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Spectral flow and the exact AdS_3/CFT_2 chiral ring

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Abstract: We compute all worldsheet three-point functions involving spectrally-flowed operators in chiral multiplets of the space-time theory for strings in $AdS_3 \times S^3 \times T^4$, thus completing the analysis of the full AdS_3/CFT_2 chiral ring. We make use of techniques recently developed for the bosonic sector, based on holomorphic covering maps from the worldsheet to the AdS_3 boundary. We highlight the role of the so-called series identifications when dealing with the complications originated by picture-changing spectrally-flowed states. We find an exact agreement with the predictions from the holographic CFT at the symmetric orbifold point.

Keywords: AdS-CFT Correspondence, Conformal Field Models in String Theory, Supersymmetry and Duality

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Contents

1 Introduction

Exact descriptions are available on both sides of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence. On the one hand, the holographic CFT is best understood at the symmetric orbifold point in moduli space. On the other, for pure Neveu-Schwarz (NS) fluxes the worldsheet theory is solvable, the main ingredient being the Wess-Zumino-Witten (WZW) model based on the universal cover of $\mathrm{SL}(2,\mathbb{R})$. However, for general values of n_5 — the number of NS5-brane sources — the latter is *not* dual to a symmetric orbifold CFT,[1] and therefore, when comparing these two theories, one must restrict to observables which are protected by non-renormalization theorems [7, 8]. The most relevant examples are the spectrum of chiral primary operators and their three-point functions.

[1]The exception corresponds to the $n_5 = 1$ case, for which the strings become tensionless and the holographic CFT reduces to $\mathrm{Sym}^{n_1}\left(T^4\right)$, where n_1 is the number of F1-string sources [1–4]. BPS correlators in this context were studied in [5, 6].

This program was initiated in [9, 10], where the authors determined some worldsheet three-point functions matching those predicted in [11–13]. The computation was done only for states in the spectrally unflowed sector of the SL(2,$\mathbb{R}$)-WZW model, which captures the low-lying spacetime chiral primaries with weights $h \leq (n_5+1)/2$. Worldsheet vertex operators with larger values of h have non-trivial spectral flow charges [14] and require a different treatment. Spectral flow constitutes one of the distinctive aspects of the model, and it introduces important complications for studying the corresponding correlation functions [15, 16], especially in the supersymmetric context [17–19].

Obtaining the full set of superstring three-point functions in $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ (or K3), including those where spectral flow is not conserved, remains an open problem. In this paper, we complete the holographic matching (at all orders in α') for the full $\mathrm{AdS}_3/\mathrm{CFT}_2$ chiral ring, building on [19]. For this, we focus on the short-string sector of the theory and make use of recently developed techniques for computing bosonic three-point functions involving vertex operators with arbitrary spectral flow charges [20–23]. We extend these methods in order to compute descendant correlators appearing after the ghost picture-changing procedure, mandatory for supersymmetric correlators. We further obtain all relevant fermionic correlation functions. The analysis of non-protected correlators with long-string insertions and the corresponding matching with the proposal of [24] are left for future work.

As the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence constitutes one of the few scenarios where we can study holography far away from the supergravity regime, our results provide an important contribution to understanding the precise mechanism at play behind this duality. Furthermore, the techniques we present in this work will also be instrumental when considering applications to black holes and some of their microstates [25–30]. The computation of the first string correlators for these more involved worldsheet models was carried out recently in [31, 32]. These results can also be connected to the study of holography beyond AdS, little string theory, and the so-called single-trace $T\bar{T}$ deformations of two-dimensional CFTs [33–40].

The paper is organized as follows. In section 2 we introduce the worldsheet theory for superstrings in $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ and provide the definition of all vertex operators corresponding to spacetime chiral primaries, relegating some details to appendix A. In section 3 we review the recent results on correlators of primary fields with non-trivial spectral flow charges in the bosonic SL(2,$\mathbb{R}$)-WZW model [21, 23], and then extend these results to the SU(2) and fermionic sectors. The main results of this paper are obtained in section 4, where we compute all fusion rules and structure constants of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ chiral ring from the worldsheet theory, and show that they agree with the predictions from the holographic CFT exactly for all $n_5 > 1$. This includes both extremal and non-extremal correlators, in the sense of [17, 41]. Section 5 contains our concluding remarks and outlook.

2 Brief review of superstrings in $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$

We start by briefly reviewing the basic definitions for type IIB superstrings on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$. We focus on the holographic dictionary for spacetime scalar chiral primaries and define the

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corresponding vertex operators in the NSNS and RR sectors of the worldsheet theory for $n_5 > 1$. More details can be found in [9, 17, 19].

2.1 Basic definitions

String propagation in $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ with NSNS fluxes is characterized by the supersymmetric WZW model based on $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SU}(2)\times\mathrm{U}(1)^4$. In this section we discuss this model, mostly following the notation of [19].

The affine SL(2,$\mathbb{R}$) and SU(2) currents and fermions are denoted as J^a, ψ^a, K^a and χ^a, respectively, with $a = 0, 1, 2$. They satisfy the following OPEs:

$$J^a(z)J^b(w) \sim \frac{\frac{n_5}{2}\eta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c J^c(w)}{z-w}, \quad K^a(z)K^b(w) \sim \frac{\frac{n_5}{2}\delta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c K^c(w)}{z-w}, \tag{2.1}$$

$$J^a(z)\psi^b(w) \sim \frac{i\epsilon^{ab}{}_c\psi^c(w)}{z-w}, \qquad K^a(z)\chi^b(w) \sim \frac{i\epsilon^{ab}{}_c\chi^c(w)}{z-w}, \tag{2.2}$$

$$\psi^a(z)\psi^b(w) \sim \frac{\frac{n_5}{2}\eta^{ab}}{z-w}, \qquad \chi^a(z)\chi^b(w) \sim \frac{\frac{n_5}{2}\delta^{ab}}{z-w}, \tag{2.3}$$

where n_5 is the level in both cases, while $\epsilon^{012} = 1$, $\eta^{ab} = \eta_{ab} = (-++)$ and $\delta^{ab} = \delta_{ab} = (+++)$. As usual, we define the ladder operators as $J^\pm = J^1 \pm iJ^2$, and similarly for $K^\pm$, $\psi^\pm$ and $\chi^\pm$. The supersymmetric currents split as

$$J^a = j^a + \hat{\jmath}^a\,, \qquad K^a = k^a + \hat{k}^a, \tag{2.4}$$

where j^a and k^a generate bosonic affine algebras $\mathrm{SL}(2,\mathbb{R})_k$ and $\mathrm{SU}(2)_{k'}$ with shifted levels $k = n_5 + 2$ and $k' = n_5 - 2$, while

$$\hat{\jmath}^a = -\frac{i}{n_5}\epsilon^a{}_{bc}\psi^b\psi^c\,, \qquad \hat{k}^a = -\frac{i}{n_5}\epsilon^a{}_{bc}\chi^b\chi^c\,, \tag{2.5}$$

generate fermionic $\mathrm{SL}(2,\mathbb{R})_{-2}$ and $\mathrm{SU}(2)_2$ algebras, decoupled from the bosonic sector. The free bosons and fermions associated with the T^4 directions are written as Y^i and λ^i, respectively, with $i = 6,\dots,9$. It becomes very convenient to bosonize the fermions by introducing bosonic fields H_I, $I = 1,\dots 5$, such that[2]

$$\psi^\pm = \sqrt{n_5}\,e^{\pm iH_1}\,, \qquad \chi^\pm = \sqrt{n_5}\,e^{\pm iH_2}\,, \qquad \lambda^6 \pm i\lambda^7 = e^{\pm iH_4}\,, \qquad \lambda^8 \pm i\lambda^9 = e^{\pm iH_5}\,, \tag{2.6a}$$

$$\psi^0 = \frac{\sqrt{n_5}}{2}\left(e^{iH_3} - e^{-iH_3}\right) \qquad \chi^0 = \frac{\sqrt{n_5}}{2}\left(e^{iH_3} + e^{-iH_3}\right)\,. \tag{2.6b}$$

with $H_I^\dagger = H_I$ for $I \neq 3$ and $H_3^\dagger = -H_3$.

The stress tensor T and supercurrent G characterizing the matter sector of the worldsheet CFT read

$$T = \frac{1}{n_5}\left(j^a j_a - \psi^a\partial\psi_a + k^a k_a - \chi^a\partial\chi_a\right) + \frac{1}{2}\left(\partial Y^i\partial Y_i - \lambda^i\partial\lambda_i\right), \tag{2.7}$$

$$G = \frac{2}{n_5}\left(\psi^a j_a + \frac{2i}{n_5}\psi^0\psi^1\psi^2 + \chi^a k_a - \frac{2i}{n_5}\chi^0\chi^1\chi^2\right) + i\,\lambda^i\partial Y_i\,. \tag{2.8}$$

[2]To be precise, one has to refine this definition slightly in order to keep track of the cocycle factors. This amounts to replacing H_I by $\hat{H}_I$ with $\hat{H}_I = H_I + \pi\sum_{J<I} N_J$, where $N_J \equiv \oint i\partial H_J$ so that $e^{ia\hat{H}_I}e^{ib\hat{H}_J} = e^{ib\hat{H}_J}e^{ia\hat{H}_I}\,e^{i\pi ab}$ if $I > J$ [9].

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We also need to consider the standard bc and $\beta\gamma$ ghost systems, leading to the BRST charge

$$\mathcal{Q} = \oint dz \left[c\,(T + T_{\beta\gamma}) - \gamma\, G + c(\partial c)b - \frac{1}{4} b\gamma^2 \right] . \tag{2.9}$$

The $\beta\gamma$ system is further bosonized as

$$\beta = e^{-\varphi}\partial\xi\,, \qquad \gamma = \eta e^{\varphi}\,, \tag{2.10}$$

where φ has a background charge $Q_\varphi = -2$, while $\xi(z)\eta(w) \sim (z-w)^{-1}$. The spacetime supercharges can thus be written as

$$Q_\varepsilon = \oint dz\, e^{-\varphi/2} S_\varepsilon\,, \qquad S_\varepsilon = \exp\left(\frac{i}{2} \sum_{I=1}^{5} \varepsilon_I H_I \right), \tag{2.11}$$

where S_ε are spin fields and $\varepsilon_I = \pm 1$. BRST-invariance and the mutual locality constraints impose $\varepsilon_1\varepsilon_2\varepsilon_3 = \varepsilon_4\varepsilon_5 = 1$, leading to the supercharges of the spacetime $\mathcal{N} = (4,4)$ superconformal algebra. Finally, the zero modes of the worldsheet SU(2) currents are identified with those of the R-symmetry of the boundary theory.

2.2 Short string spectrum and vertex operators: the NS sector

We first focus on the NS sector of the theory. The (holomorphic part of) vertex operators holographically dual to the low-lying chiral primaries of the holographic CFT are expressed as [9, 10]

$$\mathcal{V}_j(x,u,z) \;= e^{-\varphi(z)}\psi(x,z)V_j(x,z)W_{j-1}(u,z), \tag{2.12a}$$
$$\mathcal{W}_j(x,u,z) \;= e^{-\varphi(z)}V_j(x,z)\chi(u,z)W_{j-1}(u,z). \tag{2.12b}$$

in the canonical (-1) ghost picture where x is the holographic coordinate, u is the SU(2) isospin variable, and z is the worldsheet coordinate. Given the weights $\Delta_j = -j(j-1)/n_5$ and $\Delta'_l = l(l+1)/n_5$, the Virasoro condition relates the $\mathrm{SL}(2,\mathbb{R})_k$ and $\mathrm{SU}(2)_{k'}$ spins j and l, setting $l = j-1$. The bosonic primaries $V_j(x,z)$ and $W_l(u,z)$ satisfy

$$j^a(z)V_j(x,w) \sim \frac{D^a_{x,j}V_j(x,w)}{z-w}, \quad k^a(z)W_l(u,w) \sim \frac{P^a_{u,l}W_l(u,w)}{z-w}, \tag{2.13}$$

where

$$D^+_{x,j} = \partial_x, \qquad D^0_{x,j} = x\partial_x + j, \qquad D^-_{x,j} = x^2\partial_x + 2jx \tag{2.14}$$
$$P^+_{u,l} \;= \partial_u, \qquad P^0_{u,l} \;= u\partial_u - l, \qquad P^-_{u,l} \;= -u^2\partial_u + 2lu. \tag{2.15}$$

Here we work in the so-called x-basis for SL(2,$\mathbb{R}$) and the isospin u-basis for SU(2). Operators $W_l(u,z)$ have integer and half-integer spins in the range $0 \leq l \leq k'/2$, while the corresponding modes $W_{l,n}(z)$ have spin projections taking values $n = -l, -l+1, \dots, l-1, l$ [42]. Operators $V_j(x,z)$ are associated to normalizable short string states for real spins satisfying $1/2 < j < (k-1)/2$, and, for the corresponding $\mathcal{D}^+_j$ ($\mathcal{D}^-_j$) representations, the modes $V_{j,m}$ have projections $m = j, j+1, \dots$ ($m = -j, -j-1, \dots$) [15].

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The complex label x is identified with the holomorphic coordinate on the AdS$_3$ boundary. Eqs. (2.14) show that in the bosonic theory j_0^a realize the global sector of the spacetime Virasoro algebra, such that j is identified with the spacetime weight h. In the supersymmetric model an analogous statement holds for the zero-modes J_0^a, so that the spacetime weight H, identified with the spin of the corresponding $\mathrm{SL}(2,\mathbb{R})_{n_5}$ representation, is defined as $H = j$ for $\mathcal{W}_j$, while for $\mathcal{V}_j$ the fermions induce the shift $H = j - 1$.

Worldsheet operators associated to spacetime chiral primaries with $H > n_5/2$ belong to the so-called spectrally-flowed sectors of the theory and were constructed in [17]. In terms of the mode algebra, spectral flow refers to the automorphisms

$$
\begin{aligned}
j_n^{0,\omega} &= j_n^0 - \frac{k}{2}\omega\delta_{n,0}\,, \qquad & k_n^{0,\omega} &= k_n^0 + \frac{k'}{2}\omega\delta_{n,0}\,, \qquad & j_n^{\pm,\omega} &= j_{n\pm\omega}^{\pm}\,, \qquad & k_n^{\pm,\omega} &= k_{n\pm\omega}^{\pm}\,,\\
\psi_n^{0,\omega} &= \psi_n^0\,, & \chi_n^{0,\omega} &= \chi_n^0\,, & \psi_n^{\pm,\omega} &= \psi_{n\pm\omega}^{\pm}\,, & \chi_n^{\pm,\omega} &= \chi_{n\pm\omega}^{\pm}.
\end{aligned}
$$

with $\omega \in \mathbb{Z}$. Let us first focus on the SL(2,$\mathbb{R}$) sector. Due to the fact that AdS$_3$ is both Lorentzian and non-compact, spectral flow gives rise to additional physical states, i.e. it generates representations inequivalent to the unflowed ones considered above. The corresponding bosonic vertex operators are built upon the m-basis spectrally-flowed primaries $V_{j,m}^{\omega}$. The latter are affine primary fields with respect to the flowed currents $j^{a,\omega}$. Hence, for $\omega > 0$, we have

$$
j^0(z)V_{j,m}^{\omega}(w) = (m + \frac{k}{2}\omega)\frac{V_{j,m}^{\omega}(w)}{z-w} + \cdots\,, \tag{2.16a}
$$

$$
j^-(z)V_{j,m}^{\omega}(w) = (z-w)^{\omega-1}(m+j-1)V_{j,m-1}^{\omega}(w) + \cdots\,, \tag{2.16b}
$$

$$
j^+(z)V_{j,m}^{\omega}(w) = (m-j+1)\frac{V_{j,m+1}^{\omega}(w)}{(z-w)^{1+\omega}} + \sum_{l=0}^{\omega-1}\frac{\left[j_l^+V_{j,m}^{\omega}\right](w)}{(z-w)^{l+1}} + \cdots\,, \tag{2.16c}
$$

where the ellipses denote higher-order terms. As a consequence, these are the lowest-weight states in a (discrete) spin $h = m + k\omega/2$ representation of the zero-mode algebra, while the worldsheet weight becomes $\Delta_j^{\omega} = -\frac{j(j-1)}{k-2} - \omega m - \frac{k}{4}\omega^2$. Importantly, $V_{j,m}^{\omega}$ and $V_{j,-m}^{-\omega}$ lead to the same spin, and they both contribute to the same x-basis operator, such that one can freely restrict to positive values of ω in this picture. The precise definition can be written as

$$
V_{j,h}^{\omega}(x,z) = e^{xj_0^+}V_{j,m}^{\omega}(z)e^{-xj_0^+}\,, \qquad h = m + \frac{k}{2}\omega\,, \tag{2.17}
$$

which can be understood as translating the operator at the origin, namely $V_{j,m}^{\omega}(z) = V_{j,h}^{\omega}(0,z)$. Note that, unlike in the unflowed sector, here we have to keep track of the $\mathrm{SL}(2,\mathbb{R})$ unflowed spin projection m, which specifies the spin h. A similar story holds for the SL(2,$\mathbb{R}$) fermions. Fermionic spectrally-flowed m-basis primaries are nicely expressed in bosonized form as

$$
\psi^{-,\omega}(z) = \sqrt{n_5}e^{-i(1+\omega)H_1(z)}\,, \quad \psi^{+,\omega}(z) = \sqrt{n_5}e^{i(1-\omega)H_1(z)}\,, \quad \psi^{0,\omega}(z) = \psi^0(z)e^{-i\omega H_1(z)}. \tag{2.18}
$$

Note that, $\psi^{0,\omega=1}(z) = \frac{\sqrt{n_5}}{2}\hat{\jmath}^-(z)$, hence $\psi^{0,\omega+1}(z) = \frac{\sqrt{n_5}}{2}\hat{\jmath}^{-,\omega}(z)$. We define the x-basis spectrally-flowed fermion and fermionic current that will be useful below,

$$\psi^\omega(x,z) = e^{x\hat{\jmath}_0^+}\psi^{-,\omega}(z)e^{-x\hat{\jmath}_0^+}\,,\ \hat{\jmath}^\omega(x,z) = e^{x\hat{\jmath}_0^+}\hat{\jmath}^{-,\omega}(z)e^{-x\hat{\jmath}_0^+}. \tag{2.19}$$

Although including spectral flow in the SU(2) sector is, strictly speaking, unnecessary, for short string states it turns out to be useful. Hence, we also introduce the operators $W^\omega_{l,q}(u,z)$, $\chi^\omega(u,z)$, $\hat{k}^\omega(u,z)$, which are completely analogous to their SL(2,$\mathbb{R}$) cousins.

We now have all the relevant ingredients to write down the supersymmetric vertex operators corresponding to the spectrally-flowed versions of those in eq. (2.12), which read

$$\mathcal{V}^\omega_j(x,u,z) = \frac{1}{\sqrt{n_5}}e^{-\varphi(z)}\psi^\omega(x,z)V^\omega_j(x,z)\chi^{\omega-1}(u,z)W^\omega_{j-1}(u,z), \tag{2.20a}$$

$$\mathcal{W}^\omega_j(x,u,z) = \frac{1}{\sqrt{n_5}}e^{-\varphi(z)}\psi^{\omega-1}(x,z)V^\omega_j(x,z)\chi^{\omega}(u,z)W^\omega_{j-1}(u,z). \tag{2.20b}$$

Here we have used that, in the (discrete) flowed case, the BRST constraints force the bosonic primary operators to be lowest-weight [17], and further set $l = j-1$. We have also introduced the shorthands

$$V^\omega_j(x,z) \equiv V^\omega_{j,h=j_\omega}(x,z)\,, \quad W^\omega_l(u,z) \equiv W^\omega_{l,q=l_\omega}(u,z) \tag{2.21}$$

with $j_\omega = j + k\omega/2$ and $l_\omega = l + k'\omega/2$. As a consequence, the spacetime weights are given by

$$H\left[\mathcal{V}^\omega_j\right] = j-1+n_5\omega/2\,, \qquad H\left[\mathcal{W}^\omega_j\right] = j+n_5\omega/2\,. \tag{2.22}$$

There are actually two additional families of supersymmetric flowed vertex operators one can construct, starting from highest-weight operators that we denote as $V^\omega_{-j}(x,z) \equiv V^\omega_{j,-j+k\omega/2}(x,z)$ and $W^\omega_{-l}(u,z) \equiv W^\omega_{l,-l+k'\omega/2}(u,z)$. Considering them separately will not be necessary. Indeed, the statement that representations in different spectral flow sectors are inequivalent only holds up to the so-called SL(2,$\mathbb{R}$) series identifications [15]. At the bosonic level, we have equivalence relations for highest/lowest-weight flowed primaries taking the following form:

$$V^\omega_j(x,z) = \mathcal{N}(j)V^{\omega+1}_{-\left(\frac{k}{2}-j\right)}(x,z)\,, \qquad W^\omega_l(u,z) = W^{\omega+1}_{-\left(\frac{k'}{2}-l\right)}(u,z)\,, \tag{2.23}$$

where we have also included the SU(2) counterpart. The coefficient $\mathcal{N}(j)$ is defined in terms of the reflection coefficient appearing in the SL(2,$\mathbb{R}$) two-point function, see appendix A. Eq. (2.23) implies that, in the supersymmetric theory, the extra families of operators alluded to above do not lead to additional states, so it is enough to work with those in (2.20). We refer the reader to [19] for more details.

Fermionic representations also satisfy a similar identification. The $\hat{\jmath} = -1$ flowed field $\psi^\omega(x,z)$ is mapped to a $\hat{\jmath} = 0$ flowed field that we denote by $\hat{\psi}^\omega(x,z)$. More precisely, this reads

$$\psi^\omega(x,z) = \sqrt{n_5}\hat{\psi}^{\omega+1}(x,z). \tag{2.24}$$

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The unflowed field $\hat{\psi}(x,z)$ is nothing but the fermionic identity. This can be done similarly for the SU(2) fermions.

The identities presented in eq. (2.23) played a prominent role in the computation of bosonic spectrally-flowed primary three-point functions [23], which we review in section 3 below. As it turns out, the series identifications will also be a crucial ingredient in the present paper, as they will allow us to bypass the additional technical difficulties that arise when computing short-string three-point functions in the supersymmetric model involving vertex operators with non-trivial spectral flow charges [17, 19].

2.3 The R sector

Vertex operators in the Ramond sector of the theory are constructed by using the spin fields defined in eq. (2.11). The $\mathrm{AdS}_3\times\mathrm{S}^3$ chirality is defined as $\varepsilon = \varepsilon_1\varepsilon_2\varepsilon_3$, and the GSO projection imposes $\varepsilon_4\varepsilon_5 = \varepsilon$. The relevant unflowed vertex operators then involve two $\mathrm{SL}(2,\mathbb{R})_{-2}\times\mathrm{SU}(2)_2$ fields of spins $(j,l) = (-1/2, 1/2)$, namely

$$s_\varepsilon(x,u,z) = e^{u\hat{k}_0^0} e^{x\hat{\jmath}_0^0} e^{\frac{i}{2}(-H_1(z)-H_2(z)+\varepsilon H_3(z))} e^{-x\hat{\jmath}_0^0} e^{-u\hat{k}_0^0}\,, \qquad \varepsilon=\pm 1. \tag{2.25}$$

Cocycle factors are important for computing the r.h.s. of (2.25). More explicitly, BRST invariant unflowed states have spins $(j-1/2, l+1/2 = j-1/2)$ with respect to the supersymmetric currents J^a, K^a, and take the form [9, 43]

$$\mathcal{Y}_j^\epsilon(x,u,z) = e^{-\frac{1}{2}\varphi(z)} s_-(x,u,z) V_j(x,z) W_{j-1}(u,z) e^{i\frac{\epsilon}{2}(H_4(z)-H_5(z))}, \tag{2.26}$$

where we have renamed $\varepsilon_4 \to \epsilon$ for simplicity. The corresponding spectrally-flowed states are constructed as in the NS sector. One obtains

$$\mathcal{Y}_j^{\epsilon,\omega}(x,u,z) = e^{-\frac{1}{2}\varphi(z)} s_-^\omega(x,u,z) V_j^\omega(x,z) W_{j-1}^\omega(u,z) e^{i\frac{\epsilon}{2}(H_4(z)-H_5(z))}, \tag{2.27}$$

where $s_-^\omega(x,u)$ is built upon the m-basis flowed primary

$$s_{---}^\omega = e^{-i\left(\frac{1}{2}+\omega\right)H_1(z)-i\left(\frac{1}{2}+\omega\right)H_2(z)-\frac{i}{2}H_3(z)}, \tag{2.28}$$

which is the extremal state in a spin $(-1/2-\omega, 1/2+\omega)$ representation of the fermionic zero-mode algebra. As a consequence, the spacetime weights are

$$H\left[\mathcal{Y}_j^{\epsilon,\omega}\right] = j - 1/2 + n_5\omega/2. \tag{2.29}$$

2.4 Holographic dictionary

We now discuss the identification of each of these operators in terms of the boundary theory at the symmetric orbifold point, namely $\mathrm{Sym}^N\left(T^4\right)$ with $N = n_1 n_5$, following [9, 14, 17]. At large N, the holographic dictionary identifies single string states in the bulk with single cycle fields of the dual CFT. Hence, one needs to start from the twist fields, usually denoted σ_n, which must be dressed appropriately. In the T^4 case, each twist sector contains four types of chiral primary operators, whose explicit expression can be found in [13]. We

will denote them as $O_n^-(x)$, $O_n^a(x)$ and $O_n^+(x)$, respectively, and with $a = 1, 2$. Their holomorphic weights are given by

$$H\left[O_n^-\right] = \frac{n-1}{2}\,, \quad H\left[O_n^a\right] = \frac{n}{2}\,, \quad H\left[O_n^+\right] = \frac{n+1}{2}\,; \quad n = 1, 2, \ldots . \tag{2.30}$$

These are local operators on the boundary, which are associated with z-integrated x-basis operators of the worldsheet theory. Moreover, one can combine the individual states with fixed R-charge in a given R-symmetry chiral multiplet by using the $\mathrm{SU}(2)_{\mathrm{R}}$ currents, leading to the isospin variables introduced in [42]. As the zero modes of the R-symmetry currents are identified with those of the worldsheet SU(2) currents, the isospin variable is identified with the coordinate u used in the previous sections.

Similar considerations hold in the anti-holomorphic sector of the theory. In this paper we focus on chiral primary operators that are spacetime scalars. The NS and R sector operators of the worldsheet theory constructed above are promoted to NSNS and RR vertex operators by including the analogous polarization in the anti-holomorphic sector. For instance, we have

$$\begin{aligned} &\mathbb{V}_j^\omega(x,\bar{x},u,\bar{u},z,\bar{z}) \equiv \\ &\frac{1}{n_5} e^{-\varphi(z)-\bar{\varphi}(\bar{z})}\psi^\omega(x,z)\bar{\psi}^\omega(\bar{x},\bar{z})V_j^\omega(x,\bar{x},z,\bar{z})\chi^{\omega-1}(u,z)\bar{\chi}^{\omega-1}(\bar{u},\bar{z})W_{j-1}^\omega(u,\bar{u},z,\bar{z}), \end{aligned} \tag{2.31}$$

where we have momentarily reinstated the dependence of the bosonic primaries in the anti-holomorphic variables, and similarly for $\mathbb{W}_j^\omega$ and $\mathbb{Y}_j^\omega$. Consequently, and up to the normalization, which will be discussed below, the holographic dictionary for the chiral primary sector is

$$\begin{aligned} O_n^-(x,\bar{x},u,\bar{u}) &\leftrightarrow \mathbb{V}_j^\omega(x,\bar{x},u,\bar{u},z,\bar{z}), \\ O_n^a(x,\bar{x},u,\bar{u}) &\leftrightarrow \mathbb{Y}_j^\omega(x,\bar{x},u,\bar{u},z,\bar{z}), \\ O_n^+(x,\bar{x},u,\bar{u}) &\leftrightarrow \mathbb{W}_j^\omega(x,\bar{x},u,\bar{u},z,\bar{z}), \end{aligned} \tag{2.32}$$

together with the identification

$$n \;=\; 2j - 1 + n_5\,\omega\,. \tag{2.33}$$

From the worldsheet point of view, the allowed ranges are

$$j = 1, \frac{3}{2}, \ldots, \frac{n_5}{2}, \qquad \omega = 0, 1, \ldots . \tag{2.34}$$

This shows that the worldsheet theory accounts for all chiral primaries of the holographic CFT, except for those in the twisted sectors where n is a (non-zero) multiple of n_5 [9, 17]. These would sit exactly at the lower boundary of the allowed range for j. However, at this point, the spectrum degenerates due to the presence of the zero-momentum states belonging to the continuous representations [44, 45]. The fact that the worldsheet theory for strings in AdS_3 fails to describe these states indicates that the NS5-F1 model sits at a singular point in moduli space [46]. This was shown to be resolved when RR fluxes are included, thus lifting the long string sector [47].

2.5 Picture changing

Given the ghost background charge, for three-point functions, it is necessary to compute the ghost picture (0) version of the NSNS operators defined in the previous sections. Given a ghost picture (-1) operator $\mathcal{O}^{(-1)}(z)$, we have

$$\mathcal{O}^{(0)}(z) = \lim_{w\to z} \left(e^{\varphi(w)}G(w)\right)\mathcal{O}^{(-1)}(z)\,. \tag{2.35}$$

Then,

$$\mathcal{V}_j^{\omega,(0)}(x,u,z) = \mathcal{A}_j^{\omega,1}(x,u,z) + (-1)^{\omega}\mathcal{A}_j^{\omega,2}(x,u,z)\,, \tag{2.36}$$

with

$$\mathcal{A}_j^{\omega,1}(x,u,z) = \left[j^-_{-1-\omega}(x,z) - H\hat{\jmath}^-_{-1-\omega}(x,z)\right]\hat{\psi}^{\omega}(x,z)V_j^{\omega}(x,z)\hat{\chi}^{\omega}(u,z)W^{\omega}_{j-1}(u,z) \tag{2.37}$$

$$\mathcal{A}_j^{\omega,2}(x,u,z) = -\frac{1}{n_5}\left[k^+_{\omega}(u,z) - H\hat{k}^+_{\omega}(u,z)\right]\psi^{\omega}(x,z)V_j^{\omega}(x,z)\chi^{\omega}(u,z)W^{\omega}_{j-1}(u,z), \tag{2.38}$$

and

$$\mathcal{W}_j^{\omega,(0)}(x,u,z) = \mathcal{B}_j^{\omega,1}(x,u,z) + (-1)^{\omega}\mathcal{B}_j^{\omega,2}(x,u,z)\,, \tag{2.39}$$

with

$$\mathcal{B}_j^{\omega,1}(x,u,z) = \left[k^-_{-1-\omega}(u,z) - H\hat{k}^-_{-1-\omega}(u,z)\right]\hat{\psi}^{\omega}(x,z)V_j^{\omega}(x,z)\hat{\chi}^{\omega}(u,z)W^{\omega}_{j-1}(u,z), \tag{2.40}$$

$$\mathcal{B}_j^{\omega,2}(x,u,z) = \frac{1}{n_5}\left[j^+_{\omega}(x,z) + H\hat{\jmath}^+_{\omega}(x,z)\right]\psi^{\omega}(x,z)V_j^{\omega}(x,z)\chi^{\omega}(u,z)W^{\omega}_{j-1}(u,z), \tag{2.41}$$

where in each equation H corresponds to the spacetime weight of the corresponding vertex operator. These were derived in [17] and reviewed in [19], although here we have rewritten them in a slightly more symmetric way. Note that although $\mathcal{A}_j^{\omega,1}$ and $\mathcal{A}_j^{\omega,2}$ have the same total fermion parity, they differ in one unit between the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ fermion sector, therefore only one of them can contribute in a given three-point function. The same holds for $\mathcal{B}_j^{\omega,1}$ and $\mathcal{B}_j^{\omega,2}$.

In the R sector, we need to express the field (2.27) in the $(-3/2)$ ghost picture for computing two-point functions. They are given by

$$\mathcal{Y}_j^{\epsilon,\omega,(-\frac{3}{2})}(x,u,z) = -\frac{\sqrt{n_5}}{2j-1+n_5\omega}e^{-\frac{3}{2}\varphi}s^{\omega}_{+}(x,u,z)V_j^{\omega}(x,z)W^{\omega}_{j-1}(u,z)e^{i\frac{\epsilon}{2}(H_4-H_5)}\,, \tag{2.42}$$

where $s^{\omega}_{+}(x,u)$ is the x- and u-basis version of the field

$$s^{\omega}_{--+} = e^{-i\left(\frac{1}{2}+\omega\right)H_1 - i\left(\frac{1}{2}+\omega\right)H_2 + \frac{i}{2}H_3}\,. \tag{2.43}$$

Indeed, one can check that (2.42) satisfies

$$\mathcal{Y}_j^{\epsilon,\omega}(x,u,z) = \lim_{w\to z}\left(e^{\varphi}G\right)(w)\mathcal{Y}_j^{\epsilon,\omega,(-\frac{3}{2})}(x,u,z)\,. \tag{2.44}$$

3 Spectrally-flowed primary correlators

In this section, we compute the primary correlators needed for the supersymmetric three-point functions. We first review the recent results obtained for bosonic SL(2,ℝ) correlators involving vertex operators with arbitrary spectral flow charges [20–23]. We then extend these results to the SU(2) sector of the theory and also provide the relevant correlators involving fermions and spin fields. Spectrally-flowed correlators with at least one unflowed insertion were discussed in [19]. Here we restrict to cases with $\omega_i \geq 1$ for $i = 1, 2, 3$, and take $\omega_3 \geq \omega_{1,2}$ for concreteness.

3.1 $\mathbf{SL(2,\mathbb{R})}$ sector and the y-basis

An integral formula for spectrally-flowed three-point functions in the bosonic SL(2,ℝ)-WZW model at level $k > 3$ was conjectured in [21] and proved recently in [23].[3]

The intuition comes partly from the $k = 3$ case, which corresponds to the background with a single NS5-brane in the supersymmetric context.[4] There, the fundamental strings become effectively tensionless, and the holographic CFT is identified with the symmetric orbifold model $\mathrm{Sym}^{n_1}(T^4)$ [1–4, 20]. In particular, worldsheet spectral flow is mapped to the spacetime twist via AdS/CFT. Twisted correlators in the symmetric orbifold theory are non-zero only when one can construct a holomorphic covering map $\Gamma[\omega_1, \omega_2, \omega_3](z) \equiv \Gamma(z)$ satisfying

$$\Gamma(z) \sim x_i + a_i (z - z_i)^{\omega_i} + \cdots \qquad \text{when} \quad z \sim z_i\,, \tag{3.1}$$

where the ellipsis indicates higher order terms in $(z - z_i)$. Holographically, the covering space is identified with the string worldsheet itself. For three-point functions, such covering map exists and can be constructed explicitly when [12]

$$\omega_1 + \omega_2 + \omega_3 \in 2\mathbb{Z} + 1\,, \qquad \omega_1 + \omega_2 > \omega_3 - 1\,. \tag{3.2}$$

At large n_1 the main contribution comes from covering surfaces of genus zero. For these three-point functions, such a map is unique. After fixing $(z_1, z_2, z_3) = (x_1, x_2, x_3) = (0, 1, \infty)$ as usual, the coefficients a_i appearing in eq. (3.1) take the following combinatorial form

$$a_i = \begin{pmatrix} \frac{\omega_i + \omega_{i+1} + \omega_{i+2} - 1}{2} \\ \frac{-\omega_i + \omega_{i+1} + \omega_{i+2} - 1}{2} \end{pmatrix} \begin{pmatrix} \frac{-\omega_i + \omega_{i+1} - \omega_{i+2} - 1}{2} \\ \frac{\omega_i + \omega_{i+1} - \omega_{i+2} - 1}{2} \end{pmatrix}^{-1}, \tag{3.3}$$

where the subscripts are understood to be mod 3. The use of the map $\Gamma(z)$ combined with the *local* Ward identities derived from the OPEs (2.16) allowed the authors of [20] to derive the corresponding three-point functions in the $n_5 = 1$ worldsheet theory.

As mentioned in the introduction, for $k > 3$ ($n_5 > 1$) the holographic CFT does *not* have a symmetric orbifold structure [24, 51]. Nevertheless, the same local Ward identities imply a set of recursion relations for x-basis correlators [20]. These become differential equations in the y variable introduced in [21]. For a given spin j and spectral flow charge

[3]A similar conjecture for four-point functions was then put forward in [48] and further analyzed in [49].

[4]Here the RNS formalism breaks down since the bosonic SU(2) level $k' = n_5 - 2$ becomes negative, and one needs to use the so-called hybrid formalism [50].

ω, the so-called y-basis operators $\tilde{V}_j^\omega(x,y,z)$ are constructed by summing all $V_{jh}^\omega(x,z)$ over the allowed values of h. The inverse relationship can be written as a Mellin-type transform, namely

$$V_{j,h}^{\omega}(x,z) = \int d^2y \, y^{j-m-1}\bar{y}^{j-\bar{m}-1}\tilde{V}_j^\omega(x,y,z)\,, \tag{3.4}$$

which mirrors the relation between the m-basis and the x-basis for unflowed operators [16]. For operators $\tilde{V}_j^\omega(x,y,z)$, eq. (3.4) shows that the modes $j^\pm_{\pm\omega}$ and j_0^0 act as differential operators in the y variable, namely

$$j^+_\omega \sim D^+_{y,j}\,, \qquad j^0_0 \sim D^0_{y,j} + \frac{k}{2}\omega\,, \qquad j^-_{-\omega} \sim D^-_{y,j}\,, \tag{3.5}$$

where we have used the notation of eq. (2.14).

When (3.2) is satisfied, the existence of the covering map $\Gamma(z)$ in eq. (3.1) can be used to show that the differential equations in the variables y_i implied by the local Ward identities are solved by the following expression [21]:

$$\begin{aligned}\langle \tilde{V}_{j_1}^{\omega_1}(y_1)\tilde{V}_{j_2}^{\omega_2}(y_2)\tilde{V}_{j_3}^{\omega_3}(y_3)\rangle &= N_{\text{odd}}\,(y_1-a_1)^{-2j_1}(y_2-a_2)^{-2j_2}(y_3-a_3)^{-2j_3} \\ &\quad\times\left(\omega_1\frac{y_1+a_1}{y_1-a_1}+\omega_2\frac{y_2+a_2}{y_2-a_2}+\omega_3\frac{y_3+a_3}{y_3-a_3}-1\right)^{\frac{k}{2}-j_1-j_2-j_3}.\end{aligned} \tag{3.6}$$

Here and from now on, we omit the explicit dependence in x_i and z_i unless necessary. The constant N_{odd}, which fixes the normalization, was derived in [22], and will be given below. Eq. (3.6) is written in terms of the so-called y-basis operators, defined as follows. The more familiar x-basis three-point functions can then be extracted by means of (3.4). For discrete states this can be computed in terms of holomorphic and anti-holomorphic contour integrals, while for complex j one needs to integrate over the full complex plane.

Moreover, short strings are an important part of the spectrum, and, as discussed above, the series identifications in eq. (2.23) show that, for such states, ω is not defined uniquely. As a consequence, correlators can be non-trivial even when no associated covering map exists. This is consistent with the fact that the holographic CFT is not expected to be an exact symmetric orbifold [24, 51]. Nevertheless, one can use the adjacent covering maps of [23] to compute even parity correlators, namely those with

$$\omega_1+\omega_2+\omega_3 \in 2\mathbb{Z}\,, \qquad \omega_1+\omega_2 > \omega_3\,. \tag{3.7}$$

This leads to

$$\begin{aligned}\langle \tilde{V}_{j_1}^{\omega_1}(y_1)\tilde{V}_{j_2}^{\omega_2}(y_2)\tilde{V}_{j_3}^{\omega_3}(y_3)\rangle &= N_{\text{even}}\left(1-\frac{y_2}{a_2[\Gamma_3^+]}-\frac{y_3}{a_3[\Gamma_2^+]}+\frac{y_2y_3}{a_2[\Gamma_3^-]a_3[\Gamma_2^+]}\right)^{j_1-j_2-j_3} \\ &\quad\times\left(1-\frac{y_1}{a_1[\Gamma_3^+]}-\frac{y_3}{a_3[\Gamma_1^+]}+\frac{y_1y_3}{a_1[\Gamma_3^-]a_3[\Gamma_1^+]}\right)^{j_2-j_3-j_1} \\ &\quad\times\left(1-\frac{y_1}{a_1[\Gamma_2^+]}-\frac{y_2}{a_2[\Gamma_1^+]}+\frac{y_1y_2}{a_1[\Gamma_2^+]a_2[\Gamma_1^-]}\right)^{j_3-j_1-j_2},\end{aligned} \tag{3.8}$$

where $a_i[\Gamma_j^+]$ denotes the coefficient a_i of the covering map in which ω_j is shifted upwards by one unit, while N_{even} is discussed below.

As it turns out, one can show that eqs. (3.6) and (3.8) are also valid for the so-called edge cases,

$$\omega_3 = \omega_1 + \omega_2 \quad \text{or} \quad \omega_3 = \omega_1 + \omega_2 + 1\,, \tag{3.9}$$

which correspond to correlators which have a well-behaved m-basis limit [23, 52]. Roughly speaking, this also holds for correlators with unflowed insertions, where some of the y-variables are absent. Hence, these results provide an integral expression for all non-zero correlators satisfying the fusion rules derived in [16],

$$\omega_1 + \omega_2 \geq \omega_3 - 1, \tag{3.10}$$

where, again, we have $\omega_3 \geq \omega_{1,2}$. As discussed in [21], these selection rules are encoded in the normalization factors $N_{\rm odd}$ and $N_{\rm even}$ in eqs. (3.6) and (3.8), which are defined in terms of the unflowed three-point functions $C(j_1, j_2, j_3)$ [16, 53] combined with certain combinatorial factors related to spectral flow [21, 22]. Importantly, they factorize as follows

$$N_{\rm even} = C\left(j_1, j_2, j_3\right) \tilde{N}_{\rm even}(j_i, \omega_i)\,, \qquad N_{\rm odd} = \mathcal{N}(j_1) C\left(k/2 - j_1, j_2, j_3\right) \tilde{N}_{\rm odd}(j_i, \omega_i)\,. \tag{3.11}$$

We have collected the definitions of these objects as well as some of their properties in appendix A.

3.2 SU(2) and fermionic sectors

In the SU(2) sector, spectrally-flowed correlators are merely complicated linear combinations of primary and descendant unflowed correlators. However, we can use techniques analogous to those of [20, 21, 23] in order to compute them directly. Indeed, using the same covering maps, we find that three-point functions of flowed SU(2) primaries satisfy the same recursion relations as those of the SL(2,$\mathbb{R}$) model with the replacements $k \to -k'$ and $j_i \to -l_i$. Hence, we conclude that, after fixing the overall dependence in the worldsheet and isospin variables by means of the global Ward identities, we have

$$\begin{aligned}\langle \tilde{W}_{l_1}^{\omega_1}(v_1) \tilde{W}_{l_2}^{\omega_2}(v_2) \tilde{W}_{l_3}^{\omega_3}(v_3)\rangle = N'_{\rm odd}\,&(v_1 - a_1)^{2l_1} (v_2 - a_2)^{2l_2} (v_3 - a_3)^{2l_3} \\ &\times \left(\omega_1 \frac{v_1 + a_1}{v_1 - a_1} + \omega_2 \frac{v_2 + a_2}{v_2 - a_2} + \omega_3 \frac{v_3 + a_3}{v_3 - a_3} - 1\right)^{-\frac{k'}{2} + l_1 + l_2 + l_3}\,,\end{aligned} \tag{3.12}$$

for odd parity while correlators, and

$$\begin{aligned}\langle \tilde{W}_{l_1}^{\omega_1}(v_1) \tilde{W}_{l_2}^{\omega_2}(v_2) \tilde{W}_{l_3}^{\omega_3}(v_3)\rangle = N'_{\rm even} &\left(1 - \frac{v_2}{a_2[\Gamma_3^+]} - \frac{v_3}{a_3[\Gamma_2^+]} + \frac{v_2 v_3}{a_2[\Gamma_3^-] a_3[\Gamma_2^+]}\right)^{-l_1 + l_2 + l_3} \\ \times &\left(1 - \frac{v_1}{a_1[\Gamma_3^+]} - \frac{v_3}{a_3[\Gamma_1^+]} + \frac{v_1 v_3}{a_1[\Gamma_3^-] a_3[\Gamma_1^+]}\right)^{-l_2 + l_3 + l_1} \\ \times &\left(1 - \frac{v_1}{a_1[\Gamma_2^+]} - \frac{v_2}{a_2[\Gamma_1^+]} + \frac{v_1 v_2}{a_1[\Gamma_2^+] a_2[\Gamma_1^-]}\right)^{-l_3 + l_1 + l_2}\,,\end{aligned} \tag{3.13}$$

for even parity correlators. The normalizations are given by

$$N'_{\rm even}(l_i, \omega_i) = C'\left(l_1, l_2, l_3\right) \tilde{N}'_{\rm even}\,, \qquad N'_{\rm odd}(l_i, \omega_i) = C'\left(k'/2 - l_1, l_2, l_3\right) \tilde{N}'_{\rm odd}\,, \tag{3.14}$$

where $C'(l_1,l_2,l_3)$ are the unflowed SU(2) three-point functions, defined in appendix A together with the factors $\tilde{N}'_{\rm odd}$ and $\tilde{N}'_{\rm even}$. In the expressions above we have introduced the v-basis for SU(2) operators, analogous to the y-basis used in the SL(2,$\mathbb{R}$) case, i.e.

$$\tilde{W}_l(u,v,z) = \sum_{n,\bar{n}=-l}^{l} v^{l+n}\bar{v}^{l+\bar{n}} W_{l,n+\frac{k'}{2}\omega,\bar{n}+\frac{k'}{2}\omega}(u,z)\,. \tag{3.15}$$

Correlators involving flowed fermions and spin fields are also necessary for computing three-point functions in the supersymmetric model. We note that $\psi^\omega(x,z)$ ($\chi^\omega(u,z)$) is the flowed version of a spin $\hat{\jmath}=-1$ ($\hat{l}=1$) unflowed fermion, and belongs to a spin $\hat{\jmath}_\omega=-1-\omega$ ($\hat{l}_\omega=1+\omega$) representation of the zero-mode algebra of $\mathrm{SL}(2,\mathbb{R})_{-2}$ (respectively $\mathrm{SU}(2)_2$). Similarly, the fermionic field $\hat{\psi}^\omega(x,z)$ ($\hat{\chi}^\omega(u,z)$) can be seen as the flowed version of the fermionic identity, with flowed spin $\hat{\jmath}_\omega=-\omega$ ($\hat{l}_\omega=\omega$). Finally, the spin fields $s^\omega_\pm(x,u,z)$ belong to $\mathrm{SL}(2,\mathbb{R})_{-2}\times\mathrm{SU}(2)_2$ representations with flowed spins $(\hat{\jmath}_\omega,\hat{l}_\omega)=(-1/2-\omega,1/2+\omega)$, obtained by the application of spectral flow on a spin $(\hat{\jmath},\hat{l})=(-1/2,1/2)$ state. Consequently, all relevant spectrally-flowed fermionic correlators can be obtained from the bosonic formulas given above by inserting the corresponding spins and levels and taking care of the parity of the different fermion numbers.

As a check, we can recover some of the expressions obtained in [17] using free field techniques. We first compute the correlator with three $\hat{\psi}^\omega(x,z)$ insertions, which is non-vanishing only for even $\omega_1+\omega_2+\omega_3$. The quickest way to compute this is to take the *even* parity case, namely eq. (3.8), and set $j_1=j_2=j_3=0$ and $k=-2$. The desired x-basis result is then obtained by computing the relevant residue, which can be done simply by setting $y_i=0$ for $i=1,2,3$, see eq. (3.4). We obtain

$$\langle \hat{\psi}^{\omega_1}\hat{\psi}^{\omega_2}\hat{\psi}^{\omega_3}\rangle = P^2_{(\omega_1+1,\omega_2+1,\omega_3+1)}, \tag{3.16}$$

where we have fixed the worldsheet and boundary insertion points as usual, and $P(\omega_1,\omega_2,\omega_3)$ is defined in appendix A. We can also work out the slightly more complicated case

$$\begin{aligned}\langle \hat{\psi}^{\omega_1}\psi^{0,\omega_2}\psi^{0,\omega_3}\rangle &= \tfrac{1}{4}\partial_{y_2}\partial_{y_3}\,\langle \hat{\tilde{\psi}}^{\omega_1}(y_1)\tilde{\psi}^{\omega_2}(y_2)\tilde{\psi}^{\omega_3}(y_3)\rangle\,|_{y_i=0}\\ &= P_{(\omega_1+1,\omega_2+1,\omega_3+1)}P_{(\omega_1+1,\omega_2-1,\omega_3-1)} - P_{(\omega_1+1,\omega_2-1,\omega_3+1)}P_{(\omega_1+1,\omega_2+1,\omega_3-1)},\end{aligned} \tag{3.17}$$

where we identified

$$\psi^{0,\omega}(x,z) = -\frac{1}{2}(j^+_\omega\psi^\omega)(x,z) = -\frac{1}{2}\partial_y\tilde{\psi}^\omega(x,y,z)|_{y=0}. \tag{3.18}$$

Our results (3.16) and (3.17) precisely reproduce the functions $f^{(0)}$ and $f^{(2)}$ of [17].[5]

4 Chiral primary three-point functions and holographic matching

This section contains the main results of this paper. We compute all relevant short-string three-point functions with arbitrary spectral flow insertions and match our results with the

[5] We note that our y-basis methods give slightly different results for the functions $f^{(1)}$ and $f^{(3)}$ of [17]. The formulas we derive lead to the correct holographic matching, as will be shown in the next section.

predictions from the chiral ring of the holographic CFT at the symmetric orbifold point. For this, we first compute a family of descendant correlators, which appear as a result of the picture-changing procedure. These correlators, presented as the main technical obstacle in this context [17, 19], are obtained by the SL(2,$\mathbb{R}$) series identifications, in combination with the y-basis results reviewed above.

4.1 Spectrally-flowed correlators involving current insertions

Let us first focus on the NS-NS-NS short-string three-point functions. The simplest way to compute these is by inserting two vertex operators with ghost picture (-1) and one with ghost picture (0). For states polarized in the AdS$_3$ directions, the corresponding operators were given in eq. (2.36). From their expression, one finds that it is not enough to know the SL(2,$\mathbb{R}$) primary correlators. Indeed, we must also compute correlators of the form

$$\langle V^{\omega_1}_{j_1}(x_1,z_1)V^{\omega_2}_{j_2}(x_2,z_2)\left(j^{\omega_3}V^{\omega_3}_{j_3}\right)(x_3,z_3)\rangle\,, \tag{4.1}$$

where we use the shorthand defined in eq. (2.21), such that $V^{\omega}_{j}(x,z)\equiv V^{\omega}_{j,h}(x,z)$ with $h=j+\frac{k}{2}\omega$, which belongs to the $\mathcal{D}^{+,\omega}_{j}$ representation, while $\left(j^{\omega}V^{\omega}_{j}\right)(x,z)$ stands for

$$\left(j^{\omega}V^{\omega}_{j}\right)(x,z)=e^{xj_0^+}\oint_z dw\frac{j^-(w)V^{\omega}_{j,j}(z)}{(w-z)^{1+\omega}}e^{-xj_0^+}\,. \tag{4.2}$$

Here $V^{\omega}_{j,j}(z)$ is a flowed primary m-basis operator, derived from an unflowed lowest-weight state, as opposed to an x-basis operator.

Three-point functions such as those in eq. (4.1) are usually thought of as descendant correlators in the sense that they involve the action of the mode $j^-_{-1-\omega}$ on a vertex operator $V^{\omega}_{j}(x,z)$, which is a negative mode of the current j^- in the corresponding spectrally-flowed frame. However, by using the series identification (2.23) we can interpret a given (short string) built upon an unflowed lowest-weight state as an operator with an extra unit of spectral flow charge built upon an unflowed highest-weight state, including the usual spin replacement $j\to k/2-j$. Hence, we can write

$$\left(j^{\omega}V^{\omega}_{j}\right)(x,z)=\mathcal{N}(j)e^{xj_0^+}\oint_z dw\frac{1}{(w-z)^{1+\omega}}j^-(w)V^{\omega+1}_{\frac{k}{2}-j,-(\frac{k}{2}-j)}(z)e^{-xj_0^+}\,, \tag{4.3}$$

where $V^{\omega+1}_{\frac{k}{2}-j,-(\frac{k}{2}-j)}(z)$ is the highest weight m-basis operator of the $\mathcal{D}^{-,\omega}_{\frac{k}{2}-j}$ representation. Crucially, the current mode $j^-_{-1-\omega}$ can be seen as a zero mode in the spectrally-flowed frame associated to an operator with charge $\omega+1$. Inserting the OPE (2.16b) in the above expression leads to

$$\left(j^{\omega}V^{\omega}_{j}\right)(x,z)=-\mathcal{N}(j)e^{xj_0^+}V^{\omega+1}_{\tilde{\jmath},-\tilde{\jmath}-1}(z)e^{-xj_0^+}=-\mathcal{N}(j)V^{\omega+1}_{\tilde{\jmath},\tilde{h}-1}(x,z), \tag{4.4}$$

where

$$\tilde{\jmath}=k/2-j\,,\quad \tilde{h}=-\tilde{\jmath}+k(\omega+1)/2\,. \tag{4.5}$$

In eq. (4.4), $V^{\omega+1}_{\tilde{\jmath},-\tilde{\jmath}-1}(z)$ is an m-basis operator with spin projection $m=-\tilde{\jmath}-1$ belonging to the $\mathcal{D}^{-,\omega+1}_{\tilde{\jmath}}$, while $V^{\omega+1}_{\tilde{\jmath},\tilde{h}-1}(x,z)$ is its x-basis counterpart. We can then express the relevant

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correlation functions with current insertions in terms of flowed primary correlators, namely

$$\begin{aligned}\langle V_{j_1}^{\omega_1}(x_1,z_1)V_{j_2}^{\omega_2}(x_2,z_2)\left(j^{\omega_3}V_{j_3}^{\omega_3}\right)(x_3,z_3)\rangle &= \\ -\,\mathcal{N}(j_3)\,\langle V_{j_1}^{\omega_1}(x_1,z_1)V_{j_2}^{\omega_2}(x_2,z_2)&V_{\tilde{j}_3,\tilde{h}_3-1}^{\omega_3+1}(x_3,z_3)\rangle\,. \end{aligned} \tag{4.6}$$

Note that we have not imposed any condition on the spectral flow charges involved in the correlator. We are only taking advantage of the constraints imposed by the Virasoro condition for spectrally-flowed 1/2-BPS vertex operators, i.e. $m_i = j_i$. Even though the strategy at play could have been used before, its applicability would have been limited, as flowed primary three-point functions were only computed in full generality quite recently [21, 23].

In order to calculate the r.h.s. of (4.6) we go back to the y-basis operators used in the previous section. Eq. (3.5) implies that, after fixing the insertions as $(z_1,z_2,z_3) = (x_1,x_2,x_3) = (0,1,\infty)$, the relevant correlator can be expressed in terms of its y-basis cousin as follows:

$$\langle V_{j_1}^{\omega_1}V_{j_2}^{\omega_2}V_{\tilde{j}_3,\tilde{h}_3-1}^{\omega_3+1}\rangle = \lim_{y_3\to\infty} y_3^{k-2j_3}\,\langle \tilde{V}_{j_1}^{\omega_1}(y_1=0)\tilde{V}_{j_2}^{\omega_2}(y_2=0)D^{-}_{y_3,\tilde{j}_3}\tilde{V}_{\tilde{j}_3}^{\omega_3+1}(y_3)\rangle\,. \tag{4.7}$$

By using eqs. (3.6) and (3.8) we find that

$$\langle V_{j_1}^{\omega_1}V_{j_2}^{\omega_2}V_{\tilde{j}_3,\tilde{h}_3-1}^{\omega_3+1}\rangle = \alpha_{\boldsymbol{\omega}}\lim_{y_3\to\infty} y_3^{k-2j_3}\,\langle \tilde{V}_{j_1}^{\omega_1}(y_1=0)\tilde{V}_{j_2}^{\omega_2}(y_2=0)\tilde{V}_{\tilde{j}_3}^{\omega_3+1}(y_3)\rangle\,, \tag{4.8}$$

where the coefficient is given by

$$\alpha_{\boldsymbol{\omega}} \equiv \begin{cases} \dfrac{2a_3[\Gamma_{13}^{++}]\left[(\omega_1-\omega_2)(j_1-j_2)+(\omega_3+1)(\frac{k}{2}-j_3)\right]}{\omega_1+\omega_3-\omega_2+1} & \text{if } \displaystyle\sum_{i=1}^{3}\omega_i \in 2\mathbb{Z}+1, \\ \dfrac{2a_3[\Gamma_3^{+}]\left[(1+\omega_1+\omega_2)j_3-(1+\omega_3)(j_1+j_2)-\frac{k}{2}(\omega_1+\omega_2-\omega_3)\right]}{\omega_3-\omega_2-\omega_1} & \text{if } \displaystyle\sum_{i=1}^{3}\omega_i \in 2\mathbb{Z}. \end{cases} \tag{4.9}$$

Here $a_3[\Gamma_{13}^{++}]$ denotes the coefficient a_3 of the covering map $\Gamma[\omega_1+1,\omega_2,\omega_3+1](z)$. Finally, by using the series identification once more we can express the result in terms of correlators of operators in the $\mathcal{D}_j^{+,\omega}$ representations. We conclude that

$$\langle V_{j_1}^{\omega_1}V_{j_2}^{\omega_2}\left(j^{\omega_3}V_{j_3}^{\omega_3}\right)\rangle = \alpha_{\boldsymbol{\omega}}\,\langle V_{j_1}^{\omega_1}V_{j_2}^{\omega_2}V_{j_3}^{\omega_3}\rangle\,. \tag{4.10}$$

Recall that this result holds only for the discrete states considered in this paper. Analogous formulas hold for the SL(2,$\mathbb{R}$) fermionic sector, where we have to set $k=-2$, and also for the SU(2) bosons and fermions, for which the levels and spins appear with the opposite sign.

4.2 NS-NS-NS three-point functions

We now have all the necessary ingredients for computing the NS-NS-NS string correlators with arbitrary spectral flow charges. This includes not only the extremal correlators (in

the holographic CFT language), some of which were briefly discussed in [17], but also the non-extremal ones.

For concreteness, we start with all three vertex operators polarized in the AdS_3 directions, namely $\langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{V}^{\omega_3,(0)}_{j_3} \rangle$. We take the picture (0) operator to be the one with the largest spectral flow charge, $\omega_3 \geq \omega_{1,2}$, and further set $(x_1,x_2,x_3) = (u_1,u_2,u_3) = (z_1,z_2,z_3) = (0,1,\infty)$. Additionally, we will also assume that

$$\omega_3 < \omega_1 + \omega_2 + 1. \tag{4.11}$$

The edge cases $\omega_3 = \omega_1 + \omega_2 + 1$ will be discussed below. When (4.11) holds, all y-basis (and v-basis) three-point functions involved in the supersymmetric computation turn out to be regular in the limits $y_i \to 0$ (and $v_i \to 0$). We can therefore pick up the relevant poles in the corresponding transforms (3.4) by setting $y_i = v_i = 0$ in all relevant y-basis and v-basis correlators.

Let us go back to the correlator $\langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{V}^{\omega_3,(0)}_{j_3} \rangle$. The vertex operators involved in this three-point function were given in eqs. (2.20) and (2.36). Fermion number counting shows that for even parity correlators, i.e. those with $\sum_i \omega_i \in 2\mathbb{Z}$, only the first term in (2.36) gives a non-zero contribution, while only the second one is relevant when $\sum_i \omega_i \in 2\mathbb{Z}+1$. In other words, in the former case, we have

$$\langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{V}^{\omega_3,(0)}_{j_3} \rangle = \langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{A}^{\omega_3,1}_{j_3} \rangle \,, \qquad \sum_i \omega_i \in 2\mathbb{Z}. \tag{4.12}$$

This factorizes into ghost, bosonic and fermionic correlators. The latter include a current insertion, giving

$$\langle \psi^{\omega_1} \psi^{\omega_2} \left(\hat{\jmath}^{\omega_3} \hat{\psi}^{\omega_3} \right) \rangle = -\frac{2a_3[\Gamma_3^+](2+\omega_1+\omega_2+\omega_3)}{\omega_3-\omega_2-\omega_1} \langle \psi^{\omega_1} \psi^{\omega_2} \hat{\psi}^{\omega_3} \rangle, \tag{4.13}$$

where we have used (4.10) with $k \to \hat{k} = -2$, $j_{1,2} \to \hat{\jmath}_{1,2} = -1$ and $j_3 \to \hat{\jmath}_3 = 0$. Hence, we get

$$\begin{aligned} \langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{V}^{\omega_3,(0)}_{j_3} \rangle &= (h_1+h_2+h_3-2) \\ &\times \frac{2(1+\omega_3)a_3[\Gamma_3^+]}{\omega_1+\omega_2-\omega_3} \langle V^{\omega_1}_{j_1} V^{\omega_2}_{j_2} V^{\omega_3}_{j_3} \rangle \langle \psi^{\omega_1} \psi^{\omega_2} \hat{\psi}^{\omega_3} \rangle \langle W^{\omega_1}_{j_1-1} W^{\omega_2}_{j_2-1} W^{\omega_3}_{j_3-1} \rangle \langle \hat{\chi}^{\omega_1} \hat{\chi}^{\omega_2} \hat{\chi}^{\omega_3} \rangle \,, \end{aligned} \tag{4.14}$$

where $h_i = j_i + n_5\omega_i/2 = H_i + 1$. The spectrally-flowed primary three-point functions in the different sectors can be evaluated explicitly by using the results of section 3. Although the individual factors look somewhat complicated, the final result becomes extremely simple:

$$\langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{V}^{\omega_3,(0)}_{j_3} \rangle_{\text{even}} = (h_1+h_2+h_3-2) n_5 \mathcal{C}(j_i). \tag{4.15}$$

where $\mathcal{C}(j_i)$ is the product of the $\mathrm{SL}(2,\mathbb{R})_k$ and $\mathrm{SU}(2)_{k'}$ *unflowed* three-point functions.

Let us pause and briefly discuss this expression. It was shown in [9] that for short strings the relation between the SL(2,$\mathbb{R}$) and SU(2) spins stemming from the Virasoro condition leads to important cancellations for the product of unflowed three-point functions contained in $\mathcal{C}(j_i)$, whose final expressions is given in eq. (A.16) above. Here we have found

that such cancellations extend non-trivially to the spectrally-flowed sectors of the theory. Indeed, as a result of the structural similarities between spectrally-flowed correlators for SL(2,$\mathbb{R}$) and SU(2), all combinatorial factors coming from the different contributions in the second line of (4.14) exactly cancel each other. The only dependence of the final expression (4.15) on the spectral flow charges ω_i is contained in the overall prefactor. This is consistent with the partial results of [19], which were obtained by very different methods.

We now show that the result in eq. (4.15) holds also for odd parity correlators with $\sum_i \omega_i \in 2\mathbb{Z}+1$. In these cases we have

$$\langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{V}^{\omega_3,(0)}_{j_3} \rangle = \langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{A}^{\omega_3,2}_{j_3} \rangle, \tag{4.16}$$

By using that

$$\begin{aligned} \langle W^{\omega_1}_{l_1} W^{\omega_2}_{l_2} \left(k^+_{\omega_3} W^{\omega_3}_{l_3} \right) \rangle &= \langle \tilde{W}^{\omega_1}_{l_1}(v_1=0) \tilde{W}^{\omega_2}_{l_2}(v_2=0) \partial_{v_3} \tilde{W}^{\omega_3}_{l_3}(v_3)|_{v_3=0} \rangle \\ &= -\frac{2\left[\omega_3(l_1+l_2-\frac{k'}{2}) - (1+\omega_1+\omega_2)l_3\right]}{(1+\omega_1+\omega_2+\omega_3)a_3} \langle W^{\omega_1}_{l_1} W^{\omega_2}_{l_2} W^{\omega_3}_{l_3} \rangle \,, \end{aligned} \tag{4.17}$$

where a_3 is from the map $\Gamma[\omega_1, \omega_2, \omega_3]$, and

$$\langle \hat{\chi}^{\omega_1} \hat{\chi}^{\omega_2} \left(\hat{k}^+_{\omega_3} \chi^{\omega_3} \right) \rangle = \frac{2}{a_3} \langle \hat{\chi}^{\omega_1} \hat{\chi}^{\omega_2} \chi^{\omega_3} \rangle \,, \tag{4.18}$$

we find

$$\begin{aligned} \langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{A}^{\omega_3,2}_{j_3} \rangle = &-(h_1+h_2+h_3-2) \\ &\times 2 \frac{\langle V^{\omega_1}_{j_1} V^{\omega_2}_{j_2} V^{\omega_3}_{j_3} \rangle \langle W^{\omega_1}_{j_1-1} W^{\omega_2}_{j_2-1} W^{\omega_3}_{j_3-1} \rangle \langle \psi^{\omega_1} \psi^{\omega_2} \psi^{\omega_3} \rangle \langle \hat{\chi}^{\omega_1} \hat{\chi}^{\omega_2} \chi^{\omega_3} \rangle}{(1+\omega_1+\omega_2+\omega_3)a_3}. \end{aligned} \tag{4.19}$$

As in the even parity case, and after using the identity eq. (A.17), inserting the explicit expressions for each factor leads to

$$\langle \mathcal{V}^{\omega_1}_{j_1} \mathcal{V}^{\omega_2}_{j_2} \mathcal{V}^{\omega_3,(0)}_{j_3} \rangle_{\text{odd}} = (h_1+h_2+h_3-2) n_5 \mathcal{C}(j_i). \tag{4.20}$$

A similar computation can be carried out for NS-NS-NS correlators involving one, two or three states polarized in the SU(2) directions. The final results for these cases will be given in section 4.4 below.

4.2.1 Edge Cases

The y-basis three-point functions (3.6) and (3.8) are singular in the limits $y_i \to a_i$ [22–24, 49]. For the edge case, i.e. when $\omega_3 = \omega_1 + \omega_2 + 1$, we see from (3.3) that $a_3 = 0$. As a consequence, we must be careful when picking up the relevant residues at $y_3 = 0$, and use the Mellin-like transform (3.4), which for discrete states can be performed as a contour integral, namely

$$V^{\omega_3}_{j_3}(x_3, z_3) = \oint_0 \frac{dy_3}{y_3} \tilde{V}^{\omega_3}_{j_3}(x_3, y_3, z_3), \tag{4.21}$$

where, as is done throughout the paper, we have ignored the anti-holomorphic variables. Analogous formulas hold for the SU(2) and fermionic sectors.

The three-point functions we are interested in are given by

$$
\begin{aligned}
&\langle \mathcal{V}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\mathcal{V}^{\omega_3,(0)}_{j_3}\rangle = \langle \mathcal{V}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\mathcal{A}^{\omega_3,2}_{j_3}\rangle = \\
&= \langle \mathcal{V}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\left\{\frac{\psi^{\omega_3}(x)}{n_5}V^{\omega_3}_{j_3}(x)\left[k^+_{\omega_3}-\left(j_3-1+\frac{n_5\omega_3}{2}\right)W^{\omega_3}_{j_3-1}\hat{k}^+_{\omega_3}\right]\right\}W^{\omega_3}_{j_3-1}\chi^{\omega_3}\rangle\,. \qquad (4.22)
\end{aligned}
$$

Looking at the first term, the bosonic $\mathrm{SL}(2,\mathbb{R})_k$ and $\mathrm{SU}(2)_{k'}$ correlators involved are expressed as

$$
\langle V^{\omega_1}_{j_1}V^{\omega_2}_{j_2}V^{\omega_3}_{j_3}\rangle = \mathcal{N}(j_1)C(k/2-j_1,j_2,j_3)\oint_0 dy_3 y_3^{-1+\alpha}\left(1+y_3\frac{(\omega_3+1)!}{\omega_1!\omega_2!}\right)^{\beta} \qquad (4.23)
$$

and

$$
\begin{aligned}
&\langle W^{\omega_1}_{j_1-1}W^{\omega_2}_{j_2-1}\left(k^+_{\omega}W^{\omega_3}_{j_3-1}\right)\rangle = \qquad (4.24)\\
&= C'(k'/2-j_1+1,j_2-1,j_3-1)\oint_0 dv_3 v_3^{-3-\alpha}\left(1+v_3\frac{(\omega_3+1)!}{\omega_1!\omega_2!}\right)^{-(1+\beta)}
\end{aligned}
$$

where we have identified $k^+_{\omega_3}\to\partial_{v_3}$ and defined $\alpha=j_1+j_2-j_3-\frac{k}{2}$ and $\beta=\frac{k}{2}-j_1-j_2-j_3$. Due to the SU(2) selection rules contained in the structure constants (and reviewed in appendix A) α and β must be integer numbers, hence the integrals involved in these bosonic correlators will be finite. On the other hand, the fermionic SU(2) correlator gives

$$
\langle\hat{\chi}^{\omega_1}\hat{\chi}^{\omega_2}\chi^{\omega_3}\rangle = n_5\oint_0 dv_3 v_3 = 0. \qquad (4.25)
$$

The first term in the second line of (4.22) thus vanishes. The same happens with the second term after identifying $\hat{k}^+_{\omega_3}$ as a v_3-derivative for the SU(2) fermionic three-point function. Therefore, we conclude that

$$
\langle\mathcal{V}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\mathcal{V}^{\omega_1+\omega_2+1,(0)}_{j_3}\rangle = 0. \qquad (4.26)
$$

A similar procedure shows that

$$
\langle\mathcal{W}^{\omega_1}_{j_1}\mathcal{W}^{\omega_2}_{j_2}\mathcal{V}^{\omega_1+\omega_2+1,(0)}_{j_3}\rangle = \langle\mathcal{W}^{\omega_1}_{j_1}\mathcal{W}^{\omega_2}_{j_2}\mathcal{W}^{\omega_1+\omega_2+1,(0)}_{j_3}\rangle = 0. \qquad (4.27)
$$

Finally, we can show that the last possible edge correlator vanishes as well, i.e.

$$
\langle\mathcal{W}^{\omega_1}_{j_1}\mathcal{V}^{\omega_2}_{j_2}\mathcal{V}^{\omega_1+\omega_2+1,(0)}_{j_3}\rangle = 0. \qquad (4.28)
$$

This involves a factor of the form

$$
\langle V^{\omega_1}_{j_1}V^{\omega_2}_{j_2}\left(j^{\omega_3}V^{\omega_3}_{j_3}\right)\rangle. \qquad (4.29)
$$

For this specific choice of spectral flow charges, this can be computed by means of techniques similar to those used in section 3.3 of [23]. We have

$$
\langle\oint_{\mathcal{C}} dz j(x_3,z)V^{\omega_1}_{j_1}(x_1,z_1)V^{\omega_2}_{j_2}(x_2,z_2)V^{\omega_3}_{j_3}(x_3,z_3)\rangle\frac{(z-z_1)^{\omega_1+1}(z-z_2)^{\omega_2+1}}{(z-z_3)^{\omega_3+1}} = 0, \qquad (4.30)
$$

for any contour $\mathcal{C}$ that encircles all three insertion points z_i, as the integrand is regular at infinity. Moreover, the OPEs (2.16) show that the integrand is regular at $z=z_1$ and $z=z_2$, and only the residue z_3 contributes. Hence,

$$
\langle V^{\omega_1}_{j_1}V^{\omega_2}_{j_2}\left(j^{\omega_3}V^{\omega_3}_{j_3}\right)\rangle|_{\omega_3=\omega_1+\omega_2+1} = 0. \qquad (4.31)
$$

Since the corresponding fermionic correlator vanishes analogously, eq. (4.28) holds.

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4.3 R-R-NS correlators

We now move to the R-R-NS three-point functions. While the extremal cases were obtained in [17], here we complete the analysis by computing all the non-extremal ones. These three-point functions

$$\langle \mathcal{Y}_{j_1}^{\epsilon_1,\omega_1} \mathcal{Y}_{j_2}^{\epsilon_2,\omega_2} \mathcal{V}_{j_3}^{\omega_3} \rangle \qquad \text{and} \qquad \langle \mathcal{Y}_{j_1}^{\epsilon_1,\omega_1} \mathcal{Y}_{j_2}^{\epsilon_2,\omega_2} \mathcal{W}_{j_3}^{\omega_3} \rangle \tag{4.32}$$

are technically simpler since no picture changing is necessary. Hence, they can be obtained directly from the spectrally-flowed primary correlators of section 3.

We start with the non-edge cases as before, i.e. with $\omega_3 < \omega_1 + \omega_2 + 1$. The only new pieces of information we need are the fermionic correlators involving spectrally-flowed spin fields, namely

$$\langle s_-^{\omega_1} s_-^{\omega_2} \psi^{\omega_3} \hat{\chi}^{\omega_3} \rangle \, \langle e^{\frac{i\epsilon_1}{2}(H_4 - H_5)} e^{\frac{i\epsilon_2}{2}(H_4 - H_5)} \rangle \, . \tag{4.33}$$

The different sectors factorize up to an overall phase coming from the cocycle factors (see footnote 2 above). However, this can be ignored since there is a single contribution, and the phase will cancel out upon including the contributions from the anti-holomorphic sector. The torus correlators involving H_4 and H_5 impose $\epsilon_1 = -\epsilon_2$. On the other hand, the $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SU}(2)$ contributions give a product of a flowed three-point function with $\mathrm{SL}(2,\mathbb{R})_{-2}$ spins $(\hat{\jmath}_1, \hat{\jmath}_2, \hat{\jmath}_3) = (-1/2, -1/2, -1)$ and $\mathrm{SU}(2)_2$ spins $(\hat{l}_1, \hat{l}_2, \hat{l}_3) = (1/2, 1/2, 0)$. As for the NS-NS-NS cases, we find that all combinatorial factors related to spectral flow cancel out, leading to

$$\langle \mathcal{Y}_{j_1}^{\epsilon_1,\omega_1} \mathcal{Y}_{j_2}^{\epsilon_2,\omega_2} \mathcal{V}_{j_3}^{\omega_3} \rangle = \langle \mathcal{Y}_{j_1}^{\epsilon_1,\omega_1} \mathcal{Y}_{j_2}^{\epsilon_2,\omega_2} \mathcal{W}_{j_3}^{\omega_3} \rangle = \sqrt{n_5}\, \mathcal{C}(j_i) \xi^{\epsilon_1,\epsilon_2} \, , \tag{4.34}$$

where ξ is defined as

$$\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{4.35}$$

Finally, we find that R-R-NS three-point functions with $\omega_3 = \omega_1 + \omega_2 + 1$ vanish as in the previous section.

4.4 Normalized correlators

As it was argued in [16, 17] and proved recently in [19], in order to obtain a precise holographic matching, the NSNS vertex operators must be normalized as

$$\mathbb{O}_j^{\omega}(x,\bar{x},u,\bar{u},z,\bar{z}) = \frac{\mathcal{O}_j^{\omega}(x,\bar{x},u,\bar{u},z,\bar{z})}{\sqrt{2c_\nu^{-1} n_5 Q^2 \,(2h-1)\, B(j) v_4}}, \tag{4.36}$$

where $h = j + n_5\omega/2$, while c_ν, Q and $B(j)$ are given in appendix A, and v_4 is the T^4 volume. A similar computation shows that, due to the extra factor appearing in eq. (2.42), for vertex operators in the RR-sector of the worldsheet theory we have

$$\mathbb{Y}_j^{\epsilon,\bar{\epsilon},\omega}(x,\bar{x},u,\bar{u},z,\bar{z}) = \sqrt{\frac{2h-1}{2c_\nu^{-1} n_5^2 Q^2 B(j) v_4}}\, \mathcal{Y}_j^{\epsilon,\bar{\epsilon},\omega}(x,\bar{x},u,\bar{u},z,\bar{z}). \tag{4.37}$$

In these expressions we have re-inserted the anti-holomorphic dependence. String three-point functions are then obtained directly from the results of the previous sections, and one

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must also include a factor of the string coupling $g_s = \sqrt{\frac{n_5 v_4}{n_1}}$ together with an additional factor of v_4. Consequently, we find that the full set of normalized (spacetime) chiral primary three-point functions takes the following form:

$$\langle \mathbb{V}_{j_1}^{\omega_1} \mathbb{V}_{j_3}^{\omega_2} \mathbb{V}_{j_3}^{\omega_3,(0)} \rangle = \frac{1}{\sqrt{N}} \left[\frac{(h_1+h_2+h_3-2)^4}{(2h_1-1)(2h_2-1)(2h_3-1)} \right]^{1/2}, \tag{4.38a}$$

$$\langle \mathbb{W}_{j_1}^{\omega_1} \mathbb{V}_{j_2}^{\omega_2} \mathbb{V}_{j_3}^{\omega_3,(0)} \rangle = \frac{1}{\sqrt{N}} \left[\frac{(1+h_1-h_2-h_3)^4}{(2h_1-1)(2h_2-1)(2h_3-1)} \right]^{1/2}, \tag{4.38b}$$

$$\langle \mathbb{W}_{j_1}^{\omega_1} \mathbb{W}_{j_2}^{\omega_2} \mathbb{V}_{j_3}^{\omega_3,(0)} \rangle = \frac{1}{\sqrt{N}} \left[\frac{(h_1+h_2-h_3)^4}{(2h_1-1)(2h_2-1)(2h_3-1)} \right]^{1/2}, \tag{4.38c}$$

$$\langle \mathbb{W}_{j_1}^{\omega_1} \mathbb{W}_{j_2}^{\omega_2} \mathbb{W}_{j_3}^{\omega_3,(0)} \rangle = \frac{1}{\sqrt{N}} \left[\frac{(h_1+h_2+h_3-1)^4}{(2h_1-1)(2h_2-1)(2h_3-1)} \right]^{1/2}, \tag{4.38d}$$

and

$$\langle \mathbb{Y}_{j_1}^{\epsilon_1,\bar{\epsilon}_1,\omega_1} \mathbb{Y}_{j_2}^{\epsilon_2,\bar{\epsilon}_2,\omega_2} \mathbb{V}_{j_3}^{\omega_3} \rangle = \frac{1}{\sqrt{N}} \left[\frac{(2h_1-1)\,(2h_2-1)}{(2h_3-1)} \right]^{1/2} \xi^{\epsilon_1,\epsilon_2} \xi^{\bar{\epsilon}_1,\bar{\epsilon}_2}, \tag{4.39a}$$

$$\langle \mathbb{Y}_{j_1}^{\epsilon_1,\bar{\epsilon}_1,\omega_1} \mathbb{Y}_{j_2}^{\epsilon_2,\bar{\epsilon}_2,\omega_2} \mathbb{W}_{j_3}^{\omega_3} \rangle = \frac{1}{\sqrt{N}} \left[\frac{(2h_1-1)\,(2h_2-1)}{(2h_3-1)} \right]^{1/2} \xi^{\epsilon_1,\epsilon_2} \xi^{\bar{\epsilon}_1,\bar{\epsilon}_2}, \tag{4.39b}$$

where $h_i = j_i + n_5\omega_i/2$. The overall scaling with $N = n_1 n_5$ is obtained from [9]

$$\frac{1}{\sqrt{N}} = \frac{g_s}{n_5\sqrt{v_4}} \sqrt{\frac{2\pi^5}{\nu b^4 \gamma(1+b^2)}}. \tag{4.40}$$

Here $b^2 = n_5^{-1}$, while ν can be seen as a free parameter of the WZW model, which was fixed holographically in [9][6] and is given in the appendix. Note that $g_s/\sqrt{v_4}$ is independent of the T^4 volume. Finally, the selection rules on the SU(2) spins l_i and spectral flow charges ω_i can be summarized as follows:

$$l_i \leq l_k + l_j \quad \text{and} \quad \omega_i \leq \omega_k + \omega_j \quad \forall \quad i,j,k = 1,2,3. \tag{4.41}$$

The fusion rules (4.41) and the final expressions for the structure constants of the AdS$_3$/CFT$_2$ chiral ring presented in eqs. (4.38) and (4.39) are in exact agreement with all previous worldsheet results [9, 10, 17–19]. Moreover, they precisely reproduce the holographic CFT computations at the symmetric orbifold point [11–13]. This concludes our analysis of short-string correlators with arbitrary spectral flow charges.

5 Discussion

In this work, we studied short-string three-point functions of type IIB superstrings in AdS$_3\times$S$^3\times$T^4. More precisely, we focused on the worldsheet description of the spacetime

[6] See however footnote 4 in [19] and the discussion in [24].

chiral ring in this instance of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, and when the number of NS5-brane sources is strictly larger than one. It was discussed previously in the literature [17–19] that important complications arise when computing the three-point functions involving vertex operator insertions with non-zero spectral flow charges in the RNS formalism. Here we have shown how to overcome these difficulties.

Bosonic primary correlators with arbitrary spectral flow charges were obtained recently in [20–23]. Nevertheless, supersymmetric correlators remained elusive, mainly due to the appearance of current insertions arising from the picture-changing procedure. These descendant correlators cannot be obtained by the usual techniques involving contour integrals due to the rather non-trivial OPEs, presented in eq. (2.16).

Recently, in [19] all short-string x-basis correlators involving at least one unflowed vertex operator were computed by using an m-basis approach based on [52]. However, such a strategy turned out to be somewhat restrictive.

In this paper, we computed all remaining short-string supersymmetric correlators. This was done by generalizing the y-basis techniques developed in [21, 23], in order to apply them not only to the relevant descendant correlators but also to the SU(2) and fermionic sectors. As was the case for [23], the SL(2,$\mathbb{R}$) series identifications (2.23) are instrumental in our calculations.

The individual bosonic and fermionic correlators give complicated expressions as a function of the spectral flow charges, which moreover depend on the parity of the total spectral flow. We showed that, in all cases, the supersymmetric computation conspires so that almost all of these combinatorial factors disappear from the final result. This extends the analysis of [9, 10] to the spectrally-flowed sectors of the theory. The final expressions show no trace of the difference between even and odd total spectral flow appearing in the intermediate steps, as was expected from a boundary CFT perspective.

The main results of this paper are the fusion rules and structure constants presented in eqs. (4.38), (4.39) and (4.41). They precisely agree with the predictions from the holographic CFT at the symmetric orbifold point [11–13], and complete the exact (large N) holographic matching of the full $\mathrm{AdS}_3/\mathrm{CFT}_2$ chiral ring.

The techniques developed in this paper will be useful for studying more general supersymmetric correlators for strings in AdS_3, i.e. those involving long string states. Unfortunately, for such states, the Virasoro condition is less restrictive, and in order to obtain the x-basis three-point functions it will be necessary to carry out the integrals appearing in the Mellin transform (3.4) over the full complex plane [21]. Long-string correlators are not protected by non-renormalization theorems. Consequently, and as opposed to the short string case, a holographic comparison only makes sense with the holographic CFT at the same point in the moduli space. An exact matching with the results derived from the holographic CFT put forward in [24] would provide conclusive evidence for this proposal. We leave this computation for future work.

Our results are also important in the context of black holes in AdS_3 and the description of some of their microstates, as the correlators we have computed here constitute the main building blocks of those in the gauged WZW models constructed in [25–32]. These

models are also related to the discussions of [33–40] concerning little string theory, $T\bar{T}$ deformations, and holography beyond AdS.

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A Some additional definitions and properties

Here we provide some definitions related to the SL(2,$\mathbb{R}$) and SU(2) WZW models. We first focus on the unflowed sectors. For the SU(2)$_{k'}$ model, the zero-mode representations are the highest-weight states $W_l(u,z)$, with integer or half-integer spins $0 \leq l \leq k'/2$ [42]. For strings in AdS$_3$, one considers states $V_j(x,z)$ in the continuous and discrete principal series of SL(2,$\mathbb{R}$)$_k$, more precisely, those with spins $j \in 1/2 + i\mathbb{R}$ and $1/2 < j < (k-1)/2$, respectively [15]. The corresponding two-point functions (omitting the anti-holomorphic dependence as in the bulk of the paper) are given by

$$\langle W_{l_1}(u_1,z_1) W_{l_2}(u_2,z_2)\rangle = \delta_{l_1,l_2} \frac{u_{12}^{2l_1}}{z_{12}^{2\Delta'_1}} \,, \tag{A.1}$$

with

$$\Delta'_l = \frac{l(l+1)}{k'+2} \,, \tag{A.2}$$

and

$$\langle V_{j_1}(x_1,z_1) V_{j_2}(x_2,z_2)\rangle = \frac{1}{z_{12}^{2\Delta_1}} \left[\delta^2(x_1-x_2)\delta(j_1+j_2-1) + B(j_1)\frac{\delta(j_1-j_2)}{x_{12}^{2j_1}}\right] \,, \tag{A.3}$$

with

$$\Delta_j = -\frac{j(j-1)}{k-2} \,, \tag{A.4}$$

and where

$$B(j) = -\frac{\nu^{1-2j}}{\pi b^2}\gamma(1-b^2(2j-1)) \,, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-\bar{x})} \,, \quad b^2 = (k-2)^{-1} \,. \tag{A.5}$$

The parameter ν is fixed holographically as in [9, 17, 19][7] as

$$\nu = \frac{2\pi^5}{b^4\gamma(1+b^2)}. \tag{A.6}$$

Recall that $k = n_5 + 2$ and $k' = n_5 - 2$ in the supersymmetric context. The function $B(j)$ defines the coefficient $\mathcal{N}(j)$ appearing in the series identifications (2.23),

$$\mathcal{N}(j) = \sqrt{\frac{B(j)}{B(k/2-j)}}. \tag{A.7}$$

[7] See footnote 4 in [19].

For short-string states such as those considered in this paper, only the second term in the two-point function (A.3) contributes. Finally, the coefficient c_ν appearing in eqs. (4.36) and (4.37) takes the form

$$c_\nu = \frac{\pi\gamma(1-b^2)}{\nu b^2}. \tag{A.8}$$

As for the three-point functions, in the SU(2) case we have

$$\langle W_{l_1}(u_1,z_1)W_{l_2}(u_2,z_2)W_{l_3}(u_3,z_3)\rangle = C'(l_1,l_2,l_3)\frac{u_{12}^{l_1+l_2-l_3}u_{23}^{l_2+l_3-l_1}u_{13}^{l_1+l_3-l_2}}{z_{12}^{\Delta_1'+\Delta_2'-\Delta_3'}z_{23}^{\Delta_2'+\Delta_3'-\Delta_1'}z_{13}^{\Delta_1'+\Delta_3'-\Delta_2'}}\,, \tag{A.9}$$

with

$$C'(l_1,l_2,l_3) = \sqrt{\gamma(b'^2)}P(l+1)\prod_{i=1}^{3}\frac{P(l-2l_i)}{P(2l_i)\sqrt{\gamma(b'^2(2l_i+1))}}\,, \tag{A.10}$$

where $l = l_1+l_2+l_3$, $b'^2 = (k'+2)^{-1}$ and

$$P(l) = \prod_{n=1}^{l}\gamma(nb^2)\,, \qquad P(0)=1. \tag{A.11}$$

These expressions hold if $2l_i \leq l \leq k'$ and $l \in 2\mathbb{Z}$, otherwise the three-point function vanishes. In the SL(2,$\mathbb{R}$) case the three-point function is given by

$$\langle V_{j_1}(x_1,z_1)V_{j_2}(x_2,z_2)V_{j_3}(x_3,z_3)\rangle = C(j_1,j_2,j_3)\frac{x_{12}^{j_3-j_1-j_2}x_{23}^{j_1-j_2-j_3}x_{13}^{j_2-j_1-j_3}}{z_{12}^{\Delta_1'+\Delta_2'-\Delta_3'}z_{23}^{\Delta_2'+\Delta_3'-\Delta_1'}z_{13}^{\Delta_1'+\Delta_3'-\Delta_2'}}\,, \tag{A.12}$$

with, following the conventions of [9],

$$C(j_1,j_2,j_3) = -\frac{b^{1+b^2}\Upsilon(b)}{2\pi^2\gamma(1+b^2)}\frac{\left(\nu b^{2b^2}\right)^{1-j}}{\Upsilon(b(j-1))}\prod_{i}^{3}\frac{\Upsilon\left(b(2j_i-1)\right)}{\Upsilon\left(b(j-2j_i)\right)} \tag{A.13}$$

where $j = j_1+j_2+j_3$, and the upsilon function $\Upsilon(x)$ has an integral representation as[8]

$$\ln\left(\Upsilon(x)\right) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left(\left(\frac{q}{2}-x\right)\frac{t}{2}\right)}{\sinh\left(\frac{bt}{2}\right)\sinh\left(\frac{t}{2b}\right)}\right],\,, \qquad q = b+b^{-1}\,, \tag{A.14}$$

if $0 < \mathrm{Re}(x) < q$, and it is extended outside this range by means of the following properties:

$$\Upsilon(x+b) = b^{1-2bx}\gamma\left(bx\right)\Upsilon(x)\,, \qquad \Upsilon(x+\frac{1}{b}) = b^{-1+\frac{2x}{b}}\gamma\left(\frac{x}{b}\right)\Upsilon(x). \tag{A.15}$$

As pointed out in [9], a crucial simplification occurs for the product of SU(2) and SL(2,$\mathbb{R}$) structure constants appearing in (unflowed) supersymmetric short-string correlators. More explicitly, we have

$$\mathcal{C}(j_i) \equiv C(j_i)C'(j_i-1) = \sqrt{\frac{b^2\gamma(-b^2)}{4\pi\nu}}\prod_{i=1}^{3}\sqrt{B(j_i)} \equiv Q\prod_{i=1}^{3}\sqrt{B(j_i)}. \tag{A.16}$$

[8]The relation between the upsilon function and the function G used in [16] can be found in [53].

Furthermore, the product $\mathcal{C}(j_i)$ satisfies the identities

$$\mathcal{N}(j_1)\mathcal{C}(k/2-j_1,j_2,j_3)=\mathcal{N}(j_2)\mathcal{C}(j_1,k/2-j_2,j_3)=\mathcal{N}(j_3)\mathcal{C}(j_1,j_2,k/2-j_3). \tag{A.17}$$

This ensures that the three-point functions in eq. (3.6) satisfy have the correct exchange symmetry.

The spectrum of the SL(2,ℝ) model also contains states in spectrally-flowed representations. The two-point functions of the corresponding operators can be obtained by means of the so-called parafermion decomposition, giving [16]

$$\langle V^{\omega}_{j_1h_1}(x_1,z_1)V^{\omega}_{j_2h_2}(x_2,z_2)\rangle=\frac{\delta^2(h_1-h_2)}{z_{12}^{2\Delta_1}x_{12}^{2h_1}}\left[\delta(j_1+j_2-1)+\frac{\pi\delta(j_1-j_2)B(j_1)\gamma(j_1+m_1)}{\gamma(2j_1)\gamma(1-j_1+m_1)}\right]. \tag{A.18}$$

Recall that for a given vertex $V^{\omega}_{jh}(x,z)$, m is defined in terms of the spacetime weight and the spectral flow charge as $m=h-k\omega/2$. As in the unflowed sector, the first term in (A.18) is irrelevant for our short-string vertex operators. Three-point functions of vertex operators with $\omega>0$ are much more involved, and were derived only recently [20–23]. We have reviewed these results in section 3, see eqs. (3.6) and (3.8). The corresponding normalizations include the factors

$$\tilde{N}_{\text{even}}(j_i,\omega_i)=P^{j_1+j_2+j_3-k}_{(\omega_1,\omega_2,\omega_3)}P^{j_3-j_2-j_1}_{(\omega_1+1,\omega_2+1,\omega_3)}P^{j_1-j_2-j_3}_{(\omega_1,\omega_2+1,\omega_3+1)}P^{j_2-j_3-j_1}_{(\omega_1+1,\omega_2,\omega_3+1)} \tag{A.19}$$

and

$$\tilde{N}_{\text{odd}}(j_i,\omega_i)=\left(\frac{P_{(\omega_1-1,\omega_2-1,\omega_3-1)}}{\omega_1+\omega_2+\omega_3-1}\right)^{\frac{k}{2}-j_1-j_2-j_3}P^{j_3+j_2-j_1-\frac{k}{2}}_{(\omega_1-1,\omega_2,\omega_3)}P^{j_3-j_2+j_1-\frac{k}{2}}_{(\omega_1,\omega_2-1,\omega_3)}P^{-j_3+j_2+j_1-\frac{k}{2}}_{(\omega_1,\omega_2,\omega_3-1)}, \tag{A.20}$$

where

$$P_{\boldsymbol{\omega}}=0 \qquad \text{for} \qquad \sum_j\omega_j<2\max_{i=1,2,3}\omega_i \quad \text{or} \quad \sum_i\omega_i\in 2\mathbb{Z}+1\,, \tag{A.21}$$

with $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$, otherwise

$$P_{\boldsymbol{\omega}}=S_{\boldsymbol{\omega}}\frac{G\left(\frac{-\omega_1+\omega_2+\omega_3}{2}+1\right)G\left(\frac{\omega_1-\omega_2+\omega_3}{2}+1\right)G\left(\frac{\omega_1+\omega_2-\omega_3}{2}+1\right)G\left(\frac{\omega_1+\omega_2+\omega_3}{2}+1\right)}{G(\omega_1+1)G(\omega_2+1)G(\omega_3+1)}, \tag{A.22}$$

where $G(n)$ is the Barnes G function

$$G(n)=\prod_{i=1}^{n-1}\Gamma(i) \tag{A.23}$$

for positive integer values, while $S_{\boldsymbol{\omega}}$ is a phase depending on $\boldsymbol{\omega}$ mod 2. For more details, see [21]. We also need spectrally-flowed correlators in the SU(2) sector. Two-point functions read

$$\langle W^{\omega_1}_{l_1}(u_1,z_1)W^{\omega_2}_{l_2}(u_2,z_2)\rangle=\delta_{l_1,l_2}\delta_{\omega_1,\omega_2}\frac{u_{12}^{2l_1}}{z_{12}^{2\Delta'_1}}\,, \tag{A.24}$$

where we have restricted to the vertex operators relevant for this paper, i.e. those appearing in supersymmetric spectrally-flowed short-string states, built from unflowed lowest-weight

states. The extension of the methods of [20–23] for three-point functions to the SU(2) case was also given in section 3, leading to eqs. (3.12) and (3.13). In that context, one obtains the normalization factors

$$\tilde{N}'_{\rm even}(l_i,\omega_i) = P^{-l_1-l_2-l_3+k'}_{(\omega_1,\omega_2,\omega_3)} P^{-l_3+l_2+l_1}_{(\omega_1+1,\omega_2+1,\omega_3)} P^{-l_1+l_2+l_3}_{(\omega_1,\omega_2+1,\omega_3+1)} P^{-l_2+l_3+l_1}_{(\omega_1+1,\omega_2,\omega_3+1)} \tag{A.25}$$

and

$$\tilde{N}'_{\rm odd}(l_i,\omega_i) = \left(\frac{P_{(\omega_1-1,\omega_2-1,\omega_3-1)}}{\omega_1+\omega_2+\omega_3-1}\right)^{-\frac{k'}{2}+l_1+l_2+l_3} P^{-l_3-l_2+l_1+\frac{k'}{2}}_{(\omega_1-1,\omega_2,\omega_3)} P^{-l_3+l_2-l_1+\frac{k'}{2}}_{(\omega_1,\omega_2-1,\omega_3)} P^{l_3-l_2-l_1+\frac{k'}{2}}_{(\omega_1,\omega_2,\omega_3-1)} . \tag{A.26}$$

Supersymmetric short-string three-point functions are greatly simplified by the relation between SL(2,$\mathbb{R}$) and SU(2) spins $l_i = j_i - 1$. Indeed, the product between the normalizations $\tilde{N}_{\rm odd/even}$ and $\tilde{N}'_{\rm odd/even}$ leads to several cancellations, and it even becomes spin-independent:

$$\tilde{N}_{\rm odd}(j_i,\omega_i)\tilde{N}'_{\rm odd}(j_i-1,\omega_i) \equiv \mathcal{N}_{\rm odd} = \left[\frac{P(\omega_1-1,\omega_2-1,\omega_3-1)\prod_i^3 P(\boldsymbol{\omega}-\hat{e}_i)}{\omega_1+\omega_2+\omega_3-1}\right]^{-1}, \tag{A.27}$$

$$\tilde{N}_{\rm even}(j_i,\omega_i)\tilde{N}'_{\rm even}(j_i-1,\omega_i) \equiv \mathcal{N}_{\rm even} = \left[P(\omega_1,\omega_2,\omega_3)\prod_{i<j} P(\boldsymbol{\omega}+\hat{e}_i+\hat{e}_j)\right]^{-1}, \tag{A.28}$$

where $\hat{e}_1 = (1,0,0)$, $\hat{e}_2 = (0,1,0)$ and $\hat{e}_3 = (0,0,1)$. This mirrors the simplifications for the product of the SL(2, $\mathbb{R}$) and SU(2) unflowed three-point functions highlighted in eq. (A.16) above. These identities are crucial for the analysis of section 4.

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