Potential Anal (2013) 38:711–739
DOI 10.1007/s11118-012-9294-9

Variation Operators for Semigroups and Riesz Transforms on *BMO* in the Schrödinger Setting

**Jorge J. Betancor · Juan C. Fariña ·
Eleonor Harboure · Lourdes Rodríguez-Mesa**

Received: 15 April 2011 / Accepted: 31 May 2012 / Published online: 17 June 2012

Abstract In this paper we prove that the variation operators of the heat semigroup and the truncations of Riesz transforms associated to the Schrödinger operator are bounded on a suitable BMO type space.

Keywords Schrödinger operators · Variation · Riesz transforms · Heat semigroups

Mathematics Subject Classifications (2010) 42B20 · 42B25 · 40A30

1 Introduction

We consider the Schrödinger operator $\mathscr{L}$ defined by $\mathscr{L} = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$. Here V is a nonnegative and not identically zero function satisfying, for some $q \geq n/2$, the following reverse Hölder inequality:

(RH_q) There exists $C > 0$ such that, for every ball $B \subset \mathbb{R}^n$,

$$\Big(\frac{1}{|B|}\int_B V(x)^q dx\Big)^{1/q} \leq C\frac{1}{|B|}\int_B V(x)dx.$$

This paper is partially supported by MTM2010/17974 and by Consejo Nacional de Investigaciones Científicas y Técnicas (Argentina), PIP 2303.

J. J. Betancor · J. C. Fariña · L. Rodríguez-Mesa (✉)
Departamento de Análisis Matemático, Universidad de la Laguna, Campus de Anchieta, Avda. Astrofísico Francisco Sánchez, s/n, 38271, La Laguna (Sta. Cruz de Tenerife), Spain
e-mail: lrguez@ull.es

J. J. Betancor
e-mail: jbetanco@ull.es

J. C. Fariña
e-mail: jcfarina@ull.es

E. Harboure
Instituto de Matemática Aplicada del Litoral, IMAL, CONICET-Universidad Nacional del Litoral, Güemes, 3450 Santa Fe, Argentina
e-mail: harbour@santafe-conicet.gov.ar

We write $V \in RH_q$ when V verifies the property (RH_q). Note that if V is a nonnegative polynomial, then $V \in RH_q$ for every $1 < q < \infty$. Also, if $V_\alpha(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, V_α belongs to RH_q provided that $\alpha q > -n$. Hence, $V_\alpha \in RH_{n/2}$ when $\alpha > -2$, and $V_\alpha \in RH_n$ if $\alpha > -1$.

Harmonic analysis operators derived from Schrödinger operator (Riesz transforms, maximal operators associated with heat and Poisson semigroups for $\mathscr{L}$, Littlewood–Paley g-functions, fractional integrals,...) have been extensively studied in last years. The papers of Shen [27] and Zhong [33] can be considered as starting points. In [27] and [33] Riesz transforms in the Schrödinger setting were studied on L^p-spaces. The behaviour on L^p of other operators related to $\mathscr{L}$ has been investigated on L^p-spaces in [5, 18, 24, 25, 29], amongst others.

Dziubański and Zienkiewicz introduced appropriate Hardy spaces associated with $\mathscr{L}$ (see [12, 13, 15]). A function $f \in L^1(\mathbb{R}^n)$ is said to be in $H_1^{\mathscr{L}}(\mathbb{R}^n)$ if and only if $W_*^{\mathscr{L}}(f) \in L^1(\mathbb{R}^n)$, where

$$W_*^{\mathscr{L}}(f) = \sup_{t>0} |W_t^{\mathscr{L}}(f)|,$$

and $W^{\mathscr{L}} = \{W_t^{\mathscr{L}}\}_{t>0}$ denotes the heat semigroup generated by $-\mathscr{L}$.

The dual space of $H_1^{\mathscr{L}}(\mathbb{R}^n)$ was investigated in [14]. This dual space, denoted by $BMO^{\mathscr{L}}(\mathbb{R}^n)$, was characterized as the natural space of bounded mean oscillation functions in this setting. More precisely, a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be in $BMO^{\mathscr{L}}(\mathbb{R}^n)$ provided that there exists $C > 0$ such that the following two properties are satisfied:

(i) For every $x \in \mathbb{R}^n$ and $r > 0$,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy \leq C,$$

where, as usual, $f_{B(x,r)} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy$, and $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$; and

(ii) For every $x \in \mathbb{R}^n$ and $r \geq \gamma(x)$,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy \leq C.$$

Here, for any $x \in \mathbb{R}^n$, the critical radius $\gamma(x)$ is defined by

$$\gamma(x) = \sup \left\{ r > 0 : r^{2-n} \int_{B(x,r)} V(y) dy \leq 1 \right\}.$$

Since V is not identically zero and $V \in RH_q$ with $q \geq n/2$, it follows that $0 < \gamma < \infty$. The norm $\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)}$ of $f \in BMO^{\mathscr{L}}(\mathbb{R}^n)$ is defined by

$$\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)} = \inf\{C > 0 : \text{(i) and (ii) hold}\}.$$

In [14] the behavior of certain maximal operators, Littlewood–Paley g functions and fractional integrals on $BMO^{\mathscr{L}}(\mathbb{R}^n)$ were studied. Also, the $BMO^{\mathscr{L}}(\mathbb{R}^n)$-boundedness of the Riesz transforms has been analyzed in [4, 11, 31, 32].

Suppose that $\{T_t\}_{t>0}$ is a family of operators defined for functions in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$. If $\rho > 2$, the ρ-variation operator associated with $\{T_t\}_{t>0}$, $V_\rho(T_t)$, is defined by

$$V_\rho(T_t)(f)(x) = \sup_{\{t_j\}_{j=1}^{\infty}\downarrow 0} \left(\sum_{j=1}^{\infty} |T_{t_j}(f)(x) - T_{t_{j+1}}(f)(x)|^\rho \right)^{1/\rho},$$

where the supremum is taken over all the real decreasing sequences $\{t_j\}_{j=1}^{\infty}$ that converge to zero. The operator $V_\rho(T_t)$ is related to the convergence of T_t, as $t \to 0^+$, and it estimates the fluctuations near the origin of the family $\{T_t\}_{t>0}$.

We consider the linear space E_ρ that consists of all those real functions F defined on $(0, \infty)$ such that

$$\|F\|_{E_\rho} = \sup_{\{t_j\}_{j=1}^{\infty}\downarrow 0} \left(\sum_{j=1}^{\infty} |F(t_j) - F(t_{j+1})|^\rho \right)^{\frac{1}{\rho}} < \infty,$$

where the supremum is taken over all the real decreasing sequence $\{t_j\}_{j=1}^{\infty}$ that converge to zero. $\|.\|_{E_\rho}$ is a seminorm on E_ρ. The variation operator $V_\rho(T_t)$ can be rewritten in the following way

$$V_\rho(T_t)(f)(x) = \|T_t(f)(x)\|_{E_\rho}.$$

The variation operator V_ρ was introduced in the ergodic context by Bourgain [6] (see also Jones et al. [21]). In last years many authors have investigated the variation operator associated to semigroups of operators and singular integrals [7–10, 17, 19, 20, 23]. Recently, Oberlin et al. [26] have analyzed the variation norm related to Carleson Theorem.

In a previous paper (see [2]) the authors studied the L^p-boundedness properties of the variation operators for the heat semigroup $\{W_t^{\mathcal{L}}\}_{t>0}$ and the family of truncated Riesz transforms $\{R_\ell^{\mathcal{L},\varepsilon}\}_{\varepsilon>0}$, $\ell = 1, \cdots, n$, in the Schrödinger context. Here our aim is to study the behavior of the variation operators $V_\rho(W_t^{\mathcal{L}})$ and $V_\rho(R_\ell^{\mathcal{L},\varepsilon})$ acting on functions in $BMO^{\mathcal{L}}(\mathbb{R}^n)$. Previously, we shall analyze the variation operators $V_\rho(W_t)$ and $V_\rho(R_\ell^\varepsilon)$ over the classical $BMO(\mathbb{R}^n)$, where $\{W_t\}_{t>0}$ and $\{R_\ell^\varepsilon\}_{\varepsilon>0}$, $\ell = 1, \ldots, n$, stand for the classical heat semigroup and truncated Riesz transforms, respectively. As usual, by $BMO(\mathbb{R}^n)$ we denote the well known space of bounded mean oscillation functions in $\mathbb{R}^n$. We believe that these results in the classical setting are of independent interest.

This paper is organized as follows. In Section 2 we state our results. The proof of the theorems are shown in Sections 3 (classical setting) and 4 (Schrödinger context).

Throughout this paper we denote by C and c positive constants that can change from one line to another. Moreover, if $B(x_0, r_0)$ with $x_0 \in \mathbb{R}^n$ and $r_0 > 0$, we define $B^* = B(x_0, 2r_0)$ and $B^{**} = (B^*)^*$.

2 Main Results

In this section we present the main results of the paper, stated as Theorems 2.2, 2.4, 2.6, and 2.8 below.

As it is well known the heat semigroup $\{W_t\}_{t>0}$ generated by $-\Delta$ is defined, for every $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, by

$$W_t(f)(x) = \int_{\mathbb{R}^n} W_t(x-y) f(y)dy, \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

where $W_t(z) = (4\pi t)^{-n/2} e^{-|z|^2/4t}$, $z \in \mathbb{R}^n$ and $t > 0$.

The L^p-boundedness properties of the variation operator $V_\rho(W_t)$, $\rho > 2$, were studied in [22, Theorem 3.3] and [9, Theorem 1.1]. We provide here the precise statement.

Theorem 2.1 ([22, Theorem 3.3] and [9, Theorem 1.1]) *If $\rho > 2$, the variation operator $V_\rho(W_t)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

In [9] it was shown that the variation operator $V_\rho(W_t)$ is not bounded on $L^\infty(\mathbb{R}^n)$. In fact, in [9, Section 5] , the authors give an example of a function $f \in L^\infty(\mathbb{R})$ such that $V_\rho(W_t)(f)(x) = \infty$, a.e. $x \in \mathbb{R}$, for every $\rho > 2$. As it is well known $L^\infty(\mathbb{R}^n)$ is a subset of the space $BMO(\mathbb{R}^n)$ of bounded mean oscillation functions. In the next result we take care of the behavior of $V_\rho(W_t)$ on $BMO(\mathbb{R}^n)$.

Theorem 2.2 *Let $\rho > 2$. Then, if $f \in BMO(\mathbb{R}^n)$ and $V_\rho(W_t)(f)(x) < \infty$, a.e. $x \in \mathbb{R}^n$, $V_\rho(W_t)(f) \in BMO(\mathbb{R}^n)$ and $\|V_\rho(W_t)(f)\|_{BMO(\mathbb{R}^n)} \le C\|f\|_{BMO(\mathbb{R}^n)}$.*

For every $\ell = 1, \cdots, n$, the Riesz transform $R_\ell(f)$ of $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, is given by

$$R_\ell(f)(x) = \lim_{\varepsilon \to 0^+} R_\ell^\varepsilon(f)(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

where

$$R_\ell^\varepsilon(f)(x) = c_n \int_{|x-y|>\varepsilon} \frac{x_\ell - y_\ell}{|x-y|^{n+1}} f(y)dy,$$

and $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$.

Regarding the variation operator for R_ℓ, $\ell = 1, \ldots, n$, their L^p-boundedness was investigated in [7] and [8]. We reproduce here their precise statement.

Theorem 2.3 ([7, Theorem 1.2] and [8, Theorem A and Corollary 1.4]) *Let $\ell = 1, \ldots, n$. If $\rho > 2$, the variation operator $V_\rho(R_\ell^\varepsilon)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

By using transference methods Gillespie and Torrea [17, Theorem B], obtained dimension free $L^p(\mathbb{R}^n, |x|^\alpha dx)$ norm inequalities, for every $1 < p < \infty$ and $-1 < \alpha < p-1$, for variation operators of the Riesz transform R_ℓ, $\ell = 1, \ldots, n$. Using the idea developed in the proof of [17, Lemma 1.4], we are able to analyze the behavior of the operators $V_\rho(R_\ell^\varepsilon)$ on the space $BMO(\mathbb{R}^n)$.

First notice that for $\ell = 1, \ldots, n$, $f \in BMO(\mathbb{R}^n)$ and $\varepsilon > 0$, the integral $\int_{|x-y|>\varepsilon} f(y) \frac{y_\ell - x_\ell}{|y-x|^{n+1}} dy$ may be non-convergent. Indeed, for instance, the function $f(x) = \frac{1}{\log(x+2)} \chi_{(0,\infty)}(x)$, $x \in \mathbb{R}$, belongs to $L^\infty(\mathbb{R}) \subset BMO(\mathbb{R})$ but the limit

$\lim_{N\to\infty}\int_{\varepsilon<|x-y|<N}\frac{f(y)}{x-y}dy$ does not exist, for any $x\in\mathbb{R}$ and $\varepsilon>0$. However, it is clear that, for every $0<\varepsilon<\eta$, $\int_{\varepsilon<|x-y|<\eta}\frac{|f(y)|}{|x-y|^n}dy<\infty$ for any $f\in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$. Therefore, in this situation, the operators $V_\rho(R^\varepsilon_\ell)$ can be defined on $BMO(\mathbb{R}^n)$ in the obvious way, that is, by replacing $R^{\varepsilon_j}_\ell(f)(x)-R^{\varepsilon_{j+1}}_\ell(f)(x)$ by $c_n\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j}f(y)\frac{y_\ell-x_\ell}{|y-x|^{n+1}}dy$, $\ell=1,\cdots,n$ and $j\in\mathbb{N}$. Let us mention that in [10, Theorem B] it was proved that if $f\in L^\infty(\mathbb{R})$ and $\rho>2$, then either $V_\rho(H^\varepsilon)(f)(x)=\infty$, a.e. $x\in\mathbb{R}$, or $V_\rho(H^\varepsilon)(f)(x)<\infty$, a.e. $x\in\mathbb{R}$, where H denotes the Riesz transform on $\mathbb{R}$, that is, the Hilbert transform. Moreover, as it can be seen in [10, Section 1], if $f(x)=\text{sgn}(x)$, $x\in\mathbb{R}$, then $V_\rho(H^\varepsilon)(f)(x)=\infty$, a.e. $x\in\mathbb{R}$.

We can say that the above continuous results have their antecedents in other discrete results due to Jones et al. [21]. Suppose that $\{I_m\}_{m\in\mathbb{N}}$ denotes a nested sequence of intervals in $\mathbb{Z}$ such that $I_1=\{0\}$ and $\sharp I_m=m$, where $\sharp A$ means, as usual, the cardinal of $A\subset\mathbb{Z}$. For every $m\in\mathbb{N}$, we define

$$M_m(f)(x)=\frac{1}{m}\sum_{j\in I_m}f(x+j).$$

In [21, p. 915] the authors considered the variation operator $V_\rho(M_m)$, $\rho>2$, defined by

$$V_\rho(M_m)(f)(x)=\sup_{\{m_k\}}\left(\sum_{i=1}^{\infty}|M_{m_i}(f)(x)-M_{m_{i+1}}(f)(x)|^p\right)^{1/p}.$$

They proved that, for every $\rho>2$, the operator $V_\rho(M_m)$ is bounded from $L^\infty(\mathbb{R})$ into $BMO(\mathbb{R})$ [21, Theorem 4.4]. Other oscillation operators and square sums type operators associated with M_m are also considered in [21, Section 4]. Moreover, they defined a square function that does not map $L^\infty(\mathbb{R})$ into $BMO(\mathbb{R})$ (see [21, Remark 4.5]).

In the next result we establish the behavior of the variation operators $V_\rho(R^\varepsilon_\ell)$ for functions in $BMO(\mathbb{R}^n)$.

Theorem 2.4 *Let $\ell=1,\ldots,n$ and $\rho>2$. Then, if $f\in BMO(\mathbb{R}^n)$ and $V_\rho(R^\varepsilon_\ell)(f)(x)<\infty$, a.e. $x\in\mathbb{R}^n$, then $V_\rho(R^\varepsilon_\ell)(f)\in BMO(\mathbb{R}^n)$ and $\|V_\rho(R^\varepsilon_\ell)(f)\|_{BMO(\mathbb{R}^n)}\le C\|f\|_{BMO(\mathbb{R}^n)}$.*

We turn now to the Schrödinger operator setting. Let us denote by $\{W^{\mathcal{L}}_t\}_{t>0}$ the heat semigroup associated with $\mathcal{L}$. For every $t>0$, they can be written in the integral form

$$W^{\mathcal{L}}_t(f)(x)=\int_{\mathbb{R}^n}W^{\mathcal{L}}_t(x,y)f(y)dy,\quad f\in L^2(\mathbb{R}^n).$$

Even though we do not have an explicit formula for the kernels $W^{\mathcal{L}}_t(x,y)$, many properties are known and can be encountered, for instance, in [14].

L^p-boundedness properties of the variation operator $V_\rho(W^{\mathcal{L}}_t)$ were studied in [2]. We reproduce here the exact result.

Theorem 2.5 [2, Theorem 1.1]. *Let $V\in RH_q$ where $q>n/2$ and let $\rho>2$. Then, the variation operator $V_\rho(W^{\mathcal{L}}_t)$ is bounded from $L^p(\mathbb{R}^n)$ into itself, for every $1<p<\infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.*

Our next result shows the behavior of the variation operator $V_\rho(W_t^{\mathscr{L}})$ on $\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$.

Theorem 2.6 *Let $V \in RH_q$ where $q > n/2$ and let $\rho > 2$. Then, the variation operator $V_\rho(W_t^{\mathscr{L}})$ is bounded from $\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$ into itself.*

Let $\ell = 1, \ldots, n$. The Riesz transform $R_\ell^{\mathscr{L}}$ is defined by

$$R_\ell^{\mathscr{L}}(f) = \frac{\partial}{\partial x_\ell}\mathscr{L}^{-1/2} f, \quad f \in C_c^\infty(\mathbb{R}^n),$$

where $C_c^\infty(\mathbb{R}^n)$ denotes the space of smooth functions with compact support in $\mathbb{R}^n$. Here, the negative square root $\mathscr{L}^{-1/2}$ of $\mathscr{L}$ is defined in terms of the heat semigroup by

$$\mathscr{L}^{-1/2}(f)(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty W_t^{\mathscr{L}}(f)(x) t^{-1/2} dt.$$

Fractional powers of the Schrödinger operator $\mathscr{L}$ have been studied in [3].

In [2, Proposition 1.1] it was proved that $R_\ell^{\mathscr{L}}$ can be extended to $L^p(\mathbb{R}^n)$ as the principal value operator

$$R_\ell^{\mathscr{L}}(f)(x) = \lim_{\varepsilon \to 0^+}\int_{|x-y|>\varepsilon} R_\ell^{\mathscr{L}}(x, y) f(y) dy, \quad \text{a.e. } x \in \mathbb{R}^n, \tag{1}$$

where

$$R_\ell^{\mathscr{L}}(x, y) = -\frac{1}{2\pi}\frac{\partial}{\partial x_\ell}\int_{\mathbb{R}}(-i\tau)^{-1/2}\Gamma(x, y, \tau) d\tau, \quad x, y \in \mathbb{R}^n, \ x \neq y,$$

and, for every $\tau \in \mathbb{R}$, $\Gamma(x, y, \tau)$, $x, y \in \mathbb{R}^n$, represents the fundamental solution for the operator $\mathscr{L} + i\tau$, provided that

(i) $1 \leq p < \infty$, and $V \in RH_n$;
(ii) $1 < p < p_0$, where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, and $V \in RH_q$, $n/2 \leq q < n$.

Moreover, $R_\ell^{\mathscr{L}}$ is bounded from $L^p(\mathbb{R}^n)$ into itself when $1 < p < \infty$ and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, provided that $V \in RH_n$. Also, $R_\ell^{\mathscr{L}}$ is bounded from $L^p(\mathbb{R}^n)$ into itself when $1 < p < p_0$ and $V \in RH_q$, with $n/2 \leq q < n$ [27, Theorems 0.5 and 0.8].

The formal adjoint operator $\mathscr{R}_\ell^{\mathscr{L}}$ of $R_\ell^{\mathscr{L}}$ defined by

$$\mathscr{R}_\ell^{\mathscr{L}}(f)(x) = \lim_{\varepsilon \to 0^+}\int_{|x-y|>\varepsilon} R_\ell^{\mathscr{L}}(y, x) f(y) dy, \quad \text{a.e. } x \in \mathbb{R}^n,$$

is bounded from $L^p(\mathbb{R}^n)$ into itself when $p_0' < p < \infty$ and $V \in RH_q$, with $n/2 \leq q < n$, where, as usual, p_0' denotes the exponent conjugated to p_0. In the case that $V \in RH_n$, it is bounded in $L^p(\mathbb{R}^n)$, for $1 < p < \infty$, and it also maps $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

By defining the truncated Riesz transforms $R_\ell^{\mathscr{L},\varepsilon}$, $\varepsilon > 0$, in the natural way, the L^p-boundedness properties for the variation operator $V_\rho(R_\ell^{\mathscr{L},\varepsilon})$ were established in [2].

Theorem 2.7 [2, Theorem 1.2] *Let $\ell = 1, \dots, n$. Assume that $\rho > 2$. Then, the variation operator $V_\rho(R_\ell^{\mathscr{L},\varepsilon})$ is bounded*

(i) *from $L^p(\mathbb{R}^n)$ into itself, when $1 < p < \infty$, and from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, provided that $V \in RH_n$.*
(ii) *from $L^p(\mathbb{R}^n)$ into itself, when $1 < p < p_0$, where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$, and $V \in RH_q$, $n/2 \leq q < n$.*

If $f \in BMO^{\mathscr{L}}(\mathbb{R}^n)$ and $\ell = 1, \cdots, n$, the limit in Eq. 1 exists for a.e. $x \in \mathbb{R}^n$ (see [2, Proposition 1.1]). Thus, the Riesz transforms $R_\ell^{\mathscr{L}}$ are defined by Eq. 1 in $BMO^{\mathscr{L}}(\mathbb{R}^n)$. As it was remarked earlier, the situation is quite different in the classical case. In the next result we describe the behavior on $BMO^{\mathscr{L}}(\mathbb{R}^n)$ of the variation operators associated with the Riesz transforms $R_\ell^{\mathscr{L}}$ and their adjoints.

Theorem 2.8 *Let $\rho > 2$, $\ell = 1, \dots, n$. If $V \in RH_q$ where $q \geq n$, then the variation operator $V_\rho(R_\ell^{\mathscr{L},\varepsilon})$ is bounded from $BMO^{\mathscr{L}}(\mathbb{R}^n)$ into itself. Also, the variation operator $V_\rho(\mathscr{R}_\ell^{\mathscr{L},\varepsilon})$ is bounded from $BMO^{\mathscr{L}}(\mathbb{R}^n)$ into itself, provided that $V \in RH_q$ where $q > n/2$.*

Note that there is a remarkable difference between the results in the classical and in the Schrödinger settings. In the latter, the operators are defined in the whole $BMO^{\mathscr{L}}(\mathbb{R}^n)$, while in the classical case it is necessary to impose an additional "finiteness hypothesis". This fact was observed by the first time in [14].

In order to analyze operators in the Schrödinger context on $\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$ we shall use some ideas developed in [14] and we will again exploit that the Schrödinger operator $\mathscr{L} = -\Delta + V$, where $V \in RH_q$, with $q \geq n/2$, is actually a nice perturbation of the Laplacian operator $-\Delta$.

Throughout the proof of the results that we have just stated, the following properties will play an important role.

According to [14, Proposition 5] it is possible to choose a sequence $\{x_k\}_{k=1}^\infty \subset \mathbb{R}^n$, such that if $Q_k = B(x_k, \gamma(x_k))$, $k \in \mathbb{N}$, the following properties hold:

(i) $\cup_{k=1}^\infty Q_k = \mathbb{R}^n$;
(ii) For every $m \in \mathbb{N}$ there exist $C, \beta > 0$ such that, for every $k \in \mathbb{N}$,

$$\#\{l \in \mathbb{N} : 2^m Q_l \cap 2^m Q_k \neq \varnothing\} \leq C 2^{m\beta}.$$

Also, from [14, p. 346, after Lemma 9], for any operator H and $f \in \mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$, $Hf \in \mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$ provided that there exists a positive constant C such that, for every $k \in \mathbb{N}$,

(i_k) $\frac{1}{|Q_k|} \int_{Q_k} |H(f)(x)| dx \leq C ||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}$, and
(ii_k) $Hf \in \mathrm{BMO}(Q_k^*)$ and $||Hf||_{\mathrm{BMO}(Q_k^*)} \leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}$. Here $\mathrm{BMO}(Q_k^*)$ denotes the usual BMO-space over Q_k^*.

Moreover, we have that

$$\|Hf\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)} \leq M \|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)},$$

where the constant $M > 0$ depends only on the constant C.

Note that if the property (i_k) above holds for every $k \in \mathbb{N}$ then, $|H(f)(x)| < \infty$ for almost every $x \in \mathbb{R}^n$. This fact is quite different from what happens in the classical Euclidean case (see Theorem 2.2 and Theorem 2.4, and [1]).

3 Proof of Theorems 2.2 and 2.4

In this section we show our results about the behavior of the variation operator for the classical heat semigroup and Riesz transforms on $BMO(\mathbb{R}^n)$.

3.1 Proof of Theorem 2.2

Let $\rho > 2$. Assume that $f \in BMO(\mathbb{R}^n)$ and $V_\rho(W_t)(f)(x) < \infty$, a.e. $x \in \mathbb{R}^n$. Let $B = B(x_0, r_0)$, with $x_0 \in \mathbb{R}^n$ and $r_0 > 0$. We write

$$f = (f - f_B)\chi_{B^*} + (f - f_B)\chi_{(B^*)^c} + f_B = f_1 + f_2 + f_3.$$

Note that this type of decomposition allows us to see that $W_t(|f|) < \infty$, $t > 0$. According to Theorem 2.1, we have

$$\int_{\mathbb{R}^n} \left|V_\rho(W_t)(f_1)(x)\right|^2 dx \leq C \int_{B^*} |f(x) - f_B|^2 \, dx \leq C|B| \|f\|^2_{BMO(\mathbb{R}^n)}. \tag{2}$$

In particular this means that $V_\rho(W_t)(f_1)(x) < \infty$, a.e. $x \in \mathbb{R}^n$. Moreover, since $\{W_t\}_{t>0}$ is Markovian, $V_\rho(W_t)(f_3) = 0$. Then, using the hypothesis, we may choose $x_1 \in B(x_0, r_0)$ such that $V_\rho(W_t)(f_2)(x_1) < \infty$.

If E_ρ denotes the space introduced in Section 1, we can write

$$\begin{aligned}
&\frac{1}{|B|} \int_B \left|V_\rho(W_t)(f)(x) - V_\rho(W_t)(f_2)(x_1)\right| dx \\
&\quad = \frac{1}{|B|} \int_B \Big| \|W_t(f)(x)\|_{E_\rho} - \|W_t(f_2)(x_1)\|_{E_\rho} \Big| \, dx \\
&\quad \leq \frac{1}{|B|} \int_B \|W_t(f)(x) - W_t(f_2)(x_1)\|_{E_\rho} \, dx \\
&\quad = \frac{1}{|B|} \int_B \|W_t(f_1)(x) + W_t(f_2)(x) - W_t(f_2)(x_1)\|_{E_\rho} \, dx \\
&\quad \leq \frac{1}{|B|} \int_B \|W_t(f_1)(x)\|_{E_\rho} \, dx + \frac{1}{|B|} \int_B \|W_t(f_2)(x) - W_t(f_2)(x_1)\|_{E_\rho} \, dx.
\end{aligned} \tag{3}$$

Therefore, according to Eq. 2 we get

$$\frac{1}{|B|} \int_B \|W_t(f_1)(x)\|_{E_\rho} \, dx \leq \left(\frac{1}{|B|} \int_B \left|V_\rho(W_t)(f_1)(x)\right|^2 dx \right)^{\frac{1}{2}} \leq C\|f\|_{BMO(\mathbb{R}^n)}. \tag{4}$$

Also, Minkowski inequality and [9, p. 88] lead to

$$\begin{aligned}
&\frac{1}{|B|}\int_B \|W_t(f_2)(x)-W_t(f_2)(x_1)\|_{E_\rho}\,dx \\
&\quad=\frac{1}{|B|}\int_B \left\|\int_{\mathbb{R}^n} W_t(x-y)\,f_2(y)dy-\int_{\mathbb{R}^n} W_t(x_1-y)\,f_2(y)dy\right\|_{E_\rho} dx \\
&\quad\le C\frac{1}{|B|}\int_B\int_{\mathbb{R}^n} \|W_t(x-y)-W_t(x_1-y)\|_{E_\rho}\,|f_2(y)|dydx \\
&\quad\le \frac{C}{|B|}\int_B\int_{(B^*)^c} \frac{|x-x_1|}{|x-y|^{n+1}}|f(y)-f_B|dydx \\
&\quad\le C\frac{r_0}{|B|}\int_B\int_{(B^*)^c} \frac{1}{(|y-x_0|-|x_0-x|)^{n+1}}|f(y)-f_B|dydx \\
&\quad\le C\frac{r_0}{|B|}\int_B\sum_{k=1}^{\infty}\int_{2^k r_0\le|y-x_0|<2^{k+1}r_0} \frac{1}{(|y-x_0|-|x_0-x|)^{n+1}}|f(y)-f_B|dydx \\
&\quad\le C\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{(2^k r_0)^n}\int_{|y-x_0|<2^{k+1}r_0} |f(y)-f_B|dy\le C\|f\|_{BMO(\mathbb{R}^n)}. \qquad (5)
\end{aligned}$$

In the last inequality we have used the well known property (see [[30], VIII, Proposition 3.2])

$$\frac{1}{|2^m B|}\int_{2^m B}|f(y)-f_B|dy\le Cm\|f\|_{BMO(\mathbb{R}^n)},\quad m\in\mathbb{N}.$$

From Eqs. 3, 4 and 5 we conclude that

$$\frac{1}{|B|}\int_B\left|V_\rho(W_t)(f)(x)-V_\rho(W_t)(f_2)(x_1)\right|dx\le C\|f\|_{BMO(\mathbb{R}^n)}.$$

Thus, we prove that $V_\rho(W_t)(f)\in BMO(\mathbb{R}^n)$.

Remark 3.1 After a careful reading of the proof of Theorem 2.2 we can deduce the following result that will be useful in the proof of Theorem 2.6.

Proposition 3.1 *Let $\rho>2$ and $\mathscr{A}$ be a set of decreasing real sequences converging to zero. Assume that*

$$V_{\rho,\mathscr{A}}(W_t)(f)(x)=\sup_{\{t_j\}_{j=1}^\infty\in\mathscr{A}}\left(\sum_{j=1}^{\infty}|W_{t_j}(f)(x)-W_{t_{j+1}}(f)(x)|^\rho\right)^{1/\rho}<\infty,\quad a.e.\ x\in Q,$$

where Q is a ball in $\mathbb{R}^n$. Suppose that $f\in BMO(\mathbb{R}^n)$ and B is a ball contained in Q. If we define $f_2=(f-f_B)\chi_{(B^*)^c}$ and choose $x_1\in B$ such that $V_{\rho,\mathscr{A}}(W_t)(f_2)(x_1)<\infty$, then there is a constant $C>0$ independent of $\mathscr{A}$, f, and B such that

$$\frac{1}{|B|}\int_B\|W_t(f)(x)-W_t(f_2)(x_1)\|_{E_{\rho,\mathscr{A}}}dx\le C\|f\|_{BMO(\mathbb{R}^n)},$$

where, for every function $h:(0,\infty)\mapsto\mathbb{C}$, $\|h\|_{E_{\rho,\mathscr{A}}}$ *means*

$$\|h\|_{E_{\rho,\mathscr{A}}}=\sup_{\{t_j\}_{j=1}^{\infty}\in\mathscr{A}}\left(\sum_{j=1}^{\infty}|h(t_j)-h(t_{j+1})|^{\rho}\right)^{1/\rho}.$$

3.2 Proof of Theorem 2.4

Let $\rho>2$ and $\ell=1,\dots,n$. Assume that $f\in BMO(\mathbb{R}^n)$ and that $V_\rho(R_\ell^\varepsilon)(f)(x)<\infty$, a.e. $x\in\mathbb{R}^n$. To see that $V_\rho(R_\ell^\varepsilon)(f)\in BMO(\mathbb{R}^n)$ we extend to $\mathbb{R}^n$ the technique developed in the proof of [17, Lemma 1.4].

Let $B=B(x_0,r_0)$ be a ball in $\mathbb{R}^n$. We decompose f setting $f=f_1+f_2+f_3$, where $f_1=(f-f_B)\chi_{B^{**}}$, $f_2=(f-f_B)\chi_{(B^{**})^c}$ and $f_3=f_B$. According to Theorem 2.3, we have

$$\int_{\mathbb{R}^n}|V_\rho\left(R_\ell^\varepsilon\right)(f_1)(x)|^2dx\le C\int_{B^{**}}|f(x)-f_B|^2dx\le C|B|\|f\|^2_{BMO(\mathbb{R}^n)}. \tag{6}$$

Then, $V_\rho(R_\ell^\varepsilon)(f_1)(x)<\infty$, a.e. $x\in\mathbb{R}^n$. Moreover, $V_\rho(R_\ell^\varepsilon)(f_3)=0$. Then, we can choose $x_1\in B$ such that $V_\rho(R_\ell^\varepsilon)(f_2)(x_1)<\infty$.

If E_ρ denotes the space defined in Section 1, by Eq. 6 we can write

$$\begin{aligned}
&\frac{1}{|B|}\int_B\left|V_\rho\left(R_\ell^\varepsilon\right)(f)(x)-V_\rho\left(R_\ell^\varepsilon\right)(f_2)(x_1)\right|dx\\
&\quad=\frac{1}{|B|}\int_B\left|\|R_\ell^\varepsilon(f)(x)\|_{E_\rho}-\|R_\ell^\varepsilon(f_2)(x_1)\|_{E_\rho}\right|dx\\
&\quad\le\frac{1}{|B|}\int_B\|R_\ell^\varepsilon(f)(x)-R_\ell^\varepsilon(f_2)(x_1)\|_{E_\rho}dx\\
&\quad\le\frac{1}{|B|}\int_B\|R_\ell^\varepsilon(f_1)(x)\|_{E_\rho}dx+\frac{1}{|B|}\int_B\|R_\ell^\varepsilon(f_2)(x)-R_\ell^\varepsilon(f_2)(x_1)\|_{E_\rho}dx\\
&\quad\le C\|f\|_{BMO(\mathbb{R}^n)}+\frac{1}{|B|}\int_B\|R_\ell^\varepsilon(f_2)(x)-R_\ell^\varepsilon(f_2)(x_1)\|_{E_\rho}dx.
\end{aligned}\tag{7}$$

Here, the expressions with $\|.\|_{E_\rho}$ have the obvious meaning.

Denoting $R_\ell(z)=c_n\frac{z_\ell}{|z|^{n+1}}$, $z=(z_1,\cdots,z_n)\in\mathbb{R}^n\setminus\{0\}$, we have that

$$\|R_\ell^\varepsilon(f_2)(x)-R_\ell^\varepsilon(f_2)(x_1)\|_{E_\rho}\le A_1(x)+A_2(x),\quad x\in B, \tag{8}$$

where, for every $x\in B$,

$$A_1(x)=\left\|\int_{|x-y|>\varepsilon}(R_\ell(x-y)-R_\ell(x_1-y))f_2(y)dy\right\|_{E_\rho}$$

and

$$A_2(x)=\left\|\int_{\mathbb{R}^n}\left(\chi_{\{\varepsilon<|x-y|\}}(y)-\chi_{\{\varepsilon<|x_1-y|\}}(y)\right)R_\ell(x_1-y)f_2(y)dy\right\|_{E_\rho}.$$

By using Minkowski inequality and well known properties of the function $R_\ell(z)$ we get

$$\begin{aligned}
A_1(x) &\le \int_{\mathbb{R}^n} |R_\ell(x-y)-R_\ell(x_1-y)||f(y)-f_B|\chi_{(B^{**})^c}(y)dy \\
&\le C\sum_{k=2}^{\infty}\int_{2^k r_0\le|x_0-y|\le 2^{k+1}r_0}\frac{|x-x_1|}{|x-y|^{n+1}}|f(y)-f_B|dy \\
&\le C\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{(2^k r_0)^n}\int_{2^{k+1}B}|f(y)-f_B|dy \\
&\le C\|f\|_{BMO(\mathbb{R}^n)}, \quad x\in B. \qquad (9)
\end{aligned}$$

In order to analyze A_2 we split, for every $j\in\mathbb{N}$, the integral there in four terms as follows. Let $\{\varepsilon_j\}_{j=1}^\infty$ be a real decreasing sequence that converges to zero. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)-\chi_{\{\varepsilon_{j+1}<|x_1-y|<\varepsilon_j\}}(y)\right||R_\ell(x_1-y)||f_2(y)|dy \\
&\le C\Bigg(\int_{\mathbb{R}^n}\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_{j+1}+2r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{1}{|x_1-y|^n}|f_2(y)|dy \\
&\qquad+\int_{\mathbb{R}^n}\chi_{\{\varepsilon_j\le|x_1-y|<\varepsilon_j+2r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{1}{|x_1-y|^n}|f_2(y)|dy \\
&\qquad+\int_{\mathbb{R}^n}\chi_{\{\varepsilon_{j+1}<|x_1-y|<\varepsilon_{j+1}+2r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x_1-y|<\varepsilon_j\}}(y)\frac{1}{|x_1-y|^n}|f_2(y)|dy \\
&\qquad+\int_{\mathbb{R}^n}\chi_{\{\varepsilon_j<|x-y|<\varepsilon_j+2r_0\}}(y)\chi_{\{\varepsilon_{j+1}<|x_1-y|<\varepsilon_j\}}(y)\frac{1}{|x_1-y|^n}|f_2(y)|dy\Bigg) \\
&= C\left(A_{2,1}^j(x)+A_{2,2}^j(x)+A_{2,3}^j(x)+A_{2,4}^j(x)\right), \quad x\in B \text{ and } j\in\mathbb{N}. \qquad (10)
\end{aligned}$$

Observe that if $x\in B$, then $A_{2,m}^j(x)=0$, when $m=1,3$, $j\in\mathbb{N}$ and $r_0\ge\varepsilon_{j+1}$. Also, if $x\in B$, then $A_{2,m}^j(x)=0$, when $m=2,4$, $j\in\mathbb{N}$ and $r_0\ge\varepsilon_j$.

Since $2|x-y|\ge|x_1-y|\ge\frac{1}{2}|x-y|$, $y\notin B^{**}$ and $x\in B$, Hölder inequality leads, for every $j\in\mathbb{N}$, to

$$A_{2,1}^j(x)\le C\left(\int_{\mathbb{R}^n}\chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{1}{|x-y|^{ns}}|f_2(y)|^s dy\right)^{\frac{1}{s}} v_{j+1}^{\frac{1}{s'}}, \quad x\in B,$$

$$A_{2,2}^j(x)\le C\left(\int_{\mathbb{R}^n}\chi_{\{\max\{\varepsilon_{j+1},\frac{1}{2}\varepsilon_j\}<|x-y|<\varepsilon_j\}}(y)\frac{1}{|x-y|^{ns}}|f_2(y)|^s dy\right)^{\frac{1}{s}} v_j^{\frac{1}{s'}}, \quad x\in B,$$

$$A_{2,3}^j(x)\le C\left(\int_{\mathbb{R}^n}\chi_{\{\varepsilon_{j+1}<|x_1-y|<\varepsilon_j\}}(y)\frac{1}{|x_1-y|^{ns}}|f_2(y)|^s dy\right)^{\frac{1}{s}} v_{j+1}^{\frac{1}{s'}}, \quad x\in B,$$

and

$$A_{2,4}^j(x) \leq C\left(\int_{\mathbb{R}^n} \chi_{\{\max\{\varepsilon_{j+1},\frac{1}{2}\varepsilon_j\}<|x_1-y|<\varepsilon_j\}}(y)\frac{1}{|x_1-y|^{ns}}|f_2(y)|^s dy\right)^{\frac{1}{s}} v_j^{\frac{1}{s'}}, \quad x \in B.$$

Here $1 < s < \infty$, $s' = \frac{s}{s-1}$, and $v_j = (\varepsilon_j + 2r_0)^n - \varepsilon_j^n$, $j \in \mathbb{N}$. Note that $v_j \leq C \max\{r_0, \varepsilon_j\}^{n-1} r_0$, $j \in \mathbb{N}$, for a certain $C > 0$.

We define the set $\mathscr{G} = \{j \in \mathbb{N} : r_0 < \varepsilon_j\}$. We have that

$$\begin{aligned} A_{2,1}^j(x) &\leq C\frac{v_{j+1}^{1/s'}}{\varepsilon_{j+1}^{(n-1)/s'}}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{|f_2(y)|^s}{|x-y|^{n+s-1}}dy\right)^{1/s} \\ &\leq Cr_0^{1/s'}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{|f_2(y)|^s}{|x-y|^{n+s-1}}dy\right)^{1/s}, \end{aligned}$$

for every $x \in B$ and $j+1 \in \mathscr{G}$. In a similar way we can see that

$$A_{2,2}^j(x) \leq Cr_0^{1/s'}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{|f_2(y)|^s}{|x-y|^{n+s-1}}dy\right)^{1/s}, \quad x \in B \text{ and } j \in \mathscr{G},$$

$$A_{2,3}^j(x) \leq Cr_0^{1/s'}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x_1-y|<\varepsilon_j\}}(y)\frac{|f_2(y)|^s}{|x_1-y|^{n+s-1}}dy\right)^{1/s}, \quad x \in B \text{ and } j+1 \in \mathscr{G},$$

and

$$A_{2,4}^j(x) \leq Cr_0^{1/s'}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x_1-y|<\varepsilon_j\}}(y)\frac{|f_2(y)|^s}{|x_1-y|^{n+s-1}}dy\right)^{1/s}, \quad x \in B \text{ and } j \in \mathscr{G}.$$

Hence, we get

$$\begin{aligned} &\left(\sum_{j=1}^{\infty}\left(A_{2,1}^j(x) + A_{2,2}^j(x)\right)^\rho\right)^{1/\rho} \\ &\leq C\left(\sum_{j+1\in\mathscr{G}} |A_{2,1}^j(x)|^\rho + \sum_{j\in\mathscr{G}} |A_{2,2}^j(x)|^\rho\right)^{1/\rho} \\ &\leq C\left(\sum_{j\in\mathbb{N}}\left(\int_{\mathbb{R}^n} \chi_{\{\varepsilon_{j+1}<|x-y|<\varepsilon_j\}}(y)\frac{|f_2(y)|^s}{|x-y|^{n+s-1}}dy\right)^{\rho/s} r_0^{\rho/s'}\right)^{1/\rho} \\ &\leq C\left(\int_{\mathbb{R}^n} \frac{|f_2(y)|^s}{|x-y|^{n+s-1}}dy\right)^{1/s} r_0^{1/s'} \\ &\leq C\left(\left(\sum_{k=1}^{\infty}\frac{1}{(2^k r_0)^n}\int_{|x_0-y|<2^{k+1}r_0} |f(y) - f_B|^s dy \frac{1}{2^{k(s-1)}}\right)^{1/s} \\ &\leq C||f||_{\mathrm{BMO}(\mathbb{R}^n)}, \quad x \in B. \end{aligned} \tag{11}$$

In a similar way we get

$$\left(\sum_{j=1}^{\infty}|A_{2,3}^{j}(x)+A_{2,4}^{j}(x)|^{\rho}\right)^{1/\rho}\leq C||f||_{\mathrm{BMO}(\mathbb{R}^n)},\quad x\in B. \tag{12}$$

From Eqs. 10, 11 and 12 we infer that

$$A_2(x)\leq C||f||_{\mathrm{BMO}(\mathbb{R}^n)},\quad x\in B. \tag{13}$$

Altogether Eqs. 7, 8, 9 and 13 imply that

$$\frac{1}{|B|}\int_B |V_\rho\left(R_\ell^\varepsilon\right)(f)(x)-V_\rho(R_\ell^\varepsilon)(f_2)(x_1)|dx\leq C||f||_{\mathrm{BMO}(\mathbb{R}^n)}. \tag{14}$$

Thus, we prove that $V_\rho(R_\ell^\varepsilon)(f)\in \mathrm{BMO}(\mathbb{R}^n)$.

By proceeding as in the above proof we can establish the following result that will be useful in the sequel.

Proposition 3.2 *Let $a>0$ and $\ell=1,...,n$. We define, for every $\varepsilon>0$ and $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,*

$$R_{\ell,a}^{\varepsilon}(f)(x)=\int_{\varepsilon<|x-y|<a}\frac{x_\ell-y_\ell}{|x-y|^{n+1}}f(y)dy.$$

Then, if $\rho>2$, $V_\rho(R_{\ell,a}^\varepsilon)(f)\in \mathrm{BMO}(\mathbb{R}^n)$, provided that $f\in \mathrm{BMO}(\mathbb{R}^n)$.

Note that, $V_\rho(R_{\ell,a}^\varepsilon)(f)(x)<\infty$, a.e. $x\in\mathbb{R}^n$, for every $a>0$, $f\in \mathrm{BMO}(\mathbb{R}^n)$ and $\ell=1,\dots,n$. Indeed, let $a>0$, $f\in\mathrm{BMO}(\mathbb{R}^n)$ and $\ell=1,\dots,n$. Suppose that $m\in\mathbb{N}$. Since $V_\rho(R_{\ell,a}^\varepsilon)(f)\leq V_\rho(R_\ell^\varepsilon)(f)$ and $f\in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, according to Theorem 2.3, we have that

$$\begin{aligned}\int_{B(0,m)}V_\rho\left(R_{\ell,a}^\varepsilon\right)(f)(x)dx&=\int_{B(0,m)}V_\rho\left(R_{\ell,a}^\varepsilon\right)(f\chi_{B(0,m+a)})(x)dx\\&\leq\int_{B(0,m)}V_\rho\left(R_\ell^\varepsilon\right)(f\chi_{B(0,m+a)})(x)dx\\&\leq Cm^{n/2}\Big(\int_{B(0,m)}\left(V_\rho\left(R_\ell^\varepsilon\right)\left(f\chi_{B(0,m+a)}\right)(x)\right)^2dx\Big)^{1/2}\\&\leq Cm^{n/2}\Big(\int_{B(0,m+a)}|f(x)|^2dx\Big)^{1/2}<\infty.\end{aligned}$$

Hence, $V_\rho(R_{\ell,a}^\varepsilon)(f)(x)<\infty$, a.e. $x\in B(0,m)$.

4 Proof of Theorems 2.6 and 2.8

In this section we establish the boundedness in $BMO^{\mathcal{L}}(\mathbb{R}^n)$ of the variation operators for the heat semigroup and Riesz transforms in the Schrödinger setting.

4.1 Proof of Theorem 2.6

Let $\rho > 2$. Assume that $f \in \mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$. Our goal is to show that $V_\rho(W_t^{\mathscr{L}})(f)$ satisfies the properties (i_k) and (ii_k), for every $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$. We now prove (i_k), that is, there exists $C > 0$, independent of k, such that

$$\frac{1}{|Q_k|}\int_{Q_k} |V_\rho\left(W_t^{\mathscr{L}}\right)(f)(x)|dx \leq C\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)}.$$

We decompose $W_t^{\mathscr{L}}(f)$ as follows

$$W_t^{\mathscr{L}}(f)(x) = H_{k,t}^{\mathscr{L}}(f)(x) + L_{k,t}^{\mathscr{L}}(f)(x), \quad x \in Q_k \text{ and } t > 0,$$

where

$$H_{k,t}^{\mathscr{L}}(f)(x) = W_t^{\mathscr{L}}(f)(x)\chi_{\{t>\gamma(x_k)^2\}}(t), \quad x \in Q_k \text{ and } t > 0.$$

It is clear that

$$V_\rho\left(W_t^{\mathscr{L}}\right)(f)(x) \leq V_\rho\left(H_{k,t}^{\mathscr{L}}\right)(f)(x) + V_\rho\left(L_{k,t}^{\mathscr{L}}\right)(f)(x), \quad x \in Q_k. \tag{15}$$

Let $\{t_j\}_{j=1}^\infty$ be a real decreasing sequence that converges to zero. Suppose that $j_k \in \mathbb{N}$ is such that $t_{j_k+1} \leq \gamma(x_k)^2 < t_{j_k}$. We can write

$$\begin{aligned}
&\left(\sum_{j=1}^\infty |H_{k,t_j}^{\mathscr{L}}(f)(x) - H_{k,t_{j+1}}^{\mathscr{L}}(f)(x)|^\rho\right)^{1/\rho} \\
&\quad \leq \sum_{j=1}^{j_k-1} |W_{t_j}^{\mathscr{L}}(f)(x) - W_{t_{j+1}}^{\mathscr{L}}(f)(x)| + |W_{t_{j_k}}^{\mathscr{L}}(f)(x)| \\
&\quad \leq \sum_{j=1}^{j_k-1} \left|\int_{t_{j+1}}^{t_j} \frac{\partial}{\partial t} W_t^{\mathscr{L}}(f)(x)dt\right| + |W_{t_{j_k}}^{\mathscr{L}}(f)(x)| \\
&\quad \leq \int_{\gamma(x_k)^2}^\infty \int_{\mathbb{R}^n} \left|\frac{\partial}{\partial t} W_t^{\mathscr{L}}(x,y)\right| |f(y)|dydt + \sup_{t \geq \gamma(x_k)^2} |W_t^{\mathscr{L}}(f)(x)| \\
&\quad = \Omega_{1,k}(f)(x) + \Omega_{2,k}(f)(x), \quad x \in Q_k.
\end{aligned}$$

Hence

$$V_\rho\left(H_{k,t}^{\mathscr{L}}\right)(f)(x) \leq \Omega_{1,k}(f)(x) + \Omega_{2,k}(f)(x), \quad x \in Q_k. \tag{16}$$

According to [14, (5.4)], we get

$$\frac{1}{|Q_k|}\int_{Q_k} \Omega_{2,k}(f)(x)dx \leq C\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)}. \tag{17}$$

By [14, (2.7)] we have that

$$\left|\frac{\partial}{\partial t} W_t^{\mathscr{L}}(x,y)\right| \leq C\frac{e^{-c|x-y|^2/t}}{t^{1+n/2}}\left(1 + \frac{t}{\gamma(x)^2}\right)^{-1}, \quad x, y \in \mathbb{R}^n \text{ and } t > 0.$$

Hence, since $\gamma(x) \sim \gamma(x_k)$, when $x \in Q_k$, we obtain

$$\begin{aligned}
\Omega_{1,k}(f)(x) &\leq C \int_{\gamma(x_k)^2}^{\infty} \int_{\mathbb{R}^n} \frac{|f(y)|}{t^{1+n/2}} e^{-c|x-y|^2/t} \left(1 + \frac{t}{\gamma(x_k)^2}\right)^{-1} dy dt \\
&\leq C \int_{\gamma(x_k)^2}^{\infty} \frac{1}{t^{1+n/2}} \left(1 + \frac{t}{\gamma(x_k)^2}\right)^{-1} \left(\int_{|x-y|<\sqrt{t}} + \sum_{m=0}^{\infty} \int_{\sqrt{t}2^m \leq |x-y| < \sqrt{t}2^{m+1}} \right) \\
&\quad \times |f(y)| \left(1 + \frac{|x-y|}{\sqrt{t}}\right)^{-1-n} dy dt \\
&\leq C \int_{\gamma(x_k)^2}^{\infty} \frac{1}{t^{1+n/2}} \left(1 + \frac{t}{\gamma(x_k)^2}\right)^{-1} \sum_{m=0}^{\infty} \frac{1}{2^{m(1+n)}} \int_{|x-y|<2^m\sqrt{t}} |f(y)| dy dt, \\
&\quad x \in Q_k.
\end{aligned}$$

Moreover, by [14, Lemma 2], since $|f| \in BMO^{\mathcal{L}}(\mathbb{R}^n)$, there exists $C > 0$ (which does not depend on f) such that, for every $B = B(x, r)$, with $x \in \mathbb{R}^n$ and $r < \gamma(x)$,

$$\frac{1}{|B(x, 2r)|} \int_{B(x,2r)} |f(y)| dy \leq C \left(1 + \log \frac{\gamma(x)}{r}\right) ||f||_{BMO^{\mathcal{L}}(\mathbb{R}^n)}.$$

Then, it follows that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{2^{m(1+n)} t^{n/2}} \int_{|x-y|<2^m\sqrt{t}} |f(y)| dy \\
&\quad \leq \sum_{m \in \mathbb{N},\, 2^m\sqrt{t} < \gamma(x)} \frac{1}{2^{m(1+n)} t^{n/2}} \int_{|x-y|<2^m\sqrt{t}} |f(y)| dy \\
&\quad\quad + \sum_{m \in \mathbb{N},\, 2^m\sqrt{t} \geq \gamma(x)} \frac{1}{2^{m(1+n)} t^{n/2}} \int_{|x-y|<2^m\sqrt{t}} |f(y)| dy \\
&\quad \leq C \|f\|_{BMO^{\mathcal{L}}(\mathbb{R}^n)} \left(\sum_{m \in \mathbb{N},\, 2^m\sqrt{t} < \gamma(x)} \frac{1}{2^m} \left(1 + \log \frac{\gamma(x)}{2^m\sqrt{t}}\right) + \sum_{m \in \mathbb{N}} \frac{1}{2^m} \right) \\
&\quad \leq C \left(1 + \log \frac{\sqrt{t}}{\gamma(x_k)}\right) \|f\|_{BMO^{\mathcal{L}}(\mathbb{R}^n)}, \quad t \geq \gamma(x_k)^2 \text{ and } x \in Q_k.
\end{aligned}$$

Then,

$$\begin{aligned}
\Omega_{1,k}(f)(x) &\leq C \|f\|_{BMO^{\mathcal{L}}(\mathbb{R}^n)} \int_{\gamma(x_k)^2}^{\infty} \left(1 + \frac{t}{\gamma(x_k)^2}\right)^{-1} \left(1 + \log \frac{\sqrt{t}}{\gamma(x_k)}\right) \frac{dt}{t} \\
&\leq C \|f\|_{BMO^{\mathcal{L}}(\mathbb{R}^n)} \int_{1}^{\infty} (1 + \log(u)) \frac{du}{u(1+u)} \\
&\leq C \|f\|_{BMO^{\mathcal{L}}(\mathbb{R}^n)}, \quad x \in Q_k.
\end{aligned}$$

Hence, we obtain

$$\frac{1}{|Q_k|}\int_{Q_k}\Omega_{1,k}(f)(x)dx \leq C\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)}. \tag{18}$$

By combining Eqs. 16, 17 and 18 we get

$$\frac{1}{|Q_k|}\int_{Q_k}V_\rho\left(H^{\mathscr{L}}_{k,t}\right)(f)(x)dx \leq C\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)}. \tag{19}$$

Here $C > 0$ does not depend on k.

We now decompose f as follows

$$f = f\chi_{Q_k^*} + f\chi_{(Q_k^*)^c} = f_1 + f_2.$$

It is clear that

$$V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f) \leq V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f_1) + V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f_2). \tag{20}$$

By proceeding as above we get

$$\begin{aligned}V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f_1)(x) &\leq \sup_{\{t_j\}_{j=1}^\infty\downarrow 0}\left(\sum_{t_j\leq\gamma(x_k)^2}|W^{\mathscr{L}}_{t_j}(f_1)(x) - W^{\mathscr{L}}_{t_{j+1}}(f_1)(x)|^\rho\right)^{1/\rho}\\ &\quad + \sup_{0<t\leq\gamma(x_k)^2}|W^{\mathscr{L}}_t(f_1)(x)|\\ &\leq V_\rho\left(W^{\mathscr{L}}_t\right)(f_1)(x) + W^{\mathscr{L}}_*(f_1)(x),\quad x\in Q_k.\end{aligned}$$

Since $W^{\mathscr{L}}_*$ and $V_\rho(W^{\mathscr{L}}_t)$ are bounded operators from $L^2(\mathbb{R}^n)$ into itself (see Theorem 2.5) it follows that

$$\begin{aligned}\frac{1}{|Q_k|}\int_{Q_k}V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f_1)(x)dx &\leq \left(\frac{1}{|Q_k|}\int_{\mathbb{R}^n}\left(V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f_1)(x)\right)^2dx\right)^{1/2}\\ &\leq C\left(\frac{1}{|Q_k|}\int_{Q_k^*}|f(x)|^2dx\right)^{1/2}.\end{aligned}$$

Then, from [14, Corollary 3], we deduce that

$$\frac{1}{|Q_k|}\int_{Q_k}V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f_1)(x)dx \leq C\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)}. \tag{21}$$

On the other hand, we can write

$$\begin{aligned}V_\rho\left(L^{\mathscr{L}}_{k,t}\right)(f_2)(x) &\leq \int_0^{\gamma(x_k)^2}\int_{(Q_k^*)^c}\left|\frac{\partial}{\partial t}W^{\mathscr{L}}_t(x,y)\right||f(y)|dydt + \sup_{0<t\leq\gamma(x_k)^2}|W^{\mathscr{L}}_t(f_2)(x)|\\ &= \Omega_{3,k}(f)(x) + \Omega_{4,k}(f)(x),\quad x\in Q_k.\end{aligned} \tag{22}$$

According to [14, (2.7)], for certain $C, c > 0$, we get

$$\begin{aligned}
\Omega_{3,k}(f)(x) &\leq \int_0^{\gamma(x_k)^2} \int_{|x-y|>\gamma(x_k)} \left| \frac{\partial}{\partial t} W_t^{\mathscr{L}}(x,y) \right| |f(y)| dy dt \\
&\leq C \int_0^{\gamma(x_k)^2} \int_{|x-y|>\gamma(x_k)} |f(y)| \frac{e^{-c|x-y|^2/t}}{t^{n/2+1}} dy dt \\
&\leq C \int_0^{\gamma(x_k)^2} \frac{1}{t^{n/2+1}} \sum_{j=0}^{\infty} \int_{2^j\gamma(x_k)<|x-y|\leq 2^{j+1}\gamma(x_k)} |f(y)| \left(\frac{t}{|x-y|^2} \right)^{(n+1)/2} dy dt \\
&\leq C \int_0^{\gamma(x_k)^2} \frac{1}{\sqrt{t}} \sum_{j=0}^{\infty} \frac{1}{(2^j\gamma(x_k))^{n+1}} \int_{|x-y|\leq 2^{j+1}\gamma(x_k)} |f(y)| dy dt \\
&\leq C \sum_{j=0}^{\infty} \frac{1}{2^j(2^j\gamma(x_k))^n} \int_{|x_k-y|\leq 2^{j+2}\gamma(x_k)} |f(y)| dy \\
&\leq C\|f\|_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}, \quad x \in Q_k. \qquad (23)
\end{aligned}$$

Then, by Eq. 23

$$\frac{1}{|Q_k|} \int_{Q_k} \Omega_{3,k}(f)(x) dx \leq C\|f\|_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}.$$

Moreover, since $|f| \in \mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$, [14, Theorem 6] implies that

$$\frac{1}{|Q_k|} \int_{Q_k} \Omega_{4,k}(f)(x) dx \leq \frac{1}{|Q_k|} \int_{Q_k} W_*^{\mathscr{L}}(|f|)(x) dx \leq C\|f\|_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}.$$

Hence, we conclude that

$$\frac{1}{|Q_k|} \int_{Q_k} V_\rho \left(L_{k,t}^{\mathscr{L}} \right) (f_2)(x) dx \leq C\|f\|_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}. \qquad (24)$$

By combining Eqs. 20, 21 and 24 we deduce

$$\frac{1}{|Q_k|} \int_{Q_k} V_\rho \left(L_{k,t}^{\mathscr{L}} \right) (f)(x) dx \leq C\|f\|_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}. \qquad (25)$$

Finally, Eqs. 15, 19 and 25 imply that

$$\frac{1}{|Q_k|} \int_{Q_k} V_\rho \left(W_t^{\mathscr{L}} \right) (f)(x) dx \leq C\|f\|_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}.$$

Note that $C > 0$ does not depend on k.

Thus the property (i_k) is established.

Now, we are going to prove assertion (ii_k). Assume that $B = B(x_0, r_0) \subset Q_k^*$, with $x_0 \in \mathbb{R}^n$ and $r_0 > 0$. Our purpose is to check that

$$\frac{1}{|B|} \int_B \left| V_\rho \left(W_t^{\mathscr{L}} \right) (f)(x) - c^B \right| dx \leq C\|f\|_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}, \qquad (26)$$

for a certain constant c^B and with $C>0$ independent of k and B. To this end we decompose $W_t^{\mathscr{L}}(f)$ as follows

$$W_t^{\mathscr{L}}(f)(x)=H_{k,t}^{\mathscr{L}}(f)(x)+\left(L_{k,t}^{\mathscr{L}}(f)-L_{k,t}(f)\right)(x)+L_{k,t}(f)(x),\quad x\in Q_k^* \text{ and } t>0, \tag{27}$$

where $H_{k,t}^{\mathscr{L}}$ and $L_{k,t}^{\mathscr{L}}$ are defined as above, and

$$L_{k,t}(f)(x)=W_t(f)(x)\chi_{\{0<t\leq\gamma(x_k)^2\}}(t),\quad x\in Q_k^*,\quad t>0.$$

Suppose that $c^B=\|h^B\|_{E_\rho}$, where $h^B:(0,\infty)\mapsto\mathbb{C}$ is a function that will be specified later. Then, we can write

$$\begin{aligned}|V_\rho\left(W_t^{\mathscr{L}}\right)(f)(x)-c^B|&=|\|W_t^{\mathscr{L}}(f)(x)\|_{E_\rho}-\|h^B\|_{E_\rho}|\\&\leq\|W_t^{\mathscr{L}}(f)(x)-h^B(t)\|_{E_\rho}\\&\leq\|H_{k,t}^{\mathscr{L}}(f)(x)\|_{E_\rho}+\|L_{k,t}^{\mathscr{L}}(f)(x)\\&\quad-L_{k,t}(f)(x)\|_{E_\rho}+\|L_{k,t}(f)(x)-h^B(t)\|_{E_\rho}.\end{aligned}$$

Therefore, Eq. 26 will be proved if we are able to show the following three inequalities:

(A1) $\frac{1}{|B|}\int_B\|H_{k,t}^{\mathscr{L}}(f)(x)\|_{E_\rho}dx\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}$;
(A2) $\frac{1}{|B|}\int_B\|L_{k,t}^{\mathscr{L}}(f)(x)-L_{k,t}(f)(x)\|_{E_\rho}dx\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}$; and
(A3) $\frac{1}{|B|}\int_B\|L_{k,t}(f)(x)-h^B(t)\|_{E_\rho}dx\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}$,

for a certain function $h^B:(0,\infty)\mapsto\mathbb{C}$, and a constant $C>0$ independent of k and B.

According to (16) we have

$$V_\rho\left(H_{k,t}^{\mathscr{L}}\right)(f)\leq\Omega_{1,k}(f)+\Omega_{2,k}(f).$$

By proceeding as above we get

$$|\Omega_{1,k}(f)(x)|\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)},\quad x\in Q_k^*. \tag{28}$$

Moreover, by [14, (5.4)],

$$|\Omega_{2,k}(f)(x)|\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)},\quad x\in Q_k^*. \tag{29}$$

Then, from Eqs. 28 and 29, (A1) holds.

To establish $(A2)$ we firstly observe that

$$\begin{aligned}V_\rho\left(L_{k,t}^{\mathscr{L}}-L_{k,t}\right)(f)(x)&\leq\int_0^{\gamma(x_k)^2}\int_{\mathbb{R}^n}\left|\frac{\partial}{\partial t}(W_t^{\mathscr{L}}(x,y)-W_t(x-y))\right||f(y)|dydt\\&\quad+\sup_{0<t\leq\gamma(x_k)^2}|W_t^{\mathscr{L}}(f)(x)-W_t(f)(x)|\\&=\Omega_{5,k}(f)(x)+\Omega_{6,k}(f)(x),\quad x\in Q_k^*.\end{aligned}$$

By [14, (5.5)] we get

$$\Omega_{6,k}(f)(x)\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)},\quad x\in Q_k^*. \tag{30}$$

The perturbation formula ([14, (5.25)]) allows us to write

$$\begin{aligned}\frac{\partial}{\partial t}\Big(W_t(x-y)-W_t^{\mathscr{L}}(x,y)\Big) &= \int_{\mathbb{R}^n} V(z)W_{t/2}^{\mathscr{L}}(x,z)W_{t/2}(z-y)dz \\ &\quad+\int_0^{t/2}\int_{\mathbb{R}^n} V(z)\frac{\partial}{\partial t}W_{t-s}(x-z)W_s^{\mathscr{L}}(z,y)dzds \\ &\quad+\int_{t/2}^{t}\int_{\mathbb{R}^n} V(z)W_{t-s}(x-z)\frac{\partial}{\partial s}W_s^{\mathscr{L}}(z,y)dzds \\ &=\sum_{j=1}^{3}K_j(x,y,t),\quad x,y\in\mathbb{R}^n \text{ and } t>0.\end{aligned}$$

According to [14, (2.2) and (2.8)], we get

$$\begin{aligned}|K_1(x,y,t)| &\leq Ct^{-n}\int_{\mathbb{R}^n} V(z)e^{-\frac{|x-z|^2+|z-y|^2}{4t}}dz \\ &\leq Ct^{-n/2}e^{-\frac{|x-y|^2}{16t}}\int_{\mathbb{R}^n} V(z)t^{-n/2}e^{-\frac{|x-z|^2}{8t}}dz \\ &\leq C\gamma(x)^{-\delta}t^{-1+(\delta-n)/2}e^{-\frac{|x-y|^2}{16t}},\quad x,y\in Q_k^* \text{ and } 0<t<\gamma(x_k)^2.\end{aligned}$$

Here and in the sequel δ represents a positive constant.

Moreover, by using [14, (2.2) and (2.8)], since $t/2<t-s<t$ when $0<s<t/2$, it follows that

$$\begin{aligned}|K_2(x,y,t)| &\leq C\int_0^{t/2}\int_{\mathbb{R}^n} V(z)\frac{1}{(t-s)^{1+n/2}}e^{-c\frac{|x-z|^2}{t-s}}\frac{1}{s^{n/2}}e^{-c\frac{|z-y|^2}{s}}dzds \\ &\leq C\int_0^{t/2}\int_{\mathbb{R}^n} V(z)\frac{1}{t^{1+n/2}}e^{-c\frac{|x-z|^2}{t}}\frac{1}{s^{n/2}}e^{-c\frac{|z-y|^2}{s}}dzds \\ &\leq C\frac{1}{t^{1+n/2}}e^{-c\frac{|x-y|^2}{t}}\int_0^{t/2}\int_{\mathbb{R}^n} V(z)\frac{1}{s^{n/2}}e^{-c\frac{|z-y|^2}{s}}dzds \\ &\leq C\frac{1}{t^{1+n/2}}e^{-c\frac{|x-y|^2}{t}}\int_0^{t/2}\frac{s^{-1+\delta/2}}{\gamma(y)^{\delta}}ds \\ &\leq C\gamma(y)^{-\delta}t^{-1+(\delta-n)/2}e^{-c\frac{|x-y|^2}{t}}, \\ &\qquad x\in Q_k^*,\ |y-x|\leq\gamma(x_k) \text{ and } 0<t<\gamma(x_k)^2.\end{aligned}$$

By proceeding in a similar way we obtain

$$\begin{aligned}&|K_3(x,y,t)| \\ &\quad\leq C\gamma(y)^{-\delta}t^{-1+(\delta-n)/2}e^{-c\frac{|x-y|^2}{t}},\quad x\in Q_k^*,\ |x-y|\leq\gamma(x_k) \text{ and } 0<t<\gamma(x_k)^2.\end{aligned}$$

Hence, since $\gamma(x)\sim\gamma(y)\sim\gamma(x_k)$, provided that $|x-y|\leq\gamma(x_k)$ and $x\in Q_k^*$, we conclude that

$$\Big|\frac{\partial}{\partial t}\Big(W_t^{\mathscr{L}}(x,y)-W_t(x-y)\Big)\Big|$$
$$\leq C\gamma(x_k)^{-\delta}t^{-1+(\delta-n)/2}e^{-c\frac{|x-y|^2}{t}},\quad x\in Q_k^*,\ |x-y|\leq\gamma(x_k)\ \text{and}\ 0<t<\gamma(x_k)^2.$$

Therefore for some constants $C, c>0$ we get

$$\begin{aligned}\Omega_{5,k}(f)(x)&\leq C\int_0^{\gamma(x_k)^2}\frac{t^{\delta/2-1}}{\gamma(x_k)^\delta}\int_{\mathbb{R}^n}\frac{e^{-c|x-y|^2/t}}{t^{n/2}}|f(y)|dydt\\&\leq C\int_0^{\gamma(x_k)^2}\frac{t^{\delta/2-1}}{\gamma(x_k)^\delta}\sum_{j=0}^\infty\frac{e^{-c2^{2j}}}{t^{n/2}}\int_{|x-y|\leq 2^j\sqrt{t}}|f(y)|dydt\\&\leq C\int_0^{\gamma(x_k)^2}\frac{t^{\delta/2-1}}{\gamma(x_k)^\delta}\sum_{j=0}^\infty\frac{2^{jn}e^{-c2^{2j}}}{(2^j\sqrt{t})^n}\int_{|x-y|\leq 2^j\sqrt{t}}|f(y)|dydt,\quad x\in Q_k^*.\end{aligned}$$

Moreover, by [14, Lemma 2], since $\gamma(x)\sim\gamma(x_k)$, $x\in Q_k^*$,

$$\begin{aligned}&\sum_{j=0}^\infty\frac{2^{jn}e^{-c2^{2j}}}{(2^j\sqrt{t})^n}\int_{|x-y|\leq 2^j\sqrt{t}}|f(y)|dy\\&=\sum_{j\in\mathbb{N},\,2^j\sqrt{t}\leq\gamma(x)}\frac{2^{jn}e^{-c2^{2j}}}{(2^j\sqrt{t})^n}\int_{|x-y|\leq 2^j\sqrt{t}}|f(y)|dy\\&\quad+\sum_{j\in\mathbb{N},\,2^j\sqrt{t}>\gamma(x)}\frac{2^{jn}e^{-c2^{2j}}}{(2^j\sqrt{t})^n}\int_{|x-y|\leq 2^j\sqrt{t}}|f(y)|dy\\&\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}\left(\sum_{j\in\mathbb{N},\,2^j\sqrt{t}\leq\gamma(x)}2^{jn}e^{-c2^{2j}}\left(1+\log\frac{\gamma(x)}{2^j\sqrt{t}}\right)+\sum_{j\in\mathbb{N},\,2^j\sqrt{t}>\gamma(x)}2^{jn}e^{-c2^{2j}}\right)\\&\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}\left(\frac{\gamma(x_k)}{\sqrt{t}}\right)^\varepsilon,\quad x\in Q_k^*\ \text{and}\ 0<t<\gamma(x_k)^2,\end{aligned}$$

where $\varepsilon\in(0,\delta)$.

With this estimate we have that

$$\Omega_{5,k}(f)(x)\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}\int_0^{\gamma(x_k)^2}\frac{t^{\delta/2-1-\varepsilon/2}}{\gamma(x_k)^{\delta-\varepsilon}}dt\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)},\quad x\in Q_k^*.\quad(31)$$

Putting together Eqs. 30 and 31, we infer $(A2)$.

Next we notice that by Eq. 27 it follows that

$$V_\rho(L_{k,t})(f) \leq V_\rho\left(W_t^{\mathscr{L}}\right)(f) + V_\rho\left(H_{k,t}^{\mathscr{L}}\right)(f) + V_\rho\left(L_{k,t}^{\mathscr{L}} - L_{k,t}\right)(f).$$

By proceeding as in the proof of (i_k) we get

$$\int_{Q_k^*} V_\rho\left(W_t^{\mathscr{L}}\right)(f)(x)dx < \infty.$$

Then, $V_\rho(W_t^{\mathscr{L}})(f)(x) < \infty$, a.e. $x \in Q_k^*$.

From Eqs. 30 and 31 we deduce $V_\rho(L_{k,t}^{\mathscr{L}} - L_{k,t})(f)(x) < \infty$, a.e. $x \in Q_k^*$. Also, by Eqs. 28 and 29, $V_\rho(H_{k,t}^{\mathscr{L}})(f)(x) < \infty$, a.e. $x \in Q_k^*$.

Hence, $V_\rho(L_{k,t})(f)(x) < \infty$, a.e. $x \in Q_k^*$. We consider the following decomposition of f

$$f = (f - f_B)\chi_{B^*} + (f - f_B)\chi_{(B^*)^c} + f_B = f_1 + f_2 + f_3.$$

Note that

$$\begin{aligned} V_\rho(L_{k,t})(f_1)(x) &\leq C\left(\sup_{\{t_j\}_{j=1}^\infty \downarrow 0,\, t_j \leq \gamma(x_k)^2} \left(\sum_{j=1}^\infty |W_{t_j}(f_1)(x) - W_{t_{j+1}}(f_1)(x)|^\rho\right)^{1/\rho}\right. \\ &\qquad \left. + \sup_{0<t\leq\gamma(x_k)^2} |W_t(f_1)(x)|\right) \\ &\leq C(V_\rho(W_t)(f_1)(x) + W_*(f_1)(x)), \end{aligned}$$

where W_* represents the maximal operator defined by $W_*(g) = \sup_{t>0}|W_t(g)|$.

Then, since W_* and $V_\rho(W_t)$ are bounded operators from $L^2(\mathbb{R}^n)$ into itself (see Theorem 2.1), we obtain

$$\begin{aligned} \int_{Q_k^*} |V_\rho(L_{k,t})(f_1)(x)|dx &\leq C\left(|Q_k|\int_{B^*} |f(x) - f_B|^2 dx\right)^{1/2} \\ &\leq C(|B||Q_k|)^{1/2}\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)} < \infty. \end{aligned}$$

Hence $V_\rho(L_{k,t})(f_1)(x) < \infty$, a.e. $x \in Q_k^*$.

Also, since $\int_{\mathbb{R}^n} W_t(x, y)dy = 1$, $x \in \mathbb{R}^n$ and $t > 0$, we get

$$V_\rho(L_{k,t})(f_3)(x) = |f_B| < \infty, \quad x \in \mathbb{R}^n.$$

Therefore, we deduce that $V_\rho(L_{k,t})(f_2)(x) < \infty$, a.e. $x \in Q_k^*$.

Choosing $z_1 \in B$ such that $V_\rho(L_{k,t})(f_2)(z_1) < \infty$, we define $h^B(t) = L_{k,t}(f_2)(z_1)$, $t \in (0,\infty)$.

Suppose that $\{t_j\}_{j=1}^\infty$ is a real decreasing sequence that converges to zero and let $j_k \in \mathbb{N}$ be such that $t_{j_k} \le \gamma(x_k)^2$ and $t_{j_k-1} > \gamma(x_k)^2$. We can write

$$\left(\sum_{j=1}^\infty \left| L_{k,t_j}(f)(x) - L_{k,t_j}(f_2)(z_1) - (L_{k,t_{j+1}}(f)(x) - L_{k,t_{j+1}}(f_2)(z_1))\right|^\rho\right)^{1/\rho}$$

$$= \left(\sum_{j=j_k}^\infty \left| W_{t_j}(f)(x) - W_{t_j}(f_2)(z_1) - (W_{t_{j+1}}(f)(x) - W_{t_{j+1}}(f_2)(z_1))\right|^\rho \right.$$

$$\left. + \left| W_{t_{j_k}}(f)(x) - W_{t_{j_k}}(f_2)(z_1)\right|^\rho\right)^{1/\rho}$$

$$\le C\left(\left(\sum_{j=j_k}^\infty \left| W_{t_j}(f)(x) - W_{t_j}(f_2)(z_1) - (W_{t_{j+1}}(f)(x) - W_{t_{j+1}}(f_2)(z_1))\right|^\rho\right)^{1/\rho}\right.$$

$$\left. + \sup_{0<t\le\gamma(x_k)^2} |W_t(f)(x) - W_t(f_2)(z_1)|\right), \quad x \in Q_k^*,$$

and then

$$\|L_{k,t}(f)(x) - h^B(t)\|_{E_\rho}$$

$$\le C\left(\sup_{\{t_j\}_{j=1}^\infty \downarrow 0,\, 0<t_j\le\gamma(x_k)^2} \left(\sum_{j=1}^\infty \left| W_{t_j}(f)(x) - W_{t_j}(f_2)(z_1)\right.\right.\right.$$

$$\left.\left. - (W_{t_{j+1}}(f)(x) - W_{t_{j+1}}(f_2)(z_1))\right|^\rho\right)^{1/\rho}$$

$$\left. + \sup_{0<t\le\gamma(x_k)^2} |W_t(f)(x) - W_t(f_2)(z_1)|\right), \quad x \in Q_k^*.$$

By taking into account Proposition 3.1 with $\mathscr{A} = \{\{t_j\}_{j=1}^\infty \subset (0,\infty)^{\mathbb{N}} : \{t_j\}_{j=1}^\infty \downarrow 0,\ 0 < t_j \le \gamma(x_k)^2\}$, we obtain

$$\frac{1}{|B|}\int_B \sup_{\{t_j\}_{j=1}^\infty \downarrow 0,\, 0<t_j\le\gamma(x_k)^2} \left(\sum_{j=1}^\infty \left| W_{t_j}(f)(x) - W_{t_j}(f_2)(z_1)\right.\right.$$

$$\left.\left. - (W_{t_{j+1}}(f)(x) - W_{t_{j+1}}(f_2)(z_1))\right|^\rho\right)^{1/\rho} dx$$

$$\le C\|f\|_{BMO^{\mathscr{L}}(\mathbb{R}^n)}. \tag{32}$$

Also, according to [14, pp. 348–349] it follows that

$$\frac{1}{|B|}\int_B \sup_{0<t\leq\gamma(x_k)^2} |W_t(f)(x) - W_t(f_2)(z_1)|dx \leq C\|f\|_{BMO^{\mathcal{L}}(\mathbb{R}^n)}. \tag{33}$$

From Eqs. 32 and 33 we deduce (A3).

Note that the constant $C > 0$ does not depend on k and B in all the occurrences.

Thus the proof of (ii_k) is finished.

4.2 Proof of Theorem 2.8

Let $\rho > 2$. We may assume without lost of generality that $V \in RH_q$ with $q > n$. In fact, from [16], reverse Hölder inequalities are open, i.e., if $g \in RH_s$, then it is also true that $g \in RH_{s+\varepsilon}$ for some $\varepsilon > 0$.

In order to prove that the variation operator $V_\rho(R_\ell^{\mathcal{L},\varepsilon})$ is bounded from $BMO^{\mathcal{L}}(\mathbb{R}^n)$ into itself, we consider, for every $k \in \mathbb{N}$, the local operators defined as

$$R_{\ell,k}^{\mathcal{L}}(f)(x) = PV\int_{|x-y|<\gamma(x_k)} R_\ell^{\mathcal{L}}(x, y)f(y)dy,$$

and

$$R_{\ell,k}(f)(x) = PV\int_{|x-y|<\gamma(x_k)} R_\ell(x-y)f(y)dy.$$

Note that $|y - x_k| \leq 3\gamma(x_k)$ when $x \in Q_k^*$ and $|x-y| < \gamma(x_k)$. Then, if $f \in \mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)$,

$$R_{\ell,k}(f)(x) = \lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x-y|<\gamma(x_k)} R_\ell(x-y)f(y)\chi_{3Q_k}(y)dy, \quad \text{a.e. } x \in Q_k^*,$$

that is, this limit exists for almost all $x \in Q_k^*$ when $f \in \mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)$. Also, $R_{\ell,k}^{\mathcal{L}}(f)(x)$ is defined for almost every $x \in Q_k^*$ when $f \in \mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)$ (see [2, Proposition 1.1]).

Let $f \in \mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)$. We are going to analyze the properties (i_k) and (ii_k) when $H = V_\rho(R_\ell^{\mathcal{L},\varepsilon})$. Let $k \in \mathbb{N}$. We can write

$$\begin{aligned}
V_\rho\left(R_\ell^{\mathcal{L},\varepsilon}\right)(f) &= \left(V_\rho\left(R_\ell^{\mathcal{L},\varepsilon}\right)(f) - V_\rho\left(R_{\ell,k}^{\mathcal{L},\varepsilon}\right)(f)\right)\\
&\quad + \left(V_\rho\left(R_{\ell,k}^{\mathcal{L},\varepsilon}\right)(f) - V_\rho\left(R_{\ell,k}^{\varepsilon}\right)(f)\right) + V_\rho\left(R_{\ell,k}^{\varepsilon}\right)(f)\\
&= F_{1,k} + F_{2,k} + V_\rho\left(R_{\ell,k}^{\varepsilon}\right)(f).
\end{aligned}$$

For $x \in Q_k^*$ we have

$$
\begin{aligned}
|F_{1,k}(x)| &\leq V_\rho\left(R_\ell^{\mathcal{L},\varepsilon} - R_{\ell,k}^{\mathcal{L},\varepsilon}\right)(f)(x) \\
&= \sup_{\{\varepsilon_j\}_{j=1}^\infty \downarrow 0}\left(\sum_{j=1}^\infty \left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j} R_\ell^{\mathcal{L}}(x,y)f(y)dy \right.\right. \\
&\qquad \left.\left. - \int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j,\, |x-y|<\gamma(x_k)} R_\ell^{\mathcal{L}}(x,y)f(y)dy\right|^\rho\right)^{1/\rho} \\
&= \sup_{\{\varepsilon_j\}_{j=1}^\infty \downarrow 0}\left(\sum_{j=1}^\infty \left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j, |x-y|\geq\gamma(x_k)} R_\ell^{\mathcal{L}}(x,y)f(y)dy\right|^\rho\right)^{1/\rho} \\
&\leq \int_{|x-y|>\gamma(x_k)} |R_\ell^{\mathcal{L}}(x,y)||f(y)|dy.
\end{aligned}
$$

Then, according to [4, Lemma 3, (a)], since $\gamma(x_k) \geq M\gamma(x)$, $x \in Q_k^*$, for a certain $0 < M < 1$ that does not depend on $k \in \mathbb{N}$, it follows that

$$
\begin{aligned}
|F_{1,k}(x)| &\leq C\int_{|x-y|>M\gamma(x)} \frac{1}{|x-y|^n}\frac{1}{1+|x-y|/\gamma(x)}|f(y)|dy \\
&\leq C\sum_{j=0}^\infty \int_{M2^j\gamma(x)<|x-y|<M2^{j+1}\gamma(x)} \frac{1}{|x-y|^n}\frac{1}{1+|x-y|/\gamma(x)}|f(y)|dy \\
&\leq C\sum_{j=0}^\infty \frac{1}{2^j}\frac{1}{(2^j\gamma(x))^n}\int_{|x-y|<M2^{j+1}\gamma(x)} |f(y)|dy \leq C||f||_{BMO^{\mathcal{L}}(\mathbb{R}^n)}, \quad x \in Q_k^*.
\end{aligned}
$$

Also, by using [4, Lemma 3, (b)], we obtain

$$
\begin{aligned}
|F_{2,k}(x)| &\leq V_\rho\left(R_{\ell,k}^{\mathcal{L},\varepsilon} - R_{\ell,k}^{\varepsilon}\right)(f)(x) \\
&= \sup_{\{\varepsilon_j\}_{j=1}^\infty \downarrow 0}\left(\sum_{j=1}^\infty \left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j,\, |x-y|<\gamma(x_k)} (R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y))f(y)dy\right|^\rho\right)^{1/\rho} \\
&\leq \int_{|x-y|<\gamma(x_k)} |R_\ell^{\mathcal{L}}(x,y) - R_\ell(x-y)||f(y)|dy \\
&\leq C\int_{|x-y|<\gamma(x_k)} \frac{1}{|x-y|^n}\left(\frac{|x-y|}{\gamma(x)}\right)^{2-n/q}|f(y)|dy, \quad x \in Q_k^*.
\end{aligned}
$$

Then, using Hölder inequality and that $\gamma(x) \sim \gamma(x_k)$, $x \in Q_k^*$, we arrive to

$$
\begin{aligned}
|F_{2,k}(x)| &\leq C\left(\int_{|x-y|<\gamma(x_k)} |x-y|^{(2-n/q-n)r}dy\right)^{1/r}\frac{1}{\gamma(x_k)^{2-n/q}}\left(\int_{|x-y|<\gamma(x_k)} |f(y)|^{r'}dy\right)^{1/r'} \\
&\leq C\left(\frac{1}{\gamma(x_k)^n}\int_{|x-y|<\gamma(x_k)} |f(y)|^{r'}dy\right)^{1/r'} \leq C||f||_{BMO^{\mathcal{L}}(\mathbb{R}^n)}, \quad x \in Q_k^*.
\end{aligned}
$$

Here, $1 < r < n/(n-2+n/q)$.

Since, for $i = 1, 2$, $F_{i,k} \in L^\infty(Q_k^*)$ and $\|F_{i,k}\|_{L^\infty(Q_k^*)} \leq C\|f\|_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)}$, where C does not depend on $k \in \mathbb{N}$, in order to see that the properties (i_k) and (ii_k) hold for $H = V_\rho(R_\ell^{\mathcal{L},\varepsilon})$ it is sufficient to establish those properties for $H = V_\rho(R_{\ell,k}^\varepsilon)$.

Fix again $k \in \mathbb{N}$. Then

$$\begin{aligned}
V_\rho\left(R_{\ell,k}^\varepsilon\right)(f)(x) &= \sup_{\{\varepsilon_j\}_{j=1}^\infty \downarrow 0} \left(\sum_{j=1}^\infty \left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j, |x-y|<\gamma(x_k)} R_\ell(x-y) f(y)dy\right|^\rho\right)^{1/\rho} \\
&= \sup_{\{\varepsilon_j\}_{j=1}^\infty \downarrow 0} \left(\sum_{j=1}^\infty \left|\int_{\varepsilon_{j+1}<|x-y|<\varepsilon_j, |x-y|<\gamma(x_k)} R_\ell(x-y) f(y)\chi_{Q_k^{**}}(y)dy\right|^\rho\right)^{1/\rho} \\
&\leq V_\rho\left(R_{\ell,k}^\varepsilon\right)\left(f\chi_{Q_k^{**}}\right)(x), \quad x \in Q_k^*.
\end{aligned}$$

Hence, according to Theorem 2.3, we have

$$\begin{aligned}
\frac{1}{|Q_k|}\int_{Q_k} |V_\rho\left(R_{\ell,k}^\varepsilon\right)(f)(x)|dx &\leq \left(\frac{1}{|Q_k|}\int_{Q_k} |V_\rho(R_{\ell,k}^\varepsilon)(f\chi_{Q_k^{**}})(x)|^2 dx\right)^{1/2} \\
&\leq C\left(\frac{1}{|Q_k|}\int_{Q_k^{**}} |f(x)|^2 dx\right)^{1/2} \\
&\leq C\|f\|_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)}. \qquad (34)
\end{aligned}$$

Let now $x_0 \in \mathbb{R}^n$ and $r_0 > 0$ such that $B = B(x_0, r_0) \subset Q_k^*$. Then, by using Proposition 3.2 we can see

$$\frac{1}{|B|}\int_B |V_\rho\left(R_{\ell,k}^\varepsilon\right)(f)(x) - V_\rho\left(R_{\ell,k}^\varepsilon\right)(f_2)(z_1)|dx \leq C\|f\|_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)},$$

where $f_2 = (f - f_B)\chi_{(B^{**})^c}$ and $z_1 \in B$ is such that $V_\rho(R_{\ell,k}^\varepsilon)(f_2)(z_1) < \infty$. Hence, $V_\rho(R_{\ell,k}^\varepsilon)(f) \in BMO(Q_k^*)$ and

$$\|V_\rho\left(R_{\ell,k}^\varepsilon\right)(f)\|_{\mathrm{BMO}(Q_k^*)} \leq C\|f\|_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)}. \qquad (35)$$

Note that the constants $C > 0$ appearing in Eqs. 34 and 35 do not depend on $k \in \mathbb{N}$. Thus the proof of the desired result is finished.

Assume now that $V \in RH_q$ where $n/2 < q$. In fact it is sufficient to consider $n/2 < q < n$. We have to show that the variation operator for the adjoint Riesz transform, $V_\rho(\mathcal{R}_\ell^{\mathcal{L},\varepsilon})$, is bounded from $BMO^{\mathcal{L}}(\mathbb{R}^n)$ into itself. Looking at the proof above, this result will be established when we see that, for every $k \in \mathbb{N}$, the operators defined by

$$T_{1,k}(f)(x) = \int_{|x-y|>\gamma(x_k)} |R_\ell^{\mathcal{L}}(y, x)||f(y)|dy,$$

and

$$T_{2,k}(f)(x) = \int_{|x-y|<\gamma(x_k)} |R_\ell^{\mathcal{L}}(y, x) - R_\ell(y-x)||f(y)|dy,$$

map $\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$ into $L^\infty(Q_k^*)$, and, for $i = 1, 2$,

$$||T_{i,k}(f)||_{L^\infty(Q_k^*)} \leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}, \quad f \in \mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n),$$

where $C > 0$ does not depend on $k \in \mathbb{N}$.

Let $k \in \mathbb{N}$ and $f \in \mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)$. According to [27, p. 538], we have that

$$\begin{aligned}|T_{1,k}(f)(x)| &\leq C\Bigg(\int_{|x-y|>\gamma(x_k)} \frac{1}{|x-y|^n} \frac{1}{(1+|x-y|/\gamma(x))^\alpha}|f(y)|dy \\ &\quad + \int_{|x-y|>\gamma(x_k)} \frac{1}{|x-y|^{n-1}} \frac{|f(y)|}{(1+|x-y|/\gamma(x))^\alpha} \int_{B(y,\frac{|x-y|}{4})} \frac{V(z)}{|z-y|^{n-1}}dzdy\Bigg) \\ &= C(T_{1,1,k}(f)(x) + T_{1,2,k}(f)(x)), \quad x \in Q_k^*,\end{aligned}$$

where $\alpha > 0$ will be chosen later large enough.

As it was shown earlier, we have

$$||T_{1,1,k}(f)||_{L^\infty(Q_k^*)} \leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}, \tag{36}$$

provided that $\alpha \geq 1$.

On the other hand, since $\gamma(x) \sim \gamma(x_k)$ when $x \in Q_k^*$, we can write

$$\begin{aligned}|T_{1,2,k}(f)(x)| &\leq C\sum_{j=0}^\infty \frac{1}{2^{j\alpha}(2^j\gamma(x_k))^{n-1}} \int_{2^j\gamma(x_k)<|x-y|\leq 2^{j+1}\gamma(x_k)} |f(y)| \\ &\quad \times \int_{B(y,\frac{|x-y|}{4})} \frac{V(z)}{|z-y|^{n-1}}dzdy \\ &\leq C\sum_{j=0}^\infty \frac{1}{2^{j\alpha}(2^j\gamma(x_k))^{n-1}} \left(\int_{|x-y|\leq 2^{j+1}\gamma(x_k)} |f(y)|^{p_0'}dy\right)^{1/p_0'} \\ &\quad \times \left(\int_{\mathbb{R}^n} \left|\int_{|x-z|<2^{j+2}\gamma(x_k)} \frac{V(z)}{|z-y|^{n-1}}dz\right|^{p_0} dy\right)^{1/p_0}, \quad x \in Q_k^*,\end{aligned}$$

where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$. Then, the L^p-boundedness properties of the fractional integrals [28, p. 354] lead us to

$$\begin{aligned}|T_{1,2,k}(f)(x)| &\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)} \sum_{j=0}^\infty \frac{1}{2^{j\alpha}(2^j\gamma(x_k))^{n-1-n/p_0'}} \\ &\quad \times \left(\int_{|x-z|<2^{j+2}\gamma(x_k)} V(z)^q dz\right)^{1/q}, \; x \in Q_k^*.\end{aligned}$$

By using the properties of V and γ [4, Lemma 1] we obtain, for a certain $\mu > 0$,

$$\left(\int_{|x-z|<2^{j+2}\gamma(x_k)} V(z)^q dz\right)^{1/q} \leq C(2^j\gamma(x_k))^{-n/q'}2^{j\mu}\gamma(x_k)^{n-2}, \quad x \in Q_k^*.$$

By choosing $\alpha > 0$ large enough, it follows that

$$|T_{1,2,k}(f)(x)| \leq C||f||_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)} \sum_{j=0}^{\infty} \frac{1}{2^{j(\alpha+n/p_0-1+n/q'-\mu)}}$$
$$\leq C||f||_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)}, \quad x \in Q_k^*. \tag{37}$$

We conclude from Eqs. 36 and 37 that

$$||T_{1,k}(f)||_{L^\infty(Q_k^*)} \leq C||f||_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)},$$

where $C > 0$ does not depend on $k \in \mathbb{N}$.

According to [27, (5.9)] we get

$$|T_{2,k}(f)(x)| \leq C\left(\int_{|x-y|<\gamma(x_k)} \frac{1}{|x-y|^n}\left(\frac{|x-y|}{\gamma(x)}\right)^{2-n/q}|f(y)|dy\right.$$
$$\left. + \int_{|x-y|<\gamma(x_k)} \frac{1}{|x-y|^{n-1}} \int_{|y-z|<\frac{|x-y|}{4}} \frac{V(z)}{|z-y|^{n-1}}dz|f(y)|dy\right)$$
$$= C(T_{2,1,k}(f)(x) + T_{2,2,k}(f)(x)), \quad x \in Q_k^*.$$

As in the proof of the first part of this theorem we have that

$$||T_{2,1,k}(f)||_{L^\infty(Q_k^*)} \leq C||f||_{\mathrm{BMO}^{\mathcal{L}}(\mathbb{R}^n)}. \tag{38}$$

Also, we can write

$$|T_{2,2,k}(f)(x)| \leq C\sum_{j=0}^{\infty}\int_{2^{-j-1}\gamma(x_k)\leq|x-y|<2^{-j}\gamma(x_k)} \frac{|f(y)|}{(2^{-j}\gamma(x_k))^{n-1}}$$
$$\times \int_{|x-z|<2^{-j+1}\gamma(x_k)} \frac{V(z)}{|y-z|^{n-1}}dzdy$$
$$\leq C\sum_{j=0}^{\infty}\frac{1}{(2^{-j}\gamma(x_k))^{n-1}}\left(\int_{|x-y|<2^{-j}\gamma(x_k)}|f(y)|^{p_0'}dy\right)^{1/p_0'}$$
$$\times\left(\int_{\mathbb{R}^n}\left(\int_{|x-z|<2^{-j+1}\gamma(x_k)}\frac{V(z)}{|y-z|^{n-1}}dz\right)^{p_0}dy\right)^{1/p_0}$$
$$\leq C\sum_{j=0}^{\infty}\frac{1}{(2^{-j}\gamma(x_k))^{n-1-n/p_0'}}\left(\int_{|x-z|<2^{-j+1}\gamma(x_k)}V(z)^q dz\right)^{1/q}$$
$$\times\left(\frac{1}{(2^{-j}\gamma(x_k))^n}\int_{|x-y|<2^{-j}\gamma(x_k)}|f(y)|^{p_0'}dy\right)^{1/p_0'}, \quad x \in Q_k^*,$$

where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{n}$.

Since $V \in RH_q$ and $\gamma(x) \sim \gamma(x_k)$, when $x \in Q_k^*$, we have [4, Lemma 1]

$$\left(\int_{B(x,2^{-j+1}\gamma(x_k))}V(z)^q dz\right)^{1/q} \leq C\gamma(x_k)^{n/q-2}, \quad x \in Q_k^*.$$

Moreover, an argument like the one used to show [14, Lemma 2] allows us to get

$$\left(\frac{1}{(2^{-j}\gamma(x_k))^n}\int_{|x-y|<2^{-j}\gamma(x_k)}|f(y)|^{p_0'}dy\right)^{1/p_0'}\leq Cj||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}.$$

Then,

$$\begin{aligned}|T_{2,2,k}(f)(x)| &\leq C\sum_{j=0}^{\infty}\frac{j}{(2^{-j}\gamma(x_k))^{n/p_0-1}}\gamma(x_k)^{n/q-2}||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}\\ &\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)}, \quad x\in Q_k^*. \qquad (39)\end{aligned}$$

Note that $\frac{n}{p_0}-1=\frac{n}{q}-2<0$.

By combining Eqs. 38 and 39 we conclude that

$$||T_{2,k}(f)||_{L^\infty(Q_k^*)}\leq C||f||_{\mathrm{BMO}^{\mathscr{L}}(\mathbb{R}^n)},$$

where $C>0$ does not depend on $k\in\mathbb{N}$.

Thus the proof is finished.

References

1. Bennet, C., DeVore, R.A., Sharpley, R.: Weak-L^∞ and BMO. Ann. Math. (2) **113**(3), 601–611 (1981)
2. Betancor, J.J., Fariña, J.C., Harboure, E., Rodríguez-Mesa, L.: L^p-boundedness properties of variation and oscillation operators in the Schrödinger setting. Rev. Mat. Complut. (2012). doi:10.1007/s13163-012-0094-y
3. Bongioanni, B., Harboure, E., Salinas, O.: Weighted inequalities for negative powers of Schrödinger operators. J. Math. Anal. Appl. **348**(1), 12–27 (2008)
4. Bongioanni, B., Harboure, E., Salinas, O.: Riesz transform related to Schrödinger operators acting on BMO type spaces. J. Math. Anal. Appl. **357**, 115–131 (2009)
5. Bongioanni, B., Harboure, E., Salinas, O.: Commutators of Riesz transform related to Schrödinger operators. J. Fourier Anal. Appl. **17**(1), 115–134 (2011)
6. Bourgain, J.: Pointwise ergodic theorems for arithmetic sets. Publ. Math. IHES **69**, 5–45 (1989)
7. Campbell, J.T., Jones, R.L., Reinhold, K., Wierdl, M.: Oscillation and variation for the Hilbert transform. Duke Math. J. **105**(1), 59–83 (2000)
8. Campbell, J.T., Jones, R.L., Reinhold, K., Wierdl, M.: Oscillation and variation for singular integrals in higher dimensions. Trans. Am. Math. Soc. **355**(5), 2115–2137 (2003)
9. Crescimbeni, R., Macías, R.A., Menárguez, T., Torrea, J.L., Viviani, B.: The ρ-variation as an operator between maximal operators and singular integrals. J. Evol. Equ. **9**(1), 81–102 (2009)
10. Crescimbeni, R., Martín-Reyes, F., de la Torre, A., Torrea, J.L.: The ρ-variation of the Hermitian Riesz transform. Acta Math. Sin. Engl. Ser. **26**(10), 1827–1838 (2010)
11. Dong, J.-F., Liu, H.-P.: The BMO_L space and Riesz transforms associated with Schrödinger operators. Acta Math. Sin. **26**(4), 659–668 (2010)
12. Dziubański, J., Zienkiewicz, J.: Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. Rev. Mat. Iberoam. **15**(2), 279–296 (1999)
13. Dziubański, J., Zienkiewicz, J.: H^p spaces for Schrödinger operators. In: Fourier Analysis and Related Topics, vol. 56, pp. 45–53. Banach Center Publications (2002)
14. Dziubański, J., Garrigós, G., Martínez, T., Torrea, J.L., Zienkiewicz, J.: BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. Math. Z. **249**(2), 329–356 (2005)
15. Dziubański, J., Zienkiewicz, J.: H^p spaces associated with Schrödinger operators with potentials from reverse Hölder classes. Colloq. Math. **98**(1), 5–38 (2003)
16. Gehring, F.W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. **130**, 265–277 (1973)

17. Gillespie, T.A., Torrea, J.L.: Dimension free estimates for the oscillation of Riesz transforms. Isr. J. Math. **141**, 125–144 (2004)
18. Guo, Z., Li, P., Peng, L.: L^p boundedness of commutators of Riesz transforms associated to Schrödinger operator. J. Math. Anal. Appl. **341**, 421–432 (2008)
19. Harboure, E., Macías, R., Menárguez, T., Torrea, J.L.: Oscillation and variation for the Gaussian Riesz transforms and Poisson integral. Proc. R. Soc. Edinb. Sect. A **135**(1), 85–104 (2005)
20. Jones, R.L., Wang, G.: Variation inequalities for the Fejer and Poisson kernels. Trans. Am. Math. Soc. **356**(11), 4493–4518 (2004)
21. Jones, R.L., Kaufman, R., Rosenblatt, J., Wierdl, M.: Oscillation in ergodic theory. Ergod. Theory Dyn. Syst. **18**(4), 889–936 (1998)
22. Jones, R.L., Reinhold, K.: Oscillation and variation inequalities for convolution powers. Ergod. Theory Dyn. Syst. **21**(6), 1809–1829 (2001)
23. Jones, R.L., Seeger, A., Wright, J.: Strong variational and jump inequalities in harmonic analysis. Trans. Am. Math. Soc. **360**(12), 6711–6742 (2008)
24. Liu, Y., Ding, Y.: Some estimates of Schrödinger-type operators with certain nonnegative potentials. Int. J. Math. Math. Sci., article ID 214030, 8 pp. (2008). doi:10.1155/2008/214030
25. Liu, Y., Dong, J.: Some estimates of higher order Riesz transform related to Schrödinger-type operators. Potential Anal. **32**(1), 41–55 (2010)
26. Oberlin, R., Seeger, A., Tao, T., Thiele, C., Wright, J.: A variation norm Carleson theorem. J. Eur. Math. Soc. **14**(2), 421–464 (2012)
27. Shen, Z.W.: L^p estimates for Schrödinger operators with certain potentials. Ann. Inst. Fourier (Grenoble) **45**(2), 513–546 (1995)
28. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, vol. 43, xiv+695 (1993). With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III
29. Sugano, S.: L^p estimates for some Schrödinger operators and a Calderón-Zygmund operator of Schrödinger type. Tokyo J. Math. **30**, 179–197 (2007)
30. Torchinsky, A.: Real-Variable Methods in Harmonic Analysis. Dover, Mineola (2004)
31. Yang, D., Yang, D.: Characterizations of localized BMO($\mathbb{R}^n$) via commutators of localized Riesz transforms and fractional integrals associated to Schrödinger operators. Collect. Math. **61**(1), 65–79 (2010)
32. Yang, D., Yang, D., Zhou, Y.: Endpoint properties of localized Riesz transforms and fractional integrals associated to Schrödinger operators. Potential Anal. **30**, 271–300 (2009)
33. Zhong, J.: Harmonic analysis for some Schrödinger type operators. Ph.D. thesis, Princeton University (1993)