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LEARNING THE MODEL FROM THE DATA

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Abstract. The task of approximating data with a concise model comprising only a few parameters is a key concern in many applications, particularly in signal processing. These models, typically subspaces belonging to a specific class, are carefully chosen based on the data at hand. In this survey, we review the latest research on data approximation using models with few parameters, with a specific emphasis on scenarios where the data is situated in finite-dimensional vector spaces, functional spaces such as $L^2(\mathbb{R}^d)$, and other general situations. We highlight the invariant properties of these subspace-based models that make them suitable for diverse applications, particularly in the field of image processing.

Cuando entras en el corazón de un amigo,
no importa el lugar que ocupes,
lo importante es que nunca salgas de ahí.

Anónimo

1. Introduction

In this note, we will provide an overview of recent developments in the field of optimal subspaces, which has gained recently significant attention due to its application in signal and image models. We refer the reader to the references for more details and proofs.

The proliferation of available data has transformed the process of extracting meaningful information from it. As each type of data possesses specific characteristics, the design of tailored algorithms can take advantage of these shared attributes, leading to improved efficiency.

Therefore, it is crucial to construct a model for each type of data that relies on the fewest possible parameters while capturing their common features. One potential approach to achieving this is by assuming certain hypotheses about the

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device or phenomenon that generated the data, such as assuming that the signals under consideration are band-limited.

However, given the vast diversity of data available today, this approach may not be suitable in many cases, particularly when considering for example, data as internet traffic or stock market values. Instead, our strategy is to generate the model *from the data itself*, using a set of subspaces as models, from which we can choose the best fit for our data. The subspaces that we select and the data are all from the same vector space.

In signal and image processing, there are often certain transformations that are known to leave important features of the data, invariant. For example, in image processing, translations, rotations, and scaling are common transformations that preserve the spatial structure of an image.

To build effective models for such data, it is important to incorporate these known invariances into the model. This can be done by explicitly including transformation parameters and optimizing them along with the other parameters.

Incorporating invariances into the model can lead to more robust and accurate performance on real-world data, as the model is better equipped to handle variations and changes in the input data.

We will take into account subspaces that are invariant under both translations and rotations. To simplify the model, we will only consider discrete sets of translations and rotations.

We want the subspaces in the class to be "small" in a sense that will be specified in each case. This condition will be essential for the applications.

So the general scheme will be the following: Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{M}$ a family of subspaces of H. Consider a finite set of data $\mathcal{F} = \{f_1, \dots, f_m\}$ and define

$$\mathcal{E}(\mathcal{F}, \mathcal{S}) = \sum_{j=1}^{m} \|f_j - P_{\mathcal{S}} f_j\|^2, \tag{1}$$

where $\mathcal{S} \in \mathfrak{M}$ and $P_{\mathcal{S}}$ denote the orthogonal projection into the subspace $\mathcal{S}$. The functional $\mathcal{E}$ will be our gauge that will measure the fitness of the data to the subspace. We analyze the existence and construction of an optimal subspace in the class $\mathfrak{M}$ that minimizes the functional $\mathcal{E}(\mathcal{F}, \mathcal{S})$ over $\mathfrak{M}$.

Section 2 will focus on the case of a finite dimensional Hilbert space $\mathcal{H}$ and a class $\mathfrak{M}$, which consists of all the subspaces of $\mathcal{H}$ with dimensions smaller than a fixed positive integer ℓ. Next, in Section 3, we will examine the prototypical scenario of subspaces that are invariant under integer translations (SIS). We will consider optimality for the subclass of SIS that exhibits additional invariance in Section 4. Lastly, in Section 5, we will present the outcomes for models that are invariant under translation and rotation.

2. Optimality for the class of finite dimensional subspaces.

When the approximation class $\mathfrak{M}$ is the class of the finite dimensional subspaces, the problem can be solved using Singular Value Decomposition techniques. The next theorem is an adaptation of the Eckart–Young theorem ([12, 18]) and will be used throughout the paper.

Given a set of vectors $\mathcal{F} = \{f_1, \dots, f_m\}$ of a Hilbert space $\mathcal{H}$ define the Gramian matrix of $\mathcal{F}$ by $[G_{\mathcal{F}}]_{i,j} = \langle f_i, f_j \rangle_{\mathcal{H}}$, $X = \operatorname{span}\{f_1, \dots, f_m\}$, and let $r = \dim X = \operatorname{rank} G_{\mathcal{F}}$.

With this notation we have:

Theorem 2.1 ([1, Theorem 4.1]). *Let $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, and let $n \le r$ be a positive integer. Let $\lambda_1 \ge \dots \ge \lambda_m \in \mathbb{R}$ be the eigenvalues of the matrix $G_{\mathcal{F}}$ and $y_1, \dots, y_m \in \mathbb{C}^m$, with $y_i = (y_{i_1}, \dots, y_{i_m})^t$ the associated left orthonormal eigenvectors. Define the vectors $q_1, \dots, q_n \in \mathcal{H}$ by*

$$q_i = \theta_i \sum_{j=1}^{m} y_{i_j} f_j, \quad i = 1, \dots, \ell,$$

where $\theta_i = \lambda_i^{-1/2}$ if $\lambda_i \neq 0$ and $\theta_i = 0$ otherwise. Then $\{q_1, \dots, q_\ell\}$ is a Parseval frame of $W^* =$ span $\{q_1, \dots, q_\ell\}$ and the subspace W^* is optimal in the sense that, if W is any subspace with $\dim(W) \le \ell$, *we have*

$$\mathcal{E}(\mathcal{F}, W^*) = \sum_{i=1}^{m} \|f_i - P_{W^*} f_i\|^2 \le \mathcal{E}(\mathcal{F}, \mathcal{W}) = \sum_{i=1}^{m} \|f_i - P_W f_i\|^2.$$

Furthermore we have the following formula for the error:

$$\mathcal{E}(\mathcal{F}, W^*) = \sum_{i=\ell+1}^{m} \lambda_i.$$

3. Optimality for the class of SIS in $L^2(\mathbb{R}^d)$

In [1] the authors give a solution for the case where the approximation class is the class of shift-invariant spaces (SIS) of $L^2(\mathbb{R}^d)$. A closed subspace $V \in L^2(\mathbb{R}^d)$ is *shift-invariant* if it is invariant under the translations along $\mathbb{Z}^d$. A shift invariant space V always has a set of generators, i.e. a set $\Phi \subseteq L^2(\mathbb{R}^d)$ finite or countable such that

$$V = S(\Phi) = \overline{\operatorname{span}}\{T_k\phi \colon \phi \in \Phi, k \in \mathbb{Z}^d\}.$$

Here T_k denotes the translation along k, i.e. $(T_k f)(x) = f(x - k)$, $k \in \mathbb{Z}^d$. The length of a SIS is the cardinal of the minimun set of generators.

Shift-invariant spaces can be seen, using a theorem of Helson [15] (and a unitary transformation from $L^2(\mathbb{R}^d)$ onto $L^2([0,1]^d, \ell^2(\mathbb{Z}^d))$, see [9]), as a continuous of subspaces of $\ell^2(\mathbb{Z}^d)$. When the SIS is finitely generated, these subspaces are finite-dimensional, and Theorem 2.1 can be used to obtain in each of them a solution. The generators of the optimal SIS are then constructed by measurably gluing the solution in each component. (see [1] for details and a proof).

For a set of functions $\mathcal{F} = \{f_1, \dots, f_m\}$ in $L^2(\mathbb{R}^d)$. we define the Gramian as

$$[G_{\mathcal{F}}]_{i,j}(\omega) = \sum_{k \in \mathbb{Z}^d} \hat{f}_i(\omega + k)\overline{\hat{f}_j(\omega + k)}, \quad \omega \in \mathbb{U}.$$

Here $\hat{f}$ denotes the Fourier transform of f and $\mathbb{U} = [0,1]^d$.

Theorem 3.1 ([1, Theorem 2.3])**.** *Let $\mathcal{F} = \{f_1, \dots, f_m\}$ be a set of functions in $L^2(\mathbb{R}^d)$. Let $\lambda_1(\omega) \geq \cdots \geq \lambda_m(\omega)$ be the eigenvalues of the Gramian $G_{\mathcal{F}}(\omega)$. Then, there exists $V^* \in \mathcal{V}^\ell = \{V : V \text{ is a SIS of length at most } \ell\}$ such that*

$$\sum_{i=1}^{m} \|f_i - P_{V^*} f_i\|^2 \leq \sum_{i=1}^{m} \|f_i - P_V f_i\|^2, \quad \forall\, V \in \mathcal{V}^\ell.$$

Moreover, we have that

(1) *The eigenvalues $\lambda_i(\omega)$, $1 \leq i \leq m$ are $\mathbb{Z}^d$-periodic, measurable functions in $L^2(\mathbb{U})$ and*

$$\mathcal{E}(\mathcal{F}, \ell) = \sum_{i=\ell+1}^{m} \int_{\mathbb{U}} \lambda_i(\omega)\, d\omega.$$

(2) *Let $\theta_i(\omega) = \lambda_i^{-1}(\omega)$ if $\lambda_i(\omega)$ is different from zero, and zero otherwise. Then, there exists a choice of measurable left eigenvectors $Y^1(\omega), \dots, Y^\ell(\omega)$ with $Y^i = (y_1^i, \dots, y_m^i)^t$, $i = 1, \dots, \ell$, associated with the first ℓ largest eigenvalues of $G_{\mathcal{F}}(\omega)$ such that the functions defined by*

$$\widehat{\varphi}_i(\omega) = \theta_i(\omega) \sum_{j=1}^{m} y_j^i(\omega) \widehat{f}_j(\omega), \quad i = 1, \dots, \ell,\, \omega \in \mathbb{R}^d$$

are in $L^2(\mathbb{R}^d)$. Furthermore, the corresponding set of functions $\Phi = \{\varphi_1, \dots, \varphi_\ell\}$ is a generator set for the optimal subspace V^* and the set $\{\varphi_i(\cdot - k), k \in \mathbb{Z}^d, i = 1, \dots, \ell\}$ is a Parseval frame for V^*.

4. Optimality for the class of SIS with extra-invariance

4.1. **Sets of invariance and extra invariance.** In this section we will use for our approximation a subclass of the class $\mathcal{V}^\ell$ defined in the previous section. We will consider the class of the extra-invariant subspaces of length ℓ. We need first some definitions.

Definition 4.1. Let $V \subseteq L^2(\mathbb{R}^d)$ be a SIS. We define the *invariance set* as follows:

$$M := \{x \in \mathbb{R}^d : T_x f \in V, \forall f \in V\}.$$

In [2] (see also [3]), the authors proved that the invariance set of a shift invariance space $V \subseteq L^2(\mathbb{R}^d)$ is a closed additive subgroup of $\mathbb{R}^d$ that contains $\mathbb{Z}^d$. For instance, in the case of the line the invariant set of a shift invariant space could be $\mathbb{Z}$, $\frac{1}{n}\mathbb{Z}$ for some $n \in \mathbb{N}$ or $\mathbb{R}$.

Definition 4.2. Let $\Phi \subseteq L^2(\mathbb{R}^d)$. We will say that a SIS V is M *extra-invariant* if $T_m f \in V$ for all $m \in M$ and for all $f \in V$. If $M = \mathbb{R}^d$ we will say that V has *total extra-invariance.*

In other words, a shift invariant space has extra invariance if the the set of invariance is bigger than $\mathbb{Z}^d$. One example of a translation invariant space in $\mathbb{R}$

is the Paley–Wiener space of functions that are bandlimited to $[-1/2, 1/2]$ defined by

$$PW = \{f \in L^2(\mathbb{R})\colon \operatorname{supp}(\widehat{f}) \subseteq [-1/2, 1/2]\}.$$

It is easy to prove that for a measurable set $\Omega \subseteq \mathbb{R}^d$, the space

$$V_\Omega := \{f \in L^2(\mathbb{R}^d)\colon \operatorname{supp}(\widehat{f}) \subseteq \Omega\} \tag{2}$$

is translation invariant. Moreover, Wiener's theorem (see [15]) proves that any closed translation invariant subspace of $L^2(\mathbb{R}^d)$ is of the form (2).

Note that if Φ is a set of generators of V, i.e. $V = S(\Phi)$, and V has extra invariance M then

$$S(\Phi) = \overline{\operatorname{span}}\{T_k\phi\colon \phi \in \Phi, k \in \mathbb{Z}^d\} = \overline{\operatorname{span}}\{T_\alpha\phi\colon \phi \in \Phi, \alpha \in M\}.$$

In [2] the authors characterize those shift invariant spaces $V \subseteq L^2(\mathbb{R})$ that have extra-invariance. They show that either V is translation invariant, or there exists a maximum positive integer n such that V is $\frac{1}{n}\mathbb{Z}$-invariant.

The d-dimensional case is considered in [3]. There, a characterization of the extra invariance of V when M is not all $\mathbb{R}^d$ is obtained.

4.2. **Optimality and extra-invariance.** Here we consider the approximation problem for the class of finitely generated SIS with extra invariance under a given *proper* subgroup M of $\mathbb{R}^d$.

For a whole treatment we refer the reader to [10, 19, 11, 6].

Let us start introducing some notation. Let $m, \ell \in \mathbb{N}$, M be a closed proper subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$, $M^* = \{x \in \mathbb{R}^d\colon \langle x, m\rangle \in \mathbb{Z} \quad \forall m \in M\}$, and $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq L^2(\mathbb{R}^d)$. Define

$$\mathcal{V}_M^\ell = \{V : V \text{ is a SIS of length at most } \ell \text{ and } V \text{ is } M\text{-invariant}\}. \tag{3}$$

Let $\mathcal{N} = \{\sigma_1, \dots, \sigma_\kappa\}$ be a section of the quotient $\mathbb{Z}^d/M^*$ and $\{B_\sigma \ : \sigma \in \mathcal{N}\}$ the partition defined by

$$B_\sigma = \Omega + \sigma + M^* = \bigcup_{m^* \in M^*} (\Omega + \sigma) + m^*,$$

where Ω is a section of the quotient $\mathbb{R}^d/\mathbb{Z}^d$. We refer the reader to [3] for more details.

For each $\sigma \in \mathcal{N}$, we consider $\mathcal{F}^\sigma = \{f_1^\sigma, \dots, f_m^\sigma\} \subseteq L^2(\mathbb{R}^d)$ where, f_j^σ is such that $\widehat{f_j^\sigma} = \widehat{f}_j\chi_{B_\sigma}$ for $j = 1, \dots, m$.

Also, let $\widetilde{\mathcal{F}} = \{f_1^{\sigma_1}, \dots, f_m^{\sigma_1}, \dots, f_1^{\sigma_\kappa}, \dots, f_m^{\sigma_\kappa}\}$.

For each $\omega \in \mathbb{U}$, let $G_{\widetilde{\mathcal{F}}}(\omega)$ be the associated Gramian matrix of the vectors in $\widetilde{\mathcal{F}}$ with eigenvalues

$$\lambda_1(\omega) \geq \cdots \geq \lambda_{m\kappa}(\omega) \geq 0.$$

that are measurable functions.

Since $f_i^{\sigma_s}$ is orthogonal to $f_i^{\sigma_t}$ if $s \neq t$, the Gramian $G_{\widetilde{\mathcal{F}}}(\omega)$ is a diagonal block matrix with blocks $G_\sigma(\omega)$, $\sigma \in \mathcal{N}$. Here $G_\sigma(\omega)$ is the $m \times m$ Gramian associated to the data $\mathcal{F}^\sigma$. On the other hand we have that

$$G_\sigma(\omega) = U_\sigma(\omega)\Lambda_\sigma(\omega)U_\sigma^*(\omega) \quad \text{a.e. } \omega \in \mathbb{U},$$

where U_σ are unitary and $\Lambda_\sigma(\omega) := \operatorname{diag}(\lambda_1^\sigma(\omega), \dots, \lambda_m^\sigma(\omega)) \in \mathbb{C}^{m\times m}$ and they are also measurable matrices. We also have $\lambda_1^\sigma(\omega) \geq \cdots \geq \lambda_m^\sigma(\omega)$ for each $\sigma \in \mathcal{N}$.

Using the decompositions of the blocks G_σ we have that

$$G_{\widetilde{\mathcal{F}}}(\omega) = U(\omega)\Lambda(\omega)U^*(\omega),$$

where U has blocks U_σ in the diagonal, and Λ is diagonal with blocks Λ_σ. We want to recall here that for almost each ω the matrix $\Lambda(\omega)$ collects all the eigenvalues of the Gramian $G_{\widetilde{\mathcal{F}}}(\omega)$ and the columns of the matrix $U(\omega)$ are the associated left eigenvectors. Note that an eigenvector associated to the eigenvalue $\lambda_j^\sigma(\omega)$ has all the components not corresponding to the block σ equal to zero.

Now for each fixed $\omega \in \mathbb{U}$, we consider $\{(i_1(\omega), j_1(\omega)), \dots, (i_n(\omega), j_n(\omega))\}$ with $i_s(\omega) \in \mathcal{N}$ and $j_s(\omega) \in \{1, \dots, m\}$ and $n = m\kappa$ such that

$$\lambda_{j_1(\omega)}^{i_1(\omega)} \geq \cdots \geq \lambda_{j_n(\omega)}^{i_n(\omega)} \geq 0$$

are the ordered eigenvalues of $G_{\widetilde{\mathcal{F}}}(\omega)$, with corresponding left eigenvectors $Y^{(i_s(\omega), j_s(\omega))} \in \mathbb{C}^n$, for $s = 1, \dots, n$.

Here $i_s(\omega)$ indicates the block of the matrix $G_{\widetilde{\mathcal{F}}}(\omega)$ in which the eigenvalue $\lambda_{j_s(\omega)}^{i_s(\omega)}(\omega)$ is found and $j_s(\omega)$ indicates the displacement in this block of the matrix $G_{\widetilde{\mathcal{F}}}(\omega)$. More precisely, we have that $\lambda_{j_s(\omega)}^{i_s(\omega)}(\omega)$ coincides with $\lambda_{(i_s(\omega)-1)m+j_s(\omega)}(\omega)$, the $((i_s(\omega)-1)m + j_s(\omega))$-th eigenvalue of $G_{\widetilde{\mathcal{F}}}(\omega)$. When $\omega \in \mathbb{U}$ is fixed, we will write i_s instead of $i_s(\omega)$ and j_s instead of $j_s(\omega)$.

It can be proven (see [10]) that $\gamma_s(\omega) := \lambda_{j_{s(\omega)}}^{i_s(\omega)}(\omega)$ is measurable as a function on ω for each $s = 1, \dots, n$, and the associated eigenvectors are also measurable.

Finally we define $h_s \colon \mathbb{R}^d \to \mathbb{C}$, for $s = 1, \dots, \ell$

$$h_s(\omega) := \theta_{j_s}^{i_s}(\omega) \sum_{k=1}^m y_{(i_s-1)m+k}^{(i_s, j_s)}(\omega) \widehat{f}_k^{i_s}(\omega), \tag{4}$$

where $\theta_{j_s}^{i_s}(\omega) = (\lambda_{j_s}^{i_s}(\omega))^{-1/2}$ if $\lambda_{j_s}^{i_s}(\omega) \neq 0$ and $\theta_{j_s}^{i_s}(\omega) = 0$ otherwise.

Now we are ready to state the main result of this section.

Theorem 4.3. *Let $m, \ell \in \mathbb{N}$, and M be a closed proper subgroup of $\mathbb{R}^d$ containing $\mathbb{Z}^d$. Assume that $\mathcal{F} = \{f_1, \dots, f_m\} \subseteq L^2(\mathbb{R}^d)$ is given data and let $\mathcal{V}_M^\ell$ be the class defined in* (3)*. Then, there exists a shift invariant space $V^* \in \mathcal{V}_M^\ell$ such that*

$$V^* = \operatorname*{argmin}_{V \in \mathcal{V}_M^\ell} \sum_{j=1}^m \|f_j - P_V f_j\|^2.$$

Furthermore, with the above notation,

(1) *The eigenvalues* $\{\lambda_j^\sigma(\omega) : \sigma \in \mathcal{N}, j = 1, \ldots, m\}$, *are* $\mathbb{Z}^d$*-periodic, measurable functions in* $L^2(\mathbb{U})$ *and the error of approximation is*

$$\mathcal{E}(\mathcal{F}, M, \ell) := \sum_{j=1}^{m} \|f_j - P_{V^*} f_j\|^2 = \int_{\mathbb{U}} \sum_{s=\ell+1}^{m\kappa} \lambda_{j_s}^{i_s}(\omega)\, d\omega.$$

(2) *The functions* $\{h_1, \ldots, h_\ell\}$ *defined in* (4) *are in* $L^2(\mathbb{R}^d)$ *and if we define* $\varphi_1, \ldots, \varphi_\ell$ *by* $\widehat{\varphi_j} = h_j$, *then* $\Phi = \{\varphi_1, \ldots, \varphi_\ell\}$ *is a generator set for the optimal subspace* V^* *and the set* $\{\varphi_i(\cdot - k), k \in \mathbb{Z}^d, i = 1, \ldots, \ell\}$ *is a Parseval frame for* V^*.

5. Approximation with translation and rotation invariant subspaces

In the previous sections, we have only considered optimization over subspaces that are translation invariant and lack other important invariances, such as rotational invariance, which are crucial for applications.

In [5] the authors study the approximation problem for subspaces that are invariant under the action of a discrete locally compact group Γ, not necessarily commutative, with some hypotheses. This class in particular includes the crystallographic groups that split. So, the spaces become invariant under rigid movements. One recent application of these results to datasets of digital images appeared in [4]. This approach turns out to be mathematically very challenging and requires many different techniques such as fiberization, grammian analysis, frame theory and group representation methods. In this survey we will describe the problem using the straightforward example of crystallographic or crystal groups, which encompasses all the vital components.

5.1. Crystal groups.

Crystal groups (crystallographic groups or space groups) are groups of isometries of $\mathbb{R}^d$ that generalize the notion of translations along a lattice, allowing to move using different (rigid) movements in $\mathbb{R}^d$ following a bounded pattern that is repeated until it fills up space. Precisely (see [13]):

Definition 5.1. A *crystal group* is a discrete subgroup $\Gamma \subseteq \mathrm{Isom}(\mathbb{R}^d)$ such that the quotient $\mathrm{Isom}(\mathbb{R}^d)/\Gamma$ is compact, where $\mathrm{Isom}(\mathbb{R}^d)$ is endowed with the pointwise convergence topology.

Equivalently, one can define a *crystal group* to be a discrete subgroup $\Gamma \subseteq \mathrm{Isom}(\mathbb{R}^d)$ such that there exists a compact *fundamental domain* P for Γ, i.e. there exists a bounded closed set P such that

$$\bigcup_{\gamma \in \Gamma} \gamma(P) = \mathbb{R}^d \text{ and } \gamma(P^\circ) \cap \gamma'(P^\circ) = \emptyset \text{ for } \gamma \neq \gamma',$$

where P° is the interior of P.

Note that the set of translations on a lattice is the simplest of the crystal groups.

It is known that d-dimensional crystal groups are intrinsically related to regular tessellations of $\mathbb{R}^d$, being $\Gamma = \{\tau_k : k \in \Lambda\}$, the group of translations ($\tau_k(x) = x+k$)

FIGURE 1. Two versions of a tiling of the ceiling in the Alhambra, using a basic tile, its translations and rotations.

on a lattice Λ the simplest example.

We have the fundamental theorem of Bieberbach [7], [20] which states the following:

Theorem 5.2 (Bieberbach)**.** *Let* Γ *be a crystal subgroup of* $\mathrm{Isom}(\mathbb{R}^d)$*. Then*

(1) $\Lambda = \Gamma \cap \mathrm{Trans}(\mathbb{R}^d)$ *is a finitely generated abelian group of rank* d *which spans* $\mathrm{Trans}(\mathbb{R}^d)$*, and*
(2) *the linear parts of the symmetries* Γ*, the* point group *of* Γ*, is finite, and is isomorphic to* Γ/Λ*.*

(See also [16, IV-4]). Here $\mathrm{Trans}(\mathbb{R}^d)$ stands for translations of $\mathbb{R}^d$.
We will denote the point group of Γ by G.

Remark 5.3.

- Note that the set Λ is not empty by Bierberach's theorem [7] and consists of translations on the lattice Λ which is isomorphic to $\mathbb{Z}^d$, and we will denote by T_k for $k \in \Lambda$.
- The Point Group G of Γ is a finite subgroup of $\mathbf{O}(d)$, the orthogonal group of $\mathbb{R}^d$, that preserves the lattice of translations, i.e. $G\Lambda = \Lambda$. The simplest examples are if G is a group of rotations, so we will abuse notation, and denote the action of G on $L^2(\mathbb{R}^d)$ by R_g for $g \in G$.

General results on crystal groups can be found for example in [14], [21], [17], [7], and [8].

Note that the simplest example of a crystal group is the group of translations on a lattice Λ, i.e. $\Gamma = \{T_k : \ k \in \Lambda\}$, where $T_k(x) = x + k$.

One very important class of crystal groups are the *splitting crystal groups*:

Definition 5.4. Γ is called a *splitting crystal group* if it is the semidirect product of the subgroups Λ and G. In this case $\Gamma = \Lambda \rtimes G$, and for each $\gamma, \widetilde{\gamma} \in \Gamma$, we have $\gamma \cdot \widetilde{\gamma} = (k + g\widetilde{k}, g\widetilde{g})$, for $\gamma = (k, g), \widetilde{\gamma} = (\widetilde{k}, \widetilde{g})$ with $k, \widetilde{k} \in \Lambda$ and $g, \widetilde{g} \in G$ and $\gamma(x) = g(x) + k$.

Every crystal group is naturally embedded in a splitting group, and very often arguments for general groups can be relatively easy reduced to the splitting case and then be proved for that simpler case. This justifies, that from now on Γ will always be considered to be a splitting crystal group.

5.2. **The structure of Γ-invariant spaces.** Let us recall the structure of closed subspaces of $L^2(\mathbb{R}^d)$ that are invariant under the action of $\Gamma = \Lambda \rtimes G$, the semidirect product of a uniform lattice Λ in $\mathbb{R}^d$ and a discrete and countable group G that acts on $\mathbb{R}^d$ by continuous invertible automorphisms. We will assume that $g\Lambda = \Lambda$ for all $g \in G$, which implies that the Haar measure of $\mathbb{R}^d$ is invariant under the action of G.

A closed subspace $V \subseteq L^2(\mathbb{R}^d)$ is Γ-invariant if $T_k R_g V \subseteq V$ for all $(k, g) \in \Gamma$. Here for $f \in V$, $T_k f(x) = f(x - k), k \in \Lambda$ and $R_g f(x) = f(g^{-1}x), g \in G$. Equivalently, V is Γ-invariant if

$$f \in V \;\Rightarrow\; T_k f \in V \;\forall\, k \in \Lambda \;\text{ and }\; R_g f \in V \;\forall\, g \in G.$$

For an at most countable family $\Phi \subseteq L^2(\mathbb{R}^d)$, we will write

$$S_\Gamma(\Phi) := \overline{\operatorname{span}}\{T_k R_g \varphi \,:\, k \in \Lambda, g \in G, \varphi \in \Phi\}.$$

$S_\Gamma(\Phi)$ is a Γ-invariant space and the set Φ is called a set of generators. Note that, since $T_k R_g = R_g T_{g^{-1}k}$, we also have that

$$S_\Gamma(\Phi) = \overline{\operatorname{span}}\{R_g T_k \varphi :\, k \in \Lambda, g \in G, \varphi \in \Phi\}.$$

Since $L^2(\mathbb{R}^d)$ is separable, if V is a Γ-invariant subspace of $L^2(\mathbb{R}^d)$, there always exists a countable set $\Phi \subseteq L^2(\mathbb{R}^d)$ such that $V = S_\Gamma(\Phi)$.

Let V be a Γ-invariant subspace of $L^2(\mathbb{R}^d)$. As before, we denote by $\mathcal{L}(V)$, the length of V, the minimum number of generators of V:

$$\mathcal{L}(V) = \min\{n : \exists\, \Phi = \{\varphi_1, \dots, \varphi_n\} : V = S_\Gamma(\Phi)\}.$$

If V does not have a finite number of generators we set $\mathcal{L}(V) = \infty$.

Γ-invariant closed subspaces have been characterized in [5] in terms of a covariance property of the range function associated to its Λ-invariant subspace.

Definition 5.5. Let $\Omega \subseteq \widehat{\mathbb{R}^d}$ be a Borel section of $\widehat{\mathbb{R}^d}/\Lambda^\perp \approx \widehat{\Lambda}$. A range function is a map

$$\mathcal{J} : \Omega \to \{\text{closed subspaces of } \ell_2(\Lambda^\perp)\}.$$

Theorem 5.6 ([5, Theorem 3.3]). *Let $\Omega \subseteq \widehat{\mathbb{R}^d}$ be a Borel section of $\widehat{\mathbb{R}^d}/\Lambda^\perp$ such that for $\omega \in \Omega$, $g^*\omega \in \Omega$ $\forall g \in G$. A closed subspace V of $L^2(\mathbb{R}^d)$ is Γ-invariant if and only if it is Λ-invariant (shift-invariant by Λ) and its range function $\mathcal{J}_V = \mathcal{J}$ satisfies*

$$\mathcal{J}(g^*\omega) = r_{g^{-1}}\ \mathcal{J}(\omega)\,,\ \ a.e.\ \omega \in \Omega\,,\ \forall g \in G.$$

5.3. **Approximation by Γ-invariant subspaces.** In this subsection we study the approximation problem mentioned in the introduction. The idea is to find a low dimensional model (a subspace), among all Γ-invariant subspaces that best fits a given dataset. The subspace will be optimal for the data in the sense that it minimizes the gauge function $\mathcal{E}$, defined in (1) The importance of the approach in this subsection is that our class includes subspaces that are invariant by rigid movements in $\mathbb{R}^d$, since we are able to include rotations and symmetries.

We will always assume that G is finite, and that a Borel section of $\widehat{\mathbb{R}^d}/(\Lambda^\perp \rtimes G)$ exists.

Using the previously mentioned characterization of these spaces, we can employ a strategy similar to that used for shift-invariant spaces to obtain the desired theorem (for proofs of this section, see [5]).

We start with a necessary lemma.

Lemma 5.7 ([5, Lemma 5.1]). *Let $\mathcal{F}^G$ be the family $\{R(g)f_i : (i,g) \in I_m \times G\} \subseteq L^2(\mathbb{R}^d)$ ordered with the lexicographical ordering of $I_m \times G := \{1,2,\dots,m\} \times G$, and let $G_{\mathcal{F}^G}$ be its Grammian as before.*

1. *For $\omega \in \Omega$, let $\{\sigma_{i,g}(\omega)^2 : (i,g) \in I_m \times G\}$ be the eigenvalues of $G(\omega)$ ordered decreasingly with the lexicographical ordering of $I_m \times G$, counted with their multiplicity. Then they are G-invariant, in the sense that*

$$\sigma_{i,g}(g_0^*\omega) = \sigma_{i,g}(\omega) \quad \forall\,(i,g) \in I_m \times G,\ \forall\, g_0 \in G,\ a.e.\ \omega \in \Omega.$$

2. *For $\omega \in \Omega_0$, let $\{V^{i,g}(\omega) : (i,g) \in I_m \times G\} \subseteq \mathbb{C}^{m|G|}$ be the corresponding orthonormal eigenvectors of $G(\omega)$, and denote the components of the (i,g)-th eigenvector by $\{V^{i,g}_{j,q}(\omega) : (j,q) \in I_m \times G\} \subseteq \mathbb{C}$. Then, it is possible to obtain a family of orthonormal eigenvectors of $G(\omega)$ at a.e. $\omega \in \Omega$ whose components satisfy*

$$V^{i,g}_{j,q}(g_0^*\omega) = V^{i,g}_{j,g_0 q}(\omega) \quad \forall\, g_0 \in G,\ a.e.\ \omega \in \Omega.$$

Theorem 5.8 ([5, Theorem 5.2]). *Let $\mathcal{F} = \{f_1,\dots,f_m\}$ be a set of functional data in $L^2(\mathbb{R}^d)$. Using the same notations as in Lemma 5.7, the following holds:*

1. *For all $\kappa \in \{1,\dots,m\}$ there exists a Γ-invariant space $\mathcal{W} \subseteq L^2(\mathbb{R}^d)$ generated by Γ-orbits of a family $\{\psi_i\}_{i=1}^{\kappa} \subseteq L^2(\mathbb{R}^d)$ such that*

$$\mathcal{E}(\mathcal{F},\mathcal{W}) = \min\{\mathcal{E}(\mathcal{F},\mathcal{W}) : \mathcal{V} \subseteq L^2(\mathbb{R}^d)\,, \Gamma\text{-invariant and } \mathcal{L}(\mathcal{V}) \leq \kappa\}$$

and the system $\{T_k R_g \psi_i : k \in \Lambda, g \in G, i \in \{1,\dots,\kappa\}\}$ is a Parseval frame of $\mathcal{W}$.

2. *The approximation error for the minimizing space $\mathcal{W}$ is given by*
$$\mathcal{E}(\mathcal{F},\mathcal{W}) = \sum_{i=\kappa+1}^{m} \sum_{g\in G} \int_{\Omega_0} \sigma_{(i,g)}(\omega)^2 d\omega.$$

3. *A family $\{\psi_i\}_{i=1}^{\kappa} \subseteq L^2(\mathbb{R}^d)$ that generates a minimizer $\mathcal{W}$ is given by*
$$T[\psi_i](\omega) = \sum_{(j,g')\in I_m\times G} C_i^{j,g'}(\omega)\, T[R_{g'} f_j](\omega),$$
where
$$C_i^{j,g'}(\omega) = \sum_{g\in G} \theta_{i,g}(\omega) V_{j,g'}^{i,g}(\omega) \chi_{g^*\Omega_0}(\omega)\ , \quad i = 1,\dots,\kappa$$
and $\theta_{i,g}(\omega) = (\sigma_{i,g}(\omega))^{-1}$ if $\sigma_{i,g}(\omega) \neq 0$ and 0 otherwise. All identities hold for a.e. $\omega \in \Omega$.

References

[1] A. Aldroubi, C. Cabrelli, D. Hardin, and U. Molter, Optimal shift invariant spaces and their Parseval frame generators, *Appl. Comput. Harmon. Anal.* **23** no. 2 (2007), 273–283. DOI MR Zbl

[2] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson, and U. Molter, Invariance of a shift-invariant space, *J. Fourier Anal. Appl.* **16** no. 1 (2010), 60–75. DOI MR Zbl

[3] M. Anastasio, C. Cabrelli, and V. Paternostro, Invariance of a shift-invariant space in several variables, *Complex Anal. Oper. Theory* **5** no. 4 (2011), 1031–1050. DOI MR Zbl

[4] D. Barbieri, C. Cabrelli, E. Hernández, and U. Molter, Optimal translational-rotational invariant dictionaries for images, in *Proc. SPIE 11138, Wavelets and Sparsity XVIII*, 2019. DOI

[5] D. Barbieri, C. Cabrelli, E. Hernández, and U. Molter, Approximation by group invariant subspaces, *J. Math. Pures Appl. (9)* **142** (2020), 76–100. DOI MR Zbl

[6] D. Barbieri, C. Cabrelli, E. Hernández, and U. Molter, Data approximation with time-frequency invariant systems, in *Landscapes of Time-Frequency Analysis—ATFA 2019*, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, Cham, 2020, pp. 29–42. DOI MR Zbl

[7] L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Räume. (Erste Abh.), *Math. Ann.* **70** no. 3 (1911), 297–336. DOI MR Zbl

[8] L. Bieberbach, Über die Bewegungsgruppen der Euklidischen Räume. (Zweite Abh.), *Math. Ann.* **72** no. 3 (1912), 400–412. DOI MR Zbl

[9] C. de Boor, R. A. DeVore, and A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbf{R}^d)$, *J. Funct. Anal.* **119** no. 1 (1994), 37–78. DOI MR Zbl

[10] C. Cabrelli and C. A. Mosquera, Subspaces with extra invariance nearest to observed data, *Appl. Comput. Harmon. Anal.* **41** no. 2 (2016), 660–676. DOI MR Zbl

[11] C. Cabrelli, C. A. Mosquera, and V. Paternostro, An approximation problem in multiplicatively invariant spaces, in *Functional Analysis, Harmonic Analysis, and Image Processing: A Collection of Papers in Honor of Björn Jawerth*, Contemp. Math. 693, Amer. Math. Soc., Providence, RI, 2017, pp. 143–165. DOI MR Zbl

[12] C. Eckart and G. Young, The approximation of one matrix by another of lower rank, *Psychometrika* **1** (1936), 211–218. DOI Zbl

[13] D. R. Farkas, Crystallographic groups and their mathematics, *Rocky Mountain J. Math.* **11** no. 4 (1981), 511–551. DOI MR Zbl

[14] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, New York, 1987. MR Zbl

[15] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964. MR Zbl

[16] J. S. Lomont, *Applications of Finite Groups*, Dover, New York, 1993. MR Zbl

[17] G. E. Martin, *Transformation Geometry: An Introduction to Symmetry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1982. MR Zbl

[18] E. Schmidt, Zur Theorie der linearen und nicht linearen Integralgleichungen. Zweite Abhandlung: Auflösung der allgemeinen linearen Integralgleichung, *Math. Ann.* **64** no. 2 (1907), 161–174. DOI MR Zbl

[19] R. Tessera and H. Wang, Uncertainty principles in finitely generated shift-invariant spaces with additional invariance, *J. Math. Anal. Appl.* **410** no. 1 (2014), 134–143. DOI MR Zbl

[20] J. A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, New York, 1967. MR Zbl

[21] H. Zassenhaus, Beweis eines Satzes über diskrete Gruppen, *Abh. Math. Sem. Univ. Hamburg* **12** no. 1 (1938), 289–312. DOI MR Zbl

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