\quad\times\frac{1}{|x_0-y|^n}\big|b(y)-b_B\big|\big|f_2(y)\big|dy\\&\quad+\int_{\mathbb{R}^n}\chi_{\{\varepsilon_{j+1}<|x_0-y|<\varepsilon_{j+1}+r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x_0-y|<\varepsilon_j\}}(y)\\&\quad\times\frac{1}{|x_0-y|^n}\big|b(y)-b_B\big|\big|f_2(y)\big|dy\\&\quad+\int_{\mathbb{R}^n}\chi_{\{\varepsilon_j<|x-y|<\varepsilon_j+r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x_0-y|<\varepsilon_j\}}(y)\\&\quad\times\frac{1}{|x_0-y|^n}\big|b(y)-b_B\big|\big|f_2(y)\big|dy\bigg)\\&=C\big(H^j_{2,1}(x)+H^j_{2,2}(x)+H^j_{2,3}(x)+H^j_{2,4}(x)\big),\quad x\in B\text{ and }j\in\mathbb{N}.\end{aligned}\tag{10}$$

We observe that $\frac{4}{3}|x-y|\geq|x_0-y|\geq\frac{4}{5}|x-y|$, when $y\notin4B$ and $x\in B$. Moreover, if $x\in B$, then $H^j_{2,m}(x)=0$, for $m=1,3$, when $j\in\mathbb{N}$ and $r_0\geq\varepsilon_{j+1}$ and $H^j_{2,m}(x)=0$,

when $m = 2, 4$, $j \in \mathbb{N}$ and $r_0 \geq \varepsilon_j$. For every $j \in \mathbb{N}$, Hölder inequality leads to

$$H_{2,1}^j(x) \leq C\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{1}{|x-y|^{ns}}\big(|b(y)-b_B||f_2(y)|\big)^s dy\right)^{\frac{1}{s}} v_{j+1}^{\frac{1}{s'}},$$

$x \in B$,

$$H_{2,2}^j(x) \leq C\left(\int_{\mathbb{R}^n} \chi_{\{\max\{\varepsilon_{j+1},\frac{3}{4}\varepsilon_j\}<|x-y|<\varepsilon_j\}}(y)\frac{1}{|x-y|^{ns}}\big(|b(y)-b_B||f_2(y)|\big)^s dy\right)^{\frac{1}{s}} \times v_j^{\frac{1}{s'}}, \quad x \in B,$$

$$H_{2,3}^j(x) \leq C\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x_0-y|<\varepsilon_j\}}(y)\frac{1}{|x_0-y|^{ns}}\big(|b(y)-b_B||f_2(y)|\big)^s dy\right)^{\frac{1}{s}} \times v_{j+1}^{\frac{1}{s'}}, \quad x \in B,$$

and

$$H_{2,4}^j(x) \leq C\left(\int_{\mathbb{R}^n} \chi_{\{\max\{\varepsilon_{j+1},\frac{4}{5}\varepsilon_j\}<|x_0-y|<\varepsilon_j\}}(y) \times \frac{1}{|x_0-y|^{ns}}\big(|b(y)-b_B||f_2(y)|\big)^s dy\right)^{\frac{1}{s}} v_j^{\frac{1}{s'}}, \quad x \in B.$$

Here we take $1 < s < \rho$, and $v_j = (\varepsilon_j + r_0)^n - \varepsilon_j^n$, $j \in \mathbb{N}$. Note that $v_j \leq C(\max\{r_0, \varepsilon_j\})^{n-1} r_0$, $j \in \mathbb{N}$, for a certain $C > 0$ that does not depend on j.

We define the set $\mathcal{A} = \{j \in \mathbb{N} : r_0 < \varepsilon_j\}$. We have that

$$H_{2,1}^j(x) \leq C\frac{v_{j+1}^{1/s'}}{\varepsilon_{j+1}^{(n-1)/s'}}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{(|b(y)-b_B||f_2(y)|)^s}{|x-y|^{n+s-1}}dy\right)^{1/s} \leq Cr_0^{1/s'}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{(|b(y)-b_B||f_2(y)|)^s}{|x-y|^{n+s-1}}dy\right)^{1/s},$$

for every $x \in B$ and $j+1 \in \mathcal{A}$. In a similar way we can see that

$$H_{2,2}^j(x) \leq Cr_0^{1/s'}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{(|b(y)-b_B||f_2(y)|)^s}{|x-y|^{n+s-1}}dy\right)^{1/s},$$

$x \in B$ and $j \in \mathcal{A}$,

$$H_{2,3}^j(x) \leq Cr_0^{1/s'}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x_0-y|<\varepsilon_j\}}(y)\frac{(|b(y)-b_B||f_2(y)|)^s}{|x_0-y|^{n+s-1}}dy\right)^{1/s},$$

$x \in B$ and $j+1 \in \mathcal{A}$,

and

$$H_{2,4}^j(x) \le C r_0^{1/s'} \left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x_0-y|<\varepsilon_j\}}(y) \frac{(|b(y)-b_B||f_2(y)|)^s}{|x_0-y|^{n+s-1}} dy \right)^{1/s},$$

$x \in B$ and $j \in \mathcal{A}$.

Hence, we get by using Minkowski's inequality

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty} \left| H_{2,1}^j(x) + H_{2,2}^j(x) \right|^\rho \right)^{1/\rho} \\
&\le C \left(\sum_{j+1\in\mathcal{A}} \left| H_{2,1}^j(x) \right|^\rho + \sum_{j\in\mathcal{A}} |H_{2,2}^j(x)|^\rho \right)^{1/\rho} \\
&\le C \left(\sum_{j=0}^{\infty} \left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y) \frac{(|b(y)-b_B||f_2(y)|)^s}{|x-y|^{n+s-1}} dy \right)^{\rho/s} r_0^{\rho/s'} \right)^{1/\rho} \\
&\le C \left(\int_{\mathbb{R}^n} \frac{(|b(y)-b_B||f_2(y)|)^s}{|x-y|^{n+s-1}} dy \right)^{1/s} r_0^{1/s'} \\
&\le C \left(\sum_{k=2}^{\infty} \frac{1}{(2^k r_0)^n} \int_{|x_0-y|<2^{k+1}r_0} \left(\left| b(y)-b_B \right| |f(y)| \right)^s dy \frac{1}{2^{k(s-1)}} \right)^{1/s}, \quad x \in B.
\end{aligned} \tag{11}$$

In a similar way we get

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty} \left| H_{2,3}^j(x) + H_{2,4}^j(x) \right|^\rho \right)^{1/\rho} \\
&\le C \left(\sum_{k=1}^{\infty} \frac{1}{(2^k r_0)^n} \int_{|x_0-y|<2^{k+1}r_0} \left(\left| b(y)-b_B \right| |f(y)| \right)^s dy \frac{1}{2^{k(s-1)}} \right)^{1/s}, \quad x \in B.
\end{aligned} \tag{12}$$

By combining (8), (9), (11), and (12) it follows that

$$\begin{aligned}
&\left\| A_3^\varepsilon f(x) - A_3^\varepsilon f(x_0) \right\|_{E_\rho} \\
&= \left\| R_\ell\big((b-b_B) f_2\big)(x) - R_\ell\big((b-b_B) f_2\big)(x_0) \right\|_{E_\rho} \\
&\le C \left(\sum_{k=1}^{\infty} \frac{2^{-k}}{(2^k r_0)^n} \int_{|x_0-y|<2^k r_0} \left| b(y)-b_B \right| |f(y)| dy \right. \\
&\quad \left. + \left(\sum_{k=1}^{\infty} \frac{2^{-k(s-1)}}{(2^k r_0)^n} \int_{|x_0-y|<2^k r_0} \left| \big(b(y)-b_B\big) f(y) \right|^s dy \right)^{1/s} \right),
\end{aligned}$$

for $x \in B$. Then, Hölder's inequality implies that

$$\begin{aligned}
&\left\| A_3^\varepsilon(f)(x) - A_3^\varepsilon(f)(x_0) \right\|_{E_\rho} \\
&\quad \le C \left(\sum_{k=1}^{\infty} \frac{1}{2^k} \left(\frac{1}{(2^k r_0)^n} \int_{|x_0-y|<2^k r_0} \left| b(y) - b_B \right|^{r'} dy \right)^{1/r'} \right. \\
&\qquad \times \left(\frac{1}{(2^k r_0)^n} \int_{|x_0-y|<2^k r_0} \left| f(y) \right|^r dy \right)^{1/r} \\
&\qquad + \left(\sum_{k=1}^{\infty} \frac{1}{2^{k(s-1)}} \left(\frac{1}{(2^k r_0)^n} \int_{|x_0-y|<2^k r_0} \left| b(y) - b_B \right|^{st'} dy \right)^{1/t'} \right. \\
&\qquad \left. \left. \times \left(\frac{1}{(2^k r_0)^n} \int_{|x_0-y|<2^k r_0} \left| f(y) \right|^{st} dy \right)^{1/t} \right)^{1/s} \right) \\
&\quad \le C \left(\sum_{k=1}^{\infty} \frac{k}{2^k} \|b\|_{BMO(\mathbb{R}^n)} \mathcal{M}_r(f)(z) \right. \\
&\qquad \left. + \left(\sum_{k=1}^{\infty} \frac{k^s}{2^{k(s-1)}} \right)^{1/s} \|b\|_{BMO(\mathbb{R}^n)} \mathcal{M}_{ts}(f)(z) \right) \\
&\quad \le C \|b\|_{BMO(\mathbb{R}^n)} \left(\mathcal{M}_r(f)(z) + \mathcal{M}_{ts}(f)(z) \right), \quad x, z \in B, \qquad (13)
\end{aligned}$$

where $1 < t, r < \infty$.

From (13) it follows

$$\frac{1}{|B|} \int_B \left\| A_3^\varepsilon f(x) - A_3^\varepsilon f(x_0) \right\|_{E_\rho} dx \le C \|b\|_{BMO(\mathbb{R}^n)} \mathcal{M}_r(f)(z), \quad z \in B, \qquad (14)$$

where $1 < r < \infty$.

By combining (5), (6), (7) and (14) we conclude that

$$\left(V_\rho\left(C_{b,\ell}^\varepsilon \right)(f) \right)^{\#}(x) \le C \|b\|_{BMO(\mathbb{R}^n)} \left(\mathcal{M}_r(f)(x) + \mathcal{M}_r\left(V_\rho\left(R_\ell^\varepsilon \right)(f) \right)(x) \right), \quad x \in B,$$

where $1 < r < p$.

Since $\mathcal{M}_r$ is bounded from $L^p(\mathbb{R}^n)$ into itself provided that $1 < r < p < \infty$, [7, Theorem A] and [27, Corollary 1, p. 154] allow us to conclude that $V_\rho(C_{b,\ell}^\varepsilon)$ is bounded from $L^p(\mathbb{R}^n)$ into itself. □

In [29] Dachun Yang, Dongyong Yang and Yuan Zhou introduced localized Riesz transform associated with the classical Laplacian and Schrödinger operators. Here, we consider localized commutator operators with the classical Riesz transforms.

Let $b \in BMO_\theta(\gamma)$, $\theta > 0$. We define, for every $\varepsilon > 0$, the local truncation $C_{b,\ell}^{\varepsilon,\mathrm{loc}} f$ of $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, by

$$C_{b,\ell}^{\varepsilon,\mathrm{loc}}(f)(x) = \int_{\varepsilon<|x-y|<\gamma(x)} \left(b(x) - b(y) \right) R_\ell(x-y) f(y) dy, \quad x \in \mathbb{R}^n.$$

In order to define the local commutator $C^{\mathrm{loc}}_{b,\ell}$ on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, we show the L^p boundedness properties for the maximal operator $C^{*,\mathrm{loc}}_{b,\ell}$ defined by

$$C^{*,\mathrm{loc}}_{b,\ell}(f)(x) = \sup_{\varepsilon>0} \left|C^{\varepsilon,\mathrm{loc}}_{b,\ell}(f)(x)\right|, \quad x \in \mathbb{R}^n.$$

Previously we state an auxiliary result that will be also useful in other sections of this paper. According to [12, Proposition 5] we choose a sequence $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^n$, such that if $Q_k = B(x_k, \gamma(x_k))$, $k \in \mathbb{N}$, the following properties hold:

(i) $\bigcup_{k=1}^{\infty} Q_k = \mathbb{R}^n$;
(ii) For every $m \in \mathbb{N}$ there exist $C, \beta > 0$ such that, for every $k \in \mathbb{N}$,

$$\mathrm{card}\left\{l \in \mathbb{N} : 2^m Q_l \cap 2^m Q_k \neq \varnothing\right\} \leq C 2^{m\beta}.$$

The sequence $\{Q_k\}_{k\in\mathbb{N}}$ of balls will appear in different occasions throughout this paper.

Lemma 2.1 *Let $b \in BMO_\theta(\gamma)$, $\theta \geq 0$, and $M > 0$. We define the operator $T_{b,M}$ by*

$$T_{b,M}(f)(x) = \frac{1}{\gamma(x)^n} \int_{|x-y|\leq M\gamma(x)} \left|b(x) - b(y)\right| f(y)dy, \quad x \in \mathbb{R}^n.$$

Then, for every $1 < p < \infty$, $T_{b,M}$ is bounded from $L^p(\mathbb{R}^n)$ into itself. Moreover,

$$\left\|T_{b,M}(f)\right\|_{L^p(\mathbb{R}^n)} \leq C(1+MA)^{n+\theta'} \|b\|_{BMO_\theta(\gamma)} \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p\left(\mathbb{R}^n\right).$$

Here, $A, C > 0$ and $\theta' > \theta$ do not depend on M.

Proof Let $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$. We have that

$$\left\|T_{b,M}(f)\right\|_{L^p(\mathbb{R}^n)} \leq C\left(\sum_{k=1}^{\infty} \int_{Q_k} \left|T_{b,M}(f)(x)\right|^p dx\right)^{1/p}.$$

Since $\gamma(x) \sim \gamma(x_k)$, for every $x \in Q_k$, $k \in \mathbb{N}$, by using Hölder and Minkowski inequalities and [3, Proposition 3], it follows that, for certain $A > 0$ and $\theta' > \theta$,

$$\begin{aligned}
&\int_{Q_k} \left|T_{b,M}(f)(x)\right|^p dx \\
&\quad \leq C \int_{Q_k} \left(\frac{1}{\gamma(x_k)^n} \int_{|x_k-y|\leq(1+MA)\gamma(x_k)} \left|b(x) - b(y)\right|\left|f(y)\right| dy\right)^p dx \\
&\quad \leq C \int_{Q_k} \frac{1}{\gamma(x_k)^{np}} \int_{|x_k-y|\leq(1+MA)\gamma(x_k)} \left|f(y)\right|^p dy \\
&\qquad \times \left(\int_{|x_k-y|\leq(1+MA)\gamma(x_k)} \left|b(x) - b(y)\right|^{p'} dy\right)^{p/p'} dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{Q_k}\frac{1}{\gamma(x_k)^{np}}\int_{|x_k-y|\le(1+MA)\gamma(x_k)}\big|f(y)\big|^p dy\\
&\quad\times\Bigg(\Bigg(\int_{|x_k-y|\le(1+MA)\gamma(x_k)}\big|b(x)-b_{\widehat{Q_k}}\big|^{p'}dy\Bigg)^{p/p'}\\
&\quad+\Bigg(\int_{|x_k-y|\le(1+MA)\gamma(x_k)}\big|b(y)-b_{\widehat{Q_k}}\big|^{p'}dy\Bigg)^{p/p'}\Bigg)dx\\
&\le C\int_{Q_k}\frac{1}{\gamma(x_k)^{np}}\int_{|x_k-y|\le(1+MA)\gamma(x_k)}\big|f(y)\big|^p dy\\
&\quad\times\Big(\big|b(x)-b_{\widehat{Q_k}}\big|^p+\|b\|^p_{BMO_\theta(\gamma)}(2+MA)^{p\theta'}\Big)\Big((1+MA)\gamma(x_k)\Big)^{np/p'}dx\\
&\le C\int_{|x_k-y|\le(1+MA)\gamma(x_k)}\big|f(y)\big|^p dy\|b\|^p_{BMO_\theta(\gamma)}(1+MA)^{(n+\theta')p},\quad k\in\mathbb{N},
\end{aligned}
$$

where $\widehat{Q_k}=B(x_k,(1+MA)\gamma(x_k))$, $k\in\mathbb{N}$.

Hence, by taking into account the properties of $\{Q_k\}_{k\in\mathbb{N}}$, we get

$$
\begin{aligned}
&\big\|T_{b,M}(f)\big\|_{L^p(\mathbb{R}^n)}\\
&\le C\Bigg(\sum_{k=1}^{\infty}\int_{|x_k-y|\le(1+MA)\gamma(x_k)}\big|f(y)\big|^p dy\|b\|^p_{BMO_\theta(\gamma)}(1+MA)^{(n+\theta')p}\Bigg)^{1/p}\\
&\le C(1+MA)^{n+\theta'}\|b\|_{BMO_\theta(\gamma)}\|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned}
$$

□

Our next step is to advance on the boundedness of the corresponding local operators. The case of heat semigroup and Riesz transforms are not very difficult and the will be done while proving Theorems 1.1 and 1.2 in Sects. 3 and 4, respectively. As far the commutator, it involves an additional difficulty. According to our procedure we start with the commutator $C_{b,\ell}$ with $b\in BMO_\theta(\gamma)$ and we reduce the problem to the local classical commutator but now b is not necessarily in BMO in the classical sense. In the next lemma and proposition we show how to overcome this problem.

Lemma 2.2 *Let $\theta\ge 0$ and $L>0$. There exists $C>0$ such that, for every $k\in\mathbb{N}$ and $b\in BMO_\theta(\gamma)$, we can find a function $\mathfrak{b}_k\in BMO(\mathbb{R}^n)$ for which $\mathfrak{b}_k=b$ on $LQ_k=B(x_k,L\gamma(x_k))$ and $\|\mathfrak{b}_k\|_{BMO(\mathbb{R}^n)}\le C\|b\|_{BMO_\theta(\gamma)}$.*

Proof Let $k\in\mathbb{N}$ and $b\in BMO_\theta(\gamma)$. We define $b_k=b\chi_{LQ_k}$. Assume that $z_0\in\mathbb{R}^n$ and $r_0>0$ such that $B(z_0,r_0)\subset LQ_k$. It is clear that $|z_0-x_k|\le L\gamma(x_k)$. Then, for a certain $A>0$ that does not depend on k, $\gamma(x_k)\le A\gamma(z_0)$.

We have that

$$
\begin{aligned}
\frac{1}{|B(z_0,r_0)|}\int_{B(z_0,r_0)}\big|b(x)-b_{B(z_0,r_0)}\big|dx&\le\|b\|_{BMO_\theta(\gamma)}\left(1+\frac{r_0}{\gamma(z_0)}\right)^{\theta}\\
&\le C\|b\|_{BMO_\theta(\gamma)},
\end{aligned}
$$

where $C > 0$ depends on L and θ but does not depend on k. Hence, $b \in BMO(LQ_k)$ and $\|b\|_{BMO(LQ_k)} \leq C\|b\|_{BMO_\theta(\gamma)}$.

We define $w_k = \exp(b_k)$. It is well known that $w_k \in A_2(LQ_k)$, where $A_2(LQ_k)$ denotes the Muckenhoupt class of weights. Moreover, the A_2-characteristic $[w_k]_{A_2(LQ_k)}$ satisfies that $[w_k]_{A_2(LQ_k)} \leq C\|b\|_{BMO(LQ_k)}$, where $C > 0$ does not depend on k. According to [4, Lemma 1] (see also [16]) there exists $\widetilde{w_k} \in A_2(\mathbb{R}^n)$ such that $\widetilde{w_k} = w_k$ in LQ_k, and $[\widetilde{w}_k]_{A_2(\mathbb{R}^n)} \leq C[w_k]_{A_2(LQ_k)}$, $C > 0$ being independent of k. Then, there exists $\mathfrak{b}_k \in BMO(\mathbb{R}^n)$ satisfying that $\widetilde{w}_k = \exp(\mathfrak{b}_k)$ and $\|\mathfrak{b}_k\|_{BMO(\mathbb{R}^n)} \leq C[\widetilde{w}_k]_{A_2(\mathbb{R}^n)}$, where $C > 0$ does not depend on k.

We conclude that $\mathfrak{b}_k = b$ on LQ_k and $\|\mathfrak{b}_k\|_{BMO(\mathbb{R}^n)} \leq C\|b\|_{BMO_\theta(\gamma)}$, for a certain $C > 0$ independent of k. □

Proposition 2.1 *Let $b \in BMO_\infty(\gamma)$ and $\ell = 1, \ldots, n$. Then, the maximal operator $C^{*,\mathrm{loc}}_{b,\ell}$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$.*

Proof Assume that $b \in BMO_\theta(\gamma)$, with $\theta \geq 0$. Fix $k \in \mathbb{N}$. According to [25, Lemma 1.4] $\gamma(x) \sim \gamma(x_k)$, for every $x \in Q_k$. We choose $L > 0$ independent of k such that, for any $x \in Q_k$, $B(x, \gamma(x)) \subset \mathbb{Q}_k$, where $\mathbb{Q}_k = B(x_k, L\gamma(x_k))$. We can write

$$\begin{aligned}
&\left| \int_{\varepsilon<|x-y|<\gamma(x)} \big(b(x)-b(y)\big) R_\ell(x-y) f(y) dy \right| \\
&\quad \leq \left| \int_{\varepsilon<|x-y|<\gamma(x)} \big(b(x)-b(y)\big) R_\ell(x-y) f(y) dy \right. \\
&\qquad \left. - \int_{y\in\mathbb{Q}_k, |x-y|>\varepsilon} \big(b(x)-b(y)\big) R_\ell(x-y) f(y) dy \right| \\
&\qquad + \left| \int_{y\in\mathbb{Q}_k, |x-y|>\varepsilon} \big(b(x)-b(y)\big) R_\ell(x-y) f(y) dy \right| \\
&\quad = J_{1,k}(x,\varepsilon) + J_{2,k}(x,\varepsilon), \quad x \in Q_k,\ \varepsilon > 0. \qquad (15)
\end{aligned}$$

Let us analyze $J_{i,k}(x,\varepsilon)$, for $x \in Q_k$ and $\varepsilon > 0$, $i = 1, 2$. Observe that, since $\gamma(y) \sim \gamma(x_k)$, for every $y \in \mathbb{Q}_k$, we can find $C, M > 0$ that do not depend on $k \in \mathbb{N}$ such that

$$\begin{aligned}
J_{1,k}(x,\varepsilon) &\leq C \int_{\mathbb{Q}_k \setminus B(x,\gamma(x))} \frac{|b(x)-b(y)|}{|x-y|^n} |f(y)| dy \\
&\leq C \int_{\gamma(x)/M \leq |x-y| \leq M\gamma(x)} \frac{|b(x)-b(y)|}{|x-y|^n} |f(y)| \chi_{\mathbb{Q}_k}(y) dy \\
&\leq \frac{C}{\gamma(x)^n} \int_{|y-x|\leq M\gamma(x)} |b(x)-b(y)||f(y)| \chi_{\mathbb{Q}_k}(y) dy, \quad x \in Q_k \text{ and } \varepsilon > 0.
\end{aligned}$$

From Lemma 2.1 we deduce that,

$$\int_{Q_k}\Big|\sup_{\varepsilon>0}J_{1,k}(x,\varepsilon)\Big|^p dx \le C\int_{\mathbb{R}^n}\big|T_{b,M}(f\chi_{\mathbb{Q}_k})(x)\big|^p dx$$
$$\le C\|b\|^p_{BMO_\theta(\gamma)}\int_{\mathbb{Q}_k}\big|f(y)\big|^p dy, \quad k\in\mathbb{N}, \tag{16}$$

with a constant independent of k.

On the other hand, we have that, for $x\in Q_k$ and $\varepsilon>0$,

$$J_{2,k}(x,\varepsilon)=\left|\int_{|x-y|>\varepsilon}\big(b_k(x)-b_k(y)\big)R_\ell(x-y)f(y)dy\right|,$$

where $b_k=b\chi_{\mathbb{Q}_k}$. According to Lemma 2.2, there exists a function $\widetilde{b}_k\in BMO(\mathbb{R}^n)$ such that $\widetilde{b}_k=b$, on $\mathbb{Q}_k$, and $\|\widetilde{b}_k\|_{BMO(\mathbb{R}^n)}\le C\|b\|_{BMO_\theta(\gamma)}$, where $C>0$ does not depend on $k\in\mathbb{N}$. It follows that

$$\sup_{\varepsilon>0}J_{2,k}(x,\varepsilon)=C^*_{\widetilde{b}_k,\ell}(f\chi_{\mathbb{Q}_k})(x), \quad x\in Q_k.$$

Hence, we get

$$\int_{Q_k}\Big|\sup_{\varepsilon>0}J_{2,k}(x,\varepsilon)\Big|^p dx \le C\int_{\mathbb{R}^n}\big|C^*_{\widetilde{b}_k,\ell}(f\chi_{\mathbb{Q}_k}(x)\big|^p dx$$
$$\le C\|\widetilde{b}_k\|^p_{BMO(\mathbb{R}^n)}\int_{\mathbb{Q}_k}\big|f(y)\big|^p dy$$
$$\le C\|b\|^p_{BMO_\theta(\gamma)}\int_{\mathbb{Q}_k}\big|f(y)\big|^p dy, \quad k\in\mathbb{N}. \tag{17}$$

By combining (15), (16) and (17) and using the properties of the sequence $\{Q_k\}_{k\in\mathbb{N}}$ we conclude, for every $1<p<\infty$,

$$\big\|C^{*,\mathrm{loc}}_{b,\ell}(f)\big\|_{L^p(\mathbb{R}^n)}\le\left(\sum_{k=1}^\infty\int_{Q_k}\big|C^{*,\mathrm{loc}}_{b,\ell}(f)(x)\big|^p dx\right)^{1/p}$$
$$\le C\|b\|_{BMO_\theta(\gamma)}\left(\sum_{k=1}^\infty\int_{\mathbb{Q}_k}\big|f(y)\big|^p dy\right)^{1/p}$$
$$\le C\|b\|_{BMO_\theta(\gamma)}\|f\|_{L^p(\mathbb{R}^n)}, \quad f\in L^p\big(\mathbb{R}^n\big). \qquad \square$$

Suppose now that $\ell=1,\dots,n$, $f\in C_c^\infty(\mathbb{R}^n)$ and $b\in BMO_\infty(\gamma)$. Then, $bf\in L^1(\mathbb{R}^n)$. Moreover, we can write, for $\varepsilon>0$ small enough,

$$C^{\varepsilon,\mathrm{loc}}_{b,\ell}(f)(x)=C^\varepsilon_{b,\ell}(f)(x)-\int_{|x-y|\ge\gamma(x)}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy, \quad x\in\mathbb{R}^n.$$

Hence, there exists the limit

$$\lim_{\varepsilon\to0^+}C^{\varepsilon,\mathrm{loc}}_{b,\ell}(f)(x), \quad x\in\mathbb{R}^n.$$

By using standard arguments, from Proposition 2.1 we deduce that, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, there exists the limit

$$\lim_{\varepsilon \to 0^+} C_{b,\ell}^{\varepsilon,\mathrm{loc}}(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

We define, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$,

$$C_{b,\ell}^{\mathrm{loc}}(f)(x) = \lim_{\varepsilon \to 0^+} C_{b,\ell}^{\varepsilon,\mathrm{loc}}(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

The behavior in $L^p(\mathbb{R}^n)$ of the variation operators for the family of truncations $\{C_{b,\ell}^{\varepsilon,\mathrm{loc}}\}_{\varepsilon>0}$ associated with the local commutator operator $C_{b,\ell}^{\mathrm{loc}}$ are established in the following.

Theorem 2.4 *Let $b \in BMO_\infty(\gamma)$ and $\ell = 1, \dots, n$. Assume that $\rho > 2$. Then, the variation operator $V_\rho(C_{b,\ell}^{\varepsilon,\mathrm{loc}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$.*

Proof Suppose that $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$. Let $k \in \mathbb{N}$. As in the proof of Proposition 2.1 we define $\mathbb{Q}_k = B(x_k, L\gamma(x_k))$, where $L > 0$ is such that $B(x, \gamma(x)) \subset \mathbb{Q}_k$, for every $x \in Q_k$. Moreover, L does not depend on k.

For every $x \in Q_k$, we can write

$$\begin{aligned}
&V_\rho\big(C_{b,\ell}^{\varepsilon,\mathrm{loc}}\big)(f)(x)\\
&= \sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\downarrow 0}\left(\sum_{j=1}^{\infty}\left|\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j;|x-y|<\gamma(x)}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy\right|^\rho\right)^{1/\rho}\\
&\le \sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\downarrow 0}\left(\sum_{j=1}^{\infty}\left|\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j;|x-y|<\gamma(x)}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy\right.\right.\\
&\quad\left.\left.-\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j;y\in\mathbb{Q}_k}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy\right|^\rho\right)^{1/\rho}\\
&\quad+\sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\downarrow 0}\left(\sum_{j=1}^{\infty}\left|\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j;y\in\mathbb{Q}_k}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy\right|^\rho\right)^{1/\rho}\\
&\le C\left(\sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\downarrow 0}\left(\sum_{j=1}^{\infty}\left|\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j;y\in\mathbb{Q}_k\setminus B(x,\gamma(x))}\big(b(x)-b(y)\big)\right.\right.\right.\\
&\quad\left.\left.\left.\times R_\ell(x-y)f(y)dy\right|^\rho\right)^{1/\rho}\right.\\
&\quad+\sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\downarrow 0}\left(\sum_{j=1}^{\infty}\left|\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j}\big(b(x)-b(y)\big)\right.\right.
\end{aligned}$$

$$\times R_\ell(x-y)f(y)\chi_{\mathbb{Q}_k}(y)dy\Big|^\rho\Big)^{1/\rho}\Big)$$

$$\le C\Big(\sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\downarrow 0}\Big(\sum_{j=1}^{\infty}\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j;\,y\in\mathbb{Q}_k\setminus B(x,\gamma(x))}\big|b(x)-b(y)\big|\frac{1}{|x-y|^n}|f(y)|dy\Big)$$

$$+\sup_{\{\varepsilon_j\}_{j\in\mathbb{N}}\downarrow 0}\Big(\sum_{j=1}^{\infty}\Big|\int_{\varepsilon_{j+1}\le|x-y|<\varepsilon_j;}\big(\widetilde{b}_k(x)-\widetilde{b}_k(y)\big)$$

$$\times\frac{x_\ell-y_\ell}{|x-y|^{n+1}}f(y)\chi_{\mathbb{Q}_k}(y)dy\Big|^\rho\Big)^{1/\rho}\Big)$$

$$\le C\Big(\int_{\mathbb{Q}_k\setminus B(x,\gamma(x))}\big|b(x)-b(y)\big|\frac{1}{|x-y|^n}\big|f(y)\big|dy+V_\rho\big(C_{\widetilde{b}_k,\ell}(f\chi_{\mathbb{Q}_k})(x)\big)\Big),$$

where $\widetilde{b}_k\in BMO(\mathbb{R}^n)$ satisfying that $\widetilde{b}_k=b$, in $\mathbb{Q}_k$, and $\|\mathfrak{b}_k\|_{BMO(\mathbb{R}^n)}\le C\|b\|_{BMO_\theta(\gamma)}$, being $b\in BMO_\theta(\gamma)$. Here $C>0$ is independent on k (see Lemma 2.2). Then, by using Lemma 2.1 (as in the proof of Proposition 2.1) and Theorem 2.3, we conclude that

$$\int_{Q_k}\big|V_\rho\big(C^{\varepsilon,\mathrm{loc}}_{b,\ell}\big)(f)(x)\big|^p dx\le C\|b\|^p_{BMO_\theta(\gamma)}\int_{\mathbb{Q}_k}\big|f(x)\big|^p dx,$$

where again $C>0$ does not depend on k.

Hence, according to the properties of the sequence $\{Q_k\}_{k\in\mathbb{N}}$, we get

$$\big\|V_\rho\big(C^{\varepsilon,\mathrm{loc}}_{b,\ell}\big)(f)\big\|_{L^p(\mathbb{R}^n)}\le C\|b\|_{BMO_\theta(\gamma)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Thus the proof is finished. □

3 Variation operators associated with the heat semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$

As it was mentioned in Sect. 1 (after Theorem 1.1) from [20, Corollary 3.4] we can deduce that the variation operator $V_\rho(W_t^{\mathcal{L}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1<p<\infty$. By using the general procedure described in Sect. 2, we now present a proof of Theorem 1.1. This procedure allows us to show in the same way the behavior of $V_\rho(W_t^{\mathcal{L}})$ in $L^p(\mathbb{R}^n)$, $1<p<\infty$ (obtaining in this special case a proof different than that in [20, Corollary 3.4]), and in $L^1(\mathbb{R}^n)$.

As the general procedure suggests we consider the following local operators

$$W^{\mathcal{L}}_{t,\mathrm{loc}}(f)(x)=\int_{|x-y|<\gamma(x)}W^{\mathcal{L}}_t(x,y)f(y)dy,\quad x\in\mathbb{R}^n,$$

and

$$W_{t,\mathrm{loc}}(f)(x)=\int_{|x-y|<\gamma(x)}W_t(x,y)f(y)dy,\quad x\in\mathbb{R}^n,$$

where $f\in L^p(\mathbb{R}^n)$, $1\le p<\infty$.

Observe that

$$V_\rho\big(W_t^{\mathcal{L}}\big)(f) \le V_\rho\big(W_{t,\mathrm{loc}}^{\mathcal{L}} - W_{t,\mathrm{loc}}\big)(f) + V_\rho(W_{t,\mathrm{loc}})(f) + V_\rho\big(W_t^{\mathcal{L}} - W_{t,\mathrm{loc}}^{\mathcal{L}}\big)(f). \tag{18}$$

Assume that $\{t_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero. We can write

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty}\big|W_{t_j,\mathrm{loc}}(f)(x) - W_{t_{j+1},\mathrm{loc}}(f)(x)\big|^\rho\right)^{\frac{1}{\rho}}\\
&\quad\le \left(\sum_{j=0}^{\infty}\big|W_{t_j}(f)(x) - W_{t_{j+1}}(f)(x)\big|^\rho\right)^{\frac{1}{\rho}}\\
&\qquad + \left(\sum_{j=0}^{\infty}\left|\int_{|x-y|>\gamma(x)}\big(W_{t_j}(x,y) - W_{t_{j+1}}(x,y)\big)f(y)dy\right|^\rho\right)^{\frac{1}{\rho}}\\
&\quad\le V_\rho(W_t)(f)(x) + \sup_{\varepsilon>0}\left\|\int_{|x-y|>\varepsilon} W_t(x,y)f(y)dy\right\|_{E_\rho}, \quad x\in\mathbb{R}^n,
\end{aligned}$$

where the space E_ρ is defined in Sect. 1.

Then,

$$V_\rho(W_{t,\mathrm{loc}})(f)(x) \le V_\rho(W_t)(f) + \sup_{\varepsilon>0}\left\|\int_{|x-y|>\varepsilon} W_t(x,y)f(y)dy\right\|_{E_\rho}, \quad x\in\mathbb{R}^n.$$

We consider the operator defined by

$$\begin{aligned}
T : L^2\big(\mathbb{R}^n\big) &\to L^2_{E_\rho}\big(\mathbb{R}^n\big)\\
f &\longrightarrow Tf(x) = \int_{\mathbb{R}^n} W_t(x,y)f(y)dy.
\end{aligned}$$

According to [18, Theorem 3.3] T is bounded from $L^2(\mathbb{R}^n)$ into $L^2_{E_\rho}(\mathbb{R}^n)$. Moreover, T is a Calderón-Zygmund operator associated with the E_ρ-valued kernel

$$K(x,y;t) = W_t(x,y), \quad x,y\in\mathbb{R}^n,\ t>0,$$

that satisfies the following properties (see [9]):

(1) $\big\|K(x,y;\cdot)\big\|_{E_\rho} \le \dfrac{C}{|x-y|^n}, \quad x,y\in\mathbb{R}^n,\ x\neq y,$

(2) $\left\|\dfrac{\partial}{\partial x}K(x,y;\cdot)\right\|_{E_\rho} + \left\|\dfrac{\partial}{\partial y}K(x,y;\cdot)\right\|_{E_\rho} \le \dfrac{C}{|x-y|^{n+1}}, \quad x,y\in\mathbb{R}^n,\ x\neq y.$

Then, by proceeding as in the proof of [27, Proposition 2, p. 34 and Corollary 2, p. 36], we prove that the maximal operator T^* defined by

$$T^* f(x) = \sup_{\varepsilon>0} \left\| \int_{|x-y|>\varepsilon} W_t(x,y) f(y) dy \right\|_{E_\rho}$$

is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$. By combining this fact with [18, Theorem 3.3] we conclude that the operator $V_\rho(W_{t,\mathrm{loc}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

We consider now the variation operator defined by

$$V_\rho\big(W_t^{\mathcal{L}} - W_{t,\mathrm{loc}}^{\mathcal{L}}\big)(f)(x)$$
$$= \sup_{\{t_j\}_{j\in\mathbb{N}}\searrow 0} \left(\sum_{j=0}^{\infty} \left| \int_{|x-y|>\gamma(x)} \big(W_{t_j}^{\mathcal{L}}(x,y) - W_{t_{j+1}}^{\mathcal{L}}(x,y)\big) f(y) dy \right|^\rho \right)^{\frac{1}{\rho}}.$$

Assume that $\{t_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero. We can write

$$\left(\sum_{j=0}^{\infty} \left| \int_{|x-y|>\gamma(x)} \big(W_{t_j}^{\mathcal{L}}(x,y) - W_{t_{j+1}}^{\mathcal{L}}(x,y)\big) f(y) dy \right|^\rho \right)^{\frac{1}{\rho}}$$
$$\leq \sum_{j=0}^{\infty} \int_{|x-y|>\gamma(x)} |f(y)| \int_{t_{j+1}}^{t_j} \left| \frac{\partial}{\partial t} W_t^{\mathcal{L}}(x,y) \right| dt dy$$
$$= \int_{|x-y|>\gamma(x)} |f(y)| \int_0^\infty \left| \frac{\partial}{\partial t} W_t^{\mathcal{L}}(x,y) \right| dt dy, \quad x \in \mathbb{R}^n. \tag{19}$$

According to [12, (2.7)], for every $N \in \mathbb{N}$ there exist $c, C > 0$ such that

$$\left| \frac{\partial}{\partial t} W_t^{\mathcal{L}}(x,y) \right| \leq \frac{C}{t^{\frac{n}{2}+1}} \left(1 + \frac{t}{\gamma(x)^2} + \frac{t}{\gamma(y)^2} \right)^{-N} e^{\frac{-c|x-y|^2}{t}}, \quad x, y \in \mathbb{R}^n,\ t > 0. \tag{20}$$

Estimation (20) allows us to obtain

$$\int_{|x-y|>\gamma(x)} \int_{\gamma(x)^2}^\infty \left| \frac{\partial}{\partial t} W_t^{\mathcal{L}}(x,y) \right| dt |f(y)| dy$$
$$\leq C \int_{|x-y|>\gamma(x)} |f(y)| \int_{\gamma(x)^2}^\infty \frac{1}{t^{\frac{n}{2}+1}} \left(1 + \frac{t}{\gamma(x)^2} \right)^{-n-2} e^{-\frac{c|x-y|^2}{t}} dt dy$$
$$\leq \frac{C}{\gamma(x)^n} \int_{|x-y|>\gamma(x)} |f(y)| \int_1^\infty \frac{1}{u^{\frac{n}{2}+1}} \frac{1}{(1+u)^{n+2}} e^{-\frac{c|x-y|^2}{u\gamma(x)^2}} du dy$$
$$\leq \frac{C}{\gamma(x)^n} \int_{|x-y|>\gamma(x)} |f(y)| \int_1^\infty \frac{1}{u^{\frac{n}{2}+1}} \frac{1}{(1+u)^{2+n}} \left(\frac{u\gamma(x)^2}{|x-y|^2} \right)^{\frac{n}{2}+1} du dy$$

$$\leq \frac{C}{\gamma(x)^n}\int_{|x-y|>\gamma(x)} \big|f(y)\big|\bigg(\frac{\gamma(x)}{|x-y|}\bigg)^{n+2}dy$$

$$= \frac{C}{\gamma(x)^n}\sum_{k=0}^{\infty}\int_{2^k\gamma(x)<|x-y|\leq 2^{k+1}\gamma(x)} \big|f(y)\big|\bigg(\frac{\gamma(x)}{|x-y|}\bigg)^{n+2}dy$$

$$\leq C\sum_{k=0}^{\infty}\frac{1}{2^{2k}(2^k\gamma(x))^n}\int_{|x-y|\leq 2^{k+1}\gamma(x)} \big|f(y)\big|dy \leq C\mathcal{M}(f)(x), \quad x\in\mathbb{R}^n. \quad (21)$$

Here $\mathcal{M}=\mathcal{M}_1$ denotes the Hardy-Littlewood maximal function.

Also we have that

$$\int_{|x-y|>\gamma(x)}\int_0^{\gamma(x)^2}\bigg|\frac{\partial}{\partial t}W_t^{\mathcal{L}}(x,y)\bigg|dt\big|f(y)\big|dy$$

$$\leq C\int_0^{\gamma(x)^2}\int_{|x-y|>\gamma(x)}\frac{1}{t^{\frac{n}{2}+1}}e^{-c\frac{|x-y|^2}{t}}\big|f(y)\big|dydt$$

$$\leq C\int_0^{\gamma(x)^2}\frac{e^{-c\frac{\gamma(x)^2}{t}}}{t}\int_{\mathbb{R}^n}\frac{1}{t^{\frac{n}{2}}}e^{-c\frac{|x-y|^2}{t}}\big|f(y)\big|dydt$$

$$\leq C\sup_{t>0}\frac{1}{t^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{-c\frac{|x-y|^2}{t}}\big|f(y)\big|dy \leq C\mathcal{M}(f)(x), \quad x\in\mathbb{R}^n. \quad (22)$$

From (19), (21) and (22) we conclude that

$$V_\rho\big(W_t^{\mathcal{L}}-W_{t,\mathrm{loc}}^{\mathcal{L}}\big)(f)(x)\leq C\mathcal{M}(f)(x), \quad x\in\mathbb{R}^n.$$

Hence, the operator $V_\rho(W_t^{\mathcal{L}}-W_{t,\mathrm{loc}}^{\mathcal{L}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1<p<\infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$. We now analyze the operator

$$V_\rho\big(W_{t,\mathrm{loc}}^{\mathcal{L}}-W_{t,\mathrm{loc}}\big)(f)(x)$$

$$= \sup_{\{t_j\}_{j\in\mathbb{N}}\searrow 0}\Bigg(\sum_{j=0}^{\infty}\bigg|\int_{|x-y|<\gamma(x)}\big(W_{t_j}^{\mathcal{L}}(x,y)-W_{t_j}(x,y)$$

$$-\big(W_{t_{j+1}}^{\mathcal{L}}(x,y)-W_{t_{j+1}}(x,y)\big)\big)f(y)dy\bigg|^\rho\Bigg)^{\frac{1}{\rho}}, \quad x\in\mathbb{R}^n.$$

Let us take a real decreasing sequence $\{t_j\}_{j\in\mathbb{N}}$ that converges to zero. We have that

$$\Bigg(\sum_{j=0}^{\infty}\bigg|\int_{|x-y|<\gamma(x)}\big(W_{t_j}^{\mathcal{L}}(x,y)-W_{t_j}(x,y)$$

$$-\big(W_{t_{j+1}}^{\mathcal{L}}(x,y)-W_{t_{j+1}}(x,y)\big)\big)f(y)dy\bigg|^\rho\Bigg)^{\frac{1}{\rho}}$$

$$
\begin{aligned}
&= \left(\sum_{j=0}^{\infty}\left|\int_{|x-y|<\gamma(x)} f(y)\int_{t_{j+1}}^{t_j}\frac{\partial}{\partial t}\big(W_t^{\mathcal{L}}(x,y)-W_t(x,y)\big)dtdy\right|^{\rho}\right)^{\frac{1}{\rho}}\\
&\leq \sum_{j=0}^{\infty}\int_{|x-y|<\gamma(x)}\big|f(y)\big|\int_{t_{j+1}}^{t_j}\left|\frac{\partial}{\partial t}\big(W_t^{\mathcal{L}}(x,y)-W_t(x,y)\big)\right|dtdy\\
&= \int_{|x-y|<\gamma(x)}\big|f(y)\big|\int_0^{\gamma(x)^2}\left|\frac{\partial}{\partial t}\big(W_t^{\mathcal{L}}(x,y)-W_t(x,y)\big)\right|dtdy\\
&\quad+\int_{|x-y|<\gamma(x)}\big|f(y)\big|\int_{\gamma(x)^2}^{\infty}\left|\frac{\partial}{\partial t}\big(W_t^{\mathcal{L}}(x,y)-W_t(x,y)\big)\right|dtdy\\
&= I_1(f)(x)+I_2(f)(x), \quad x\in\mathbb{R}^n. \qquad (23)
\end{aligned}
$$

Note firstly that, according to (20) and the known estimates for $\frac{\partial}{\partial t}W_t$, for certain $C, c>0$, we get

$$
\begin{aligned}
I_2(f)(x) &\leq C\int_{|x-y|<\gamma(x)}\big|f(y)\big|\int_{\gamma(x)^2}^{\infty}\frac{e^{-c\frac{|x-y|^2}{t}}}{t^{\frac{n}{2}+1}}dtdy\\
&\leq C\int_{|x-y|<\gamma(x)}\big|f(y)\big|\int_{\gamma(x)^2}^{\infty}\frac{dt}{t^{\frac{n}{2}+1}}dy\\
&\leq \frac{C}{\gamma(x)^n}\int_{|x-y|<\gamma(x)}\big|f(y)\big|dy\leq C\mathcal{M}(f)(x), \quad x\in\mathbb{R}^n. \qquad (24)
\end{aligned}
$$

On the other hand, the perturbation formula ([12, (5.25)]) leads to

$$
\begin{aligned}
&\frac{\partial}{\partial t}\big(W_t(x,y)-W_t^{\mathcal{L}}(x,y)\big)\\
&\quad=\int_{\mathbb{R}^n}V(z)W_{\frac{t}{2}}(x,z)W_{\frac{t}{2}}^{\mathcal{L}}(z,y)dz\\
&\qquad+\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}V(z)\frac{\partial}{\partial t}W_{t-s}(x,z)W_s^{\mathcal{L}}(z,y)dzds\\
&\qquad+\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n}W_{t-s}(x,z)V(z)\frac{\partial}{\partial s}W_s^{\mathcal{L}}(z,y)dzds\\
&\quad=K_1(x,y,t)+K_2(x,y,t)+K_3(x,y,t), \quad x,y\in\mathbb{R}^n \text{ and } t>0.
\end{aligned}
$$

Hence $I_1(f)=T_1(f)+T_2(f)+T_3(f)$, where, for $m=1,2,3$,

$$
T_m f(x)=\int_{|x-y|<\gamma(x)}\big|f(y)\big|\int_0^{\gamma(x)^2}\big|K_m(x,y,t)\big|dtdy, \quad x\in\mathbb{R}^n.
$$

By [12, (2.2) and (2.8)], we obtain

$$\begin{aligned}\int_0^{\gamma(x)^2}\big|K_1(x,y,t)\big|dt &\le C\int_0^{\gamma(x)^2}\int_{\mathbb{R}^n} V(z)\frac{1}{t^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{2t}}\frac{1}{t^{\frac{n}{2}}}e^{-\frac{|z-y|^2}{2t}}dzdt\\ &\le C\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}\int_{\mathbb{R}^n}\frac{1}{t^{\frac{n}{2}}}e^{-\frac{|x-z|^2}{4t}}V(z)dzdt\\ &\le C\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4t}}\left(\frac{\sqrt{t}}{\gamma(x)}\right)^{\delta}dt,\end{aligned}$$

for a certain $\delta>0$. Then,

$$\begin{aligned}\big|T_1(f)(x)\big| &\le C\int_{|x-y|<\gamma(x)}\big|f(y)\big|\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}+1}}e^{-\frac{|x-y|^2}{4t}}\left(\frac{\sqrt{t}}{\gamma(x)}\right)^{\delta}dtdy\\ &\le C\int_0^{\gamma(x)^2}\frac{t^{-1+\frac{\delta}{2}}}{\gamma(x)^{\delta}}\frac{1}{t^{\frac{n}{2}}}\int_{\mathbb{R}^n}\big|f(y)\big|e^{-\frac{|x-y|^2}{4t}}dydt\\ &\le C\sup_{t>0}\frac{1}{t^{\frac{n}{2}}}\int_{\mathbb{R}^n}\big|f(y)\big|e^{-\frac{|x-y|^2}{4t}}dy\le C\mathcal{M}(f)(x),\quad x\in\mathbb{R}^n.\end{aligned}\tag{25}$$

Also, since $\frac{t}{2}<t-s<t$ provided that $0<s<\frac{t}{2}$, [12, (2.2) and (2.8)] imply that, for some $0<c<\frac{1}{4}$,

$$\begin{aligned}&\int_0^{\gamma(x)^2}\big|K_2(x,y,t)\big|dt\\ &\le C\int_0^{\gamma(x)^2}\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n} V(z)\frac{1}{(t-s)^{\frac{n}{2}+1}}e^{-c\frac{|x-z|^2}{t-s}}\frac{1}{s^{\frac{n}{2}}}e^{-\frac{|y-z|^2}{4s}}dzdsdt\\ &\le C\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}+1}}\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n} V(z)e^{-c\frac{|x-z|^2}{t}}\frac{1}{s^{\frac{n}{2}}}e^{-\frac{|y-z|^2}{4s}}dzdsdt\\ &\le C\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}+1}}\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n} e^{-\frac{c}{2}\frac{|x-z|^2+|y-z|^2}{t}}V(z)\frac{1}{s^{\frac{n}{2}}}e^{-\frac{c}{2}\frac{|y-z|^2}{s}}dzdsdt\\ &\le C\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}+1}}\int_0^{\frac{t}{2}} e^{-\frac{c}{2}\frac{|x-y|^2}{t}}\int_{\mathbb{R}^n}\frac{1}{s^{\frac{n}{2}}}e^{-\frac{c}{2}\frac{|y-z|^2}{s}}V(z)dzdsdt\\ &\le C\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}+1}}e^{-\frac{c}{2}\frac{|x-y|^2}{t}}\int_0^{\frac{t}{2}}\frac{s^{-1+\frac{\delta}{2}}}{\gamma(y)^{\delta}}dsdt\\ &\le \frac{C}{\gamma(x)^{\delta}}\int_0^{\gamma(x)^2}\frac{1}{t^{\frac{n}{2}+1-\frac{\delta}{2}}}e^{-\frac{c}{2t}|x-y|^2}dt,\quad x,y\in\mathbb{R}^n,\ |x-y|\le\gamma(x).\end{aligned}$$

We have taken into account that $\gamma(x) \sim \gamma(y)$, provided that $|x-y| \leq \gamma(x)$. Then,

$$\begin{aligned}\left|T_2 f(x)\right| &\leq \frac{C}{\gamma(x)^\delta} \int_{|x-y|<\gamma(x)} \left|f(y)\right| \int_0^{\gamma(x)^2} \frac{e^{-\frac{c}{2t}|x-y|^2}}{t^{\frac{n}{2}+1-\frac{\delta}{2}}} dt dy \\ &\leq \frac{C}{\gamma(x)^\delta} \int_0^{\gamma(x)^2} \frac{dt}{t^{1-\frac{\delta}{2}}} \int_{\mathbb{R}^n} \frac{1}{t^{\frac{n}{2}}} e^{-\frac{c}{2t}|x-y|^2} \left|f(y)\right| dy dt \\ &\leq C \sup_{t>0} \frac{1}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{c}{2t}|x-y|^2} \left|f(y)\right| dy \leq C\mathcal{M}(f)(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{26}$$

By proceeding in a similar way and using [12, (2.7)]we see that

$$\left|T_3 f(x)\right| \leq C\mathcal{M}(f)(x), \quad x \in \mathbb{R}^n. \tag{27}$$

From (25), (26) and (27), we get

$$I_1(f)(x) \leq C\mathcal{M}(f)(x), \quad x \in \mathbb{R}^n. \tag{28}$$

Inequalities (24) and (28) imply that

$$V_\rho\left(W^{\mathcal{L}}_{t,\mathrm{loc}} - W_{t,\mathrm{loc}}\right)(f)(x) \leq C\mathcal{M}(f)(x), \quad x \in \mathbb{R}^n.$$

Hence, the operator $V_\rho(W^{\mathcal{L}}_{t,\mathrm{loc}} - W_{t,\mathrm{loc}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

Finally, by (18) we conclude that $V_\rho(W^{\mathcal{L}}_t)$ is a bounded operator from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

4 Variation operators associated with Riesz transform $R^{\mathcal{L}}$

In this section we prove Proposition 1.1 and Theorem 1.2. As it was mentioned in Sect. 1, for every $\ell = 1, \ldots, n$, the ℓ-th Riesz transform associated with $\mathcal{L}$ is defined formally by

$$R^{\mathcal{L}}_\ell = \frac{\partial}{\partial x_\ell} \mathcal{L}^{-\frac{1}{2}}. \tag{29}$$

Here $\mathcal{L}^{-\frac{1}{2}}$ denotes the negative square root of the operator $\mathcal{L}$ given by

$$\mathcal{L}^{-\frac{1}{2}} f(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \Gamma(x, y, \tau) d\tau f(y) dy,$$

where $\Gamma(x, y, \tau)$ represents the fundamental solution for the operator $\mathcal{L} + i\tau$, with $\tau \in \mathbb{R}$ (see [25, §2]).

We recall that, for every $\ell = 1, \ldots, n$, the function $R^{\mathcal{L}}_\ell(x, y)$ is defined by

$$R^{\mathcal{L}}_\ell(x, y) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_\ell} \Gamma(x, y, \tau) d\tau, \quad x, y \in \mathbb{R}^n, \ x \neq y.$$

The following estimates for the kernels $R_\ell^{\mathcal{L}}$, $\ell = 1, \dots, n$, were established in [25, Sects. 5 and 6] (see also [2, Lemma 1]) and will be very useful in the sequel.

Lemma 4.1 *Let $\ell = 1, \dots, n$ and $V \in B_q$.*

(i) *Assume that $q > n$. Then, for every $k \in \mathbb{N}$ there exists $C > 0$ such that*

$$\left|R_\ell^{\mathcal{L}}(x, y)\right| \leq C \frac{1}{(1 + |x - y|/\gamma(x))^k} \frac{1}{|x - y|^n}. \tag{30}$$

Moreover, we have

$$\left|R_\ell^{\mathcal{L}}(x, y) - R_\ell(x - y)\right| \leq C \frac{1}{|x - y|^n} \left(\frac{|x - y|}{\gamma(x)}\right)^{2-n/q}, \quad 0 < |x - y| < \gamma(x). \tag{31}$$

(ii) *Suppose that $n/2 < q < n$. Then, for every $k \in \mathbb{N}$ there exists $C > 0$ such that*

$$\begin{aligned}\left|R_\ell^{\mathcal{L}}(x, y)\right| \leq{}& C \frac{1}{(1 + |x - y|/\gamma(y))^k} \frac{1}{|x - y|^{n-1}} \\ &\times \left(\frac{1}{|x - y|} + \int_{B(x,|x-y|/4)} \frac{V(z)}{|z - x|^{n-1}} dz\right).\end{aligned} \tag{32}$$

Also, we have

$$\begin{aligned}&\left|R_\ell^{\mathcal{L}}(x, y) - R_\ell(x - y)\right| \\ &\leq C \frac{1}{|x - y|^{n-1}} \\ &\quad \times \left(\frac{1}{|x - y|} \left(\frac{|x - y|}{\gamma(x)}\right)^{2-n/q} + \int_{B(x,|x-y|/4)} \frac{V(z)}{|z - x|^{n-1}} dz\right), \\ &0 < |x - y| < \gamma(x).\end{aligned} \tag{33}$$

Remark 4.1 Note that, according to [13], if $V \in B_n$, there exists $\varepsilon > 0$ such that $V \in B_{n+\varepsilon}$. Then, the estimates in Lemma 4.1, (i), can be applied to $V \in B_n$ by taking $q = n + \varepsilon$, where $\varepsilon > 0$ is small enough and it depends on V.

We consider, for every $\ell = 1, \dots, n$ and $\varepsilon > 0$,

$$R_\ell^{\mathcal{L},\varepsilon}(f)(x) = \int_{|x-y|>\varepsilon} R_\ell^{\mathcal{L}}(x, y) f(y) dy, \quad x \in \mathbb{R}^n.$$

According to Lemma 4.1, it is not hard to see that, for every $\ell = 1, \dots, n$, $\varepsilon > 0$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the integral defining $R_\ell^{\mathcal{L},\varepsilon}(f)(x)$ is absolutely convergent for each $x \in \mathbb{R}^n$.

Before proving Proposition 1.1 we establish the L^p-boundedness properties of the maximal operator $R_\ell^{\mathcal{L},*}$ defined by

$$R_\ell^{\mathcal{L},*}(f) = \sup_{\varepsilon>0} \left|R_\ell^{\mathcal{L},\varepsilon}(f)\right|.$$

Proposition 4.1 *Let $\ell = 1, \dots, n$ and $V \in B_q$, $q \geq n/2$. Then, if $1 < p < p_0$, where $\frac{1}{p_0} = (\frac{1}{q} - \frac{1}{n})_+$, $R_\ell^{\mathcal{L},*}$ is bounded from $L^p(\mathbb{R}^n)$ into itself. Moreover, $R_\ell^{\mathcal{L},*}$ is bounded from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

Proof It is enough to prove the result when $n/2 < q < n$. Indeed, according to [13], if $V \in B_{n/2}$ then $V \in B_{\varepsilon+n/2}$ for some $\varepsilon > 0$. Moreover, $B_r \subset B_s$, when $r \geq s$. We split the operator $R_\ell^{\mathcal{L},*}$, in the spirit of the general procedure described at the beginning of this section, as follows

$$\begin{aligned} R_\ell^{\mathcal{L},*}(f)(x) &\leq \int_{|x-y|<\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)\right| |f(y)| dy \\ &\quad + \int_{|x-y|>\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y)\right| |f(y)| dy \\ &\quad + \sup_{\varepsilon>0} \left| \int_{\varepsilon<|x-y|<\gamma(x)} R_\ell(x-y) f(y) dy \right| \\ &= \tau_1\big(|f|\big)(x) + \tau_2\big(|f|\big)(x) + \tau_3(f)(x), \quad x \in \mathbb{R}^n. \end{aligned}$$

It is clear that

$$\tau_3(f)(x) \leq \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} R_\ell(x-y) f(y) dy \right|, \quad x \in \mathbb{R}^n.$$

Then, from well known results we infer that τ_3 is bounded from $L^r(\mathbb{R}^n)$ into itself when $1 < r < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

For a certain $M > 1$, $\frac{1}{M} \leq \frac{\gamma(x)}{\gamma(y)} \leq M$, when $|x-y| \leq \gamma(y)$. Moreover, if $|x-y| > \gamma(x)$, $|x-y| > \gamma(y)/M$. Indeed, it is sufficient to observe that

$$\begin{aligned} \big\{y \in \mathbb{R}^n : |x-y| > \gamma(x)\big\} &\subset \big\{y \in \mathbb{R}^n : |x-y| < \gamma(y) \text{ and } |x-y| > \gamma(x)\big\} \\ &\quad \cup \big\{y \in \mathbb{R}^N : |x-y| > \gamma(y) \text{ and } |x-y| > \gamma(x)\big\}. \end{aligned}$$

We denote by τ_j^* the adjoint operator of τ_j, $j = 1, 2$. We have that

$$\begin{aligned} \tau_1^*\big(|g|\big)(y) &= \int_{|x-y|<\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)\right| g(x) dx \\ &\leq \int_{|x-y|<M\gamma(y)} \left|R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)\right| g(x) dx, \quad y \in \mathbb{R}^n, \end{aligned}$$

and

$$\begin{aligned} \tau_2^*\big(|g|\big)(y) &= \int_{|x-y|\geq\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y)\right| |g(x)| dx \\ &\leq \int_{|x-y|\geq\gamma(y)/M} \left|R_\ell^{\mathcal{L}}(x,y)\right| |g(x)| dx, \quad y \in \mathbb{R}^n. \end{aligned}$$

According to [25, (5.9) and the proof of Lemma 5.8] and the arguments in the proof of [25, Lemma 5.7] we obtain that τ_i^*, $i = 1, 2$, are bounded from $L^r(\mathbb{R}^n)$ into itself when $p_0' < r \leq \infty$. Then, τ_i, $i = 1, 2$, are bounded from $L^r(\mathbb{R}^n)$ into itself when $1 \leq r < p_0$.

By combining the above properties we conclude that $R_\ell^{\mathcal{L},*}$ is a bounded operator from $L^p(\mathbb{R}^n)$ into itself, provided that $1 < p < p_0$, and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. □

Proof of Proposition 1.1 By Proposition 4.1, in order to show Proposition 1.1 it is sufficient to see (4) for every $\phi \in C_c^\infty(\mathbb{R}^n)$.

Suppose that $\phi \in C_c^\infty(\mathbb{R}^n)$. By $\Gamma_0(x, y, \tau)$ we denote the fundamental solution of the operator $-\Delta + i\tau$ in $\mathbb{R}^n$, with $\tau \in \mathbb{R}$. We are going to see that, for every $\ell = 1, \ldots, n$, $\mathcal{L}^{-\frac{1}{2}}\phi - (-\Delta)^{-\frac{1}{2}}\phi$ admits derivative with respect to x_ℓ, for almost all $\mathbb{R}^n$, and that a.e. $x \in \mathbb{R}^n$,

$$\frac{\partial}{\partial x_\ell}\big(\mathcal{L}^{-\frac{1}{2}}\phi(x) - (-\Delta)^{-\frac{1}{2}}\phi(x)\big)$$
$$= -\frac{1}{2\pi}\int_{\mathbb{R}^n}\phi(y)\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_\ell}\big(\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)\big)(-i\tau)^{-\frac{1}{2}}d\tau dy.$$

To simplify the notation we consider $\ell = 1$. Also we assume that $q > n/2$ ([13]). According to [25, Lemma 4.5], for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$\big|\Gamma(x, y, \tau) - \Gamma_0(x, y, \tau)\big| \leq \frac{C_k|x - y|^{2-n}}{(1 + |\tau|^{\frac{1}{2}}|x - y|)^k}\left(\frac{|x - y|}{\gamma(x)}\right)^{2-\frac{n}{q}} \tag{34}$$

provided that $\tau \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, $|x - y| \leq \gamma(x)$. Moreover, by [25, Theorem 2.7], for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$\big|\Gamma(x, y, \tau)\big| \leq \frac{C_k|x - y|^{2-n}}{(1 + |\tau|^{\frac{1}{2}}|x - y|)^k(1 + \frac{|x-y|}{\gamma(x)})^k}, \quad x, y \in \mathbb{R}^n,\ \tau \in \mathbb{R}, \tag{35}$$

and

$$\big|\Gamma_0(x, y, \tau)\big| \leq \frac{C_k|x - y|^{2-n}}{(1 + |\tau|^{\frac{1}{2}}|x - y|)^k}, \quad x, y \in \mathbb{R}^n,\ \tau \in \mathbb{R}. \tag{36}$$

We will use repeatedly without reference the following equality

$$\int_{-\infty}^{+\infty}\frac{1}{|\tau|^{\frac{1}{2}}}\frac{1}{(1 + |x - y\|\tau|^{\frac{1}{2}})^k}d\tau = \frac{C_k}{|x - y|}, \quad x, y \in \mathbb{R}^n,\ x \neq y \text{ and } k \geq 1.$$

From (35) and (36) we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{-\infty}^{+\infty} \big|\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big| |\tau|^{-\frac{1}{2}} d\tau dy \\
&\quad \leq C \int_{\mathbb{R}^n} \frac{|\phi(y)|}{|x-y|^{n-2}} \int_{-\infty}^{+\infty} \frac{1}{|\tau|^{\frac{1}{2}}} \frac{1}{(1+|x-y||\tau|^{\frac{1}{2}})^2} d\tau dy \\
&\quad \leq C \int_{\mathbb{R}^n} \frac{|\phi(y)|}{|x-y|^{n-1}} dy \\
&\quad \leq C \frac{1}{1+|x|^{n-1}}, \quad x \in \mathbb{R}^n. \qquad (37)
\end{aligned}$$

Then, the function $F = \mathcal{L}^{-\frac{1}{2}}\phi - (-\Delta)^{-\frac{1}{2}}\phi$ defines in $\mathbb{R}^n$ a distribution S_F by

$$\langle S_F, \psi \rangle = \int_{\mathbb{R}^n} F(y)\psi(y)dy, \quad \psi \in C_c^\infty\big(\mathbb{R}^n\big).$$

We have that, by writing $\overline{x} = (x_2, \ldots, x_n)$ and considering $\psi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
&\left\langle \frac{\partial}{\partial x_1} S_F, \psi \right\rangle \\
&\quad = -\int_{\mathbb{R}^n} F(x) \frac{\partial}{\partial x_1}\psi(x)dx \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) d\tau dy \frac{\partial}{\partial x_1}\psi(x)dx \\
&\quad = \frac{1}{2\pi} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) \\
&\qquad \times \frac{\partial}{\partial x_1}\psi(x)dx_1 d\tau dy d\overline{x}. \qquad (38)
\end{aligned}$$

We now apply partial integration in the following way. Since the integral is absolutely convergent (see (37)) it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) \\
&\qquad \times \frac{\partial}{\partial x_1}\psi(x)dx_1 d\tau dy d\overline{x} \\
&\quad = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \\
&\qquad \times \left(\int_{-\infty}^{y_1-\varepsilon} + \int_{y_1+\varepsilon}^{+\infty} \right) \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) \frac{\partial}{\partial x_1}\psi(x)dx_1 d\tau dy d\overline{x} \\
&\quad = \lim_{\varepsilon \to 0^+} \left(-\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \right.
\end{aligned}$$

$$\times \left(\int_{-\infty}^{y_1-\varepsilon} + \int_{y_1+\varepsilon}^{+\infty} \right) \frac{\partial}{\partial x_1} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau) \big) \psi(x) dx_1 d\tau dy d\overline{x}$$

$$+ \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \big(\big(\Gamma(y_1-\varepsilon, \overline{x}, y, \tau) - \Gamma_0(y_1-\varepsilon, \overline{x}, y, \tau) \big)$$

$$\times \psi(y_1-\varepsilon, \overline{x})$$

$$- \big(\Gamma(y_1+\varepsilon, \overline{x}, y, \tau) - \Gamma_0(y_1+\varepsilon, \overline{x}, y, \tau) \big) \psi(y_1+\varepsilon, \overline{x}) \big) d\tau dy d\overline{x} \Big). \tag{39}$$

We have that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{-\infty}^{+\infty} |\phi(y)| |\psi(x)| |\tau|^{-1/2} \left| \frac{\partial}{\partial x_1} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau) \big) \right| d\tau dx dy < \infty. \tag{40}$$

Indeed, by [25, Theorem 2.7] (see also the proof of [25, Lemma 5.7]), for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that, for every $x, y \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$,

$$\left| \frac{\partial}{\partial x_1} \Gamma(x,y,\tau) \right|$$

$$\leq C_k \frac{|x-y|^{2-n}}{(1+|\tau|^{1/2}|x-y|)^k (1+\frac{|x-y|}{\gamma(y)})^k} \left(\int_{|z-x|<\frac{|x-y|}{4}} \frac{V(z)}{|z-x|^{n-1}} dz + \frac{1}{|x-y|} \right). \tag{41}$$

Moreover, for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$\left| \frac{\partial}{\partial x_1} \Gamma_0(x,y,\tau) \right| \leq C_k \frac{|x-y|^{1-n}}{(1+|\tau|^{1/2}|x-y|)^k}, \quad x, y \in \mathbb{R}^n, \ \tau \in \mathbb{R}. \tag{42}$$

By [25, p. 541], for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$\left| \frac{\partial}{\partial x_1} \Gamma(x,y,\tau) - \frac{\partial}{\partial x_1} \Gamma_0(x,y,\tau) \right|$$

$$\leq C_k \frac{|x-y|^{2-n}}{(1+|\tau|^{1/2}|x-y|)^k}$$

$$\times \left(\int_{|z-x|<\frac{|x-y|}{4}} \frac{V(z)}{|z-x|^{n-1}} dz + \frac{1}{|x-y|} \left(\frac{|x-y|}{\gamma(y)} \right)^{2-n/q} \right),$$

$$|x-y| < \gamma(y), \ \tau \in \mathbb{R}, \tag{43}$$

where $q > n/2$.

Assume firstly that $V \in B_n$. Then $V \in B_q$, for some $q > n$ [13]. According to [2, Lemma 1], we deduce, from (41) and (43), that for every $k \in \mathbb{N}$, there exists $C_k > 0$ such that

$$\left| \frac{\partial}{\partial x_1} \Gamma(x,y,\tau) \right| \leq C_k \frac{|x-y|^{1-n}}{(1+|\tau|^{1/2}|x-y|)^k (1+\frac{|x-y|}{\gamma(y)})^k}, \quad x, y \in \mathbb{R}^n, \ \tau \in \mathbb{R}, \tag{44}$$

and, for every $\tau \in \mathbb{R}$ and $|x-y| < \gamma(y)$,

$$\left| \frac{\partial}{\partial x_1}\Gamma(x,y,\tau) - \frac{\partial}{\partial x_1}\Gamma_0(x,y,\tau) \right| \leq C_k \frac{|x-y|^{1-n}}{(1+|\tau|^{1/2}|x-y|)^k} \left(\frac{|x-y|}{\gamma(y)} \right)^{2-n/q}. \quad (45)$$

According to [25, Lemma 1.4], γ and $1/\gamma$ are bounded on any compact subset of $\mathbb{R}^n$. Since ϕ and ψ have compact support, there exists $A > 0$ such that $|x-y| \leq A\gamma(y)$, $x \in \operatorname{supp}\psi$ and $y \in \operatorname{supp}\phi$. Then, by using (42), (44) and (45) we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{-\infty}^{+\infty} |\phi(y)||\psi(x)||\tau|^{-1/2} \left| \frac{\partial}{\partial x_1}\big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) \right| d\tau dx dy \\
&\quad \leq C \int_{\operatorname{supp}\psi} |\psi(x)| \int_{\operatorname{supp}\phi} |\phi(y)| \int_{-\infty}^{\infty} \frac{|x-y|^{3-n-n/q}}{|\tau|^{1/2}(1+|\tau|^{1/2}|x-y|)^2} d\tau dy dx \\
&\quad \leq C \int_{\operatorname{supp}\psi} \int_{\operatorname{supp}\phi} \frac{1}{|x-y|^{n-2+n/q}} dy dx < \infty.
\end{aligned}$$

We consider now $\frac{n}{2} < q < n$. We recall that the 1-th Euclidean fractional integral is bounded from $L^q(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$, when $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$ ([27]). Since $q > n/2$ in order to establish (40) we only have to see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\phi(y)| \int_{|x-y|\leq\gamma(y)} |\psi(x)| \\
&\quad \times \int_{-\infty}^{\infty} \frac{|x-y|^{2-n}}{|\tau|^{1/2}(1+|\tau|^{1/2}|x-y|)^2} \int_{|z-x|<\frac{|x-y|}{4}} \frac{V(z)}{|z-x|^{n-1}} dz d\tau dy dx < \infty.
\end{aligned}$$

To do this we can proceed as follows. There exists $M > 0$ for which

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\phi(y)| \int_{|x-y|\leq\gamma(y)} |\psi(x)| \int_{-\infty}^{\infty} \frac{|x-y|^{2-n}}{|\tau|^{1/2}(1+|\tau|^{1/2}|x-y|)^2} \\
&\quad \times \int_{|z-x|<\frac{|x-y|}{4}} \frac{V(z)}{|z-x|^{n-1}} dz d\tau dy dx \\
&\quad \leq C \int_{\operatorname{supp}\psi} \int_{\operatorname{supp}\phi} \frac{1}{|x-y|^{n-1}} \int_{|z|\leq M} \frac{|V(z)|}{|z-x|^{n-1}} dz dy dx \\
&\quad \leq C \int_{\operatorname{supp}\phi} \left(\int_{\operatorname{supp}\psi} \frac{1}{|x-y|^{p_0'(n-1)}} dx \right)^{1/p_0'} \left(\int_{\mathbb{R}^n} \big| I_1(\chi_{B(0,M)}V)(x) \big|^{p_0} dx \right)^{1/p_0} dy \\
&\quad \leq C \int_{\operatorname{supp}\phi} \left(\int_{\operatorname{supp}\psi} \frac{1}{|x-y|^{p_0'(n-1)}} dx \right)^{1/p_0'} \left(\int_{B(0,M)} |V(z)|^q dz \right)^{1/q} dy < \infty,
\end{aligned}$$

because $V \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, and $(n-1)(p_0'-1) < 1$ when $q > n/2$.

Hence, (40) holds and we can write

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}\phi(y)\int_{-\infty}^{+\infty}(-i\tau)^{-\frac{1}{2}}\Big(\int_{-\infty}^{y_1-\varepsilon}+\int_{y_1+\varepsilon}^{+\infty}\Big)\frac{\partial}{\partial x_1}\big(\Gamma(x,y,\tau)$$
$$-\Gamma_0(x,y,\tau)\big)\psi(x)dx_1d\tau dyd\overline{x}$$
$$=\int_{\mathbb{R}^n}\psi(x)\int_{\mathbb{R}^n}\phi(y)\int_{-\infty}^{+\infty}(-i\tau)^{-\frac{1}{2}}\frac{\partial}{\partial x_1}\big(\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)\big)d\tau dydx.$$

Our next task is to see that $\lim_{\varepsilon\to 0^+} I(\varepsilon)=0$, where

$$I(\varepsilon)=\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}\phi(y)\int_{-\infty}^{+\infty}(-i\tau)^{-\frac{1}{2}}\big(\big(\Gamma(y_1-\varepsilon,\overline{x},y,\tau)$$
$$-\Gamma_0(y_1-\varepsilon,\overline{x},y,\tau)\big)\psi(y_1-\varepsilon,\overline{x})$$
$$-\big(\Gamma(y_1+\varepsilon,\overline{x},y,\tau)-\Gamma_0(y_1+\varepsilon,\overline{x},y,\tau)\big)\psi(y_1+\varepsilon,\overline{x})\big)d\tau dyd\overline{x},\quad \varepsilon>0.$$

We have that

$$\big(\Gamma(y_1-\varepsilon,\overline{x},y,\tau)-\Gamma_0(y_1-\varepsilon,\overline{x},y,\tau)\big)\psi(y_1-\varepsilon,\overline{x})-\big(\Gamma(y_1+\varepsilon,\overline{x},y,\tau)$$
$$-\Gamma_0(y_1+\varepsilon,\overline{x},y,\tau)\big)\psi(y_1+\varepsilon,\overline{x})$$
$$=\big(\big(\Gamma(y_1-\varepsilon,\overline{x},y,\tau)-\Gamma(y_1+\varepsilon,\overline{x},y,\tau)\big)$$
$$-\big(\Gamma_0(y_1-\varepsilon,\overline{x},y,\tau)-\Gamma_0(y_1+\varepsilon,\overline{x},y,\tau)\big)\big)\psi(y_1-\varepsilon,\overline{x})$$
$$+\big(\psi(y_1-\varepsilon,\overline{x})-\psi(y_1+\varepsilon,\overline{x})\big)\big(\Gamma(y_1+\varepsilon,\overline{x},y,\tau)-\Gamma_0(y_1+\varepsilon,\overline{x},y,\tau)\big)$$
$$=J_1(\overline{x},y,\varepsilon,\tau)+J_2(\overline{x},y,\varepsilon,\tau),\quad \overline{x}\in\mathbb{R}^{n-1},\ y\in\mathbb{R}^n,\ \varepsilon>0,\ \tau\in\mathbb{R}.$$

According to [25, p. 535], if $\overline{x}\in\mathbb{R}^{n-1}$, $y\in\mathbb{R}^n$, $0<\varepsilon<|\overline{x}-\overline{y}|/15$ and $\tau\in\mathbb{R}$,

$$\big|\Gamma(y_1-\varepsilon,\overline{x},y,\tau)-\Gamma(y_1+\varepsilon,\overline{x},y,\tau)\big|\leq C\frac{\varepsilon^\delta|\overline{x}-\overline{y}|^{2-n-\delta}}{(1+|\tau|^{1/2}|\overline{x}-\overline{y}|)^3},\tag{46}$$

and

$$\big|\Gamma_0(y_1-\varepsilon,\overline{x},y,\tau)-\Gamma_0(y_1+\varepsilon,\overline{x},y,\tau)\big|\leq C\frac{\varepsilon^\delta|\overline{x}-\overline{y}|^{2-n-\delta}}{(1+|\tau|^{1/2}|\overline{x}-\overline{y}|)^3},\tag{47}$$

for a certain $\delta>0$ that depends on q.

By using (34), (35) and (36) we can deduce that

$$\lim_{\eta\to 0^+}\int_{\mathbb{R}^n}\big|\phi(y)\big|\int_{|\overline{x}-\overline{y}|<\eta}\int_{-\infty}^{\infty}|\tau|^{-1/2}\big|J_m(\overline{x},y,\varepsilon,\tau)\big|d\tau dyd\overline{x}=0,\quad m=1,2,$$

uniformly in $\varepsilon\in(0,1)$.

Indeed, let $\varepsilon\in(0,1)$. According to (35) and (36), the mean value theorem leads to

$$\big|J_2(\overline{x},y,\varepsilon,\tau)\big|\leq C\varepsilon\frac{(\varepsilon+|\overline{x}-\overline{y}|)^{2-n}}{(1+|\tau|^{1/2}(\varepsilon+|\overline{x}-\overline{y}|))^3},\quad \overline{x},\overline{y}\in\mathbb{R}^{n-1}\text{ and }\tau\in\mathbb{R}.$$

Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \int_{-\infty}^{\infty} |\tau|^{-1/2} \big|J_2(\overline{x}, y, \varepsilon, \tau)\big| d\tau d\overline{x} dy \\
&\quad \le C \int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \int_{-\infty}^{\infty} |\tau|^{-1/2} \varepsilon \frac{(\varepsilon + |\overline{x}-\overline{y}|)^{2-n}}{(1+|\tau|^{1/2}(\varepsilon+|\overline{x}-\overline{y}|))^3} d\tau d\overline{x} dy \\
&\quad \le C \int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \varepsilon \big(\varepsilon + |\overline{x}-\overline{y}|\big)^{1-n} d\overline{x} dy \\
&\quad \le C \int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} |\overline{x}-\overline{y}|^{2-n} d\overline{x} dy, \quad \eta > 0,
\end{aligned}$$

where $C > 0$ does not depend on ε. Hence,

$$\lim_{\eta \to 0^+} \int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \int_{-\infty}^{\infty} |\tau|^{-1/2} \big|J_2(\overline{x}, y, \varepsilon, \tau)\big| d\tau dy d\overline{x} = 0,$$

uniformly in $\varepsilon \in (0, 1)$.

By using (43), since $\operatorname{supp}\phi$ is compact, we get, for a certain $a > 0$, that if $\varepsilon \in (0, 1)$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \int_{-\infty}^{\infty} |\tau|^{-1/2} \big|J_1(\overline{x}, y, \varepsilon, \tau)\big| d\tau d\overline{x} dy \\
&\quad \le C \int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \int_{-\infty}^{\infty} |\tau|^{-1/2} \\
&\qquad \times \left| \int_{y_1-\varepsilon}^{y_1+\varepsilon} \frac{\partial}{\partial u} \big(\Gamma(u, \overline{x}, y, \tau) - \Gamma_0(u, \overline{x}, y, \tau)\big) du \right| d\tau d\overline{x} dy \\
&\quad \le C \int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \int_{y_1-\varepsilon}^{y_1+\varepsilon} \int_{-\infty}^{\infty} |\tau|^{-1/2} \\
&\qquad \times \left| \frac{\partial}{\partial u} \big(\Gamma(u, \overline{x}, y, \tau) - \Gamma_0(u, \overline{x}, y, \tau)\big) \right| d\tau du d\overline{x} dy \\
&\quad \le C \int_{\mathbb{R}^n} \big|\phi(y)\big| \int_{|\overline{x}-\overline{y}|<\eta} \int_{-a}^{a} \int_{-\infty}^{\infty} |\tau|^{-1/2} \\
&\qquad \times \left| \frac{\partial}{\partial u} \big(\Gamma(u, \overline{x}, y, \tau) - \Gamma_0(u, \overline{x}, y, \tau)\big) \right| d\tau du d\overline{x} dy, \quad \eta > 0.
\end{aligned}$$

It was showed in (40) that, for every K compact subset of $\mathbb{R}^n$, we have that

$$\int_K \int_K \int_{-\infty}^{\infty} |\tau|^{-1/2} \left| \frac{\partial}{\partial u} \big(\Gamma(u, \overline{x}, y, \tau) - \Gamma_0(u, \overline{x}, y, \tau)\big) \right| d\tau dy dx < \infty.$$

Then, it follows that

$$\lim_{\eta\to 0^+}\int_{\mathbb{R}^n}\big|\phi(y)\big| \times\int_{|\overline{x}-\overline{y}|<\eta}\int_{-a}^{a}\int_{-\infty}^{\infty}|\tau|^{-1/2}\Big|\frac{\partial}{\partial u}\big(\Gamma(u,\overline{x},y,\tau)-\Gamma_0(u,\overline{x},y,\tau)\big)\Big|d\tau du d\overline{x}dy=0.$$

Hence, we conclude

$$\lim_{\eta\to 0^+}\int_{\mathbb{R}^n}\big|\phi(y)\big|\int_{|\overline{x}-\overline{y}|<\eta}\int_{-\infty}^{\infty}|\tau|^{-1/2}\big|J_1(\overline{x},y,\varepsilon,\tau)\big|d\tau d\overline{x}dy=0,$$

uniformly in $\varepsilon\in(0,1)$.

Therefore, in order to achieve our purpose it is sufficient to see that, for every $\eta>0$,

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}\int_{|\overline{x}-\overline{y}|>\eta}\big|\phi(y)\big|\int_{-\infty}^{\infty}|\tau|^{-1/2}\big|J_m(\overline{x},y,\varepsilon,\tau)\big|d\tau d\overline{x}dy=0,\quad m=1,2.$$

Assume that $\eta>0$. By (46) and (47), we get, for certain $A_1,A_2>0$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{|\overline{x}-\overline{y}|>\eta}\big|\phi(y)\big|\int_{-\infty}^{\infty}|\tau|^{-1/2}\big|J_1(\overline{x},y,\varepsilon,\tau)\big|d\tau d\overline{x}dy\\&\quad\le C\varepsilon^{\delta}\int_{B(0,A_1)}\int_{\eta<|\overline{x}-\overline{y}|<A_2}\int_{-\infty}^{\infty}\frac{|\tau|^{-1/2}|\overline{x}-\overline{y}|^{2-n-\delta}}{(1+|\tau|^{1/2}|\overline{x}-\overline{y}|)^3}d\tau d\overline{x}dy\\&\quad\le C\varepsilon^{\delta}\int_{B(0,A_1)}\int_{\eta<|\overline{x}-\overline{y}|<A_2}|\overline{x}-\overline{y}|^{1-n-\delta}d\overline{x}dy,\quad 0<\varepsilon<\eta/15.\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}\int_{|\overline{x}-\overline{y}|>\eta}\big|\phi(y)\big|\int_{-\infty}^{\infty}|\tau|^{-1/2}\big|J_1(\overline{x},y,\varepsilon,\tau)\big|d\tau d\overline{x}dy=0.$$

Finally, in order to see that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}\int_{|\overline{x}-\overline{y}|>\eta}\big|\phi(y)\big|\int_{-\infty}^{\infty}|\tau|^{-1/2}\big|J_2(\overline{x},y,\varepsilon,\tau)\big|d\tau d\overline{x}dy=0,$$

we use the mean value theorem, (35) and (36).

Thus we have proved that $\lim_{\varepsilon\to 0^+}I(\varepsilon)=0$.

Hence, coming back to (39) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^n}\phi(y)\int_{-\infty}^{+\infty}(-i\tau)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\big(\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)\big)\frac{\partial}{\partial x_1}\psi(x)dx_1d\tau dyd\overline{x}\\&\quad=-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\phi(y)\psi(x)\int_{-\infty}^{+\infty}(-i\tau)^{-\frac{1}{2}}\frac{\partial}{\partial x_1}\big(\Gamma(x,y,\tau)-\Gamma_0(x,y,\tau)\big)d\tau dydx,\end{aligned}$$

and from (38) it follows that

$$\left\langle \frac{\partial}{\partial x_1} S_F, \psi \right\rangle$$
$$= -\frac{1}{2\pi} \int_{\mathbb{R}^n} \psi(x) \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) d\tau dy dx.$$

Therefore we have proved that the distributional derivative $\frac{\partial}{\partial x_1} S_F$ of S_F is

$$\frac{\partial}{\partial x_1} S_F = -\frac{1}{2\pi} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) d\tau dy.$$

Moreover, the above argument shows that the right hand side in the last equality defines a locally integrable function in $\mathbb{R}^n$.

Now, by invoking now [23, §5, Theorem V, part (2)] we can conclude that $\mathcal{L}^{-\frac{1}{2}}\phi - (-\Delta)^{-\frac{1}{2}}\phi$ admits classical derivative with respect to x_1 for almost all $x \in \mathbb{R}^n$, and

$$\frac{\partial}{\partial x_1} \big(\mathcal{L}^{-\frac{1}{2}}\phi(x) - (-\Delta)^{-\frac{1}{2}}\phi(x)\big)$$
$$= -\frac{1}{2\pi} \int_{\mathbb{R}^n} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) d\tau dy,$$

a.e. $x \in \mathbb{R}^n$. Moreover, the last integral is absolutely convergent. Then,

$$\frac{\partial}{\partial x_1} \big(\mathcal{L}^{-\frac{1}{2}}\phi(x) - (-\Delta)^{-\frac{1}{2}}\phi(x)\big)$$
$$= -\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} \big(\Gamma(x,y,\tau) - \Gamma_0(x,y,\tau)\big) d\tau dy,$$

a.e. $x \in \mathbb{R}^n$, and we obtain that

$$\frac{\partial}{\partial x_1} \mathcal{L}^{-\frac{1}{2}}\phi(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) R_1^{\mathcal{L}}(x,y) dy, \quad \text{a.e. } x \in \mathbb{R}^n,$$

since, as it is well known,

$$\frac{\partial}{\partial x_1} (-\Delta)^{-\frac{1}{2}}\phi(x)$$
$$= \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \phi(y) \int_{-\infty}^{+\infty} (-i\tau)^{-\frac{1}{2}} \frac{\partial}{\partial x_1} \Gamma_0(x,y,\tau) d\tau dy, \quad \text{a.e. } x \in \mathbb{R}^n. \quad \square$$

We now prove L^p-boundedness properties for the variation operators associated with the Riesz transforms $R_\ell^{\mathcal{L}}$.

Proof of Theorem 1.2 As in the proof of Proposition 4.1 it is enough to assume $n/2 < q < n$. We consider the operators

$$R_{\ell,\mathrm{loc}}^{\mathcal{L}}(f)(x) = P.V. \int_{|x-y|<\gamma(x)} R_\ell^{\mathcal{L}}(x,y) f(y) dy$$

and

$$R_{\ell,\mathrm{loc}}(f)(x) = P.V. \int_{|x-y|<\gamma(x)} R_\ell(x-y) f(y) dy,$$

where $f \in L^p(\mathbb{R}^n)$ for a suitable $1 \leq p < \infty$.

Suppose that $\{\eta_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero. Following the general procedure we may write

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty} \left|R_\ell^{\mathcal{L},\eta_j}(f)(x) - R_\ell^{\mathcal{L},\eta_{j+1}}(f)(x)\right|^\rho\right)^{\frac{1}{\rho}} \\
&\quad\leq \left(\sum_{j=0}^{\infty} \left|\int_{\eta_{j+1}<|x-y|<\eta_j,\, |x-y|<\gamma(x)} \left(R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)\right) f(y) dy\right|^\rho\right)^{\frac{1}{\rho}} \\
&\qquad + \left(\sum_{j=0}^{\infty} \left|\int_{\eta_{j+1}<|x-y|<\eta_j,\, |x-y|>\gamma(x)} R_\ell^{\mathcal{L}}(x,y) f(y) dy\right|^\rho\right)^{\frac{1}{\rho}} \\
&\qquad + \left(\sum_{j=0}^{\infty} \left|\int_{\eta_{j+1}<|x-y|<\eta_j,\, |x-y|<\gamma(x)} R_\ell(x-y) f(y) dy\right|^\rho\right)^{\frac{1}{\rho}} \\
&\quad\leq \sum_{j=0}^{\infty} \int_{\eta_{j+1}<|x-y|<\eta_j, |x-y|<\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)\right| \left|f(y)\right| dy \\
&\qquad + \sum_{j=0}^{\infty} \int_{\eta_{j+1}<|x-y|<\eta_j,\, |x-y|>\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y)\right| \left|f(y)\right| dy \\
&\qquad + \left(\sum_{j=0}^{\infty} \left|\int_{\eta_{j+1}<|x-y|<\eta_j,\, |x-y|<\gamma(x)} R_\ell(x-y) f(y) dy\right|^\rho\right)^{\frac{1}{\rho}} \\
&\quad\leq \int_{|x-y|<\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)\right| \left|f(y)\right| dy \\
&\qquad + \int_{|x-y|>\gamma(x)} \left|R_\ell^{\mathcal{L}}(x,y)\right| \left|f(y)\right| dy \\
&\qquad + \sup_{\{t_j\}_{j\in\mathbb{N}}\searrow 0} \left(\sum_{j=0}^{\infty} \left|\int_{t_{j+1}<|x-y|<t_j} R_\ell(x-y) f(y) dy\right|^\rho\right)^{\frac{1}{\rho}}.
\end{aligned}$$

Hence, we get

$$V_\rho\big(R_\ell^{\mathcal{L},\varepsilon}\big)(f)(x) \le \int_{|x-y|<\gamma(x)} \big|R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)\big|\big|f(y)\big|dy$$
$$+ \int_{|x-y|>\gamma(x)} \big|R_\ell^{\mathcal{L}}(x,y)\big|\big|f(y)\big|dy + V_\rho\big(R_\ell^{\varepsilon}\big)(f)(x)$$
$$= \tau_1\big(|f|\big)(x) + \tau_2\big(|f|\big)(x) + V_\rho\big(R_\ell^{\varepsilon}\big)(f)(x), \quad x \in \mathbb{R}^n. \quad (48)$$

Note that the operators τ_1 and τ_2 are the same ones that appeared in the proof of Proposition 4.1. Then, as it was proved there, τ_1 and τ_2 are bounded from $L^r(\mathbb{R}^n)$ into itself provided that $1 \le r < p_0$.

By (48) and [7, Theorem A and Corollary 1.4] we conclude the desired L^p-boundedness properties for $V_\rho(R_\ell^{\mathcal{L},\varepsilon})$. □

5 Variation operators associated with commutators $C_b^{\mathcal{L}}$

Proposition 1.2 and Theorem 1.3 are proved in this section. Assume that $b \in BMO_\theta(\gamma)$, for some $\theta \ge 0$. Let us remind that, for every $\ell = 1, \dots, n$, the commutator operator $C_{b,\ell}^{\mathcal{L}}$ for the Riesz transform $R_\ell^{\mathcal{L}}$ is given by

$$C_{b,\ell}^{\mathcal{L}} f = bR_\ell^{\mathcal{L}} f - R_\ell^{\mathcal{L}}(bf), \quad f \in C_c^\infty\big(\mathbb{R}^n\big).$$

Some L^p-boundedness results for these operators were established in [3, Theorem 1].

From Proposition 1.1 we deduce that, for every $f \in C_c^\infty(\mathbb{R}^n)$,

$$C_{b,\ell}^{\mathcal{L}}(f)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \big(b(x) - b(y)\big) R_\ell^{\mathcal{L}}(x,y) f(y) dy, \quad x \in \mathbb{R}^n.$$

In Proposition 1.2, that is proved in the following, we extend this property for every $f \in L^p(\mathbb{R}^n)$, $1 < p < p_0$, where $\frac{1}{p_0} = (\frac{1}{q} - \frac{1}{n})_+$ and $V \in B_q$, $q \ge n/2$.

Proof of Proposition 1.2 It is enough to prove that the maximal operator $C_{b,\ell}^{\mathcal{L},*}$ defined by

$$C_{b,\ell}^{\mathcal{L},*}(f)(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \big(b(x) - b(y)\big) R_\ell^{\mathcal{L}}(x,y) f(y) dy \right|, \quad x \in \mathbb{R}^n,$$

is bounded from $L^p(\mathbb{R}^n)$ into itself when V and p satisfy the conditions specified in this proposition.

As in the proof of Theorem 1.2 it suffices to take care of the case $n/2 < q < n$. Suppose that $f \in L^p(\mathbb{R}^n)$, where $1 < p < p_0$.

Let us consider the local operators

$$C_{b,\ell}^{\mathcal{L},*,\mathrm{loc}}(f)(x) = \sup_{\varepsilon>0} \left| \int_{\varepsilon<|x-y|<\gamma(x)} \big(b(x) - b(y)\big) R_\ell^{\mathcal{L}}(x,y) f(y) dy \right|, \quad x \in \mathbb{R}^n,$$

and

$$C_{b,\ell}^{*,\mathrm{loc}}(f)(x)=\sup_{\varepsilon>0}\left|\int_{\varepsilon<|x-y|<\gamma(x)}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy\right|,\quad x\in\mathbb{R}^n.$$

With this notation we have that

$$\begin{aligned}C_{b,\ell}^{\mathcal{L},*}(f)(x)&=C_{b,\ell}^{\mathcal{L},*,\mathrm{loc}}(f)(x)-C_{b,\ell}^{*,\mathrm{loc}}(f)(x)+C_{b,\ell}^{\mathcal{L},*}(f)(x)-C_{b,\ell}^{\mathcal{L},*,\mathrm{loc}}(f)(x)\\&\quad+C_{b,\ell}^{*,\mathrm{loc}}(f)(x)\\&\leq\int_{|x-y|<\gamma(x)}\big|b(x)-b(y)\big|\big|R_\ell^{\mathcal{L}}(x,y)-R_\ell(x-y)\big|f(y)\big|dy\\&\quad+\int_{|x-y|>\gamma(x)}\big|b(x)-b(y)\big|\big|R_\ell^{\mathcal{L}}(x,y)\big|f(y)\big|dy+C_{b,\ell}^{*,\mathrm{loc}}(f)(x)\\&=T_1\big(|f|\big)(x)+T_2\big(|f|\big)(x)+C_{b,\ell}^{*,\mathrm{loc}}(f)(x),\quad x\in\mathbb{R}^n.\end{aligned}\tag{49}$$

According to Proposition 2.1 the operator $C_{b,\ell}^{*,\mathrm{loc}}$ is bounded from $L^r(\mathbb{R}^n)$ into itself, for every $1<r<\infty$.

We now analyze the L^p-boundedness properties for the operators T_1 and T_2 studying the behavior of their adjoints T_1^* and T_2^*.

The operator T_1^* adjoint of T_1 is defined by

$$T_1^*(g)(y)=\int_{|x-y|<\gamma(x)}\big|b(y)-b(x)\big|\big|R_\ell^{\mathcal{L}}(x,y)-R_\ell(x-y)\big|g(x)dx.$$

According to (33), since $\gamma(x)\sim\gamma(y)$ when $|x-y|\leq\gamma(x)$ there exists $A>0$ for which

$$\begin{aligned}&\big|T_1^*(g)(y)\big|\\&\leq C\left(\int_{|x-y|<A\gamma(y)}\frac{1}{|x-y|^n}\left(\frac{|x-y|}{\gamma(y)}\right)^{2-n/q}\big|b(y)-b(x)\big|\big|g(x)\big|dx\right.\\&\quad\left.+\int_{|x-y|<A\gamma(y)}\frac{1}{|x-y|^{n-1}}\int_{B(x,\frac{|x-y|}{4})}\frac{V(z)}{|z-x|^{n-1}}dz\big|b(y)-b(x)\big|\big|g(x)\big|dx\right)\\&=C\big(T_{1,1}^*\big(|g|\big)(y)+T_{1,2}^*\big(|g|\big)(y)\big),\quad y\in\mathbb{R}^n.\end{aligned}\tag{50}$$

We have that

$$\begin{aligned}&T_{1,1}^*\big(|g|\big)(y)\\&\leq C\sum_{j=0}^{\infty}\frac{2^{-j\delta}}{(2^{-j}\gamma(y))^n}\int_{2^{-j-1}A\gamma(y)\leq|x-y|\leq2^{-j}A\gamma(y)}\big|b(x)-b(y)\big|\big|g(x)\big|dx,\\&y\in\mathbb{R}^n,\end{aligned}$$

where $\delta=2-\frac{n}{q}>0$.

Then,

$$\left\|T^*_{1,1}\big(|g|\big)\right\|_{L^{p'}(\mathbb{R}^n)} \leq C \sum_{j=0}^{\infty} \left\|T^*_{1,1,j}(f)\right\|_{L^{p'}(\mathbb{R}^n)}, \tag{51}$$

where, for every $j \in \mathbb{N}$,

$$T^*_{1,1,j}(g)(y) = \frac{2^{-j\delta}}{(2^{-j}\gamma(y))^n} \int_{|x-y|\leq 2^{-j}A\gamma(y)} \big|b(x)-b(y)\big|\big|g(x)\big|dx, \quad y \in \mathbb{R}^n.$$

To deal with these operators we consider the covering $\{Q_k\}_{k\in\mathbb{N}}$ as given in Sect. 2. We know that there exists $C > 0$ such that $\gamma(y) \leq C\gamma(x_k)$, for every $y \in 2Q_k$. We choose the smaller $L \in \mathbb{N}$ such that $AC + 1 \leq 2^L$. It is not hard to see that we can find $M \in \mathbb{N}$ such that, for every $k, j \in \mathbb{N}$, there exist $N_j \in \mathbb{N}$ and $x^i_{k,j}$, $i = 1, \ldots, N_j$, such that, by denoting $Q^i_{k,j} = B(x^i_{k,j}, 2^{-j}\gamma(x_k))$, $i = 1, \ldots, N_j$, the following properties hold:

(i) $Q_k \subset \bigcup_{i=1}^{N_j} Q^i_{k,j} \subset 2Q_k$;
(ii) $\text{card}\,\{l \in \mathbb{N} : 2^L Q^i_{k,j} \cap 2^L Q^l_{k,j} \neq \varnothing\} \leq M$, $i = 1, \ldots, N_j$.

Clearly, we have that $B(y, 2^{-j}A\gamma(y)) \subset 2^L Q^i_{k,j} = \widetilde{Q^i_{k,j}}$, when $y \in Q^i_{k,j}$, $j, k \in \mathbb{N}$ and $i = 1, \ldots, N_j$.

We can write, for every $j \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \big|T^*_{1,1,j}(g)(y)\big|^{p'} dy \\
&\quad \leq C \sum_{k=0}^{\infty} \int_{Q_k} \left(\frac{2^{-j\delta}}{(2^{-j}\gamma(y))^n} \int_{|x-y|\leq 2^{-j}A\gamma(y)} \big|g(x)\big|\big|b(x)-b(y)\big|dx \right)^{p'} dy \\
&\quad \leq C \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \int_{Q^i_{k,j}} \left(\frac{2^{-j\delta}}{(2^{-j}\gamma(x_k))^n} \int_{\widetilde{Q^i_{k,j}}} \big|g(x)\big|\big|b(x)-b(y)\big|dx \right)^{p'} dy \\
&\quad \leq C \left(\sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{np'}} \int_{Q^i_{k,j}} \big|b(y) - b_{Q^i_{k,j}}\big|^{p'} \left(\int_{\widetilde{Q^i_{k,j}}} \big|g(x)\big|dx \right)^{p'} dy \right. \\
&\qquad \left. + \sum_{k=0}^{\infty} \sum_{i=1}^{N_j} \frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{np'}} \int_{Q^i_{k,j}} \left(\int_{\widetilde{Q^i_{k,j}}} \big|b(x) - b_{Q^i_{k,j}}\big|\big|g(x)\big|dx \right)^{p'} dy \right).
\end{aligned} \tag{52}$$

Each summand is estimated separately. For the first one, since $\gamma(x^i_{k,j}) \sim \gamma(x_k)$, by [3, Proposition 3], we have that

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{np'}}\int_{Q_{k,j}^i}\left|b(y)-b_{Q_{k,j}^i}\right|^{p'}\left(\int_{\widetilde{Q_{k,j}^i}}|g(x)|dx\right)^{p'}dy\right)^{1/p'}$$

$$\leq C\sum_{j=0}^{\infty}2^{-j\delta}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{1}{(2^{-j}\gamma(x_k))^{np'}}\int_{Q_{k,j}^i}\left|b(y)-b_{Q_{k,j}^i}\right|^{p'}dy\right.$$

$$\left.\times\int_{\widetilde{Q_{k,j}^i}}\left|g(x)\right|^{p'}dx\left(2^{-j}\gamma(x_k)\right)^{np'/p}\right)^{1/p'}$$

$$\leq C\sum_{j=0}^{\infty}2^{-j\delta}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{1}{(2^{-j}\gamma(x_k))^{n}}\int_{Q_{k,j}^i}\left|b(y)-b_{Q_{k,j}^i}\right|^{p'}dy\right.$$

$$\left.\times\int_{\widetilde{Q_{k,j}^i}}\left|g(x)\right|^{p'}dx\right)^{1/p'}$$

$$\leq C\|b\|_{BMO_\theta(\gamma)}\sum_{j=0}^{\infty}2^{-j\delta}\left(\sum_{k=0}^{\infty}\int_{2^{L+1}Q_k}\left|g(x)\right|^{p'}dx\right)^{1/p'}$$

$$\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)}. \tag{53}$$

Also, by using again [3, Proposition 3],

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{np'}}\int_{Q_{k,j}^i}\left(\int_{\widetilde{Q_{k,j}^i}}\left|b(x)-b_{Q_{k,j}^i}\right||g(x)|dx\right)^{p'}dy\right)^{1/p'}$$

$$\leq C\sum_{j=0}^{\infty}2^{-j\delta}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{1}{(2^{-j}\gamma(x_k))^{np'-n}}\left(\int_{Q_{k,j}^i}\left|b(x)-b_{Q_{k,j}^i}\right|^{p}dx\right)^{p'/p}\right.$$

$$\left.\times\int_{\widetilde{Q_{k,j}^i}}\left|g(x)\right|^{p'}dx\right)^{1/p'}$$

$$\leq C\|b\|_{BMO_\theta(\gamma)}\sum_{j=0}^{\infty}2^{-j\delta}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\int_{\widetilde{Q_{k,j}^i}}\left|g(x)\right|^{p'}dx\right)^{1/p'}$$

$$\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)}. \tag{54}$$

By combining (51), (52), (53) and (54), we obtain

$$\left\|T_{1,1}^*(|g|)\right\|_{L^{p'}(\mathbb{R}^n)}\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)}. \tag{55}$$

On the other hand, we have

$$\begin{aligned}\big|T^*_{1,2}(g)(y)\big| &\leq \sum_{j=0}^{\infty}\int_{2^{-j-1}A\gamma(y)\leq|x-y|<2^{-j}A\gamma(y)} \big|g(x)\big|\frac{|b(y)-b(x)|}{|x-y|^{n-1}}\\ &\quad\times\int_{B(x,\frac{|x-y|}{4})}\frac{V(z)}{|z-x|^{n-1}}dzdx\\ &\leq C\sum_{j=0}^{\infty}\frac{1}{(2^{-j}\gamma(y))^{n-1}}\int_{|x-y|\leq 2^{-j}A\gamma(y)}\int_{B(y,2^{-j+1}A\gamma(y))}\frac{V(z)}{|z-x|^{n-1}}dz\\ &\quad\times\big|b(y)-b(x)\big|\big|g(x)\big|dx. \end{aligned}\tag{56}$$

Since $\frac{1}{p_0}=\frac{1}{q}-\frac{1}{n}$, by using [2, Lemma 1], we get

$$\begin{aligned}&\big\|I_1(\chi_{B(y,2^{-j+1}A\gamma(y))}V)\big\|_{L^{p_0}(\mathbb{R}^n)}\\ &\quad=\left(\int_{\mathbb{R}^n}\left|\int_{B(y,2^{-j+1}A\gamma(y))}\frac{V(z)}{|z-x|^{n-1}}dz\right|^{p_0}dx\right)^{1/p_0}\\ &\quad\leq C\left(\int_{B(y,2^{-j+1}A\gamma(y))}\big|V(z)\big|^q dz\right)^{1/q}\\ &\quad\leq C\big(2^{-j}\gamma(y)\big)^{n(-1+1/q)}\int_{B(y,2^{-j+1}A\gamma(y))}V(z)dz\\ &\quad\leq C\big(2^{-j}\gamma(y)\big)^{n(-1+1/q)}2^{-j(2-n/q)}\big(2^{-j}\gamma(y)\big)^{n-2}\\ &\quad\leq C\gamma(y)^{-2+n/q},\quad y\in\mathbb{R}^n.\end{aligned}$$

Then, Hölder inequality implies that

$$\begin{aligned}&\big|T^*_{1,2}(g)(y)\big|\\ &\quad\leq C\sum_{j=0}^{\infty}\frac{1}{(2^{-j}\gamma(y))^{n-1}}\left(\int_{|x-y|\leq 2^{-j}A\gamma(y)}\big(\big|b(y)-b(x)\big|\big|g(x)\big|\big)^{p_0'}dx\right)^{1/p_0'}\\ &\qquad\times\big\|I_1(\chi_{B(y,2^{-j+1}A\gamma(y))}V)\big\|_{L^{p_0}(\mathbb{R}^n)}\\ &\quad\leq C\sum_{j=0}^{\infty}\frac{1}{(2^{-j}\gamma(y))^{n-1}}\\ &\qquad\times\left(\int_{|x-y|\leq 2^{-j}A\gamma(y)}\big(\big|b(x)-b(y)\big|\big|g(x)\big|\big)^{p_0'}dx\right)^{1/p_0'}\gamma(y)^{n/q-2},\quad y\in\mathbb{R}^n.\end{aligned}$$

We can write,

$$\left\|T^*_{1,2}(g)\right\|_{L^{p'}(\mathbb{R}^n)} \leq C\sum_{j=0}^{\infty}\left\|T^*_{1,2,j}(g)\right\|_{L^{p'}(\mathbb{R}^n)}, \tag{57}$$

where, for every $j\in\mathbb{N}$,

$$T^*_{1,2,j}(g)(y)=\frac{\gamma(y)^{n/q-2}}{(2^{-j}\gamma(y))^{n-1}}\left(\int_{|x-y|\leq 2^{-j}A\gamma(y)}\left(\left|b(x)-b(y)\right|\left|g(x)\right|\right)^{p'_0}dx\right)^{1/p'_0},$$

$y\in\mathbb{R}^n$.

As before we have, for every $j\in\mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}^n}\left|T^*_{1,2,j}(g)(y)\right|^{p'}dy \leq C\Bigg(&\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{(n-n/q+1)p'}}\\ &\times\int_{Q^i_{k,j}}\left|b(y)-b_{Q^i_{k,j}}\right|^{p'}\left(\int_{\widetilde{Q^i_{k,j}}}\left|g(x)\right|^{p'_0}dx\right)^{p'/p'_0}dy\\ &+\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{(n-n/q+1)p'}}\\ &\times\int_{Q^i_{k,j}}\left(\int_{\widetilde{Q^i_{k,j}}}\left(\left|b(y)-b_{Q^i_{k,j}}\right|\left|g(y)\right|\right)^{p'_0}dy\right)^{p'/p'_0}dx\Bigg),\end{aligned} \tag{58}$$

where $\delta=2-n/q$. Here Q_k, $k\in\mathbb{N}$, and $Q^i_{k,j}$, and $\widetilde{Q}^i_{k,j}$ $k,j\in\mathbb{N}$, $i=1,\dots,N_j$, are the same balls that we considered above.

Also, since $p'>p'_0$, we get

$$\begin{aligned}&\sum_{j=0}^{\infty}\Bigg(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{(n/q'+1)p'}}\\ &\qquad\times\int_{Q^i_{k,j}}\left|b(y)-b_{Q^i_{k,j}}\right|^{p'}\left(\int_{\widetilde{Q^i_{k,j}}}\left|g(x)\right|^{p'_0}dx\right)^{p'/p'_0}dy\Bigg)^{1/p'}\\ &\leq C\sum_{j=0}^{\infty}2^{-j\delta}\Bigg(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{1}{(2^{-j}\gamma(x_k))^{(n/q'+1)p'}}\int_{Q^i_{k,j}}\left|b(y)-b_{Q^i_{k,j}}\right|^{p'}dy\\ &\qquad\times\int_{\widetilde{Q^i_{k,j}}}\left|g(x)\right|^{p'}dx\left(2^{-j}\gamma(x_k)\right)^{\frac{np'}{p'_0}(1-\frac{p'_0}{p'})}\Bigg)^{1/p'}\\ &\leq C\|b\|_{BMO_\theta(\gamma)}\sum_{j=0}^{\infty}2^{-j\delta}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\int_{\widetilde{Q^i_{k,j}}}\left|g(x)\right|^{p'}dx\right)^{1/p'}\\ &\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)},\end{aligned} \tag{59}$$

and

$$\sum_{j=0}^{\infty}\Bigg(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{2^{-j\delta p'}}{(2^{-j}\gamma(x_k))^{(n/q'+1)p'}}$$

$$\times\int_{Q_{k,j}^i}\left(\int_{\widetilde{Q_{k,j}^i}}\left(\left|b(x)-b_{Q_{k,j}^i}\right|\left|g(x)\right|\right)^{p_0'}dx\right)^{p'/p_0'}dy\Bigg)^{1/p'}$$

$$\leq C\sum_{j=0}^{\infty}2^{-j\delta}\Bigg(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\frac{1}{(2^{-j}\gamma(x_k))^{(n/q'+1)p'-n}}$$

$$\times\left(\int_{\widetilde{Q_{k,j}^i}}\left|b(x)-b_{Q_{k,j}^i}\right|^{p_0'p'/(p'-p_0')}dx\right)^{(p'-p_0')/p_0'}\int_{\widetilde{Q_{k,j}^i}}\left|g(x)\right|^{p'}dx\Bigg)^{1/p'}$$

$$\leq C\|b\|_{BMO_\theta(\gamma)}\sum_{j=0}^{\infty}2^{-j\delta}\left(\sum_{k=0}^{\infty}\sum_{i=1}^{N_j}\int_{\widetilde{Q}_{k,j}^i}\left|g(x)\right|^{p'}dx\right)^{1/p'}$$

$$\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)}. \tag{60}$$

From (57), (58), (59) and (60) we conclude that

$$\left\|T_{1,2}^*(g)\right\|_{L^{p'}(\mathbb{R}^n)}\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)},\quad g\in L^{p'}\left(\mathbb{R}^n\right). \tag{61}$$

By invoking (50), (55), and (61) it follows that

$$\left\|T_1^*g\right\|_{L^{p'}(\mathbb{R}^n)}\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)},\quad g\in L^{p'}\left(\mathbb{R}^n\right).$$

Then, T_1 is a bounded operator from $L^p(\mathbb{R}^n)$ into itself.

The operator T_2^* adjoint of T_2 is defined by

$$T_2^*(g)(y)=\int_{|x-y|\geq\gamma(x)}\left|b(x)-b(y)\right|\left|R_\ell^{\mathcal{L}}(x,y)\right|g(x)dx,\quad y\in\mathbb{R}^n.$$

According to (32), since for a certain $A>0$, $|x-y|\geq A\gamma(y)$, when $|x-y|\geq\gamma(x)$, we have that

$$\left|T_2^*(g)(y)\right|$$

$$\leq C_\alpha\Bigg(\int_{|x-y|\geq A\gamma(y)}\frac{1}{|x-y|^n}\frac{1}{(1+\frac{|x-y|}{\gamma(y)})^\alpha}\left|b(x)-b(y)\right|\left|g(x)\right|dx$$

$$+\int_{|x-y|\geq A\gamma(y)}\frac{1}{|x-y|^{n-1}}\frac{|b(x)-b(y)||g(x)|}{(1+\frac{|x-y|}{\gamma(y)})^\alpha}\int_{B(x,\frac{|x-y|}{4})}\frac{V(z)}{|z-x|^{n-1}}dzdx\Bigg)$$

$$=C_\alpha\left(T_{2,1}^*\left(|g|\right)(y)+T_{2,2}^*\left(|g|\right)(y)\right),\quad y\in\mathbb{R}^n. \tag{62}$$

Here $\alpha>0$ will be sufficiently large and it will be fixed later.

By choosing $\alpha > \theta' + 1$, where θ' is the one appearing in Lemma 2.1, we can prove that

$$\begin{aligned}\left|T^*_{2,1}\big(|g|\big)(y)\right| &\leq C\sum_{m=0}^{\infty}\int_{2^m A\gamma(y)<|x-y|\leq 2^{m+1}A\gamma(y)}\left|b(x)-b(y)\right|\\ &\quad\times\frac{1}{|x-y|^n}\left(1+\frac{|x-y|}{\gamma(y)}\right)^{-(\theta'+1)}\left|g(x)\right|dx\\ &\leq C\sum_{m=0}^{\infty}\frac{1}{2^{m(\theta'+1)}}\frac{1}{(2^m\gamma(y))^n}\int_{|x-y|\leq 2^{m+1}A\gamma(y)}\left|b(x)-b(y)\right|\left|g(x)\right|dx\\ &\leq C\sum_{m=0}^{\infty}\frac{1}{2^{m(\theta'+1+n)}}T_{b,2^{m+1}A}\big(|g|\big)(y),\quad y\in\mathbb{R}^n,\end{aligned}$$

where the operators $T_{b,M}$ are the ones introduced in Lemma 2.1.

Then, by using Lemma 2.1 we get

$$\begin{aligned}\left\|T^*_{2,1}\big(|g|\big)\right\|_{L^{p'}(\mathbb{R}^n)} &\leq C\sum_{m=0}^{\infty}\frac{1}{2^{m(\theta'+1+n)}}\left\|T_{b,2^{m+1}A}\big(|g|\big)\right\|_{L^{p'}(\mathbb{R}^n)}\\ &\leq C\|b\|_{BMO_\theta(\gamma)}\|f\|_{L^{p'}(\mathbb{R}^n)},\quad f\in L_{p'}\big(\mathbb{R}^n\big).\end{aligned}\tag{63}$$

On the other hand, since $\gamma(y)\sim\gamma(x_k)$, $y\in Q_k$ and $k\in\mathbb{N}$, we have, for a certain $\beta>0$,

$$\begin{aligned}\left\|T^*_{2,2}g\right\|^{p'}_{L^{p'}(\mathbb{R}^n)} &\leq C\sum_{k=0}^{\infty}\int_{Q_k}\left(\int_{|x-y|\geq\beta\gamma(x_k)}\frac{|b(x)-b(y)||g(x)|}{|x-y|^{n-1}}\right.\\ &\quad\left.\times\frac{1}{(1+\frac{|x-y|}{\gamma(x_k)})^{\alpha}}\int_{B(x,\frac{|x-y|}{4})}\frac{V(z)dz}{|z-x|^{n-1}}dx\right)^{p'}dy\\ &=\sum_{k=0}^{\infty}\int_{Q_k}J_k(y)^{p'}dy.\end{aligned}\tag{64}$$

Let $k\in\mathbb{N}$. It follows that

$$\begin{aligned}J_k(y) &\leq C\sum_{j=0}^{\infty}\int_{2^j\beta\gamma(x_k)<|x-y|\leq 2^{j+1}\beta\gamma(x_k)}\frac{|b(x)-b(y)||g(x)|}{(2^j\gamma(x_k))^{n-1}}\\ &\quad\times\frac{1}{2^{j\alpha}}\int_{B(y,\beta 2^{j+2}\gamma(x_k))}\frac{V(z)dz}{|z-x|^{n-1}}dx\\ &\leq C\sum_{j=0}^{\infty}\frac{1}{2^{j\alpha}(2^j\gamma(x_k))^{n-1}}\left(\int_{\widetilde{Q}^k_j}\big(\left|b(x)-b(y)\right|\left|g(x)\right|\big)^{p'_0}dx\right)^{1/p'_0}\\ &\quad\times\left(\int_{\mathbb{R}^n}\left(\int_{\widetilde{Q}^k_j}\frac{V(z)dz}{|z-x|^{n-1}}\right)^{p_0}dx\right)^{1/p_0},\quad y\in Q_k,\end{aligned}$$

where p_0 is such that $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, and for every $j \in \mathbb{N}$ we consider $\widetilde{Q}_j^k = B(x_k, c2^j\gamma(x_k))$, where $c > 0$ is independent of $j, k \in \mathbb{N}$ and such that $B(y, \beta 2^{j+2}\gamma(x_k)) \subset \widetilde{Q}_j^k$, $y \in Q_k$.

Since the 1-th Euclidean fractional integral is bounded from $L^q(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$ we deduce

$$J_k(y) \leq C \sum_{j=0}^{\infty} \frac{2^{-j\alpha}}{(2^j\gamma(x_k))^{n-1}} \left(\int_{\widetilde{Q}_j^k} \left(\left|b(x) - b(y)\right| \left|g(x)\right| \right)^{p_0'} dx \right)^{1/p_0'} \times \left(\int_{\widetilde{Q}_j^k} V(z)^q dz \right)^{1/q}, \quad y \in Q_k.$$

Moreover, by using the doubling property of V it follows that, for some $\mu > 0$,

$$\left(\int_{\widetilde{Q}_j^k} V(z)^q dz \right)^{1/q} \leq C \left(2^j\gamma(x_k)\right)^{-n/q'} 2^{j\mu} \int_{Q_k} V(z) dz \leq C \left(2^j\gamma(x_k)\right)^{-n/q'} 2^{j\mu} \gamma(x_k)^{n-2}, \quad y \in Q_k,$$

where in the last inequality we just use the definition of γ.

Then,

$$J_k(y) \leq C \sum_{j=0}^{\infty} \frac{2^{j(\mu-\alpha-n+1-n/q)}}{\gamma(x_k)^{1+n/q'}} \left(\int_{\widetilde{Q}_j^k} \left(\left|b(x) - b(y)\right| \left|g(x)\right| \right)^{p_0'} dx \right)^{1/p_0'}, \quad y \in Q_k.$$

Since $p' > p_0'$, calling $\nu = p_0'(p'/p_0')'$, Hölder inequality and [3, Proposition 3] imply that, for some $\theta' > \theta$,

$$\begin{aligned} J_k(y) &\leq C \sum_{j=0}^{\infty} \frac{2^{j(\mu-\alpha-n+1-n/q)}}{\gamma(x_k)^{1+n/q'}} \left(\int_{\widetilde{Q}_j^k} \left|b(x) - b(y)\right|^{\nu} dx \right)^{\frac{1}{\nu}} \left(\int_{\widetilde{Q}_j^k} \left|g(x)\right|^{p'} dx \right)^{1/p'} \\ &\leq C \sum_{j=0}^{\infty} \frac{2^{j(\mu-\alpha-n+1-n/q)}}{\gamma(x_k)^{1+n/q'}} \left\{ \left(\int_{\widetilde{Q}_j^k} \left|b(x) - b_{Q_k}\right|^{\nu} dx \right)^{\frac{1}{\nu}} \right. \\ &\quad \left. + \left(\int_{\widetilde{Q}_j^k} \left|b(y) - b_{Q_k}\right|^{\nu} dx \right)^{\frac{1}{\nu}} \right\} \left(\int_{\widetilde{Q}_j^k} \left|g(x)\right|^{p'} dx \right)^{1/p'} \\ &\leq C \sum_{j=0}^{\infty} \gamma(x_k)^{-n/p'} 2^{-j(n-2+\alpha-\mu+n/p')} \\ &\quad \times \left((j+1) 2^{j\theta'} \|b\|_{BMO_\theta(\gamma)} + \left|b(y) - b_{Q_k}\right| \right) \\ &\quad \times \left(\int_{\widetilde{Q}_j^k} \left|g(x)\right|^{p'} dx \right)^{1/p'}, \quad y \in Q_k. \end{aligned}$$

Then, by taking into account the properties of the sequence $\{Q_k\}_{k\in\mathbb{N}}$ and Minkowski inequality, we can choose α large enough such that

$$\begin{aligned}
&\left(\sum_{k=0}^{\infty}\int_{Q_k} J_k(y)^{p'}dy\right)^{1/p'}\\
&\quad\leq C\|b\|_{BMO_\theta(\gamma)}\left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}2^{-j(n-2+\alpha-\mu+\theta'+\frac{n}{p'})}(j+1)\right.\right.\\
&\qquad\left.\left.\times\left(\int_{\widetilde{Q}_j^k}\big|g(x)\big|^{p'}dx\right)^{1/p'}\right)^{p'}\right)^{1/p'}\\
&\quad\leq C\|b\|_{BMO_\theta(\gamma)}\sum_{j=0}^{\infty}2^{-j-j(n-2+\alpha-\mu+\theta'+\frac{n}{p'})}(j+1)\left(\sum_{k=0}^{\infty}\int_{\widetilde{Q}_j^k}\big|g(x)\big|^{p'}dx\right)^{1/p'}\\
&\quad\leq C\|b\|_{BMO_\theta(\gamma)}\|g\|_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, we prove that

$$\big\|T_{2,2}^*(g)\big\|_{L^{p'}(\mathbb{R}^n)}\leq C\|g\|_{L^{p'}(\mathbb{R}^n)},\quad g\in L^{p'}\big(\mathbb{R}^n\big). \tag{65}$$

By combining (62), (63) and (65) we obtain that

$$\big\|T_2^*(g)\big\|_{L^{p'}(\mathbb{R}^n)}\leq C\|g\|_{L^{p'}(\mathbb{R}^n)},\quad g\in L^{p'}\big(\mathbb{R}^n\big).$$

Hence, the operator T_2 is bounded from $L^p(\mathbb{R}^n)$ into itself.

Finally, the L^p-boundedness of T_1 and T_2 allows us to conclude that the operator $C_{b,\ell}^{\mathcal{L},*}$ is bounded from $L^p(\mathbb{R}^n)$ into itself. □

We now prove the L^p-boundedness properties of the variation operators associated with $\{C_{b,\ell}^{\mathcal{L},\varepsilon}\}_{\varepsilon>0}$ that are established in Theorem 1.3.

Proof of Theorem 1.3 Assume that $\rho>2$. We consider the operators

$$C_{b,\ell}^{\mathcal{L},\mathrm{loc}}(f)(x)=\lim_{\varepsilon\to0^+}\int_{\varepsilon<|x-y|<\gamma(x)}\big(b(x)-b(y)\big)R_\ell^{\mathcal{L}}(x,y)f(y)dy,$$

and

$$C_{b,\ell}^{\mathrm{loc}}(f)(x)=\lim_{\varepsilon\to0^+}\int_{\varepsilon<|x-y|<\gamma(x)}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy,$$

and we define the truncations $C_{b,\ell}^{\mathcal{L},\varepsilon,\mathrm{loc}}(f)$ and $C_{b,\ell}^{\varepsilon,\mathrm{loc}}(f)$, $\varepsilon>0$, in the usual way.

If $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero, we can write

$$\begin{aligned}
&\left(\sum_{j=0}^{\infty}\left|C_{b,\ell}^{\mathcal{L},\varepsilon_j}(f)(x)-C_{b,\ell}^{\mathcal{L},\varepsilon_{j+1}}(f)(x)\right|^{\rho}\right)^{1/\rho}\\
&\quad\leq\left(\sum_{j=0}^{\infty}\left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j,|x-y|<\gamma(x)}\big(b(x)-b(y)\big)\right.\right.\\
&\qquad\left.\left.\times\big(R_\ell^{\mathcal{L}}(x,y)-R_\ell(x-y)\big)f(y)dy\right|^{\rho}\right)^{1/\rho}\\
&\qquad+\left(\sum_{j=0}^{\infty}\left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j,|x-y|\geq\gamma(x)}\big(b(x)-b(y)\big)R_\ell^{\mathcal{L}}(x,y)f(y)dy\right|^{\rho}\right)^{1/\rho}\\
&\qquad+\left(\sum_{j=0}^{\infty}\left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j,|x-y|<\gamma(x)}\big(b(x)-b(y)\big)R_\ell(x-y)f(y)dy\right|^{\rho}\right)^{1/\rho}\\
&\quad\leq\int_{|x-y|<\gamma(x)}\big|b(x)-b(y)\big|\big|R_\ell^{\mathcal{L}}(x,y)-R_\ell(x-y)\big|\big|f(y)\big|dy\\
&\qquad+\int_{|x-y|\geq\gamma(x)}\big|b(x)-b(y)\big|\big|R_\ell^{\mathcal{L}}(x,y)\big|\big|f(y)\big|dy+V_\rho\big(C_{b,\ell}^{\varepsilon,\mathrm{loc}}\big)(f)(x).
\end{aligned}$$

Hence,

$$V_\rho\big(C_{b,\ell}^{\mathcal{L},\varepsilon}\big)(f)\leq T_1(f)+T_2(f)+V_\rho\big(C_{b,\ell}^{\varepsilon,\mathrm{loc}}\big)(f),$$

where the operator T_1 and T_2 are the ones defined in the proof of Proposition 1.2.

According to the L^p-boundedness properties of the operators T_1 and T_2 (see the proof of Proposition 1.2) and Proposition 2.1, we conclude that the variation operator $V_\rho(C_{b,\ell}^{\mathcal{L},\varepsilon})$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

□

Remark 5.1 In [25] (see also [1] and [2]) it is considered, for every $\ell=1,\dots,n$, the adjoint $(R_\ell^{\mathcal{L}})^*$ of $R_\ell^{\mathcal{L}}$, when $V\in B_q$ with $\frac{n}{2}<q<n$. By proceeding as in the previous results of this section we can prove the following properties.

Assume that $V\in B_q$, with $\frac{n}{2}<q$, $\ell=1,2,\dots,n$, and $p_0<p<\infty$, where $\frac{1}{p_0}=(\frac{1}{q}-\frac{1}{n})_+$. For every $f\in L^p(\mathbb{R}^n)$ there exists the limit

$$\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon}R_\ell^{\mathcal{L}}(y,x)f(y)dy,\quad\text{a.e. } x\in\mathbb{R}^n,$$

and defining the operator $\mathcal{R}_\ell^{\mathcal{L}}$ on $L^p(\mathbb{R}^n)$ as

$$\mathcal{R}_\ell^{\mathcal{L}}f(x)=\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon}R_\ell^{\mathcal{L}}(y,x)f(y)dy,\quad\text{a.e. } x\in\mathbb{R}^n,$$

$\mathcal{R}_\ell^{\mathcal{L}}$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

Moreover, if $\rho > 2$ the variation operator $V_\rho(\mathcal{R}_\ell^{\mathcal{L}})$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

Suppose that $b \in BMO_\theta(\gamma)$. For every $f \in L^p(\mathbb{R}^n)$, there exists the limit

$$\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\big(b(x)-b(y)\big)R_\ell^{\mathcal{L}}(y,x)f(y)dy, \quad \text{a.e. } x\in\mathbb{R}^n,$$

and the operator $\mathcal{C}_{b,\ell}^{\mathcal{L}}$ defined on $L^p(\mathbb{R}^n)$ by

$$\mathcal{C}_{b,\ell}^{\mathcal{L}}(f)(x)=\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\big(b(x)-b(y)\big)R_\ell^{\mathcal{L}}(y,x)f(y)dy, \quad \text{a.e. } x\in\mathbb{R}^n,$$

is bounded from $L^p(\mathbb{R}^n)$ into itself.

Moreover, if $\rho > 2$ the variation operator $V_\rho(\mathcal{C}_{b,\ell}^{\mathcal{L},\varepsilon})$ is bounded from $L^p(\mathbb{R}^n)$ into itself.

Remark 5.2 The fluctuations of a family $\{T_t\}_{t>0}$ of operators when $t\to 0^+$ also can be analyzed by using oscillation operators (see, for instance, [5] and [19]). If $\{t_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero, the oscillation operator $O(T_t;\{t_j\}_{j\in\mathbb{N}})$ is defined by

$$O\big(T_t;\{t_j\}_{j\in\mathbb{N}}\big)(f)(x)$$
$$=\left(\sum_{j=0}^{\infty}\sup_{t_{j+1}\le\varepsilon_{j+1}<\varepsilon_j\le t_j}\big|T_{\varepsilon_j}f(x)-T_{\varepsilon_{j+1}}f(x)\big|^2\right)^{1/2}, \quad f\in L^p\big(\mathbb{R}^n\big).$$

L^p-boundedness properties for the oscillation operators associated with the heat semigroup, Riesz transforms and commutators with the Riesz transforms in the Schrödinger setting can be established by using the procedures developed in this paper.

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