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Extended Robust Model Predictive Control of Integrating Systems

Alejandro H. González and Jacinto L. Marchetti
Institute of Technological Development for the Chemical Industry (INTEC), CONICET–Universidad Nacional del Litoral (U.N.L.), Güemes 3450, 3000 Santa Fe – Argentina

Darci Odloak
Dept. of Chemical Engineering, University of São Paulo, 05508-900 São Paulo, Brazil

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The robust model predictive control (MPC) of systems with stable and integrating modes is addressed. The approach proposed here extends the method presented in Odloak in 2004 that can only be applied to open-loop stable systems. Here, the robust controller is developed assuming that there is model uncertainty in both stable and integrating parts of the system. The method considers a modified cost function that turns the infinite output horizon MPC globally convergent for any finite input horizon. The controller is based on a modified version of the state-space model utilized by Carrapiço and Odloak in 2005 to develop a nominally stable MPC for systems with stable and integrating modes. The approach considers the inclusion of feasible cost contracting constraints in the control optimization problem, taking into account the annulment of the integrating modes to assure a bounded infinite horizon cost. A simulation example is included to illustrate the performance and robustness of the proposed approach and to demonstrate that the controller can be implemented in real applications. *AIChE J,* 53: 1758–1769, 2007
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Introduction

The lack of guaranteed stability is still one of the weaknesses of the available linear model predictive control (MPC) commercial packages.[1] A robust controller is able to provide closed-loop stability at different process operating conditions. As most of the chemical processes are nonlinear, different operating conditions mean that different linear models should be used to represent the process. As the MPC controller is usually based on a single nominal linear model of the process, it can be expected that stability becomes an issue when the system operating point has to be changed significantly. For open-loop stable systems, this subject has been extensively treated in the control literature,[2–6] and the existing solutions to the robust MPC problem for stable systems seem already in an acceptable stage for practical implementation. For the integrating case, Zheng[7] shows that the output steady state error can be used as a cost function that can be made bounded and decreasing in the presence of model uncertainty. Lee and Cooley[8] present a min-max approach for systems with bounded uncertainty in the input matrix. Ralhan and Badgwell[9] use the cost function proposed by Zheng[7] and the cost contraction constraint approach proposed by Badgwell,[10] to construct a stable MPC for the integrating system.

Correspondence concerning this article should be addressed to D. Odloak at odloak@usp.br.

In all these approaches, it is assumed that the controller works as a regulator, or that the desired reference values for the system inputs and states are at the origin. On the basis of a modified cost function, which includes a set of suitable slack variables, Cano and Odloak[11] proposed a robust MPC for pure integrating systems with single poles. The controller was shown to be robust in the output tracking case and to be offset-free in the presence of unknown disturbances. However, the method could not be directly applied to the system in which there are also stable poles.

One of the usual forms to obtain nominal stability in MPC is to adopt an infinite prediction horizon. However, to produce an offset-free tracking operation, an integrating disturbance model should be added to the system model or the model should be written in the incremental form in the input. This formulation adds integrating modes to the system output. These modes must be zeroed at the end of the control horizon to keep the infinite horizon cost-bounded. When the system to be controlled has already integrating modes, an additional set of constraints must be added to the control problem to cancel the effect of these modes on the system output. For the uncertain system, these constraints may conflict with the input constraints and the optimization problem that produces the MPC may become infeasible. The consequence is that global convergence of the cost function may not be achieved. More recently, the class of robust MPC controllers, which is based on a terminal set constraint and a terminal set controller, has showed some fruitful results[12,13] in the direction of enlarging the convergence region of the controller, by considering a time varying terminal constraint set. However, global convergence has still not been achieved, particularly for the integrating system. Kim et al.[14] have shown that the global convergence of the terminal controller can be obtained for the nominal system. They define a nonquadratic Lyapunov function that allows the saturation of the input of the terminal controller and consequently making it global.

In this article, the methods proposed by Carrapiço and Odloak[15] and Odloak[6] will be combined to produce a globally robust MPC for systems with integrating and stable modes with uncertainty in the model parameters. In the next section, we present the state-space model considered in this work. Then, the nominal infinite horizon MPC problem is reviewed for the system with stable and integrating modes. Next, the robust controller developed by Odloak[6] to stable systems is extended to integrating systems, and the convergence and stability of the proposed approach are studied. Finally, we present some simulation results and conclude the article.

A New State Space Model for Systems with Integrating Modes

Rodrigues and Odloak[16] presented a state-space model that is based on an analytical expression for the step response of systems with nonintegrating modes. The model was used to develop a robust min-max MPC with infinite prediction horizon for open-loop stable systems. On the basis of the discrete version of the same state-space model, Odloak[6] proposed the extension of the robust regulators of Badgwell[10] and Lee and Yu[5] to the general MPC problem of output tracking of stable systems in which the steady state is unknown. Rodrigues and Odloak[17] introduced the state-space model corresponding to the analytical step response of systems with integrating and stable modes. On the basis of that model, Carrapiço and Odloak[15] proposed a globally stable MPC for the nominal system with stable and integrating nonrepeated poles. Recently, González et al.[18] extended the method of Carrapiço and Odloak[15] to the case in which there are uncertainties in the model parameters related to the stable modes of the system. However, the approach could not be extended to the general case, where uncertainty also appears in the model parameters related to the integrating modes. In this section, we develop a new version of this state-space model, which is also based on the analytical step response of systems with integrating modes. As we will show later, this new model allows the extension of the method of Carrapiço and Odloak[15] to produce a robust controller for uncertain systems with stable and integrating modes.

Consider initially a SISO system that can be represented by the following transfer function model:

$$\frac{y(z)}{u(z)} = \frac{b_0 + b_1 z + \cdots + b_{nb} z^{nb}}{(z-1)(z-r_1)\ldots(z-r_{na})} \tag{1}$$

where $r_1, \ldots, r_{na}$ are distinct stable poles. If at time 0, the system is at the origin and a step move $\Delta u(0)$ is introduced in the input, then the output response at sampling time k can be represented as follows:

$$s(k) = (d^0 + d_1^d r_1^k + \cdots + d_{na}^d r_{na}^k + k\Delta t\, d^i)\Delta u(0) \tag{2}$$

where d^0, $d^d_{i=1,\ldots,na}$ and d^i are obtained from the partial fraction expansion of (1) and Δt is the sampling period.

Suppose now that the system starts from the same steady state and the following sequence of control moves is introduced into the system: $\Delta u(0)$, $\Delta u(1)$, $\ldots$, $\Delta u(k-1)$. Then, based on Eq. 2, the corresponding system output, at time step k, can be written as follows:

$$\begin{aligned} y(k) = {} & \left[d^0 + \sum_{l=1}^{na} d_l^d r_l^k + k\Delta t\, d^i\right]\Delta u(0) \\ & + \left[d^0 + \sum_{l=1}^{na} d_l^d r_l^{k-1} + (k-1)\Delta t\, d^i\right]\Delta u(1) + \cdots \\ & + \left[d^0 + \sum_{l=1}^{na} d_l^d r_l + \Delta t\, d^i\right]\Delta u(k-1) \end{aligned}$$

The earlier equation can be rearranged and expressed in the following form:

$$\begin{aligned} y(k) = {} & d^0\Delta u(0) + (k-1)\Delta t d^i \Delta u(0) + d^0 \Delta u(1) \\ & + (k-2)\Delta t d^i \Delta u(1) + \cdots + d^0 \Delta u(k-2) + \Delta t d^i \Delta u(k-2) \\ & + \Delta t\, d^i(\Delta u(0) + \Delta u(1) + \cdots + \Delta u(k-2)) \\ & + d^0 \Delta u(k-1) + \Delta t d^i \Delta u(k-1) \\ & + \sum_{l=1}^{na}\Big(\Big[d_l^d r_l^{(k-1)}\Delta u(0) + d_l^d r_l^{(k-2)}\Delta u(1) + \cdots \\ & + d_l^d r_l \Delta u(k-2)\Big] r_l + d_l^d r_l \Delta u(k-1)\Big) \end{aligned} \tag{3}$$

Now define the following variables

$$x_1(k-1) = d^0\Delta u(0) + (k-1)\Delta t d^i \Delta u(0) + d^0 \Delta u(1)$$
$$+ (k-2)\Delta t d^i \Delta u(1) + \cdots + d^0 \Delta u(k-2) + \Delta t d^i \Delta u(k-2)$$
$$x_{l+1}(k-1) = d_l^d r_l^{(k-1)} \Delta u(0) + d_l^d r_l^{(k-2)} \Delta u(1) + \cdots$$
$$+ d_l^d r_l \Delta u(k-2) \quad l = 1, \ldots, na$$
$$x_{na+2}(k-1) = \Delta u(0) + \Delta u(1) + \cdots + \Delta u(k-2) \qquad (4)$$

Then, substituting the variables defined in (4) into (3), the following equation is obtained.

$$y(k) = x_1(k-1) + d^i \Delta t\, x_{na+2}(k-1) + \sum_{l=1}^{na} x_{l+1}(k-1)$$
$$+ \left(d^0 + \Delta t\, d^i\right)\Delta u(k-1) + \sum_{l=1}^{na} \big(r_l x_{l+1}(k-1)$$
$$+ d_l^d r_l \Delta u(k-1)\big) \qquad (5)$$

It is easy to see that the variables defined in (4) can be updated recursively as follows:

$$x_1(k) = x_1(k-1) + d^i \Delta t\, x_{na+2}(k-1) + (d^0 + \Delta t\, d^i)\Delta u(k-1)$$
$$x_{l+1}(k) = r_l x_{l+1}(k-1) + d_l^d r_l \Delta u(k-1), \quad l = 1, \ldots, na$$
$$x_{na+2}(k) = x_{na+2}(k-1) + \Delta u(k-1) \qquad (6)$$

and the system output can be represented in terms of the new state variables as follows:

$$y(k) = x_1(k) + \sum_{l=1}^{na} x_{l+1}(k)$$

In a multivariable system with ny outputs and nu inputs, if for each pair ($y_i \times u_j$) we have a transfer function as in (1), then the state-space model corresponding to (6) takes the following form:

$$x(k+1) = Ax(k) + B\Delta u(k) \qquad (7)$$
$$y(k) = C\,x(k)$$

where

$$x = \begin{bmatrix} x^s \\ x^d \\ x^i \end{bmatrix}, \quad x^s \in \square^{ny}, \quad x^d \in \square^{nd}, \quad x^i \in \square^{nu}, \quad y \in \square^{ny},$$
$$nd = ny\ nu\ na$$

$$A = \begin{bmatrix} I_{ny} & 0 & \Delta t D^i \\ 0 & F & 0 \\ 0 & 0 & I_{nu} \end{bmatrix} \in \square^{nx \times nx}, \quad B = \begin{bmatrix} D^0 + \Delta t D^i \\ D^d F N \\ I_{nu} \end{bmatrix} \in \square^{nx \times nu},$$

$$C = [I_{ny} \quad \Psi \quad 0_{ny \times nu}]$$

$$\Psi = \begin{bmatrix} \overbrace{1\,1 \cdots 1}^{nu.na} & & \overbrace{0\,0 \cdots 0}^{nu.na} \\ 0\,0 \cdots 0 & \ldots & 0\,0 \cdots 0 \\ \vdots\,\vdots \cdots \vdots & & \vdots\,\vdots \cdots \vdots \\ 0\,0 \cdots 0 & & 1\,1 \cdots 1 \end{bmatrix}, \quad nx = ny + nd + nu$$

$$F = \mathrm{diag}\big(e^{r_{1,1,1}\Delta t} \cdots e^{r_{1,1,na}\Delta t} \cdots e^{r_{1,nu,1}\Delta t} \cdots e^{r_{1,nu,na}\Delta t} \cdots e^{r_{ny,1,1}\Delta t} \cdots$$
$$e^{r_{ny,1,na}\Delta t} \cdots e^{r_{ny,nu,1}\Delta t} \cdots e^{r_{ny,nu,na}\Delta t}\big)$$

$$F \in \square^{nd \times nd}$$

$$D^d = \mathrm{diag}\big(d^d_{1,1,1} \cdots d^d_{1,1,na} \cdots d^d_{1,nu,1} \cdots d^d_{1,nu,na} \cdots d^d_{ny,1,1} \cdots$$
$$d^d_{ny,1,n1} \cdots d^d_{ny,nu,1} \cdots d^d_{ny,nu,na}\big), \quad D^d \in \square^{nd \times nd}$$

$r_{l,i,j}$ is the lth stable pole of the pair ($y_i \times u_j$) and $d_{l,i,j}$ is the corresponding step response coefficient as defined in (1).

$$N = \begin{bmatrix} J_1 \\ J_2 \\ \vdots \\ J_{ny} \end{bmatrix}, \quad N \in \square^{nd \times nu}; \quad J_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$
$$J_i \in \square^{nu.na \times nu}$$

In this model formulation, x^s corresponds to the integrating states related to the incremental form of the inputs, x^d represents the stable states, and x^i stands for the integrating states of the system. A model which is similar to the one defined earlier was utilized by Carrapiço and Odloak,[15] except for the component x^i of the state vector, which in the model presented earlier is not related to any of the model parameters that characterize the system. This means that, x^i is not affected by model uncertainty. This property of the model defined in (7) will show to be useful in the development of an infinite horizon MPC that is robust to uncertainty in any of the model parameters.

It is easy to show that the model defined in (7), will be controllable and observable as long as matrices D^0 and D^i are such that, rank(D^0) $= ny$ and rank(D^i) $= nu$. However, in most practical systems, D^i is not a full rank matrix and we need some other criteria to assure controllability and observability of the model defined in (7). For this purpose, it can be shown that, if rank of D^i is equal to the number of inputs related to integrating modes, then the model defined in (7) will be controllable and observable. Let ni be the number of integrated inputs. If $ni < nu$, then, in model (7), state x^i can be reduced and written with ni components, which will be controllable and observable. If state x^i is written with nu components, then only ni of these components will be controllable and observable. However, the remaining $(nu - ni)$ components will be fixed at the origin and consequently will not affect the controllability or the observability of model (7).

Nominal Infinite Horizon MPC

The infinite horizon MPC that is considered here is based on the following cost function:

$$V_{1,k} = \sum_{j=0}^{\infty} e(k+j)^T Q\, e(k+j) + \sum_{j=0}^{m-1} \Delta u(k+j/k)^T R \Delta u(k+j/k) \quad (8)$$

where $Q \in \square^{ny \times ny}$ is positive definite and $R \in \square^{nu \times nu}$ is positive semidefinite, $e(k+j) = y(k+j) - y^r$ is the error of the predicted output at sampling time $k + j$ including the effect of future control moves, y^r is the output reference, and m is the control horizon. It is assumed that $\Delta u(k + j/k) = 0$ for $j \geq m$. For the model defined in (7), it is easy to show that, because of the presence of the integrating modes, the cost defined in (8) will be unbounded unless the integrating states are zeroed at the end of the control horizon. This means that, for the state-space model defined in (7), we need to include in the control problem, the following constraints:

$$x^s(k+m) - y^r = 0 \quad (9)$$

$$x^i(k+m) = 0 \quad (10)$$

Using the definitions of the state components, Eqs. 9 and 10 can be expressed as Eqs. 11 and 12, respectively:

$$e^s(k) + \left[D_m^0 - D_{2m}^i\right]\Delta u_k = 0 \quad (11)$$

$$x^i(k) + \bar{I}\Delta u_k = 0 \quad (12)$$

where

$$e^s(k) = x^s(k) - y^r$$

$$D_m^0 = [D^0 \ \cdots \ D^0] \in \square^{ny \times m.nu}, \ \bar{I} = \Big[\overbrace{I_{nu} \ \cdots \ I_{nu}}^{m}\Big]$$

$$D_{2m}^i = [0 \ \ \Delta t D^i \ \cdots \ (m-1)\Delta t D^i] \in \square^{nu \times m.nu}$$

$$\Delta u_k = \left[\Delta u(k/k)^T \ \cdots \ \Delta u(k+m-1/k)^T\right]^T \in \square^{m.nu}$$

If Eqs. 11 and 12 are substituted in (8), the control cost can be expressed as follows:

$$V_{1,k} = \sum_{j=0}^{m-1} e(k+j)^T Q e(k+j) + x^d(k+m)^T \overline{Q} x^d(k+m) + \sum_{j=0}^{m-1} \Delta u(k+j/k)^T R\, \Delta u(k+j/k)$$

where $\overline{Q}$ is the solution to the equation:

$$\overline{Q} - F^T\, \overline{Q} F = F^T \Psi^T Q \Psi F$$

Now, a MPC that is nominally stable for systems with stable and integrating modes can be obtained from the solution to the following optimization problem:

$$\min_{\Delta u_k} V_{1,k} \quad (13)$$

subject to:
(11), (12), and

$$\Delta u(k+j/k) \in U, \quad j = 0, 1, \ldots, m-1 \quad (14)$$

$$U = \left\{ \Delta u(k+j/k) \,\middle|\, \begin{array}{c} -\Delta u^{\max} \leq \Delta u(k+j/k) \leq \Delta u^{\max} \\ u^{\min} \leq \Delta u(k-1) + \sum_{i=0}^{j} \Delta u(k+i/k) < u^{\max} \end{array} \right\}$$

when problem (13) is feasible, we can follow the same steps as Carrapiço and Odloak[15] to prove that the control law resulting from the solution to this problem drives the output of the undisturbed closed-loop system to the reference value. Local asymptotic convergence of the system output to the reference value is obtained only for those values of (x^e, y^r) lying in the set in which the problem defined in (13) is feasible. The infeasibility of the earlier problem may result from a conflict between constraints (11) or (12) and (14).

In order to obtain a stable MPC with global convergence for systems where the number of integrating poles is not larger than the number of outputs, we consider the control cost proposed by Cano and Odloak,[11] which is defined as follows:

$$V_{2,k} = \sum_{j=1}^{\infty} \left(e(k+j) + \delta_k^s + j\Delta t \delta_k^i\right)^T Q \left(e(k+j) + \delta_k^s + j\Delta t \delta_k^i\right) + \sum_{j=0}^{m-1} \Delta u(k+j/k)^T R \Delta u(k+j/k) + \delta_k^{s^T} S_1 \delta_k^s + \delta_k^{i^T} S_2 \delta_k^i$$

where $\delta_k^s \in \square^{ny}$ and $\delta_k^i \in \square^{ny}$ are slack variables that need to be computed as additional variables of the control problem. S_1 and S_2 are positive definite weight matrices. The aforementioned cost can also be made bounded by zeroing the integrating modes at the end of the control horizon, which results in the following constraints:

$$e^s(k) + \left[D_m^0 - D_{2m}^i\right]\Delta u_k + \delta_k^s = 0 \quad (15)$$

$$x^i(k) + \bar{I}\Delta u_k + \delta_k^i = 0 \quad (16)$$

Considering the two constraints defined earlier, the control cost becomes

$$V_{2,k} = \sum_{j=1}^{m-1} \left(e(k+j) + \delta_k^s + j\Delta t \delta_k^i\right)^T Q \left(e(k+j) + \delta_k^s + j\Delta t \delta_k^i\right) + x^d(k+m)^T \overline{Q} x^d(k+m) + \sum_{j=0}^{m-1} \Delta u(k+j/k)^T R \Delta u(k+j/k) + \delta_k^{s^T} S_1 \delta_k^s + \delta_k^{i^T} S_2 \delta_k^i$$

and a globally feasible infinite horizon MPC would be obtained by solving the following optimization problem:

$$\min_{\Delta u_k, \delta_k^s, \delta_k^i} V_{2,k} \tag{17}$$

subject to:
(14), (15), and (16)

However, the control law resulting from the solution to the problem defined in (17) does not necessarily produce an asymptotic converging closed loop for the integrating system. In order to obtain a converging system, we split the problem defined in (17) into two subproblems as follows:

Problem 1a.

$$\min_{\Delta u_{a,k}, \delta_k^i} V_{a,k} = \delta_k^{i^T} S_2 \delta_k^i \tag{18}$$

subject to:

$$\Delta u_a(k+j/k) \in U, \quad j = 0, 1, \ldots, m-1$$

$$x^i(k) + \bar{I}\Delta u_{a,k} + \delta_k^i = 0 \tag{19}$$

where

$$\Delta u_{a,k} = [\Delta u_a(k/k)^T \quad \cdots \quad \Delta u_a(k+m-1/k)^T]^T$$

Let the optimal solution to the problem defined in (18) be designated $\left(\Delta u_{a,k}^*,\ \delta_k^{i*}\right)$ and consider the total input increment corresponding to this optimal solution

$$u_a^*(k+m-1/k) - u(k-1) = \sum_{j=0}^{m-1} \Delta u_a^*(k+j/k)$$

This optimal input increment is passed to a second problem, which is solved within the same time step:

Problem 1b.

$$\min_{\Delta u_{b,k}, \delta_k^s} V_{b,k} = \sum_{j=0}^{m-1} (e(k+j) + \delta_k^s)^T Q (e(k+j) + \delta_k^s)$$
$$+ x^d(k+m)^T \, \bar{Q} x^d(k+m) + \sum_{j=0}^{m-1} \Delta u_b(k+j/k)^T R \Delta u_b(k+j/k)$$
$$+ \delta_k^{s^T} S_1 \delta_k^s$$

subject to.

$$\Delta u_{b,k}(k+j/k) \in U, \quad j = 0, 1, \ldots, m-1$$

$$e^s(k) + [D_m^0 - D_{2m}^i]\Delta u_{b,k} + \delta_k^s = 0$$

$$\sum_{j=0}^{m-1} \Delta u_{b,k}(k+j/k) = u_a^*(k+m-1/k) - u(k-1))$$

where $\Delta u_{b,k} = [\Delta u_b(k/k)^T \ldots \Delta u_b(k+m-1/k)^T]^T$

The convergence of the closed-loop system with the controller defined through Problems 1a and 1b is assured by the following theorem:

Theorem 1. *For systems with stable and integrating modes that remain controllable at the steady state corresponding to the desired output reference, Problems 1a and 1b are always feasible. Also, if weight S_1 is sufficiently large, then the control sequence obtained from the solution to Problems 1a and 1b at successive time steps drives the output of the closed-loop system asymptotically to the reference value.*

Proof: We can show that at time step $k+1$

$$\Delta u_{a,k+1} = [\Delta u_b(k+1/k)^{*T} \cdots \Delta u_b(k+m-1/k)^{*T} \quad 0]^T \quad \text{and} \quad \delta_{k+1}^i = \delta_k^{i*}$$

is a feasible solution to Problem 1a and for this feasible solution, we have $V_{a,k+1} = V_{a,k}^* = (\delta_k^{i^*})^T S_2 \delta_k^{i^*}$. Then, if the input increment is not constrained, δ_{k+1}^i can be made equal to zero by considering the following control sequence: $\Delta u_{a,k+1} = [\Delta u_b(k+1/k)^{*T} \ldots \Delta u_b(k+m-1/k)^{*T} - \delta_k^{i^*}]^T$

If the input is constrained, then it is easy to show that δ_k^i can be reduced to zero in a number of time steps not larger than $\max_j(|\delta_{j,k}^{i^*}|)/\Delta u_{j,\max}$, where index j designates the components of δ_k^{i*} and $\Delta u_{\max}$.

After convergence of $V_{a,k}$ to zero, solving Problem 1a becomes equivalent to solving Eq. 12. Consequently, the solution obtained by solving Problems 1a and 1b sequentially becomes equivalent to solving the following problem:

$$\min_{\Delta u_k, \delta_k^s} V_{3,k} = \sum_{j=0}^{m-1} (e(k+j) + \delta_k^s)^T Q (e(k+j) + \delta_k^s)$$
$$+ x^d(k+m)^T \, \bar{Q} x^d(k+m)$$
$$+ \sum_{j=0}^{m-1} \Delta u(k+j/k)^T R \Delta u(k+j/k) + \delta_k^{s^T} S_1 \delta_k^s \tag{20}$$

(12), (14), and (15)

It is easy to show that the cost defined in (20) is decreasing and converges to a minimum. However, the inclusion of the slack δ_k^s in the control problem may allow the term in $V_{3,k}$ related to the error on the output to converge to zero, while the slack has not converged to zero. This situation corresponds to the convergence of the closed-loop system to a steady state with offset in the output. This may happen when the stable modes are no longer controllable (e.g. the input becomes saturated) or parameter S_1 is not properly selected. Here, we show how to select S_1 in order to prevent output offset. For this purpose, suppose that when $k \to \bar{k}$ (large enough) the state tends to the steady state defined by $x^s(\bar{k})$, $x^d(\bar{k})$, and $x^i(\bar{k})$. Note that at this steady state, we have $\Delta u_{\bar{k}} = 0$ and, consequently, from (12) and (15), we have respectively $x^i(\bar{k}) = 0$ and $e^s(\bar{k}) = x^s(\bar{k}) - y^r = -\delta_{\bar{k}}^s$. Also, the stable part of the state tends to zero at this steady state or $x^d(\bar{k}) = 0$. Thus, at this steady state, the cost is given by $V_{3,\bar{k}} = \delta_{\bar{k}}^{s^T} S_1 \delta_{\bar{k}}^s$. Now, lets us try to find a control sequence that corresponds to a value of the cost that is smaller than $V_{3,\bar{k}}$. For this purpose, assume that $m = 2$, which is the minimum control horizon to produce an offset-free controller and assume also that we do not have any active input constraint. Then, the solution to problem (20) at k produces $\Delta\overline{u}_{\bar{k}}$ that has to satisfy constraints (12) and (15). From the constraint defined in (12) we have $x^i(\bar{k}) + \bar{I}\ \Delta\overline{u}_{\bar{k}} = 0$ and, since $x^i(\bar{k})$ is null, this condition is reduced to $\bar{I}\Delta\overline{u}_{\bar{k}} = 0$, and consequently (12) becomes equivalent to

$$\Delta\overline{u}(\bar{k}/\bar{k}) = -\Delta\overline{u}(\bar{k}+1/\bar{k}) \tag{21}$$

Also, for this control sequence the constraint defined in (15) can be written as follows:

$$-\delta^s_{\bar{k}} + \overline{\delta}^s_{\bar{k}} + (D^0_m - D^i_{2m})\Delta\overline{u}_{\bar{k}} = 0 \tag{22}$$

Now, let us find a control sequence that satisfies (21) and makes $\delta^s_{\bar{k}} = 0$. Observe that Eq. 22 with the condition defined in (21) becomes,

$$\Delta t\ D^i\Delta\overline{u}(\bar{k}/\bar{k}) = \delta^s_{\bar{k}}$$

Consequently, assuming that D^i is not singular, a possible control sequence is given by $\Delta\overline{u}_{\bar{k}} = \begin{bmatrix} (\Delta t D^i)^{-1} \\ -(\Delta t D^i)^{-1} \end{bmatrix}\delta^s_{\bar{k}}$. For this control sequence, the value of the cost is

$$\overline{V}_{3,\bar{k}} = \delta^{s^T}_{\bar{k}}\left[\left((\Delta t D^i)^{-1}\right)^T \quad \left((-\Delta t D^i)^{-1}\right)^T\right] (G^T\tilde{Q}G + \tilde{R})\begin{bmatrix} (\Delta t D^i)^{-1} \\ (-\Delta t D^i)^{-1} \end{bmatrix}\delta^s_{\bar{k}}$$

where

$$G = \begin{bmatrix} D^0 + \Delta t D^i + \Psi D^d F N & 0 \\ D^0 + 2\Delta t D^i + \Psi D^d F^2 N & D^0 + \Delta t D^i + \Psi D^d F N \end{bmatrix},$$
$$\tilde{Q} = \mathrm{diag}(Q,Q) \quad \tilde{R} = \mathrm{diag}(R,R)$$

Consequently, $\overline{V}_{3,\bar{k}}$ will be smaller than $V_{3,\bar{k}}$ if

$$S_1 > \left[\left((\Delta t D^i)^{-1}\right)^T \ \left((-\Delta t D^i)^{-1}\right)^T\right](G^T\tilde{Q}G + \tilde{R})\begin{bmatrix} (\Delta t D^i)^{-1} \\ (-\Delta t D^i)^{-1} \end{bmatrix} \tag{23}$$

Analogously, for other values of m, a similar procedure can be used to define a sufficiently large value of S_1, such that the convergence of the output of the closed-loop system to the reference is guaranteed. □

The Robust Infinite Horizon MPC for Integrating Systems

In this section we extend the controller presented in the latter section to the case where the system model is not exactly known. With the model structure presented in (7), model uncertainty is related to uncertainty in matrices F, D^0, D^d, and D^i. There are several practical ways to represent model uncertainty in MPC. One of the simple ways to represent model uncertainty is to consider the multiplant system,[10] where we have a discrete set Ω of plants, and the real plant is unknown, but it is assumed to be one of the members of this set. With this representation of model uncertainty, we can define the set of possible plants as $\Omega = \{\theta_1,\ldots,\theta_L\}$, where each θ_n corresponds to a particular plant: $\theta_n = (F_n, D^0_n, D^d_n, D^i_n)$, $n = 1, \ldots, L$.

Also, let us assume that the true plant, which lies within the set Ω is designated as θ_T and there is a most likely plant that also lies in Ω and is designated as θ_N.

Badgwell[10] developed a robust linear quadratic regulator for stable systems with the multiplant uncertainty. Odloak[6] extended the method of Badgwell[10] to the output tracking of stable systems, considering the same kind of model uncertainty. These strategies can be classified as cost-contracting strategies, since they force the cost corresponding to each of the models lying in Ω to decrease at successive time steps. In this section we combine the approach presented in the latter section with the approach proposed by Odloak,[6] to develop a robust MPC for systems with stable and integrating poles. The globally stable nominal controller produced by the sequential solution to Problems 1a and 1b can be extended to the multiplant system. The resulting robust controller is obtained from the solution to the following sequential problems:

Problem 2a.

$$\min_{\Delta u_{a,k},\delta^i_k} V_{a,k} = \delta^{i^T}_k S_2 \delta^i_k$$

subject to

$$\Delta u_a(k+j/k) \in U \quad j = 0,1,\ldots,m-1$$
$$x^i(k) + \bar{I}\Delta u_{a,k} + \delta^i_k = 0$$

Observe that this problem is the same as Problem 1a and produces the control sequence

$$\Delta u^*_{a,k} = [\Delta u^*_a(k/k)^T \ \cdots \ \Delta u^*_a(k+m-1/k)^T]^T$$

that is passed to the second problem.

Problem 2b.

$$\min_{\Delta u_{b,k},\delta^s_k(\theta_1),\ldots,\delta^s_k(\theta_L)} V_{b,k}(\Delta u_{b,k},\delta^s_k(\theta_N),\theta_N) = \sum_{j=0}^{\infty}(e(k+j) + \delta^s_k(\theta_N))^T Q(e(k+j) + \delta^s_k(\theta_N)) + \sum_{j=0}^{m-1}\Delta u_b(k+j/k)^T R\Delta u_b(k+j/k) + \delta^s_k(\theta_N)^T S_1 \delta^s_k(\theta_N)$$

subject to

$$\Delta u_b(k+j/k) \in U \quad j = 0,1,\ldots,m-1$$

$$e^s(k) + \delta^s_k(\theta_n) + \left(D^0_m(\theta_n) - D^i_{2m}(\theta_n)\right)\Delta u_{b,k} = 0 \quad n = 1,\ldots,L$$

$$\sum_{j=0}^{m-1}\Delta u_{b,k}(k+j/k) = u^*_a(k+m-1/k) - u(k-1) \tag{24}$$

$$V_{b,k}(\Delta u_{b,k}, \delta_k^s(\theta_n), \theta_n) \le V_{b,k}(\Delta \tilde{u}_{b,k}, \tilde{\delta}_k^s(\theta_n), \theta_n) \quad n = 1, \ldots, L \tag{25}$$

where

$$\Delta \tilde{u}_{b,k} = [\Delta u_b^*(k/k-1)^T \ \cdots \ \Delta u_b^*(k+m-2/k-1)^T \quad 0]^T$$

and $\tilde{\delta}_k^s(\theta_{n=1,\ldots,L})$ is such that

$$e^s(k) + \tilde{\delta}_k^s(\theta_n) + (D_m^0(\theta_n) - D_{2m}^i(\theta_n))\Delta \tilde{u}_{b,k} = 0, \qquad n = 1, \ldots, L$$

Remarks

1. The state is assumed to be measured, and corresponds to the actual plant θ_T.
2. If $\delta_k^i \neq 0$ problem P2b may not be feasible. This can be easily seen for instance if $\Delta u_{a,k}^* = [\Delta u_{\max}^T \cdots \Delta u_{\max}^T]^T$. Then, because of constraint (24), the only possible solution to Problem 2b would be $\Delta u_{b,k} = [\Delta u_{\max}^T \cdots \Delta u_{\max}^T]^T$ and this control sequence may not satisfy constraint (25). Another conflicting situation may occur when a disturbance in state x^i enters the system. In such a case, the pseudovariable $\Delta \tilde{u}_{b,k}$ may not be a feasible solution to Problem P2b, and then the feasibility of this problem cannot be assured. Then, to implement the controller defined by Problems 2a and 2b, we follow the algorithm below that makes Problem 2b feasible after a finite number of steps.
   - At time step k, solve Problem 2a that produces the control sequence $\Delta u_{a,k}^*$.
   - Try to solve Problem 2b, if it results infeasible because of a conflict between constraints (24) and (25), adopt $\Delta u_{b,k}^* = \Delta u_{a,k}^*$ and inject the first control move in the real system.
3. Problem 2a is exactly the same as Problem 1a in the nominal case, because state x^i that is related to the integrating modes does not depend on the model parameters, and consequently is not affected by model uncertainty. Thus, zeroing the integrating modes at the end of the control horizon leads to only one constraint represented in Eq. 19 even though there is uncertainty in the parameters associated with the integrating modes. This is a very convenient property of the state-space model defined in (7), as it allows the extension of the robust controller to model uncertainty in the integrating coefficients D^i. In the model formulation utilized by Carrapiço and Odloak,[15] the consideration of uncertainty in D^i would lead to L constraints (19) for the multiplant uncertainty case. These constraints could not be simultaneously satisfied at the steady state, and robustness could only be guaranteed when uncertainty was restricted to the other model parameters.[18]

The global stability of the controller resulting from the sequential solution to Problems 2a and 2b is guaranteed by the following theorem:

Theorem 2: *Consider a system with stable and integrating modes whose true model is unknown but lies within the set* Ω. *Assume that in the control objective* $V_{b,k}$, *weights Q, R, and* S_1 *are such that (23) is true for all models lying within* Ω. *Assume also that the system is controllable at the desired reference. Then, the control law obtained from the sequential solution to Problems 2a and 2b, taking into account Remark 2, is stable and drives the true system to the reference value.*

Proof: As we have seen in Theorem 1, Problem 2a converges in a finite number of time steps. Then, for a finite $\bar{k}$ we will have $\delta_k^i = 0$ and consequently, Problem 2a together with constraint (24) can be substituted by the constraint

$$x^i(k) + \bar{I}\Delta u_k = 0$$

Consequently, after convergence of Problem 2a, the sequential solution of Problems 2a and 2b becomes equivalent to solving the following problem:

$$\min_{\Delta u_k, \delta_k^s(\theta_1), \ldots, \delta_k^s(\theta_L)} V_{4,k}(\Delta u_k, \delta_k^s(\theta_N), \theta_N) = \sum_{j=0}^{\infty} (e(k+j) + \delta_k^s(\theta_N))^T Q (e(k+j) + \delta_k^s(\theta_N)) + \sum_{j=0}^{m-1} \Delta u(k+j/k)^T R \Delta u(k+j/k) + \delta_k^s(\theta_N)^T S_1 \delta_k^s(\theta_N) \tag{26}$$

subject to
(12), (14)

$$V_{4,k}(\Delta u_k, \delta_k^s(\theta_n), \theta_n) \le V_{4,k}(\Delta \tilde{u}_k, \tilde{\delta}_k^s(\theta_n), \theta_n), \quad n = 1, \ldots, L \tag{27}$$

$$e^s(k) + \delta_k^s(\theta_n) + (D_m^0(\theta_n) - D_{2m}^i(\theta_n))\Delta u_k = 0, \quad n = 1, \ldots, L \tag{28}$$

Consider now that at time step k, the problem defined in (26) is solved for the undisturbed system and the optimal solution is represented by Δu_k^*, $\delta_k^{s^*}(\theta_1), \ldots, \delta_k^{s^*}(\theta_n)$. Then, for the true plant, the corresponding cost is

$$V_{4,k}(\Delta u_k^*, \delta_k^{s^*}(\theta_T), \theta_T) = \sum_{j=1}^{\infty} (e(k+j) + \delta_k^{s^*}(\theta_T))^T Q (e(k+j) + \delta_k^{s^*}(\theta_T)) + \sum_{j=0}^{m-1} \Delta u^*(k+j/k)^T R \Delta u^*(k+j/k) + \delta_k^{s^*}(\theta_T)^T S_1 \delta_k^{s^*}(\theta_T)$$

Assume that we inject the first control action $\Delta u^*(k)$ into the true system and we move to time $k+1$. At this time step, consider the solution: $(\Delta \tilde{u}_{k+1}, \tilde{\delta}_{k+1}^s(\theta_1), \ldots, \tilde{\delta}_{k+1}^s(\theta_n))$
where

$$\Delta \tilde{u}_{k+1} = [\Delta u^*(k+1/k)^T \ \cdots \ \Delta u^*(k+m-1/k)^T \quad 0]^T$$

and $\tilde{\delta}_{k+1}^s(\theta_{n=1}, \ldots, L)$ is such that

$$e^s(k+1) + \tilde{\delta}_{k+1}^s(\theta_n) + (D_m^0(\theta_n) - D_{2m}^i(\theta_n))\Delta \tilde{u}_{k+1} = 0, \qquad n = 1, \ldots, L$$

It is easy to show that this is a feasible solution to the problem defined in (26) at $k + 1$ and, in addition, we have $\tilde{\delta}_{k+1}^s(\theta_T) = \delta_k^{s^*}(\theta_T)$. Thus, the value of the cost for the true plant with this feasible solution is given by

$$V_{4,k+1}(\Delta\tilde{u}_{k+1},\tilde{\delta}^{s}_{k+1}(\theta_T)) = V_{4,k}(\Delta u^*_k,\delta^{s^*}_k(\theta_T)) - (e(k) + \delta^{s^*}_k(\theta_T))^T Q(e(k)+\delta^{s^*}_k(\theta_T)) - \Delta u^*(k/k)^T R\Delta u^*(k/k)$$

Then

$$V_{4,k+1}(\Delta\tilde{u}_{k+1},\tilde{\delta}^{s}_{k+1}(\theta_T),\theta_T) \le V_{4,k}(\Delta u^*_k,\delta^{s^*}_k(\theta_T),\theta_T)$$

and from (27) it is clear that

$$V_{4,k+1}(\Delta u^*_{k+1},\delta^{s^*}_{k+1}(\theta_T),\theta_T) \le V_{4,k+1}(\Delta\tilde{u}_{k+1},\tilde{\delta}^{s}_{k+1}(\theta_T),\theta_T)$$

$$V_{4,k+1}(\Delta u^*_{k+1},\delta^{s^*}_{k+1}(\theta_T),\theta_T) \le V_{4,k}(\Delta u^*_k,\delta^{s^*}_k(\theta_T),\theta_T)$$

Also, if Q, R, and S_1 are selected such that condition (23) is satisfied for all the plants in Ω, then, in the earlier relation, equality will be true only if $e(k) = 0$ and $\Delta u(k/k) = 0$. This shows that the sequence of optimal cost for the true plant is decreasing and converges to zero although $V_{4,k}(\theta)_{\theta\neq\theta_T}$ is not necessarily decreasing.

We can also prove robust stability of the controller defined by Problems 2a and 2b. To simplify the proof, assume that $m = 2$ and suppose that at time k, Problem 2a has already converged and these two problems have become equivalent to Problem (26), whose optimal solution that was obtained at step $k-1$ is represented by $\Delta u^*_{k-1} = [\Delta u^*(k-1/k-1)^T\ \Delta u^*_{k-1}(k/k-1)^T]^T$ and $\delta^{s^*}_{k-1}$. Then, at time k, $\Delta\tilde{u}_k = [\Delta u^*(k/k-1)^T\ 0]^T$ and $\tilde{\delta}^s_k = \delta^{s^*}_{k-1}$ is a feasible solution to Problem (26). For this feasible solution, let us calculate the corresponding value of the cost function of Problem (26):

$$\tilde{V}_{4,k}(\Delta\tilde{u}_k,\tilde{\delta}^s_k,\theta_T) = \sum_{j=0}^{1}(e(k+j)+\delta^{s^*}_{k-1}(\theta_T))^T Q(e(k+j) + \delta^{s^*}_{k-1}(\theta_T)) + x^d(k+2)^T\bar{Q}(\theta_T)x^d(k+2) + \Delta u^*(k/k-1)^T \times R\Delta u^*(k/k-1) + \delta^{s^*}_{k-1}(\theta_T)^T S_1\delta^{s^*}_{k-1}(\theta_T) \tag{29}$$

From (12) we have

$$\Delta u^*(k/k-1) = -x^i(k) = C_u\bar{x}(k) \quad \text{where } C_u = [0\ 0\ -I_{ny}]$$

$$\text{and}\quad \bar{x}(k) = \begin{bmatrix} e^s(k)\\ x^d(k)\\ x^i(k)\end{bmatrix}$$

Also, from (28) we have

$$\delta^{s^*}_{k-1}(\theta_T) = -e^s(k) - D^0(\theta_T)x^i(k) = C_\delta(\theta_T)\bar{x}(k) \quad \text{where } C_\delta(\theta_T) = [-I_{ny}\quad 0\quad -D^0(\theta_T)]$$

Now, using the state model defined in (7), we can write the following equations:

$$e(k) = C\bar{x}(k) \quad \text{where } C = [I_{ny}\quad \Psi\quad 0]$$

$$e(k+1) = C_1(\theta_T)\bar{x}(k) \quad \text{where } C_1(\theta_T) = [I_{ny}\ \Psi F(\theta_T)\ -(D^0(\theta_T)+\Psi D^d(\theta_T)F(\theta_T)N)]$$

$$x^d(k+2) = C_d(\theta_T)\bar{x}(k) \quad \text{where } C_d(\theta_T) = [0\quad F(\theta_T)^2\ -D^d(\theta_T)F(\theta_T)^2N]$$

Then, the cost represented in (29) can be written as follows:

$$\tilde{V}_{4,k}(\Delta\tilde{u}_k,\tilde{\delta}^s_k,\theta_T) = \bar{x}(k)^T H(\theta_T)\bar{x}(k)$$

where

$$H(\theta_T) = (C+C_\delta(\theta_T))^T Q(C+C_\delta(\theta_T)) + (C_1(\theta_T)+C_\delta(\theta_T))^T Q(C_1(\theta_T)+C_\delta(\theta_T)) + C_d(\theta_T)^T\bar{Q}(\theta_T)C_d(\theta_T) + C_u^T R\, C_u + C_\delta(\theta_T)S_1C_\delta(\theta_T)$$

By a similar procedure as mentioned earlier, at time $k+n$ for any $n > 1$, based on the optimal solution at time $k+n-1$, we can find a feasible solution to Problem (26), and the cost for the true plant corresponding to this feasible solution can be written as follows:

$$\tilde{V}_{4,k+n}(\Delta\tilde{u}_{k+n},\tilde{\delta}^s_{k+n},\theta_T) = \bar{x}(k+n)^T H(\theta_T)\bar{x}(k+n)$$

By the cost contraction condition (27), it is clear that

$$\tilde{V}_{4,k+n}(\Delta\tilde{u}_{k+n},\tilde{\delta}^s_{k+n},\theta_T) \le \tilde{V}_{4,k}(\Delta\tilde{u}_k,\tilde{\delta}^s_k,\theta_T)$$

Now, let us define

$$\alpha = [\lambda_{\max}(H(\theta_T)/\lambda_{\min}(H(\theta_T)]^{1/2}$$

Consequently, if $\|\bar{x}\ (k)\| \le \rho$, where ρ is positive arbitrary, then $\|\bar{x}(k+n)\| \le \alpha\rho$, which proves stability of the proposed control law. □

In the controller described earlier, at any time step k, Problem 2a is solved first and then we solve Problem 2b, and the two problems interact. However, because Problem 2a does not consider the dynamic behavior of the system, one might speculate that the performance of the closed-loop system may not be satisfactory. An alternative controller that may remedy this problem is based on the following problem:

$$\min_{\substack{\Delta u_k,\delta^s_k(\theta_n),\delta^i_k\\ n=1,\ldots,L}} V_{5,k} = \sum_{j=1}^{\infty}(e(k+j)+\delta^s_k(\theta_N)+j\Delta t\delta^i_k)^T Q(e(k+j) + \delta^s_k(\theta_N)+j\Delta t\delta^i_k) + \sum_{j=0}^{m-1}\Delta u(k+j/k)^T R\Delta u(k+j/k) + \delta^s_k(\theta_N)^T S_1\delta^s_k(\theta_N) + \delta^{i^T}_k S_2\delta^i_k \tag{30}$$

subject to

$$e^s(k)+\delta^s_k(\theta_n) + (D^0_m(\theta_n) - D^i_{2m}(\theta_n))\Delta u_k = 0 \quad n = 1,\ldots,L$$

$$x^i(k)+\delta^i_k+\bar{I}\Delta u_k = 0$$

$$\Delta u(k+j/k)\in U,\quad j = 0,1,\ldots,m-1$$

$$V_{5,k}(\Delta u_k,\delta^s_k(\theta_n),\delta^i_k(\theta_n),\theta_n) \le V_{5,k}(\Delta\tilde{u}_k,\tilde{\delta}^s_k(\theta_n),\tilde{\delta}^i_k(\theta_n),\theta_n) \quad n = 1,\ldots,L$$

Table 1. Parameters of the Multiplant System

	Mo 1	Mo 2	Mo 3	Mo 4	Mo 5
$K_{1,1}$	−0.38	−0.38	−0.019	−0.019	−0.19
$K_{1,2}$	−4.68	−4.68	−0.17	−0.17	−1.7
$K_{2,1}$	−2.10	−0.08	−2.10	−0.08	−0.76
$K_{2,2}$	0.47	0.0235	0.47	0.0235	0.235
$\tau_{1,2}$	7.73	37.67	13.04	37.67	19.50
$\tau_{2,1}$	9.14	17.61	146.56	146.56	31.75

where
$\tilde{\delta}_k^i$ and $\tilde{\delta}_k^s$ are such that

$$\bar{\delta}_k^i = -x^i(k) - \bar{I}\,\Delta\tilde{u}_k$$

$$\Delta\tilde{u}_k = [\Delta u^*(k/k-1)^T \ \cdots \ \Delta u^*(k+m-2/k-1)^T \ 0]^T$$

$$\bar{\delta}_k^s(\theta_n) = -\bar{e}^s(\theta_n) + (D_{2m}^i(\theta_n) - D_m^0(\theta_n))\Delta\tilde{u}_k \quad n = 1, \ldots, L$$

Although we cannot guarantee the convergence of $V_{5,k}$ to zero, it is clear that if S_2 is increased, the problem defined in (30) tends to become equivalent to Problem 2a and δ_k^i converges to zero. Also, after the convergence of δ_k^i to zero, Problem (30) becomes equivalent to the problem defined in (26). Thus, if S_2 is selected large enough, we can expect that the controller resulting from the solution to Problem (30)

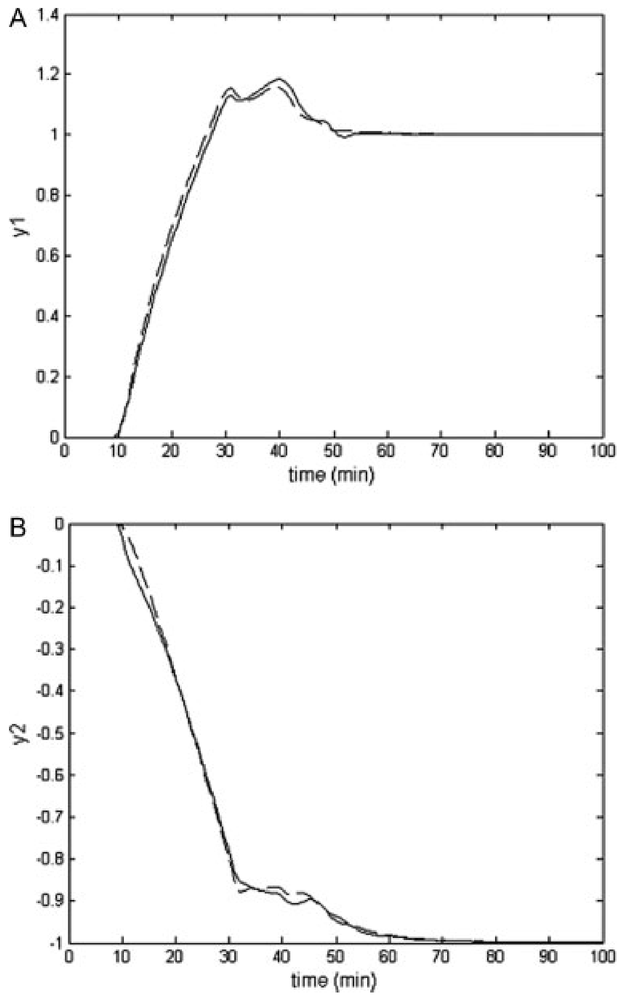

Figure 1. Outputs of the ethylene oxide system in closed loop for output tracking with the robust MPC: Controller I (—) and Controller II (– – –).

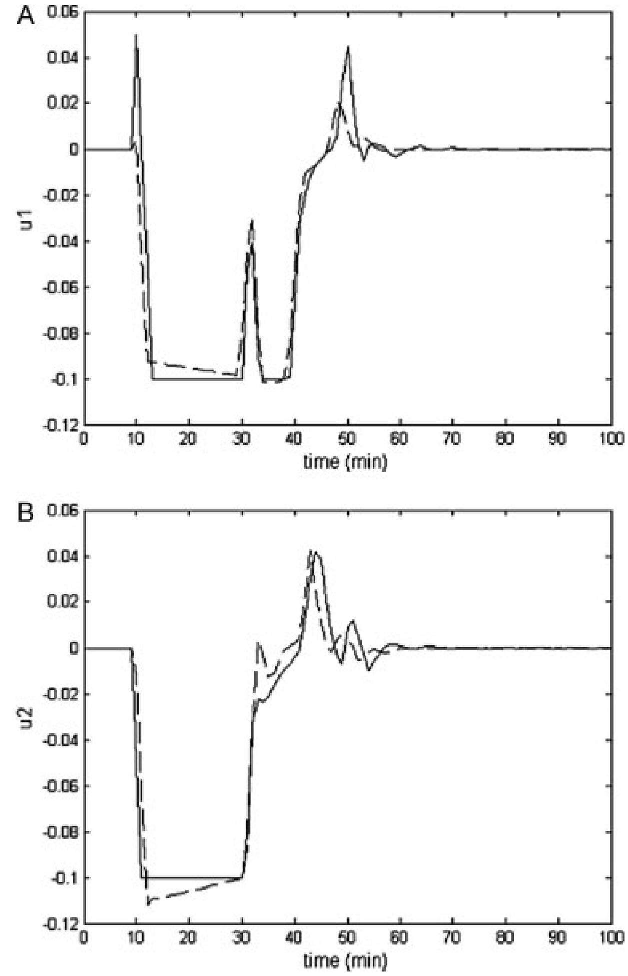

Figure 2. Inputs of the ethylene oxide system in closed loop for output tracking with the robust MPC: Controller I (—) and Controller II (– – –).

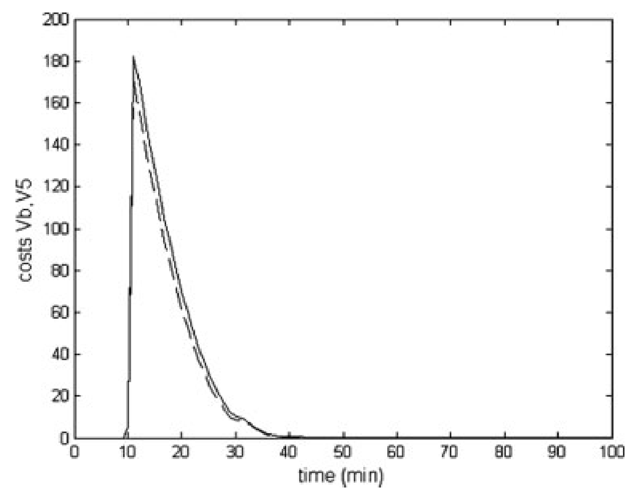

Figure 3. Control costs of Controllers I (—) and II (– – –) for output tracking.

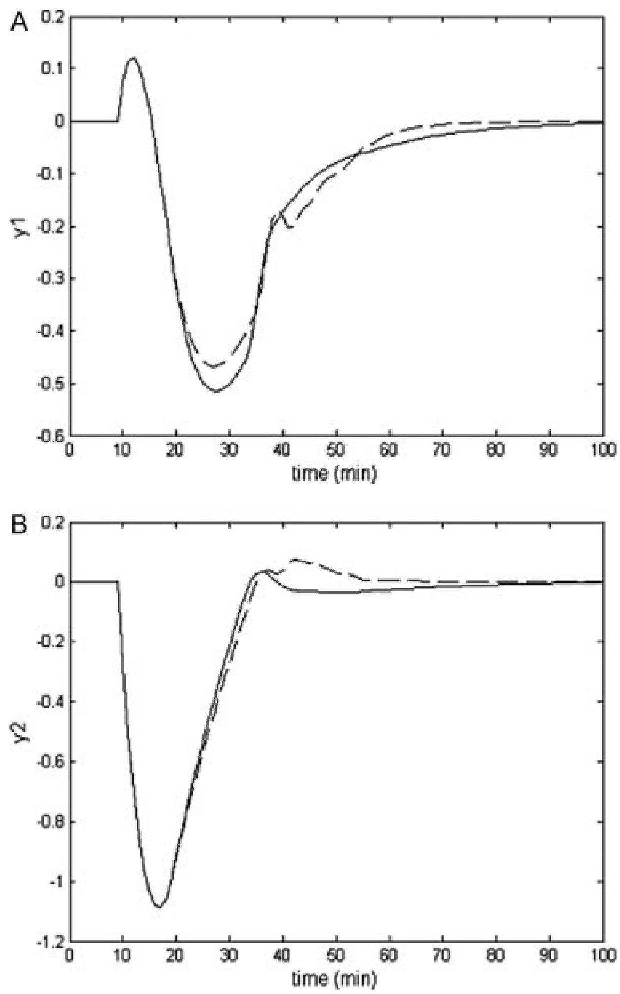

Figure 4. Outputs of the ethylene oxide system in closed loop for the regulator case with Controller I (—) and Controller II (– – –).

will perform as the globally convergent robust MPC defined in Problems 2a and 2b.

Simulation Results

The system adopted as an example to test the performance of the robust controller presented here is part of the ethylene oxide reactor system presented by Rodrigues and Odloak.[17] This is a typical example of the chemical process industry that exhibits stable and integrating poles. The simulated system is represented by the following transfer function:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{K_{11}}{s} & \dfrac{K_{12}}{\tau_{12}s+1} \\ \dfrac{K_{21}}{\tau_{21}s+1} & \dfrac{K_{22}}{s} \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}.$$

It is assumed that we have uncertainty on the gains and time constants of the system. The robust controller is designed for the case where set Ω contains five different plants. Table 1 presents the parameters corresponding to each of these plants. In the simulations presented here, the nominal plant is represented by model 5 and the true plant is represented by model 1.

In all the cases considered here, the tuning parameters of the controller are the following: $m = 3$, $\Delta t = 1$, $u_{\max} = [0.75\ 0.75]$, $u_{\min} = [-0.75\ -0.75]$, $\Delta u_{\max} = [0.05\ 0.05]$, $Q = \mathrm{diag}(1\ 1)$, $R = \mathrm{diag}(7.5\ 7.5)$, $S_1 = \mathrm{diag}(1\ 1) \times 10^2$, and $S_2 = \mathrm{diag}(1\ 1) \times 10^4$. Let us designate Controller I, the controller defined by the sequential solution to Problems 2a and 2b, and we designate Controller II, the controller defined by the solution to the problem defined in (30). The system starts from the origin and at time step 10 min, the desired output values are changed to $y^r = [1 -1]^T$.

We can see in Figures 1 and 2 that the system inputs and outputs are almost coincident for the two controllers. This is easy to justify since the system starts from steady state, in which we have $x^i(0) = 0$, and consequently, during all the simulation time, constraint (12) remains feasible with $\delta_k^i = 0$. Thus, the conditions for the asymptotic convergence of the robust controller defined by Problems 2a and 2b and the controller defined in Problem (30) are satisfied since the beginning of the simulation period. This is shown in Figure 3 where we see that $V_{b,k}$ and $V_{5,k}$ for the true plant converges asymptotically to zero. Also, as $\delta_k^i = 0$ for the output tracking case, the objective function of Problem 2a remains equal

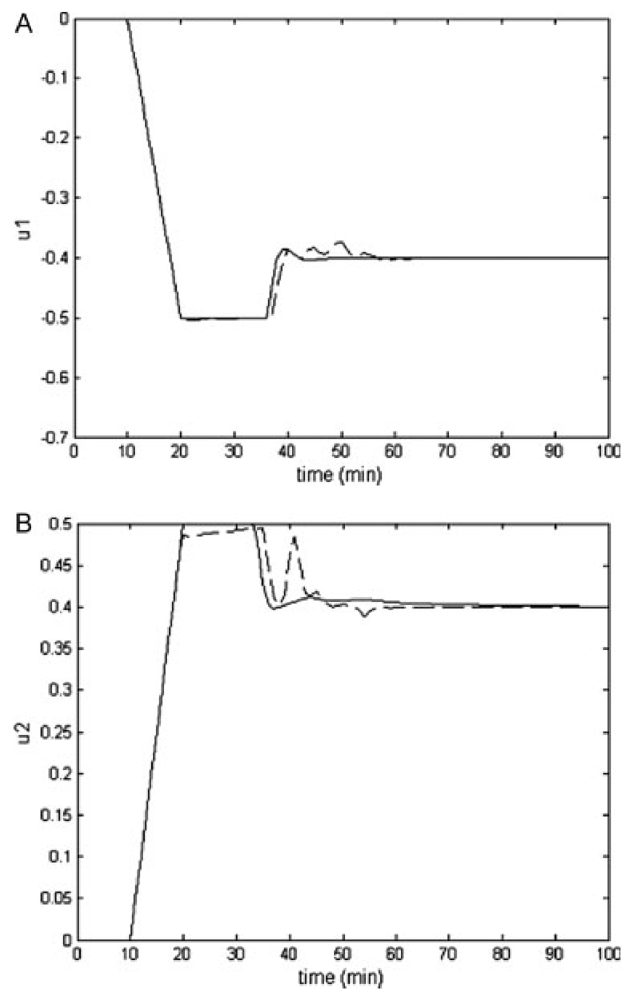

Figure 5. Inputs of the ethylene oxide system in closed loop for the regulator case with Controller I (—) and Controller II (– – –).

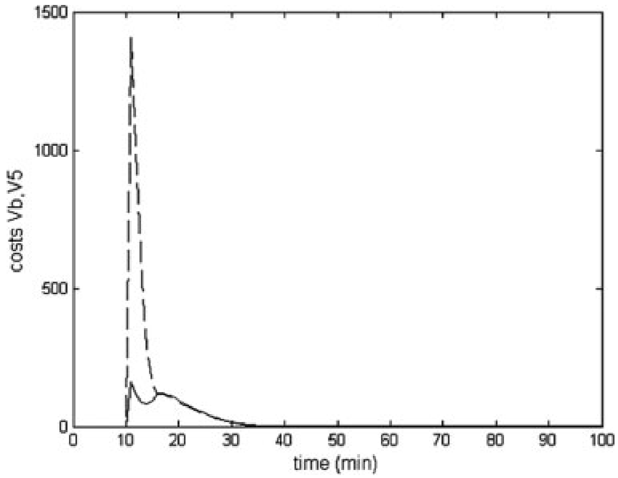

Figure 6. Control costs for cOntrollers I (—) and II (– – –) in the regulator case.

to zero throughout the simulation time, and $V_{b,k}$ tends to be equal to $V_{5,k}$ as shown in Figure 3.

The same controllers were also tested for the regulator case by simulating the closed-loop system for an unmeasured disturbance in the input. This disturbance corresponds to $\Delta u = [0.4\ \ -0.4]$. The desired output values were kept at $(y^r{}_1, y^r{}_2) = [0\ 0]$ during all the simulation time. Figures 4 and 5 show the outputs and inputs of the system, respectively, for this case. It is clear that with both controllers, the outputs tend to the desired values while the inputs converge to new steady state values that are not equal to zero. This is so because the inputs need to compensate the effect of the disturbance that is introduced in the input. However, although the input and output responses with the two controllers are quite similar, cost functions $V_{5,k}$ and $V_{b,k}$ behave differently from each other. Figure 6 shows that for Controller I, cost $V_{b,k}$ is not strictly decreasing until the components of the slack vector δ_k^i that is represented in Figure 7 are zeroed. This can only be made after eight time steps. For this simulated example, the cost of Controller II is strictly decreasing, although the components of δ_k^i tend to zero in a much slower pace than in Controller I.

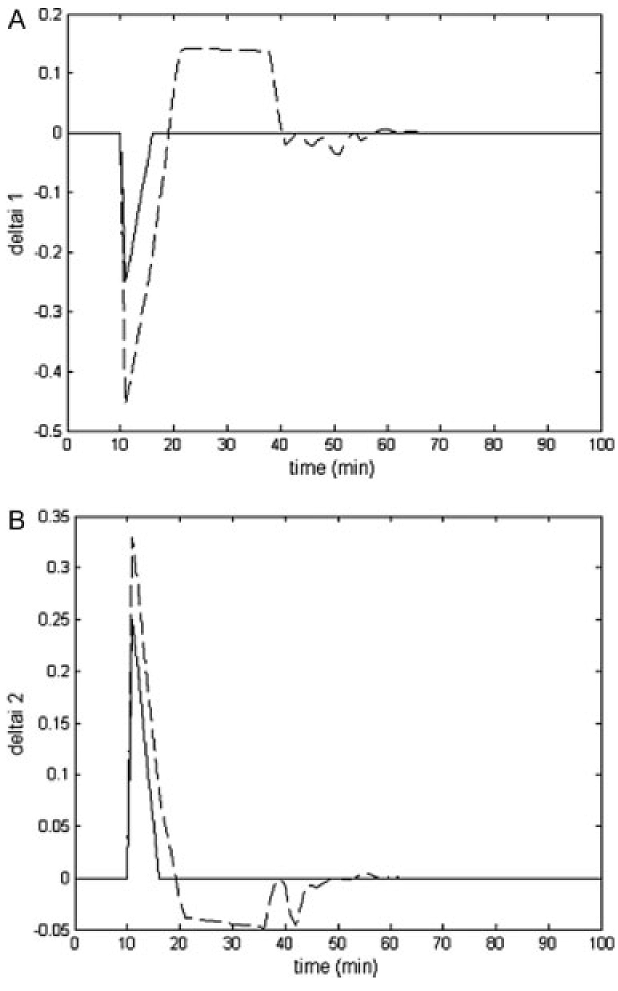

Figure 7. Slacks δ_k^i for Controllers I (—) and II (– – –) in the regulator case.

Conclusion

In this article we have discussed methods to consider model uncertainty in the infinite horizon MPC controller that is stable for systems containing stable and integrating modes. Robust stability is achieved by assembling cost-contracting constraints as well as the constraints that are necessary to cancel the effect of the integrating modes on the output prediction at time instants beyond the control horizon. On the other hand, the control formulation allows dealing with problems that cannot be reduced to the regulator problem due to unknown disturbances or model nonlinearity, and can be directly implemented in real applications. A representative example shows the capability of the controllers to handle uncertainties in all the parameters of the linear model considered in controller.

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