
Volume 319 Number 6, March 15, 2008 ISSN 0021-8693
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Available online at www.sciencedirect.com
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Journal of Algebra 319 (2008) 2295–2318
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On conformal bialgebras

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Received 9 September 2006

Communicated by Efim Zelmanov

Abstract

We study conformal algebras from the point of view of conformal dual of classical Lie coalgebra structures. We define the notions of Lie conformal coalgebra and bialgebra. We obtain a conformal analog of the CYBE, the Manin triples and Drinfeld's double. With the definition of vertex duals, we obtain a natural description of the Lie algebra associated to a conformal algebra as a convolution algebra, clarifying the classical constructions in the theory of conformal algebras and vertex algebras.

Keywords: Lie bialgebra; Lie conformal algebra

1. Introduction

The notion of conformal algebra was introduced by V. Kac as a formal language describing the singular part of the operator product expansion in two-dimensional conformal field theory. Classification problems, cohomology theory and representation theory have been developed (see [2,3,6,8,9], and references therein).

In the present work, we study conformal algebras from the point of view of conformal dual of classical Lie coalgebra structures. We introduce the notions of Lie conformal coalgebra and bialgebra (see Section 2). In Section 3, we obtain a conformal analog of the CYBE, we study coboundary Lie bialgebras, and a conformal version of Manin triples and Drinfeld's double. Usually, in the theory of conformal algebras the proofs of conformal version of classical results need to be carefully translated, as in the present work.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.11.031

Two Lie algebras are usually associated to a Lie conformal algebra R, that is $\mathrm{Lie}(R)$ and the annihilation algebra (see [8]). Their construction, at first sight, is not natural unless you look at a similar notion from vertex algebra theory. In Section 4, using the language of coalgebras, we will see them as convolution algebras of certain type, obtaining a more natural and conceptual construction of them.

A generalization to the language of H-pseudoalgebras [1] will appear in [4]. The associative version of this work is in [10].

2. Conformal bialgebra

2.1. Definitions

First we introduce the basic definitions and notations, see [3,6,8].

Definition 2.1. A *Lie conformal algebra* R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map,

$$R \otimes R \to \mathbb{C}[\lambda] \otimes R, \qquad a \otimes b \mapsto [a_\lambda b]$$

called the λ-bracket, and satisfying the following axioms $(a, b, c \in R)$,

Conformal sesquilinearity: $[\partial a_\lambda b] = -\lambda[a_\lambda b]$, $[a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b]$,
Skew-symmetry: $[a_\lambda b] = -[b_{-\lambda-\partial} a]$,
Jacobi identity: $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]$.

As usual in the theory of conformal algebras, the RHS of skew-symmetry means that we have to take $[b_\mu a]$, expand as a polynomial in μ with coefficients in R and then evaluate $\mu = -\lambda - \partial$ with the corresponding action of ∂ in the coefficients.

If we consider the expansion

$$[a_\lambda b] = \sum_n \frac{\lambda^n}{n!}(a_{(n)} b),$$

the coefficients of $\frac{\lambda^n}{n!}$ are called the (n)-products, and the definition can be written in terms of them.

A Lie conformal algebra is called *finite* if it has finite rank as $\mathbb{C}[\partial]$-module. The notions of homomorphism, ideal and subalgebras of a Lie conformal algebra are defined in the usual way.

Definition 2.2. A *module* M over a Lie conformal algebra R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes M \to \mathbb{C}[\lambda] \otimes M$, $a \otimes v \mapsto a_\lambda v$, satisfying the following axioms $(a, b \in R)$, $v \in M$,

$(M1)_\lambda$ $(\partial a)^M_\lambda v = [\partial^M, a^M_\lambda] v = -\lambda a^M_\lambda v$,

$(M2)_\lambda$ $[a^M_\lambda, b^M_\mu] v = [a_\lambda b]^M_{\lambda+\mu} v$.

An R-module M is called *finite* if it is finitely generated over $\mathbb{C}[\partial]$. Sometimes it will be convenient to consider λ-actions with values in the formal power series $M[\![\lambda]\!]$.

Definition 2.3. Given two $\mathbb{C}[\partial]$-modules U and V, a *conformal linear map* from U to V is a $\mathbb{C}$-linear map $a : U \to \mathbb{C}[\lambda] \otimes_{\mathbb{C}} V$, denoted by $a_\lambda : U \to V$, such that $[\partial, a_\lambda] = -\lambda a_\lambda$, that is $\partial^V a_\lambda - a_\lambda \partial^U = -\lambda a_\lambda$. The vector space of all such maps, denoted by $\operatorname{Chom}(U, V)$, is a $\mathbb{C}[\partial]$-module with

$$(\partial a)_\lambda := -\lambda a_\lambda. \tag{2.1}$$

We define the *conformal dual* of a $\mathbb{C}[\partial]$-module U as $U^{*c} = \operatorname{Chom}(U, \mathbb{C})$, where $\mathbb{C}$ is viewed as the trivial $\mathbb{C}[\partial]$-module, that is

$$U^{*c} = \{a : U \to \mathbb{C}[\lambda] \mid \mathbb{C}\text{-linear and } a_\lambda(\partial b) = \lambda a_\lambda(b)\}. \tag{2.2}$$

We shall see that this is the right dual notion in the category of conformal algebras and modules.

Now, we define $\operatorname{gc} V := \operatorname{Chom}(V, V)$ and, provided that V is a finite $\mathbb{C}[\partial]$-module, $\operatorname{gc} V$ has a canonical structure of a Lie conformal algebra defined by

$$[a_\lambda b]_\mu v = a_\lambda(b_{\mu-\lambda} v) - b_{-\lambda-\partial}(a_{\mu+\lambda+\partial} v), \quad a, b \in \operatorname{gc} V,\ v \in V. \tag{2.3}$$

$\operatorname{gc} V$ is called the *general Lie conformal algebra of* V [6,8].

Remark 2.4. Observe that, by definition, a structure of a conformal module over a Lie conformal algebra R in a finite $\mathbb{C}[\partial]$-module V is the same as a homomorphism of R to the Lie conformal algebra $\operatorname{gc} V$.

If U and V are modules over a Lie conformal algebra R, then $\operatorname{Chom}(U, V)$ also has an R-module structure defined by

$$\left(a_\lambda^N \varphi\right)_\mu u = a_\lambda^V(\varphi_{\mu-\lambda} u) - \varphi_{\mu-\lambda}\left(a_\lambda^U u\right), \tag{2.4}$$

where $a \in R$, $\varphi \in \operatorname{Chom}(U, V)$ and $u \in U$. Therefore, one particular case is the contragradient conformal R-module $U^{*c} = \operatorname{Chom}(U, \mathbb{C})$, also called the *conformal module*, where $\mathbb{C}$ is viewed as the trivial R-module and $\mathbb{C}[\partial]$-module.

We shall need the following proposition

Proposition 2.5. *Let $\{e_i\}_{i=1}^n$ be a $\mathbb{C}[\partial]$-basis of a Lie conformal algebra R and let $\{u_j\}_{j=1}^m$ be the $\mathbb{C}[\partial]$-basis of a conformal R-module U. Then the R-action on U^{*c} is explicitly given by*

$$e_i{}_\lambda u_j^* = -\sum_k D_{ik}^j(\lambda, -\lambda - \partial) u_k^*,$$

where $e_i{}_\lambda u_j = \sum_k D_{ij}^k(\lambda, \partial) u_k$ and $\{u_j^*\}$ is the dual $\mathbb{C}[\partial]$-basis in U^{*c}.

We also define the tensor product $U \otimes V$ of R-modules as the ordinary tensor product with $\mathbb{C}[\partial]$-module structure ($u \in U, v \in V$):

$$\partial(u \otimes v) = \partial u \otimes v + u \otimes \partial v$$

and λ-action defined by ($r \in R$):

$$r_\lambda(u \otimes v) = r_\lambda u \otimes v + u \otimes r_\lambda v.$$

Proposition 2.6. *(See [3].) Let U and V be two R-modules. Suppose that U has finite rank as a $\mathbb{C}[\partial]$-module. Then $U^{*c} \otimes V \simeq \mathrm{Chom}(U, V)$ as R-modules, with the identification $(f \otimes v)_\lambda(u) = f_{\lambda+\partial^V}(u)v$, $f \in U^{*c}$, $u \in U$ and $v \in V$.*

Example 2.7. The *Virasoro conformal algebra* is defined by:

$$\mathrm{Vir} = \mathbb{C}[\partial]l, \qquad [l_\lambda l] = (\partial + 2\lambda)l.$$

Example 2.8. Let $\mathfrak{g}$ be a Lie algebra. The *current conformal algebra* associated to $\mathfrak{g}$ is defined by:

$$\mathrm{Cur}\,\mathfrak{g} = \mathbb{C}[\partial] \otimes \mathfrak{g}, \qquad [a_\lambda b] = [a, b], \quad a, b \in \mathfrak{g}.$$

It is known [6] that the conformal algebras $\mathrm{Cur}\,\mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, and Vir exhaust all finite simple conformal algebras. The most important example of infinite Lie conformal algebra is the following.

Example 2.9. For any positive integer n, we define (see (2.3)):

$$\mathrm{gc}_n := \mathrm{gc}\,\mathbb{C}[\partial]^n.$$

There is a natural isomorphism (see [3])

$$\mathrm{gc}_n \simeq \mathrm{Mat}_n\mathbb{C}[\partial, x]$$

and the λ-bracket become:

$$\big[A(\partial, x)_\lambda B(\partial, x)\big] = A(-\lambda, x + \lambda + \partial)B(\lambda + \partial, x) - B(\lambda + \partial, -\lambda + x)A(-\lambda, x)$$

for any $A(\partial, x), B(\partial, x) \in \mathrm{Mat}_n\mathbb{C}[\partial, x]$.

The natural λ-action of gc_n on $\mathbb{C}[\partial]^n$ is

$$A(\partial, x)_\lambda v(\partial) = A(-\lambda, \lambda + \partial + \alpha)v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^n.$$

In general, given a module M over a Lie conformal algebra R and $\alpha \in \mathbb{C}$, we may construct the α-twisted module M_α by replacing ∂ by $\partial + \alpha$ in the formulas for action of R on M.

The gc_n-modules $\mathbb{C}[\partial]^n_\alpha$ and $(\mathbb{C}[\partial]^{n*})_\alpha$, where $\alpha \in \mathbb{C}$, exhaust all finite irreducible gc_n-modules. This is a result of Kac, Radul and Wakimoto. Moreover, these authors completely described all finite gc_n-modules, which amounted to prove a complete reducibility result for finite modules over the annihilation algebra (see [9]).

Natural conformal analogs to the conformal orthogonal and symplectic Lie algebras appeared in [3], and a conjecture on the classification of all infinite conformal subalgebras of gc_n that act irreducible on $\mathbb{C}[\partial]^n$ was stated.

As a motivation for the definition of conformal coalgebra and bialgebra, we use the cohomology of conformal algebras [2], in order to get to the right notion of cocycle that will be the compatibility condition between λ-bracket and coproduct.

Looking at basic complex, a 1-cochain is a map

$$\delta : R \to (R \otimes R)[\lambda]$$

such that $\delta_\lambda(\partial a) = -\lambda\delta_\lambda(a)$ (see Section 3.1 for details). For example, given $\gamma \in R \otimes R$, $\delta_\lambda(a) = a_\lambda \gamma$ is a 1-cochain. The condition $d\delta = 0$ becomes

$$a_\lambda\big(\delta_\mu(b)\big) - b_\mu\big(\delta_\lambda(a)\big) = \delta_{\lambda+\mu}\big([a_\lambda b]\big).$$

But in the reduced complex (we have to take quotient by $(\lambda + \partial_M)\widetilde{C}^1$, here $M = R \otimes R$, for details see [2, p. 570], or Section 3.1 below), a 1-cochain is

$$\delta : R \to (R \otimes R)$$

such that $\delta(\partial a) = \partial\delta(a)$ ($\mathbb{C}[\partial]$-homomorphism). This give us the following definition:

Definition 2.10. A *conformal Lie coalgebra* R is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}[\partial]$-homomorphism

$$\delta : R \to \wedge^2 R$$

such that

$$(I \otimes \delta)\delta - \tau_{12}(I \otimes \delta)\delta = (\delta \otimes I)\delta,$$

where $\tau_{12}(a \otimes b \otimes c) = b \otimes a \otimes c$.

That is, the standard definition of coalgebra, plus a compatible $\mathbb{C}[\partial]$-structure.

Definition 2.11. A *conformal Lie bialgebra* is a triple $(R, [\ {}_\lambda\ ], \delta)$ such that $(R, [\ {}_\lambda\ ])$ is a conformal algebra, (R, δ) is a conformal coalgebra, and they satisfy the cocycle condition:

$$a_\lambda\big(\delta(b)\big) - b_{-\lambda-\partial}\big(\delta(a)\big) = \delta\big([a_\lambda b]\big). \tag{2.5}$$

2.2. Duality

Now, the natural question is if the "dual" of one structure produce the other, at least in finite rank.

Proposition 2.12. *Let $\Phi : R^{*c} \otimes R^{*c} \to \mathbb{C}[\mu] \otimes (R \otimes R)^{*c}$ given by*

$$\big[\Phi_\mu(f \otimes g)\big]_\lambda(r \otimes r') = f_\mu(r) g_{\lambda-\mu}(r'). \tag{2.6}$$

Then we have:

(a) $\Phi_\mu(\partial f \otimes g) = -\mu \Phi_\mu(f \otimes g)$ *and* $\Phi_\mu(f \otimes \partial g) = (\partial + \mu)\Phi_\mu(f \otimes g)$.
(b) Φ *is a homomorphism of* $\mathbb{C}[\partial]$*-modules.*

Proof. Straightforward computation. □

We shall use the standard notation:

$$\delta(r) = \sum r_{(1)} \otimes r_{(2)}. \tag{2.7}$$

For any $f, g \in U^{*c}$ and $u, v \in U$, we define:

$$(f \otimes g)_{\mu,\lambda}(u \otimes v) = f_\mu(u) g_\lambda(v). \tag{2.8}$$

Proposition 2.13. (a) *Let* (R, δ) *be a finite Lie conformal coalgebra, then* $R^{*c} = \mathrm{Chom}(R, \mathbb{C})$ *is a Lie conformal algebra with the following bracket* ($f, g \in R^{*c}$):

$$\big([f_\mu g]\big)_\lambda(r) = \sum f_\mu(r_{(1)}) g_{\lambda-\mu}(r_{(2)}) = (f \otimes g)_{\mu,\lambda-\mu}\big(\delta(r)\big), \tag{2.9}$$

where $\delta(r) = \sum r_{(1)} \otimes r_{(2)}$.

(b) *Let* $(R, [\ _\lambda\ ])$ *be a Lie conformal algebra free of finite rank, that is* $R = \bigoplus_{i=1}^n \mathbb{C}[\partial]a^i$*, then* $R^{*c} = \mathrm{Chom}(R, \mathbb{C}) = \bigoplus_{i=1}^n \mathbb{C}[\partial]a_i$*, where* $\{a_i\}$ *is a dual* $\mathbb{C}[\partial]$*-basis in the sense that* $(a_i)_\lambda(a^j) = \delta_{ij}$*, is a Lie conformal coalgebra with the following co-bracket*:

$$\delta(f) = \sum_{i,j} f_\mu\big(\big[a^i_\lambda a^j\big]\big)(a_i \otimes a_j)|_{\lambda=\partial\otimes 1,\ \mu=-\partial\otimes 1-1\otimes\partial}. \tag{2.10}$$

More precisely, if

$$\big[a^i_\lambda a^j\big] = \sum_k P_k^{ij}(\lambda, \partial) a^k,$$

where P_k^{ij} *are some polynomials in* λ *and* ∂*, then the co-bracket is*

$$\delta(a_k) = \sum_{i,j} Q_k^{ij}(\partial \otimes 1, 1 \otimes \partial) a_i \otimes a_j,$$

where $Q_k^{ij}(x, y) = P_k^{ij}(x, -x-y)$.

Proof. (a) We are basically using the map of Proposition 2.12, that replace the classical inclusion $R^* \otimes R^* \to (R \otimes R)^*$, that is

$$\Phi : R^{*c} \otimes R^{*c} \to \mathbb{C}[\mu] \otimes (R \otimes R)^{*c}, \tag{2.11}$$

with

$$\big[\Phi_\mu(f \otimes g)\big]_\lambda(r \otimes r') = f_\mu(r) g_{\lambda-\mu}(r').$$

Conformal sesquilinearity follows by (2.1), Proposition 2.12 and the definition of the bracket in (2.9). Skew-symmetry follows by (2.1) and the skew-symmetry of the coproduct:

$$\begin{aligned}[f_\lambda g]_\mu(r) &= \sum f_\lambda(r_{(1)}) g_{\mu-\lambda}(r_{(2)}) = -\sum g_{\mu-\lambda}(r_{(1)}) f_\lambda(r_{(2)}) \\ &= -[g_{-\lambda+\mu} f]_\mu(r) = -[g_{-\lambda-\partial} f]_\mu(r).\end{aligned}$$

It remains to prove Jacobi. Let $f, g, h \in R^{*c}$, then

$$\begin{aligned}&\big([f_\lambda[g_\mu h]] - [[f_\lambda g]_{\lambda+\mu} h] - [g_\mu[f_\lambda h]]\big)_\nu(r) \\ &\quad= \sum \big(\big(f \otimes [g_\mu h]\big)_{\lambda,\nu-\lambda} - \big([f_\lambda g] \otimes h\big)_{\lambda+\mu,\nu-\lambda-\mu} - \big(g \otimes [f_\lambda h]\big)_{\mu,\nu-\mu}\big)(r_{(1)} \otimes r_{(2)}) \\ &\quad= \sum (f \otimes g \otimes h)_{\lambda,\mu,\nu-\lambda-\mu}\big(r_{(1)} \otimes (r_{(2)})_{(1)} \otimes (r_{(2)})_{(2)} - (r_{(1)})_{(1)} \otimes (r_{(1)})_{(2)} \otimes r_{(2)} \\ &\qquad - (r_{(2)})_{(1)} \otimes r_{(1)} \otimes (r_{(2)})_{(2)}\big)\end{aligned}$$

and the last term is co-Jacobi identity.

(b) It is clear that both expressions for δ are equivalent since, using (2.10), we have

$$\begin{aligned}\delta(a_k) &= \sum_{i,j} (a_k)_\mu\big(\big[a^i{}_\lambda a^j\big]\big)(a_i \otimes a_j)|_{\lambda=\partial\otimes 1,\ \mu=-\partial\otimes 1-1\otimes\partial} \\ &= \sum_{i,j} (a_k)_\mu\bigg(\sum_l P_l^{ij}(\lambda,\partial) a^l\bigg)(a_i \otimes a_j)|_{\lambda=\partial\otimes 1,\ \mu=-\partial\otimes 1-1\otimes\partial} \\ &= \sum_{i,j} P_k^{ij}(\lambda,\mu)(a_i \otimes a_j)|_{\lambda=\partial\otimes 1,\ \mu=-\partial\otimes 1-1\otimes\partial} \\ &= \sum_{i,j} Q_k^{ij}(\partial\otimes 1, 1\otimes\partial) a_i \otimes a_j.\end{aligned}$$

The skew-symmetry of R is equivalent to $P_k^{ij}(\lambda,\partial) = -P_k^{ji}(-\lambda-\partial,\partial)$, that translate to $Q_k^{ij}(x,y) = -Q_k^{ji}(y,x)$, that is, $\delta(f) \in \wedge^2 R$. Let us check co-Jacobi identity. We have

$$\begin{aligned}&(I\otimes\delta)\delta(a_k) - \tau_{12}(I\otimes\delta)\delta(a_k) \\ &\quad= \sum_{i,j,l,r} Q_k^{ij}(\partial\otimes 1\otimes 1, 1\otimes\partial\otimes 1 + 1\otimes 1\otimes\partial) \\ &\qquad\times Q_j^{lr}(1\otimes\partial\otimes 1, 1\otimes 1\otimes\partial)(a_i\otimes a_l\otimes a_r) - \tau_{12}\quad \text{(the same term)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\delta\otimes I)\delta(a_k) = \sum_{i,j,l,r} &Q_k^{ij}(\partial\otimes 1\otimes 1 + 1\otimes\partial\otimes 1, 1\otimes 1\otimes\partial) \\ &\times Q_i^{lr}(\partial\otimes 1\otimes 1, 1\otimes\partial\otimes 1)(a_l\otimes a_r\otimes a_j).\end{aligned}$$

Writing these identities in terms of P_k^{ij}, it is easy to see that it is equivalent to conformal Jacobi identity, finishing the proof. □

Remark 2.14. (a) Proposition 2.13 hold if we replace Lie conformal algebras by associative conformal algebras. The details of analogous results and definitions will appear in [10].

(b) The appearance of the relation $Q(x,y)=P(x,-x-y)$ in Proposition 2.13 shows that there is a natural interpretation in the language of Lie pseudo-algebras (cf. Eq. (1.5) in [1]). A detail study will appear in [4].

2.3. Examples

Example 2.15. Let $R=\mathbb{C}[\partial]L$ and $\delta(p(\partial)L)=p(\partial^{\otimes})(\partial L\otimes L-L\otimes\partial L)$ with $\partial^{\otimes}:=\partial\otimes 1+1\otimes\partial$, then it is a conformal coalgebra, and (R^{*c},δ^{*c}) is isomorphic to Vir as conformal algebras. But it is not a bialgebra structure on Vir since

$$L_\lambda\delta(L)-L_{-\lambda-\partial^{\otimes}}\delta(L)=5\delta[L_\lambda L].$$

Proposition 2.16. *If R is a conformal Lie coalgebra free of rank 1, then $R\simeq\mathbb{C}[\partial]L$ with $\delta\equiv 0$ or $\delta(L)=c(\partial L\otimes L-L\otimes\partial L)$, $c\in\mathbb{C}$. Therefore, there is no non-trivial conformal Lie bialgebra of rank 1.*

Proof. Since any non-trivial conformal Lie algebra of rank 1 is isomorphic to Vir (see [6, p. 386]), the result follows. □

Example 2.17. Let $(\mathfrak{g},[,],\bar{\delta})$ be a Lie bialgebra. Now $\mathrm{Cur}(\mathfrak{g})$ has a natural conformal Lie bialgebra structure defined by

$$\delta\big(p(\partial)a\big)=p(\partial\otimes 1+1\otimes\partial)\bar{\delta}(a)$$

but not all the bialgebra structures on $\mathrm{Cur}(\mathfrak{g})$ are of this form, as it is shown in the next example.

Example 2.18. Fix $p(\lambda)\in\mathbb{C}[\lambda]$ and let $R_p=\mathbb{C}[\partial]a\oplus\mathbb{C}[\partial]b$ be a rank 2 solvable Lie conformal algebra with λ-brackets given by (extend it by skew-symmetry and sesquilinearity)

$$[a_\lambda a]=0=[b_\lambda b],\qquad [a_\lambda b]=p(\lambda)b.$$

It is possible to see that $R_p\simeq R_q$ with $p,q\in\mathbb{C}[\lambda]$ if and only if $p(\lambda)=cq(\lambda)$, with $0\neq c\in\mathbb{C}$. We do not plan to give an exhaustive classification of conformal Lie bialgebra structures on R_p, instead, we shall study bialgebra structures on R_p whose underlying coalgebra structure comes from the dual of a solvable Lie conformal algebra R_q, $q\in\mathbb{C}[\lambda]$. That is, fix $q\in\mathbb{C}[\lambda]$, then by applying Proposition 2.13 to R_q we obtain a Lie conformal coalgebra structure on R_p by taking $\delta_q:R_p\to\wedge^2R_p$ given by

$$\delta_q(a)=0,\qquad \delta_q(b)=q(\partial)a\wedge b.$$

A simple computation shows that δ_q is a bialgebra structure on R_p if and only if $p(x)q(x)$ is an odd polynomial (i.e. $-p(x)q(x)=p(-x)q(-x)$). Observe that distinct q satisfying this condition produce non-isomorphic bialgebra structures in R_p.

In the special case of $p(\lambda) \equiv 1$, we have that $R_p \simeq \mathrm{Cur}(T_2)$ where T_2 is the 2-dimensional Lie algebra considered in Examples 2.2 and 3.2 in [7]. In this case every odd polynomial q produce a non-isomorphic bialgebra structure in $\mathrm{Cur}(T_2)$, obtaining bialgebra structures that do not come from bialgebra structures in T_2, as in the previous example. Moreover, in order to see how different is the situation from the classical case, observe that if $q(x) = x$, then $\delta_q(t) = (dr)_\lambda(t)|_{\lambda=-\partial^{\otimes 2}} = t_\lambda r|_{\lambda=-\partial^{\otimes 2}}$ for all $t \in R_p$, where $r = \frac{1}{2}(\partial x \otimes x - x \otimes \partial x)$ (cf. (3.2) below), showing that there are coboundary structures $\delta = dr$ (see next section for the definition) in $\mathrm{Cur}(T_2)$ with $\delta(a) = 0$ and such structures are not present in the Lie algebra T_2 (see Example 3.2 in [7]).

Example 2.19. Let $\mathrm{gc}_1 = \mathbb{C}[\partial, x]$ as in Example 2.9. By direct computations, it is non-trivial to see that

$$\delta(x^n) = x^n_\lambda(\partial \otimes 1 - 1 \otimes \partial)|_{\lambda=-(\partial\otimes 1+1\otimes\partial)}$$

gives a conformal Lie bialgebra structure on gc_1. It can be explicitly written as follows

$$\delta(x^n) = \sum_{i=1}^{n} \binom{n}{i} \left[x^{n-i} \otimes (-\partial)^{i+1} - (-\partial)^{i+1} \otimes x^{n-i}\right].$$

This is an example of coboundary conformal Lie bialgebra defined in the following section.

Remark 2.20. The examples presented here shows that this theory is richer than the classical Lie bialgebra theory. We are far from classification results in this context. Observe that it is not known if a conformal version of Whitehead's lemma holds for $\mathrm{Cur}(\mathfrak{g})$, with $\mathfrak{g}$ simple.

3. Coboundary conformal Lie bialgebras

In this section we study a very important class of conformal Lie algebras, for which the coalgebra structure comes from a 1-coboundary of the algebra.

3.1. Cohomology of conformal algebras

For completeness, we shall present the definition given in [2].

Definition 3.1. An *n-cochain* ($n \in \mathbb{Z}_+$) *of a conformal Lie algebra R with coefficients in an R-module M* is a $\mathbb{C}$-linear map

$$\gamma : R^{\otimes n} \to M[\lambda_1, \dots, \lambda_n], \qquad a_1 \otimes \cdots \otimes a_n \mapsto \gamma_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_n),$$

satisfying the following conditions:

(1) $\gamma_{\lambda_1,\dots,\lambda_n}(a_1, \dots, \partial a_i, \dots, a_n) = -\lambda_i \gamma_{\lambda_1,\dots,\lambda_n}(a_1, \dots, a_n)$,
(2) γ is skew-symmetric with respect to simultaneous permutations of a_i's and λ_i's.

As usual, we let $R^{\otimes 0} = \mathbb{C}$, so that a 0-cochain is an element of M. Sometimes, when the λ-action on the module M takes values in formal power series, we should also consider formal power series instead of polynomials in the definition of cochains.

The differential d of an n-cochain γ is defined as follows:

$$\begin{aligned}
&(d\gamma)_{\lambda_1,\dots,\lambda_{n+1}}(a_1,\dots,a_{n+1}) \\
&\quad= \sum_{i=1}^{n+1}(-1)^{i+1} a_{i\,\lambda_i}\gamma_{\lambda_1,\dots,\widehat{\lambda_i},\dots,\lambda_{n+1}}(a_1,\dots,\widehat{a_i},\dots,a_{n+1}) \\
&\quad+ \sum_{i,j=1;\, i<j}^{n+1}(-1)^{i+j}\gamma_{\lambda_i+\lambda_j,\lambda_1,\dots,\widehat{\lambda_i},\dots,\widehat{\lambda_j},\dots,\lambda_{n+1}}\big([a_{i\,\lambda_i} a_j], a_1,\dots,\widehat{a_i},\dots,\widehat{a_j},\dots,a_{n+1}\big),
\end{aligned}$$

where γ is extended linearly over the polynomials in λ_i. In particular, if $\gamma \in M$ is a 0-cochain, then $(d\gamma)_\lambda(a) = a_\lambda \gamma$.

The cochains of a conformal Lie algebra R with coefficients in a module M form a complex, which is called the *basic complex* and will be denoted by

$$\widetilde{C}^\bullet(R,M) = \bigoplus_{n\in\mathbb{Z}_+}\widetilde{C}^n(R,M).$$

In order to define the right cohomology of conformal Lie algebras, we need to define a $\mathbb{C}[\partial]$-module structure on $\widetilde{C}^\bullet(R,M)$ by letting

$$(\partial\cdot\gamma)_{\lambda_1,\dots,\lambda_n}(a_1,\dots,a_n) = \left(\partial_M + \sum_{i=1}^{n}\lambda_i\right)\gamma_{\lambda_1,\dots,\lambda_n}(a_1,\dots,a_n), \tag{3.1}$$

where ∂_M denotes the action of ∂ on M.

It is easy to see that $d\partial = \partial d$. Now define the *reduced complex* by

$$C^\bullet(R,M) = \widetilde{C}^\bullet(R,M)/\partial\widetilde{C}^\bullet(R,M) = \bigoplus_{n\in\mathbb{Z}_+}C^n(R,M).$$

Then, the *cohomology $H^\bullet(R,M)$ of a conformal Lie algebra R with coefficients in M* is the cohomology of the reduced complex. The *basic cohomology* corresponds to the basic complex.

Remark 3.2. Note that the cocycle condition for the cocommutator $\delta : R \to \wedge^2 R$ in the definition of a conformal Lie algebra is indeed the condition that δ be a 1-cocycle of R with coefficients in $\wedge^2 R$ in the reduced complex.

3.2. Definitions and conformal CYBE

Among the 1-cocycles of R with values in $\wedge^2 R$ are the 1-coboundaries δ that comes from the differential of an element $r \in \wedge^2 R$, that is

$$\delta(a) = (dr)_\lambda(a)\big|_{\lambda=-\partial^{\otimes 2}} = a_\lambda r\big|_{\lambda=-\partial^{\otimes 2}}, \tag{3.2}$$

for all $a \in R$ and with $\partial^{\otimes^2} := \partial \otimes 1 + 1 \otimes \partial$. In this case we shall use the following abbreviated notation:

$$\delta = dr := (dr)_{-\partial^{\otimes 2}}. \tag{3.3}$$

The main difference with the classical Lie algebra case is the appearance of λ that should be evaluated in $-\partial$. Sometimes we will simply use ∂ instead of $\partial^{\otimes^2}$. As in the Lie algebra case, it will be convenient to consider the more general situation with $r \in R \otimes R$.

Definition 3.3. A *coboundary conformal Lie bialgebra* is a triple $(R, [\ {}_\lambda\ ], r)$, with $r \in R \otimes R$, such that $(R, [\ {}_\lambda\ ], dr)$ is a conformal Lie bialgebra. In this case, the element $r \in R \otimes R$ is said to be a *coboundary structure*.

Let $r \in R \otimes R$, with $r = \sum_i a_i \otimes b_i$. Now define

$$\begin{aligned} [\![r, r]\!] = \sum_{i,j} \big(& [a_{i\,\mu} a_j] \otimes b_i \otimes b_j |_{\mu = 1 \otimes \partial \otimes 1} \\ & - a_i \otimes [a_{j\,\mu} b_i] \otimes b_j |_{\mu = 1 \otimes 1 \otimes \partial} - a_i \otimes a_j \otimes [b_{j\,\mu} b_i] |_{\mu = 1 \otimes \partial \otimes 1} \big). \end{aligned} \tag{3.4}$$

Now, we can state one of the main results of this article.

Theorem 3.4. *Let R be a conformal Lie algebra and let $r \in R \otimes R$. The map*

$$\delta(a) = (dr)_\lambda(a)|_{\lambda = -\partial^{\otimes 2}} = a_\lambda r|_{\lambda = -\partial^{\otimes 2}},$$

is the cocommutator of a conformal Lie bialgebra structure on R if and only if the following conditions are satisfied:

(a) *the symmetric part of r is R-invariant, that is:*

$$a_\lambda\big(r + r^{21}\big)\big|_{\lambda = -\partial^{\otimes 2}} = 0$$

where $r^{21} = \sum b_i \otimes a_i$, if $r = \sum a_i \otimes b_i$.

(b) $a_\lambda [\![r, r]\!]|_{\lambda = -\partial^{\otimes 3}} = 0$, *where $\partial^{\otimes^3} = \partial \otimes 1 \otimes 1 + 1 \otimes \partial \otimes 1 + 1 \otimes 1 \otimes \partial$.*

Remark 3.5. (1) This result have been recently generalized in [4] in the context of H-pseudoalgebras, introduced in [1]. The analogous to the CYBE becomes more symmetric in the language of H-pseudoalgebras, and it is a generalization of the classical case that is obtained with $H = \mathbb{C}$.

(2) Condition (b) in previous theorem means that we have to take $[\![r, r]\!]$ with the evaluation of μ included, that is an element of $R \otimes R \otimes R$, and *then* apply the λ-action of a, take the expansion in powers of λ followed by the corresponding evaluation of λ.

In order to prove the theorem, we shall need the following result

Lemma 3.6. *Let R be a conformal Lie algebra and let $r = \sum_i a_i \otimes b_i \in R \otimes R$. If we take $\delta(a) = (dr)_\lambda(a)|_{\lambda=-\partial^{\otimes}} = a_\lambda r|_{\lambda=-\partial^{\otimes}}$, then*

$$\begin{aligned}(\delta\otimes 1)\big(\delta(x)\big) = \sum_{i,j}\big(\big[[x_\lambda a_i]_\mu a_j\big]\otimes b_j\otimes b_i + a_j\otimes\big[[x_\lambda a_i]_\mu b_j\big]\otimes b_i \\ + [a_{i\,\mu} a_j]\otimes b_j\otimes[x_\lambda b_i] + a_j\otimes[a_{i\,\mu} b_j]\otimes[x_\lambda b_i]\big)\big|_{\lambda=-\partial^{\otimes 3},\ \mu=-(\partial^{\otimes 2}\otimes 1)},\end{aligned}$$

where in this case (cf. Remark 3.5(2)) the right-hand side is understood in the following way: take the λ and μ-brackets, expand the four terms as a polynomial in λ and μ with coefficients in $R \otimes R \otimes R$, and then make the corresponding evaluation of λ and μ.

Proof. We have

$$\begin{aligned}(\delta\otimes 1)\big(\delta(x)\big) &= (\delta\otimes 1)\Big(\sum_i\big([x_\lambda a_i]\otimes b_i + a_i\otimes[x_\lambda b_i]\big)\big|_{\lambda=-\partial^{\otimes 2}}\Big) \\ &= (\delta\otimes 1)\Big(\sum_i\sum_{k\geqslant 0}\frac{(-1)^k}{k!}(\partial\otimes 1 + 1\otimes\partial)^k\big((x_{(k)}a_i)\otimes b_i + a_i\otimes(x_{(k)}b_i)\big)\Big).\end{aligned}$$

Now, using that $(\delta\otimes 1)(\partial^{\otimes^2}(a\otimes b)) = \partial^{\otimes^3}(\delta\otimes 1(a\otimes b))$, we obtain

$$\begin{aligned}(\delta\otimes 1)\big(\delta(x)\big) &= \sum_i\sum_{k\geqslant 0}\frac{(-1)^k}{k!}\big(\partial^{\otimes^3}\big)^k\big(\delta(x_{(k)}a_i)\otimes b_i + \delta(a_i)\otimes(x_{(k)}b_i)\big) \\ &= \sum_{i,j}\sum_{k\geqslant 0}\frac{(-1)^k}{k!}\big(\partial^{\otimes^3}\big)^k\big(\big[(x_{(k)}a_i)_\mu a_j\big]\otimes b_j\otimes b_i + a_j\otimes\big[(x_{(k)}a_i)_\mu b_j\big]\otimes b_i \\ &\quad + [a_{i\,\mu}a_j]\otimes b_j\otimes(x_{(k)}b_i) + a_j\otimes[a_{i\,\mu}b_j]\otimes(x_{(k)}b_i)\big)\big|_{\mu=-(\partial^{\otimes 2}\otimes 1)} \\ &= \sum_{i,j}\big(\big[[x_\lambda a_i]_\mu a_j\big]\otimes b_j\otimes b_i + a_j\otimes\big[[x_\lambda a_i]_\mu b_j\big]\otimes b_i \\ &\quad + [a_{i\,\mu}a_j]\otimes b_j\otimes[x_\lambda b_i] + a_j\otimes[a_{i\,\mu}b_j]\otimes[x_\lambda b_i]\big)\big|_{\lambda=-\partial^{\otimes 3},\ \mu=-(\partial^{\otimes 2}\otimes 1)},\end{aligned}$$

completing the proof. □

Proof of Theorem 3.4. We shall translate the proof given in [5] for the classical case. Many details and cancellations are not obvious in the conformal case.

The proof that condition (a) is equivalent to the skew-symmetry of δ is straightforward. So, we need to prove that (b) (in the presence of (a)) is equivalent to co-Jacobi identity. In fact we shall see that

$$\sum_{c.p.}(\delta\otimes 1)\delta(x) + x_\lambda[\![r,r]\!]|_{\lambda=-\partial^{\otimes 3}} = 0,$$

where $\sum_{c.p.}$ means that we also have to add the two similar terms obtained by the cyclic permutation of the factors in $R\otimes R\otimes R$.

Using the previous lemma and taking care of the interpretation of the evaluation of λ and μ, and the corresponding cyclic permutation, we have

$$\begin{aligned}
&\sum_{c.p.}(\delta\otimes 1)\big(\delta(x)\big)\\
&\quad=\sum_{i,j}\Big(\big[[x_\lambda a_i]_\mu a_j\big]\otimes b_j\otimes b_i+a_j\otimes\big[[x_\lambda a_i]_\mu b_j\big]\otimes b_i\\
&\qquad+[a_{i\,\mu}a_j]\otimes b_j\otimes[x_\lambda b_i]+a_j\otimes[a_{i\,\mu}b_j]\otimes[x_\lambda b_i]\Big)\Big|_{\lambda=-\partial^{\otimes^3},\ \mu=-(\partial^{\otimes^2}\otimes 1)}\\
&\qquad+\sum_{i,j}\Big(b_j\otimes b_i\otimes\big[[x_\lambda a_i]_\mu a_j\big]+\big[[x_\lambda a_i]_\mu b_j\big]\otimes b_i\otimes a_j\\
&\qquad+b_j\otimes[x_\lambda b_i]\otimes[a_{i\,\mu}a_j]+[a_{i\,\mu}b_j]\otimes[x_\lambda b_i]\otimes a_j\Big)\Big|_{\lambda=-\partial^{\otimes^3},\ \mu=-(1\otimes 1\otimes\partial+\partial\otimes 1\otimes 1)}\\
&\qquad+\sum_{i,j}\Big(b_i\otimes\big[[x_\lambda a_i]_\mu a_j\big]\otimes b_j+b_i\otimes a_j\otimes\big[[x_\lambda a_i]_\mu b_j\big]\\
&\qquad+[x_\lambda b_i]\otimes[a_{i\,\mu}a_j]\otimes b_j+[x_\lambda b_i]\otimes a_j\otimes[a_{i\,\mu}b_j]\Big)\Big|_{\lambda=-\partial^{\otimes^3},\ \mu=-(1\otimes\partial^{\otimes^2})}.
\end{aligned}\tag{3.5}$$

We shall assign a number to the twelve terms in (3.5), for example the term (3) is $[a_{i\,\mu}a_j]\otimes b_j\otimes[x_\lambda b_i]$ with the corresponding evaluation in λ and μ.

On the other hand, taking care of the order of the evaluation of μ and λ, and using skew-symmetry of the λ-bracket (see Remark 3.5(2) and (3.4)), we have

$$\begin{aligned}
&x_\lambda[\![r,r]\!]\big|_{\lambda=-\partial^{\otimes^3}}\\
&\quad=\sum_{i,j}\Big[x_\lambda\big([a_{i\,\mu}a_j]\otimes b_i\otimes b_j|_{\mu=1\otimes\partial\otimes 1}\big)\\
&\qquad-x_\lambda\big(a_i\otimes[a_{j\,\mu}b_i]\otimes b_j|_{\mu=1\otimes 1\otimes\partial}\big)-x_\lambda\big(a_i\otimes a_j\otimes[b_{j\,\mu}b_i]|_{\mu=1\otimes\partial\otimes 1}\big)\Big]\Big|_{\lambda=-\partial^{\otimes^3}}\\
&\quad=\sum_{i,j}\Big(\big[x_\lambda[a_{i\,\mu}a_j]\big]\otimes b_i\otimes b_j|_{\mu=1\otimes\partial\otimes 1}+[a_{i\,\mu}a_j]\otimes[x_\lambda b_i]\otimes b_j|_{\mu=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes\partial)}\\
&\qquad+[a_{i\,\mu}a_j]\otimes b_i\otimes[x_\lambda b_j]|_{\mu=1\otimes\partial\otimes 1}+[x_\lambda a_i]\otimes[b_{i\,\mu}a_j]\otimes b_j|_{\mu=-(1\otimes\partial\otimes 1+1\otimes 1\otimes\partial)}\\
&\qquad+a_i\otimes\big[x_\lambda[b_{i\,\mu}a_j]\big]\otimes b_j|_{\mu=-(1\otimes\partial\otimes 1+1\otimes 1\otimes\partial)}+a_i\otimes[b_{i\,\mu}a_j]\otimes[x_\lambda b_j]|_{\mu=\partial\otimes 1\otimes 1}\\
&\qquad+[x_\lambda a_i]\otimes a_j\otimes[b_{i\,\mu}b_j]|_{\mu=-(1\otimes\partial\otimes 1+1\otimes 1\otimes\partial)}+a_i\otimes[x_\lambda a_j]\otimes[b_{i\,\mu}b_j]|_{\mu=\partial\otimes 1\otimes 1}\\
&\qquad+a_i\otimes a_j\otimes\big[x_\lambda[b_{i\,\mu}b_j]\big]\big|_{\mu=-(1\otimes\partial\otimes 1+1\otimes 1\otimes\partial)}\Big)\Big|_{\lambda=-\partial^{\otimes^3}}.
\end{aligned}\tag{3.6}$$

We shall assign a number with a tilde to the nine terms at the RHS of (3.6), for example the term $(\widetilde{3})$ is $[a_{i\,\mu}a_j]\otimes b_i\otimes[x_\lambda b_j]$ with the corresponding evaluation in λ and μ. Observe that in both Eqs. (3.5) and (3.6), the expansions and evaluations are understood as in Lemma 3.6. Now, the study of the sum of both equations is divided in several steps.

First, observe that $(3)+(\widetilde{3})=0$. Indeed, using skew-symmetry, we get for $\lambda=-\partial^{\otimes^3}$ (with a summation over repeated indices understood)

$$\begin{aligned}(3) &= [a_{i\,\mu} a_j] \otimes b_j \otimes [x_\lambda b_i]|_{\mu=-(\partial\otimes 1\otimes 1+1\otimes\partial\otimes 1)}\\ &= -[a_{j-\mu-\partial\otimes 1\otimes 1} a_i] \otimes b_j \otimes [x_\lambda b_i]|_{\mu=-(\partial\otimes 1\otimes 1+1\otimes\partial\otimes 1)}\\ &= -[a_{j\,\mu} a_i] \otimes b_j \otimes [x_\lambda b_i]|_{\mu=1\otimes\partial\otimes 1} = -(\widetilde{3}).\end{aligned}$$

Similarly, we have $(4) + (\widetilde{6}) = 0$.

Interchanging the indices i and j and using Jacobi identity, we have ($\lambda = -\partial^{\otimes^3}$)

$$\begin{aligned}(1) + (\widetilde{1}) &= \big[[x_\lambda a_j]_\mu a_i\big] \otimes b_i \otimes b_j|_{\mu=-\partial^{\otimes 2}\otimes 1} + \big[x_\lambda [a_{i\,\mu} a_j]\big] \otimes b_i \otimes b_j|_{\mu=1\otimes\partial\otimes 1}\\ &= \big[[x_\lambda a_i]_\mu a_j\big] \otimes b_i \otimes b_j|_{\mu=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes\partial)}.\end{aligned} \tag{3.7}$$

Now, using the invariance property

$$\sum_i x_\lambda(a_i \otimes b_i + b_i \otimes a_i)\big|_{\lambda=-\partial^{\otimes 2}} = 0 \tag{3.8}$$

of part (a) of this theorem, and (3.7), we obtain for $\lambda = -\partial^{\otimes^3}$

$$\begin{aligned}(6) + (1) + (\widetilde{1}) &= \big(\big[[x_\lambda a_i]_\mu b_j\big] \otimes b_i \otimes a_j + \big[[x_\lambda a_i]_\mu a_j\big] \otimes b_i \otimes b_j\big)\big|_{\mu=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes\partial)}\\ &= \big(-a_j \otimes b_i \otimes \big[[x_\lambda a_i]_\mu b_j\big] - b_j \otimes b_i \otimes \big[[x_\lambda a_i]_\mu a_j\big]\big)\big|_{\mu=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes\partial)}\\ &:= (A) + (B),\end{aligned}$$

observe that we could apply the invariance in the previous equation because we had μ evaluated in the right way. This is a detail that we have to take care in the conformal case! It is easy to see that $(B) + (5) = 0$, hence it remains to cancel (A).

Now, if we denote $ad\, x_\lambda$ the conformal adjoint, that is $ad\, x_\lambda(y) := [x_\lambda y]$, then using skew-symmetry we get

$$\begin{aligned}&(11) + (\widetilde{4})\\ &\quad= [x_\lambda b_i] \otimes [a_{i\,\mu} a_j] \otimes b_j + [x_\lambda a_i] \otimes [b_{i\,\mu} a_j] \otimes b_j|_{\lambda=-\partial^{\otimes 3}, \mu=-1\otimes\partial^{\otimes 2}}\\ &\quad= -[x_\lambda b_i] \otimes [a_{j\ -\mu-1\otimes\partial\otimes 1} a_i] \otimes b_j - [x_\lambda a_i] \otimes [a_{j\ -\mu-1\otimes\partial\otimes 1} b_i] \otimes b_j|_{\lambda=-\partial^{\otimes 3},\ \mu=-1\otimes\partial^{\otimes 2}}\\ &\quad= -\big[(1 \otimes ad\, a_{j\ -\mu-1\otimes\partial})\big([x_\lambda b_i] \otimes\ a_i + [x_\lambda a_i] \otimes b_i\big)\big|_{\lambda=-\partial^{\otimes 2}}\big] \otimes b_j|_{\mu=-1\otimes\partial^{\otimes 2}}\end{aligned} \tag{3.9}$$

where the last term has to be understood in the following way: first take the λ-brackets, expand in λ-powers and evaluate λ (which is different from the λ in the second term), then take $1 \otimes ad$, expand in μ-powers and finally evaluate μ. Now, since λ is the right one in (3.9), we can use the invariance property (3.8), obtaining

$$\begin{aligned}&(11) + (\widetilde{4})\\ &\quad= \big[(1 \otimes ad\, a_{j\ -\mu-1\otimes\partial})\big(a_i \otimes [x_\lambda b_i] + b_i \otimes [x_\lambda a_i]\big)\big|_{\lambda=-\partial^{\otimes 2}}\big] \otimes b_j|_{\mu=-1\otimes\partial^{\otimes 2}}\\ &\quad= a_i \otimes \big[a_{j\ -\mu-1\otimes\partial\otimes 1}[x_\lambda b_i]\big] \otimes b_j + b_i \otimes \big[a_{j-\mu-1\otimes\partial\otimes 1}[x_\lambda a_i]\big] \otimes b_j|_{\lambda=-\partial^{\otimes 3},\ \mu=-1\otimes\partial^{\otimes 2}}\\ &\quad= a_i \otimes \big[a_{j\,\mu}[x_\lambda b_i]\big] \otimes b_j + b_i \otimes \big[a_{j\,\mu}[x_\lambda a_i]\big] \otimes b_j|_{\lambda=-\partial^{\otimes 3},\ \mu=1\otimes 1\otimes\partial} := (C) + (D).\end{aligned}$$

and it is obvious that $(D) + (9) = 0$, hence it remains (C).

Similarly, we have

$$\begin{aligned}(12) + (\widetilde{7}) &= -(1 \otimes 1 \otimes ad\, b_j\,_{-\mu-1\otimes 1\otimes \partial})\big([x_\lambda b_i] \otimes a_j \otimes a_i \\ &\quad + [x_\lambda a_i] \otimes a_j \otimes b_i|_{\lambda=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes \partial)}\big)\big|_{\mu=-1\otimes \partial^{\otimes 2}} \\ &= (1 \otimes 1 \otimes ad\, b_j\,_{-\mu-1\otimes 1\otimes \partial})\big(a_i \otimes a_j \otimes [x_\lambda b_i] \\ &\quad + b_i \otimes a_j \otimes [x_\lambda a_i]|_{\lambda=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes \partial)}\big)\big|_{\mu=-1\otimes \partial^{\otimes 2}} \\ &= a_i \otimes a_j \otimes \big[b_{j\,\mu}[x_\lambda b_i]\big] + b_i \otimes a_j \otimes \big[b_{j\,\mu}[x_\lambda a_i]\big]\big|_{\lambda=-\partial^{\otimes 3},\ \mu=1\otimes \partial\otimes 1} := (E) + (F)\end{aligned}$$

and it is obvious that $(F) + (10) = 0$, hence it remains (E). In a similar way, it is easy to see that

$$\begin{aligned}(8) + (\widetilde{2}) &= -b_j \otimes [x_\lambda b_i] \otimes [a_{i\,\mu} a_j] - a_j \otimes [x_\lambda b_i] \otimes [a_{i\,\mu} b_j]|_{\lambda=-\partial^{\otimes 3},\ \mu=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes \partial)} \\ &:= (G) + (H)\end{aligned}$$

and we have $(G) + (7) = 0$, hence it remains (H).

By a simple computation and taking care of the different values of λ and μ, it is easy to see that $(2) + (\widetilde{5}) + (C) = 0$ by Jacobi identity. Now, we can write, using skew-symmetry and invariance property,

$$\begin{aligned}&(\widetilde{8}) + (A) + (H) \\ &\quad = a_i \otimes \big([x_\lambda a_j] \otimes [b_i\,_{-\mu-1\otimes 1\otimes \partial} b_j] \\ &\qquad - b_j \otimes \big[[x_\lambda a_j]_\mu b_i\big] - [x_\lambda b_j] \otimes [a_{j\,\mu} b_i]\big)\big|_{\lambda=-\partial^{\otimes 3},\ \mu=-(\partial\otimes 1\otimes 1+1\otimes 1\otimes \partial)} \\ &\quad = a_i \otimes (1 \otimes ad\, b_{i\,\mu})\big([x_\lambda a_j] \otimes b_j + b_j \otimes [x_\lambda a_j] + [x_\lambda b_j] \otimes a_j|_{\lambda=-\partial^{\otimes 2}}\big)\big|_{\mu=\partial\otimes 1\otimes 1} \\ &\quad = -a_i \otimes (1 \otimes ad\, b_{i\,\mu})\big(a_j \otimes [x_\lambda b_j]|_{\lambda=-\partial^{\otimes 2}}\big)\big|_{\mu=\partial\otimes 1\otimes 1} \\ &\quad = -a_i \otimes a_j \otimes \big[b_{i\,\mu}[x_\lambda b_j]\big]\big|_{\lambda=-\partial^{\otimes 3},\ \mu=\partial\otimes 1\otimes 1}.\end{aligned} \tag{3.10}$$

Finally, a simple computation shows that $(\widetilde{9}) + (E) + (3.10) = 0$ by Jacobi identity, and it is easy to check that we have canceled all the terms, finishing the proof. □

Definition 3.7. A *quasitriangular conformal Lie bialgebra* is a coboundary Lie bialgebra $(R, [\ _\lambda\ ], r)$ with $r \in R \otimes R$ such that $[\![r, r]\!] = 0 \mod (\partial^{\otimes 3})$ and r is R-invariant: $x_\lambda(r + r^{21})|_{\lambda=-\partial^{\otimes 2}} = 0$.

Observe that instead of $[\![r, r]\!] = 0$, we put $[\![r, r]\!] = 0 \mod (\partial^{\otimes 3})$, and this condition automatically implies $a_\lambda [\![r, r]\!]|_{\lambda=-\partial^{\otimes 3}} = 0$ by sesquilinearity.

3.3. Conformal Manin triples

Let us recall some basic notions defined in [3]. Let V be a $\mathbb{C}[\partial]$-module. A *conformal bilinear form* on V is a $\mathbb{C}$-bilinear map $\langle\, ,\rangle_\lambda : V \times V \to \mathbb{C}[\lambda]$ such that

$$\langle \partial v, w\rangle_\lambda = -\lambda \langle v, w\rangle_\lambda = -\langle v, \partial w\rangle_\lambda \quad \text{for all } v, w \in V.$$

The conformal bilinear form is *symmetric* if $\langle v, w\rangle_\lambda = \langle w, v\rangle_{-\lambda}$ for all $v, w \in V$.

The conformal bilinear form in a conformal Lie algebra R is called *invariant* if

$$\big\langle [a_\mu b], c\big\rangle_\lambda = \big\langle a, [b_{\lambda-\partial} c]\big\rangle_\mu = -\big\langle a, [c_{-\lambda} b]\big\rangle_\mu \tag{3.11}$$

for all $a, b, c \in R$.

Given a conformal bilinear form on a $\mathbb{C}[\partial]$-module V, we have a homomorphism of $\mathbb{C}[\partial]$-modules, $L : V \to V^{*c}$, $v \mapsto L_v$, given as usual by

$$(L_v)_\lambda w = \langle v, w\rangle_\lambda, \quad v \in V. \tag{3.12}$$

Let V be a free finite rank $\mathbb{C}[\partial]$-module and fix $\beta = \{e_1, \ldots, e_N\}$ a $\mathbb{C}[\partial]$-basis of V. Then *the matrix of* $\langle\, ,\rangle_\lambda$ *with respect to* β is defined as $P_{i,j}(\lambda) = \langle e_i, e_j\rangle_\lambda$. Hence, identifying V with $\mathbb{C}[\partial]^N$, we have

$$\big\langle v(\partial), w(\partial)\big\rangle_\lambda = v^t(-\lambda) P(\lambda) w(\lambda). \tag{3.13}$$

Observe that $P^t(-x) = P(x)$ if the conformal bilinear form is symmetric. We also have that $\operatorname{Im} L = P(-\partial)V^{*c}$, where L is defined in 3.12. Indeed, given $v(\partial) \in V$, consider $g_\lambda \in V^{*c}$ defined by $g_\lambda(w(\partial)) = v^t(-\lambda)w(\lambda)$, then by 3.13

$$(L_{v(\partial)})_\lambda w(\partial) = v^t(-\lambda)P(\lambda)w(\lambda) = g_\lambda\big(P(\partial)w(\partial)\big) = \big(P(-\partial)g\big)_\lambda\big(w(\partial)\big),$$

where in the last equality we are identifying V^{*c} with $\mathbb{C}[\partial]^N$ in the natural way, that is $f \in V^{*c}$ corresponds to $(f_{-\partial}e_1, \ldots, f_{-\partial}e_N) \in \mathbb{C}[\partial]^N$.

Now, suppose that a conformal bilinear form satisfies that $\langle v, w\rangle_\lambda = 0$ for all $w \in V$, implies $v = 0$. Then L gives an isomorphism between V and $P(-\partial)V^{*c}$, with $\det P \neq 0$, and in this case the bilinear form was called non-degenerate in [3]. But in this work, a conformal bilinear form is called *non-degenerate* if L gives an isomorphism between V and V^{*c}. Therefore, if the conformal bilinear form is non-degenerate, then $\det P$ is a non-zero scalar.

Definition 3.8. A (finite rank) *conformal Manin triple* is a triple of finite rank Lie conformal algebras (R, R_1, R_0), where R is equipped with a non-degenerate invariant symmetric bilinear form $\langle\, ,\rangle_\lambda$ such that

1. R_1, R_0 are Lie conformal subalgebras of R and $R = R_0 \oplus R_1$ as $\mathbb{C}[\partial]$-module.
2. R_0 and R_1 are isotropic with respect to $\langle\, ,\rangle_\lambda$, that is $\langle R_i, R_i\rangle_\lambda = 0$ for $i = 0, 1$.

Theorem 3.9. *Let L be a Lie conformal algebra free of finite rank. Then there is a one-to-one correspondence between Lie conformal bialgebra structures on L and conformal Manin triples (R, R_1, R_0) such that $R_1 = L$.*

Proof. Given a Lie conformal bialgebra L, we construct a Manin triple in the following way: we set $R_1 = L$, $R_0 = L^{*c}$ with the Lie conformal algebra structure given by the dual of the coalgebra structure in L, $R = L \oplus L^{*c}$, and take the non-degenerate symmetric conformal bilinear form given by

$$\langle a + f, b + g\rangle_\lambda = f_\lambda(b) + g_{-\lambda}(a).$$

Now, observe that the invariance of the bilinear form uniquely determines the bracket on $L \oplus L^{*c}$, namely: let $\{e_i\}_{i=1}^n$ be a $\mathbb{C}[\partial]$-basis of R_1 and let $\{e_i^*\}_{i=1}^n$ be the dual basis in $R_0 \simeq R_1^{*c}$. Suppose that

$$[e_i\,{}_\lambda e_j] = \sum_s A_{ij}^s(\lambda, \partial)e_s, \qquad [e_i^*\,{}_\lambda e_j^*] = \sum_s B_s^{ij}(\lambda, \partial)e_s^*. \tag{3.14}$$

Then, using invariance, we get

$$\begin{aligned}\langle [e_i^*\,{}_\lambda e_j], e_k\rangle_\mu &= \langle e_i^*, [e_j\,{}_{\mu-\partial} e_k]\rangle_\lambda \\ &= \sum_s \langle e_i^*, A_{jk}^s(\mu - \partial, \partial)e_s\rangle_\lambda \\ &= A_{jk}^i(\mu - \lambda, \lambda) = \langle A_{jk}^i(-\partial - \lambda, \lambda)e_k^*, e_k\rangle_\mu\end{aligned}$$

and

$$\begin{aligned}\langle [e_j\,{}_\lambda e_i^*], e_k^*\rangle_\mu &= \langle e_j, [e_i^*\,{}_{\mu-\partial} e_k^*]\rangle_\lambda \\ &= \sum_s \langle e_j, B_s^{ik}(\mu - \partial, \partial)e_s^*\rangle_\lambda \\ &= B_j^{ik}(\mu - \lambda, \lambda) = \langle B_j^{ik}(-\lambda - \partial, \lambda)e_k, e_k^*\rangle_\mu.\end{aligned}$$

Hence, using skew-symmetry, we have

$$\begin{aligned}[e_i^*\,{}_\lambda e_j] &= \sum_k \big(A_{jk}^i(-\partial - \lambda, \lambda)e_k^* - B_j^{ik}(\lambda, -\lambda - \partial)e_k\big) \\ &= \sum_k \big(A_{jk}^i(-\partial - \lambda, \lambda)e_k^* - C_j^{ik}(\lambda, \partial)e_k\big),\end{aligned} \tag{3.15}$$

where $C(\lambda, \partial) = B(\lambda, -\lambda - \partial)$. By Proposition 2.5, the bracket can be rewritten as follows:

$$[f_\lambda x] = ad^*(f)_\lambda(x) - ad^*(x)_{-\lambda-\partial}(f),$$

where ad^* denotes the coadjoint actions of L on L^* and L^* on L. It remains to show that this is indeed a Lie conformal algebra bracket (i.e. it satisfies the Jacobi identity, because sesquilinearity is clear).

We must show that (cf. Definition 2.1)

$$0 = [e_p^*\,{}_\lambda [e_k\,{}_\mu e_l]] - [[e_p^*\,{}_\lambda e_k]_{\lambda+\mu} e_l] - [e_k\,{}_\mu [e_p^*\,{}_\lambda e_l]]$$

together with a similar relation involving two e^*'s and one e. Expanding it, by using (3.15), we get

$$\begin{aligned}0 = \sum_{t,i} A^i_{kl}(\mu, \lambda + \partial)\big(A^p_{it}(-\lambda - \partial, \lambda)e^*_t - C^{pt}_i(\lambda, \partial)e_t\big)\\ - \sum_{t,i} A^p_{ki}(\mu, \lambda)\big(A^i_{lt}(-\lambda - \mu - \partial, \lambda + \mu)e^*_t - C^{it}_l(\lambda + \mu, \partial)e_t\big)\\ + \sum_{t,i} A^t_{il}(\lambda + \mu, \partial)C^{pi}_k(\lambda, -\lambda - \mu)e_t\\ + \sum_{t,i} A^p_{li}(-\lambda - \mu - \partial, \lambda)\big(A^i_{kt}(\mu, -\mu - \partial)e^*_t - C^{it}_k(-\mu - \partial, \partial)e_t\big)\\ + \sum_{t,i} A^t_{ki}(\mu, \partial)C^{pi}_l(\lambda, \mu + \partial)e_t. \qquad (3.16)\end{aligned}$$

The coefficients of e^*_t in (3.16) gives a relation equivalent to the Jacobi identity of L, and it is easy to see (after renaming some variables) that the coefficients of e_t in (3.16) gives a relation equivalent to (3.21) which is exactly the 1-cocycle condition of the cobracket in L (see below). In a similar way, the other Jacobi identity in $L \oplus L^*$ is equivalent to (3.21) and the Jacobi identity of L^*.

Conversely, let (R, R_1, R_0) be a conformal Manin triple. The non-degenerate form $\langle\,,\rangle_\lambda$ induces a non-degenerate pairing $R_0 \otimes R_1 \to \mathbb{C}[\lambda]$ that produce an isomorphism $R_1^{*c} \simeq R_0$ as $\mathbb{C}[\partial]$-modules, and hence a Lie algebra structure on R_1^{*c}. Denote by δ the Lie coalgebra structure induced on R_1 by Proposition 2.13. We have to show that $(R_1, [\ {}_\lambda\ ], \delta)$ is a Lie conformal bialgebra and hence R_0 is its dual Lie conformal bialgebra. Therefore, we have to check the cocycle condition

$$a_\lambda\big(\delta(b)\big) - b_{-\lambda-\partial}\big(\delta(a)\big) = \delta\big([a_\lambda b]\big). \qquad (3.17)$$

In order to do it, let $\{e_i\}_{i=1}^n$ be a $\mathbb{C}[\partial]$-basis of R_1 and let $\{e^*_i\}_{i=1}^n$ be the dual basis in $R_0 \simeq R_1^{*c}$. Let A^s_{ij} and B^{ij}_s be as in (3.14). By definition (see Proposition 2.13),

$$\delta(e_i) = \sum_{k,l} C^{kl}_i(\partial \otimes 1, 1 \otimes \partial)e_k \otimes e_l,$$

where $C^{kl}_i(x, y) = B^{kl}_i(x, -x - y)$. Then, we have

$$\begin{aligned}\delta\big([e_k\, {}_\lambda e_l]\big) &= \sum_i A^i_{kl}(\lambda, \partial \otimes 1 + 1 \otimes \partial)\delta(e_i)\\ &= \sum_{i,p,q} A^i_{kl}(\lambda, \partial \otimes 1 + 1 \otimes \partial)C^{pq}_i(\partial \otimes 1, 1 \otimes \partial)e_p \otimes e_q. \qquad (3.18)\end{aligned}$$

On the other hand, we get

$$
\begin{aligned}
e_{k\ \lambda}\delta(e_l) &= e_{k\ \lambda}\left(\sum_{p,q} C_l^{pq}(\partial\otimes 1, 1\otimes\partial)e_p\otimes e_q\right)\\
&= \sum_{p,q,i}\big[C_l^{pq}(\lambda+\partial\otimes 1, 1\otimes\partial)A_{kp}^i(\lambda,\partial\otimes 1)e_i\otimes e_q\\
&\quad + C_l^{pq}(\partial\otimes 1,\lambda+1\otimes\partial)A_{kq}^i(\lambda,1\otimes\partial)e_p\otimes e_i\big]
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
&e_{l\ -\lambda-\partial^{\otimes}}\delta(e_k)\\
&\quad= e_{l\ -\lambda-\partial^{\otimes}}\left(\sum_{p,q} C_k^{pq}(\partial\otimes 1, 1\otimes\partial)e_p\otimes e_q\right)\\
&\quad= \sum_{p,q,i}\big[C_k^{pq}(-\lambda-1\otimes\partial, 1\otimes\partial)A_{lp}^i(-\lambda-\partial\otimes 1-1\otimes\partial,\partial\otimes 1)e_i\otimes e_q\\
&\qquad + C_k^{pq}(\partial\otimes 1,-\lambda-\partial\otimes 1)A_{lq}^i(-\lambda-\partial\otimes 1-1\otimes\partial,1\otimes\partial)e_p\otimes e_i\big].
\end{aligned}
\tag{3.20}
$$

By taking the coefficients of $e_p\otimes e_q$ in (3.18), (3.19) and (3.20), the cocycle condition (3.17) become

$$
\begin{aligned}
&\sum_i A_{kl}^i(\lambda,\partial\otimes 1+1\otimes\partial)C_i^{pq}(\partial\otimes 1,1\otimes\partial)\\
&\quad= \sum_i\big[C_l^{iq}(\lambda+\partial\otimes 1,1\otimes\partial)A_{ki}^p(\lambda,\partial\otimes 1)\\
&\qquad + C_l^{pi}(\partial\otimes 1,\lambda+1\otimes\partial)A_{ki}^q(\lambda,1\otimes\partial)\big]\\
&\qquad - \sum_i\big[C_k^{iq}(-\lambda-1\otimes\partial,1\otimes\partial)A_{li}^p(-\lambda-\partial\otimes 1-1\otimes\partial,\partial\otimes 1)\\
&\qquad + C_k^{pi}(\partial\otimes 1,-\lambda-\partial\otimes 1)A_{li}^q(-\lambda-\partial\otimes 1-1\otimes\partial,1\otimes\partial)\big],
\end{aligned}
\tag{3.21}
$$

which is equivalent (after renaming the variables: $\partial\otimes 1=\lambda$, $\lambda=\mu$, $1\otimes\partial=\partial$) to the coefficients of e_t in (3.16), that is, the Jacobi identity on $R=R_1\oplus R_0\simeq R_1\oplus R_1^{*c}$, finishing the proof. □

3.4. Conformal Drinfeld's double

The correspondence between conformal bialgebras and conformal Manin triples gives us a Lie conformal algebra structure on $R\oplus R^{*c}$ if R is a conformal bialgebra. In fact, a more general result is true.

Theorem 3.10. *Let R be a finite rank Lie conformal bialgebra and let $(R\oplus R^{*c}, R, R^{*c})$ be the associated conformal Manin triple. Then there is a canonical conformal Lie bialgebra structure on $R\oplus R^{*c}$ such that the inclusions*

$$R\hookrightarrow R\oplus R^{*c}\hookleftarrow \big(R^{*c}\big)^{\mathrm{op}}$$

into the two summands are homomorphisms of Lie conformal bialgebras, that is $\delta_{R\oplus R^{*c}} = \delta_R - \delta_{R^{*c}}$.

Moreover, $R \oplus R^{*c}$ *is a quasitriangular Lie conformal bialgebra.*

The Lie conformal bialgebra $R \oplus R^{*c}$ is called the Drinfeld double of R and is denoted by $\mathcal{D}R$.

Proof. Let $\{e_i\}_{i=1}^n$ be a $\mathbb{C}[\partial]$-basis of R and let $\{e_i^*\}_{i=1}^n$ be the dual basis in R^{*c}. Suppose that

$$[e_i\,{}_\lambda e_j] = \sum_s A_{ij}^s(\lambda, \partial)e_s, \qquad \big[e_i^*\,{}_\lambda e_j^*\big] = \sum_s B_s^{ij}(\lambda, \partial)e_s^*.$$

Let $r = \sum_{i=1}^n e_i \otimes e_i^* \in R \otimes R^{*c} \subset \mathcal{D}R \otimes \mathcal{D}R$ be the canonical element corresponding to $\mathcal{I} \in \mathrm{Chom}(R, R) \simeq R \otimes R^{*c}$ (see Proposition 2.6), where $\mathcal{I}(a(\partial)) = a(\lambda + \partial)$. Now, let us see that $\delta_{R\oplus R^{*c}} := \delta_R - \delta_{R^{*c}} = dr$. Using (3.15), we have

$$\begin{aligned}
(dr)_\lambda(e_j)|_{\lambda=-\partial^\otimes} &= \sum_i \big([e_j\,{}_\lambda e_i] \otimes e_i^* + e_i \otimes \big[e_j\,{}_\lambda e_i^*\big]\big)\big|_{\lambda=-\partial^\otimes} \\
&= \sum_{i,k} \big(A_{ji}^k(\lambda, \partial \otimes 1)e_k \otimes e_i^* - A_{jk}^i(\lambda, -\lambda - 1 \otimes \partial)e_i \otimes e_k^* \\
&\quad + B_j^{ik}(-\lambda - 1 \otimes \partial, \lambda)e_i \otimes e_k\big)\big|_{\lambda=-\partial^\otimes} \\
&= \sum_{i,k} B_j^{ik}(\partial \otimes 1, -\partial \otimes 1 - 1 \otimes \partial)e_i \otimes e_k = \delta_R(e_j).
\end{aligned}$$

Similarly, by using Proposition 2.13 with (3.15), and then skew-symmetry, we get

$$\begin{aligned}
(dr)_\lambda\big(e_j^*\big)\big|_{\lambda=-\partial^\otimes} &= \sum_i \big[e_j^*\,{}_\lambda e_i\big] \otimes e_i^* + e_i \otimes \big[e_j^*\,{}_\lambda e_i^*\big] \\
&= \sum_{i,k} \big(A_{ik}^j(-\lambda - \partial \otimes 1, \lambda)e_k^* \otimes e_i^* - B_i^{jk}(\lambda, -\lambda - \partial \otimes 1)e_k \otimes e_i^* \\
&\quad + B_k^{ji}(\lambda, 1 \otimes \partial)e_i \otimes e_k^*\big)\big|_{\lambda=-\partial^\otimes} \\
&= -\sum_{i,k} A_{k,i}^j(\partial \otimes 1, -\partial \otimes 1 - 1 \otimes \partial)e_k^* \otimes e_i^* = -\delta_{R^{*c}}(e_j).
\end{aligned}$$

It remains to see that r gives us a quasitriangular structure (recall Definition 3.7). Using (3.4), we have

$$\begin{aligned}
[\![r, r]\!] &= \sum_{i,j} \big([e_i\,{}_\mu e_j] \otimes e_i^* \otimes e_j^*\big|_{\mu=1\otimes\partial\otimes1} - e_i \otimes \big[e_j\,{}_\mu e_i^*\big] \otimes e_j^*\big|_{\mu=1\otimes1\otimes\partial} \\
&\quad - e_i \otimes e_j \otimes \big[e_j^*\,{}_\mu e_i^*\big]\big|_{\mu=1\otimes\partial\otimes1}\big) \\
&= \sum_{i,j,s} \big(A_{ij}^s(1 \otimes 1 \otimes \partial, \partial \otimes 1 \otimes 1)\big(e_s \otimes e_i^* \otimes e_j^*\big)
\end{aligned}$$

$$+ A^i_{js}(1\otimes 1\otimes \partial, -1\otimes 1\otimes \partial - 1\otimes \partial\otimes 1)\big(e_i\otimes e^*_s\otimes e^*_j\big)$$
$$- B^{is}_j(-1\otimes \partial\otimes 1 - 1\otimes 1\otimes \partial, 1\otimes 1\otimes \partial)\big(e_i\otimes e_s\otimes e^*_j\big)$$
$$- B^{ji}_s(1\otimes \partial\otimes 1, 1\otimes 1\otimes \partial)\big(e_i\otimes e_j\otimes e^*_s\big)\big).$$

Now, the last two terms cancels out by skew-symmetry (after interchanging the summation indices j and s). Then, it is easy to see that $[\![r,r]\!]=0 \bmod (\partial^{\otimes^3})$, by using in the second term that $A^i_{js}(1\otimes 1\otimes \partial, -1\otimes 1\otimes \partial - 1\otimes \partial\otimes 1) = A^i_{js}(1\otimes 1\otimes \partial, \partial\otimes 1\otimes 1) \bmod (\partial^{\otimes^3})$.

Finally, by similar computations, it is possible to verify that

$$e_i\,{}_\lambda\big(r+r^{21}\big)\big|_{\lambda=-\partial^{\otimes 2}} = e_i\,{}_\lambda\bigg(\sum_j e_j\otimes e^*_j + e^*_j\otimes e_j\bigg)\bigg|_{\lambda=-\partial^{\otimes 2}} = 0,$$

finishing the proof. □

4. Lie (*R*) and the annihilation algebra

Two Lie algebras are usually associated to a Lie conformal algebra R, that is Lie(R) and the annihilation algebra (see [8, p. 42] for details). Their construction, at first sight, is not natural unless you look at a similar notion from vertex algebra theory. In this section, using the language of coalgebras, we will see them as convolution algebras of certain type, obtaining a more natural and conceptual construction of them. Another way to understand the construction of these Lie algebras in a natural way is to view them in the setting of Lie pseudo-algebras. In fact, our Theorem 4.1 (see below) is reminiscent of Eq. (7.2) from [1].

In order to recall the usual construction of Lie(R) and the annihilation algebra we need the following general result: If R is a Lie conformal algebra, then ∂R is a two-sided ideal of R with respect to the (0)-product, and $R/\partial R$ is a Lie algebra with this product. In particular, we will apply it to the *affinization* of R, that is we shall consider the conformal algebra

$$R\big[t,t^{-1}\big] = R\otimes \mathbb{C}\big[t,t^{-1}\big],$$

with $\widetilde{\partial} = \partial\otimes 1 + 1\otimes \partial_t$, and the (n)th product on $R[t,t^{-1}]$ defined by $(a,b\in R,\ f,g\in \mathbb{C}[t,t^{-1}],\ n\in\mathbb{Z}_+)$

$$(a\otimes f)_{(n)}(b\otimes g) = \sum_{j\in\mathbb{Z}_+}\binom{m}{j}(a_{(n+j)}b)\otimes\big(\big(\partial_t^{(j)}f\big)g\big).$$

Now, define

$$\mathrm{Lie}(R) = R\big[t,t^{-1}\big]/\widetilde{\partial}R\big[t,t^{-1}\big]$$

with the bracket induced by the (0)-product in $R[t,t^{-1}]$, more precisely, if we denote by $a_n = \overline{a\otimes t^n}$ the image of $a\otimes t^n$ in Lie(R), then the bracket is given by

$$[a_m,b_n] = \sum_{j\geqslant 0}\binom{m}{j}(a_{(j)}b)_{m+n-j}. \tag{4.1}$$

Another useful formula for the (0)-product in $R[t,t^{-1}]$ is the following (see [8, p. 42])

$$[a\otimes f, b\otimes g] = [a_{\partial_t} b]\otimes f(t)g(t')|_{t'=t}. \tag{4.2}$$

It is clear from the bracket formula (4.1) that

$$\mathrm{Lie}(R)_- := \mathbb{C}\text{-span of } \{a_n \mid a\in R,\ n\in\mathbb{Z}_+\}$$

is a subalgebra of $\mathrm{Lie}(R)$, which is called the *annihilation algebra* of R and it plays an important role in the theory of conformal modules.

Now, in order to give a more conceptual understanding of these Lie algebras, we define the *vertex dual* of a $\mathbb{C}[\partial]$-module V as follows

$$V^{*v} = \mathrm{Hom}_{\mathbb{C}[\partial]}\big(V, \mathbb{C}\big[t,t^{-1}\big]\big) = \big\{f = f_t : V\to\mathbb{C}\big[t,t^{-1}\big] \bigm| f(\partial a) = \partial_t\big(f(a)\big)\big\}$$

where the $\mathbb{C}[\partial]$-module structure of $\mathbb{C}[t,t^{-1}]$ is given by $\partial=\partial_t$. Similarly, we take

$$V^{*v_+} = \mathrm{Hom}_{\mathbb{C}[\partial]}\big(V,\mathbb{C}[t]\big) = \big\{f : V\to\mathbb{C}[t] \bigm| f(\partial a) = \partial_t\big(f(a)\big)\big\}.$$

Now, we can give an interpretation of $\mathrm{Lie}(R)$ and the annihilation algebra as convolution algebras.

Theorem 4.1. *Let R be a free finite Lie conformal algebra, let (R^{*c},δ) be the corresponding Lie conformal coalgebra and denote by m the usual product in $\mathbb{C}[t,t^{-1}]$. Then there is an isomorphism of Lie algebras*

$$\mathrm{Lie}(R) \simeq \big(R^{*c}\big)^{*v} = \mathrm{Hom}_{\mathbb{C}[\partial]}\big(R^{*c}, \mathbb{C}\big[t,t^{-1}\big]\big)$$

with the bracket in the space of homomorphisms given by

$$[f,g] = m\circ(f\otimes g)\circ\delta.$$

In particular, $(\mathrm{Lie}(R))_- \simeq (R^{*c})^{*v_+}$.

Remark 4.2. Obviously, if we replace in this section $\mathbb{C}[t,t^{-1}]$ by any commutative associative algebra A with a derivation, we can obtain the Lie algebra $\mathrm{Lie}_A(R)$ defined in Remark 2.7(d) in [8] as a convolution algebra as well.

Proof. Let $\varphi : \mathrm{Lie}(R)\to (R^{*c})^{*v} = \mathrm{Hom}_{\mathbb{C}[\partial]}(R^{*c}, \mathbb{C}[t,t^{-1}])$ defined by

$$\overline{a\otimes f} \longmapsto \big(b^*_\lambda \longmapsto b^*_{-\partial_t}(a) f_t\big)$$

with $a\in R$, $f\in\mathbb{C}[t,t^{-1}]$, $b^*_\lambda\in R^{*c}$. As usual, $b^*_{-\partial_t}(a)f_t$ means that we expand $b^*_{-\partial_t}(a)$ in ∂_t-powers and then we apply it to f_t.

First, we have to see that $F_t := \varphi(\overline{a\otimes f})\in(R^{*c})^{*v}$:

$$F_t\big(\partial b^*\big) = \big(\partial b^*\big)_{-\partial_t}(a)t^n = \partial_t\big(b^*_{-\partial_t}(a)t^n\big) = \partial_t\big(F_t\big(b^*\big)\big).$$

We also have that φ is well defined since

$$\varphi(\overline{\partial a \otimes f + a \otimes \partial_t f})\big(b^*_\lambda\big) = b^*_{-\partial_t}(\partial a) f_t + b^*_{-\partial_t}(a)\partial_t f_t = 0.$$

It is easy to see that φ is a bijection. Finally we have to check that it is a homomorphism of Lie algebras. Let $\{e_i\}_{i=1}^n$ be a $\mathbb{C}[\partial]$-basis of R and let $\{e_i^*\}_{i=1}^n$ be the dual basis in R^{*c}. Suppose, that

$$[e_i\, {}_\lambda e_j] = \sum_s P_{ij}^s(\lambda, \partial) e_s,$$

then we have

$$\delta\big(e_k^*\big) = \sum_{i,j} Q_{ij}^k(\partial \otimes 1, 1 \otimes \partial) e_i \otimes e_j,$$

where $Q(x, y) = P(x, -x - y)$. Now, using (4.2), we obtain (we shall simply write $a \otimes f$ for its quotient class in $\mathrm{Lie}(R)$)

$$\begin{aligned}
\varphi\big([e_i \otimes f, e_j \otimes g]\big)\big(e_k^*\big) &= \varphi\big([e_i\, {}_{\partial_t} e_j] \otimes f(t)g(t')|_{t'=t}\big)\big(e_k^*\big) \\
&= \big(e_k^*\big)_{-(\partial_t + \partial_{t'})}\Big(\sum_l P_{ij}^l(\partial_t, \partial) e_l\Big) f(t)g(t')|_{t'=t} \\
&= \sum_l P_{ij}^l(\partial_t, -\partial_t - \partial_{t'})\big(\big(e_k^*\big)_{-(\partial_t + \partial_{t'})}(e_l)\big) f(t)g(t')|_{t'=t} \\
&= Q_{ij}^k(\partial_t, \partial_{t'}) f(t)g(t')|_{t'=t}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\big[\varphi(e_i \otimes f), \varphi(e_j \otimes g)\big]\big(e_k^*\big) &= m\big(\big(\varphi(e_i \otimes f) \otimes \varphi(e_j \otimes g)\big)\delta\big(e_k^*\big)\big) \\
&= m\Big(\big(\varphi(e_i \otimes f) \otimes \varphi(e_j \otimes g)\big)\Big(\sum_{r,s} Q_{rs}^k(\partial \otimes 1, 1 \otimes \partial) e_r^* \otimes e_s^*\Big)\Big) \\
&= m\Big(\sum_{r,s} Q_{rs}^k(\partial_t \otimes 1, 1 \otimes \partial_t)\varphi(e_i \otimes f)\big(e_r^*\big) \otimes \varphi(e_j \otimes g)\big(e_s^*\big)\Big) \\
&= m\big(Q_{ij}^k(\partial_t \otimes 1, 1 \otimes \partial_t) f(t) \otimes g(t)\big) \\
&= Q_{ij}^k(\partial_t, \partial_{t'}) f(t)g(t')|_{t'=t}
\end{aligned}$$

finishing the proof. □

Acknowledgments

J. Liberati was supported in part by Conicet, ANPCyT, Agencia Cba Ciencia, Secyt-UNC and Fomec (Argentina). This project started in 1997. The main results on conformal Lie bialgebras were obtained in 2001. Special thanks go to MSRI (Berkeley) for the hospitality during my stay there, where most part of this work was done and shared with other participants of that special and wonderful semester (Spring 2002). The author would like to thanks Victor Kac for valuable discussions.

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